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Some Extended Berezin Number Inequalities

Satyajit Sahoo[a], Mojtaba Bakherad[b]

[a] *P.G. Department of Mathematics, Utkal University, Vanivihar, Bhubaneswar-751004, India.*
[b] *Department of Mathematics, Faculty of Mathematics, University of Sistan and Baluchestan, Zahedan, Iran.*

Abstract. We present generalized extensions of Berezin number inequalities involving the Euclidean Berezin number and $f-$connection of operators.

1. Introduction

Let $\mathcal{H}$ be a complex Hilbert space with an inner product $\langle\cdot,\cdot\rangle$ and the corresponding norm $\|\cdot\|$. Let $\mathcal{L}(\mathcal{H})$ be the C^*-algebra of all bounded linear operators from $\mathcal{H}$ into itself. In the case when $\dim\mathcal{H} = n$, we identify $\mathcal{L}(\mathcal{H})$ with the matrix algebra $\mathbb{M}_n$ of all $n \times n$ complex matrices. An operator $A \in \mathcal{L}(\mathcal{H})$ is said to be positive, and denoted $A \geq 0$, if $\langle Ax, x\rangle \geq 0$ for all $x \in \mathcal{H}$.

The *numerical range* of $T \in \mathcal{L}(\mathcal{H})$ is defined as

$$W(T) = \{\langle Tx, x\rangle : x \in \mathcal{H}, \|x\| = 1\}$$

and the *numerical radius* of T, denoted by $w(T)$, is defined by $w(T) = \sup\{|z| : z \in W(T)\}$.

It is well-known that the set $W(T)$ is a convex subset of the complex plane and that the numerical radius $w(\cdot)$ is a norm on $\mathcal{L}(\mathcal{H})$; being equivalent to the usual operator norm $\|T\| = \sup\{\|Tx\| : x \in \mathcal{H}, \|x\| = 1\}$. In fact, for every $T \in \mathcal{L}(\mathcal{H})$,

$$\frac{1}{2}\|T\| \leq w(T) \leq \|T\|. \tag{1}$$

Obtaining sharper lower and upper bounds of (1) have attracted numerous researchers due to its applications in the operator theory and other fields. For example, bounds for the zeros of polynomials is an interesting application of the numerical radius inequalities (see [7]). We refer the reader to [9, 11, 18, 23, 24, 28] as a sample of references treating numerical radius inequalities.
Another interesting set of applications of the quantity $w(A)$ includes the study of perturbation, convergence and approximation problems as well as iterative methods, etc; [2].

A Hilbert space $\mathcal{H} = \mathcal{H}(\Omega)$ of complex valued functions on a nonempty open set $\Omega \subset \mathbb{C}$ which has the property that point evaluations are continuous, is called a *functional Hilbert space*. The point evaluations are

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Communicated by Fuad Kittaneh
Corresponding author: Satyajit Sahoo
Email addresses: satyajitsahoo2010@gmail.com (Satyajit Sahoo), mojtaba.bakherad@yahoo.com (Mojtaba Bakherad)

continuous means for each $\lambda \in \Omega$, the map $f \longmapsto f(\lambda)$ is a continuous linear functional on $\mathcal{H}$. For each $\lambda \in \Omega$, there is a unique element k_λ of $\mathcal{H}$ such that $f(\lambda) = \langle f, k_\lambda \rangle$ for all $f \in \mathcal{H}$ by Riesz representation theorem. The collection $\{k_\lambda : \lambda \in \Omega\}$ is known as the *reproducing kernel* of $\mathcal{H}$. Problem 37 of [14] states that the reproducing kernel of a functional Hilbert space $\mathcal{H}$ with $\{e_n\}$ as an orthonormal basis is $k_\lambda(z) = \sum_n \overline{e_n(\lambda)} e_n(z)$. Let $\hat{k}_\lambda = k_\lambda / \|k_\lambda\|$ be the nomalized reproducing kernel of $\mathcal{H}$, where $\lambda \in \Omega$. The function $\tilde{A}$ defined on Ω by $\tilde{A}(\lambda) = \langle A\hat{k}_\lambda, \hat{k}_\lambda \rangle$ is the *Berezin symbol* of a bounded linear operator A on $\mathcal{H}$. The *Berezin set* and the *Berezin number* of the operator A are defined by

$$\mathrm{Ber}(A) = \{\tilde{A}(\lambda) : \lambda \in \Omega\} \text{ and } \mathrm{ber}(A) = \sup\{|\tilde{A}(\lambda)| : \lambda \in \Omega\},$$

respectively. These definitions are named in honor of Felix Berezin, who introduced these notions in [8]. For our purpose, we set the *Berezin norm* of an operator as $\|A\|_{ber} = \sup\{|\langle A\hat{k}_{\lambda_1}, \hat{k}_{\lambda_2} \rangle| : \lambda_1, \lambda_2 \in \Omega\}$. Clearly, the Berezin symbol $\tilde{A}$ is a bounded function on Ω whose values lie in the numerical range of the operator A, and hence

$$\mathrm{Ber}(A) \subseteq W(A) \text{ and } \mathrm{ber}(A) \leq w(A).$$

The Berezin number of an operator T satisfies the following properties:
(i) $\mathrm{ber}(\beta T) = |\beta| \mathrm{ber}(T)$ for all $\beta \in \mathbb{C}$.
(ii) $\mathrm{ber}(T + S) \leq \mathrm{ber}(T) + \mathrm{ber}(S)$.
Let $T_i \in \mathcal{L}(\mathcal{H}(\Omega)), 1 \leq i \leq n$. Bakherad [4] then introduced the concept of generalized Euclidean Berezin number of $T_1, \ldots, T_n$ as

$$\mathrm{ber}_r(T_1, \ldots, T_n) = \sup_{\lambda \in \Omega} \left(\sum_{i=1}^{n} |\langle T_i \hat{k}_\lambda, \hat{k}_\lambda \rangle|^r \right)^{1/r}.$$

The generalized Euclidean Berezin number ber_r, $r \geq 1$, has the following properties:
(i) $\mathrm{ber}_r(\beta T_1, \ldots, \beta T_n) = |\beta| \, \mathrm{ber}_r(T_1, \ldots, T_n)$ for all $\beta \in \mathbb{C}$;
(ii) $\mathrm{ber}_r(T_1 + S_1, \ldots, T_n + S_n) \leq \mathrm{ber}_r(T_1, \ldots, T_n) + \mathrm{ber}_r(S_1, \ldots, S_n)$,
where $T_i, S_i \in \mathcal{L}(\mathcal{H}(\Omega)), 1 \leq i \leq n$.

The Berezin symbol has been studied in detail for Toeplitz and Hankel operators on Hardy and Bergman spaces. A nice property of the Berezin symbol is mentioned next. If $\tilde{A}(\lambda) = \tilde{B}(\lambda)$ for all $\lambda \in \Omega$, then $A = B$. Therefore, the Berezin symbol uniquely determines the operator. Some excellent results about the Berezin number were found in [4, 5, 13, 25–27] very recently.

Among many techniques in obtaining numerical radius and Berezin number inequalities is the study of certain scalar ones. For example, the classical Young inequality which states that if $a, b \geq 0$ and $0 \leq \beta \leq 1$, then

$$a^\beta b^{1-\beta} \leq \beta a + (1 - \beta) b, \tag{2}$$

is an example of such important scalar inequalities.
During the last decades several generalizations, reverses, refinements and applications of the Young inequality in various setting have been given, see [3, 19–21] and the references therein. A refinement of inequality (2) is presented by Kittaneh and Manasrah [19] as follows:

$$a^\beta b^{1-\beta} \leq \beta a + (1 - \beta) b - r_0 (a^{\frac{1}{2}} - b^{\frac{1}{2}})^2, \text{ where } r_0 = \min\{\beta, 1 - \beta\}. \tag{3}$$

Later, the same authors in [1] presented the general form of (3) as follows:

$$(a^\beta b^{1-\beta})^m + r_0^m (a^{\frac{m}{2}} - b^{\frac{m}{2}})^2 \leq (\beta a + (1 - \beta) b)^m, \text{ where } r_0 = \min\{\beta, 1 - \beta\} \tag{4}$$

and for any positive integer m. Recently, Choi [10] gave a further refinement of the Young inequality as follows:

$$(a^\beta b^{1-\beta})^m + (2r_0)^m \left(\left(\frac{a + b}{2} \right)^m - (ab)^{\frac{m}{2}} \right) \leq (\beta a + (1 - \beta) b)^m, \tag{5}$$

$$(a^\beta b^{1-\beta})^m + (2R_0)^m \left(\left(\frac{a + b}{2} \right)^m - (ab)^{\frac{m}{2}} \right) \geq (\beta a + (1 - \beta) b)^m, \tag{6}$$

where $r_0 = \min\{\beta, 1-\beta\}$ and $R_0 = \max\{\beta, 1-\beta\}$.
We refer the reader also to [22, Section 2.4] for more elaboration on this refinement.

We know from [15] that for $0 \le \beta \le 1$ and $r \ge 1$,

$$\beta a + (1-\beta)b \le (\beta a^r + (1-\beta)b^r)^{\frac{1}{r}}. \tag{7}$$

It follows from (7) and inequality (5) that

$$(a^\beta b^{1-\beta})^m + (2r_0)^m \left(\left(\frac{a+b}{2} \right)^m - (ab)^{\frac{m}{2}} \right) \le (\beta a^r + (1-\beta)b^r)^{\frac{m}{r}}, \tag{8}$$

where $r_0 = \min\{\beta, 1-\beta\}$. In particular, for $\beta = \frac{1}{2}$, we get

$$(a^{\frac{1}{2}} b^{\frac{1}{2}})^m + \left(\left(\frac{a+b}{2} \right)^m - (ab)^{\frac{m}{2}} \right) \le \frac{1}{2^{\frac{m}{r}}} (a^r + b^r)^{\frac{m}{r}}. \tag{9}$$

In 1952, Kato [16] showed the mixed Schwarz inequality, which asserts

$$|\langle Ax, y\rangle|^2 \le \left\langle |A|^{2\beta} x, x \right\rangle \left\langle |A^*|^{2(1-\beta)} y, y \right\rangle, \qquad 0 \le \beta \le 1, \tag{10}$$

for the operator $A \in \mathcal{L}(\mathcal{H})$ and the vectors $x, y \in \mathcal{H}$, where $|A| = (A^*A)^{1/2}$.

The objective of this paper is to present some results of Berezin number inequalities involving f-connection of operators. Finally, we present a generalized Euclidean Berezin number inequality and refine the inequality (13).

Many related results that extend known results from the literature will be presented with an emphasize on the relation with known results in the literature. The first needed inequality is the following generalization of the mixed Cauchy-Schwarz inequality [17, Theorem 1].

Lemma 1.1. *Let $A \in \mathcal{L}(\mathcal{H})$ and let f and g be non-negative continuous functions on $[0, \infty)$ satisfying the identity $f(t)g(t) = t$ for all $t \in [0, \infty)$. Then*

$$|\langle Ax, y\rangle| \le \|f(|A|x)\| \|g(|A^*|)y\|$$

for all x, y in $\mathcal{H}$.

When dealing with inner product inequalities, the following inequality becomes handy [12, Theorem 1.2]:

$$f(\langle Ax, x\rangle) \le \langle f(A)x, x\rangle, \tag{11}$$

valid for the convex function $f : J \to \mathbb{R}$, the self adjoint operator A with spectrum in J and the unit vector $x \in \mathcal{H}$. The inequality (11) is reversed when f is concave. As a consequence of this inequality, we obtain the following celebrated McCarthy inequality.

Lemma 1.2. *Let $T \in \mathcal{L}(\mathcal{H})$, $T \ge 0$ and $x \in \mathcal{H}$ be a unit vector. Then*
(i) $\langle Tx, x\rangle^r \le \langle T^r x, x\rangle$ for $r \ge 1$;
(ii) $\langle T^r x, x\rangle \le \langle Tx, x\rangle^r$ for $0 < r \le 1$.

2. Main Results

2.1. Berezin number inequalities for f−connections of operators

For positive definite operators $T, S \in \mathcal{L}(\mathcal{H})$, the operator geometric mean is defined by

$$T \sharp S = T^{1/2}(T^{-1/2} S T^{-1/2})^{1/2} T^{1/2}.$$

Let f be a continuous function defined on the real interval J containing the spectrum of $T^{-1/2}ST^{-1/2}$, where S is a self-adjoint operator and T is a positive invertible operator. By using the continuous functional calculus, the f-connection σ_f is defined as follows

$$T\sigma_f S = T^{1/2}f(T^{-1/2}ST^{-1/2})T^{1/2}. \tag{12}$$

Note that for the functions $(1-\beta)+\beta t$ and t^{β}, the definition (12) leads to the arithmetic and geometric operator means, respectively; see [12]. The aim of this subsection is to extend and generalize main result of [6, Theorem 2].

Theorem 2.1. *Let $T, S, X \in \mathcal{L}(\mathcal{H})$ be such that T, S are positive invertible. Then for $m \in \mathbb{N}$ and $r \geq 1$,*

$$ber^m((T\sigma_f S)X) \leq 2^{-m/r} ber^{m/r}((X^*T^{1/2}f^2(T^{-1/2}ST^{-1/2})T^{1/2}X)^r + T^r) - \inf_{\lambda\in\Omega}\xi(\hat{k}_\lambda),$$

where

$$\xi(\hat{k}_\lambda) = \left\langle \frac{1}{2}(X^*T^{1/2}f^2(T^{-1/2}ST^{-1/2})T^{1/2}X + T)\hat{k}_\lambda, \hat{k}_\lambda \right\rangle^m - (\langle X^*T^{1/2}f^2(T^{-1/2}ST^{-1/2})T^{1/2}X\hat{k}_\lambda, \hat{k}_\lambda\rangle\langle T\hat{k}_\lambda, \hat{k}_\lambda\rangle)^{m/2}.$$

Proof. We have

$$\begin{aligned}
&|\langle (T\sigma_f S)X\hat{k}_\lambda, \hat{k}_\lambda\rangle|^m \\
&= |\langle T^{1/2}f(T^{-1/2}ST^{-1/2})T^{1/2}X\hat{k}_\lambda, \hat{k}_\lambda\rangle|^m \\
&= |\langle f(T^{-1/2}ST^{-1/2})T^{1/2}X\hat{k}_\lambda, T^{1/2}\hat{k}_\lambda\rangle|^m \\
&\leq \|f(T^{-1/2}ST^{-1/2})T^{1/2}X\hat{k}_\lambda\|^m \|T^{1/2}\hat{k}_\lambda\|^m \\
&= \left(\langle f(T^{-1/2}ST^{-1/2})T^{1/2}X\hat{k}_\lambda, f(T^{-1/2}ST^{-1/2})T^{1/2}X\hat{k}_\lambda\rangle^{1/2}\langle T^{1/2}\hat{k}_\lambda, T^{1/2}\hat{k}_\lambda\rangle^{1/2}\right)^m \\
&= \left(\langle X^*T^{1/2}f^2(T^{-1/2}ST^{-1/2})T^{1/2}X\hat{k}_\lambda, \hat{k}_\lambda\rangle^{1/2}\langle T\hat{k}_\lambda, \hat{k}_\lambda\rangle^{1/2}\right)^m \\
&\leq 2^{\frac{-m}{r}}\left(\langle X^*T^{1/2}f^2(T^{-1/2}ST^{-1/2})T^{1/2}X\hat{k}_\lambda, \hat{k}_\lambda\rangle^r + \langle T\hat{k}_\lambda, \hat{k}_\lambda\rangle^r\right)^{\frac{m}{r}} \\
&\quad - \left\{\left(\frac{\langle X^*T^{1/2}f^2(T^{-1/2}ST^{-1/2})T^{1/2}X\hat{k}_\lambda, \hat{k}_\lambda\rangle + \langle T\hat{k}_\lambda, \hat{k}_\lambda\rangle}{2}\right)^m \right.\\
&\quad \left. - (\langle X^*T^{1/2}f^2(T^{-1/2}ST^{-1/2})T^{1/2}X\hat{k}_\lambda, \hat{k}_\lambda\rangle\langle T\hat{k}_\lambda, \hat{k}_\lambda\rangle)^{m/2}\right\} \\
&\leq 2^{\frac{-m}{r}}\langle((X^*T^{1/2}f^2(T^{-1/2}ST^{-1/2})T^{1/2}X)^r + T^r)\hat{k}_\lambda, \hat{k}_\lambda\rangle^{m/r} \\
&\quad - \left\{\left(\frac{\langle (X^*T^{1/2}f^2(T^{-1/2}ST^{-1/2})T^{1/2}X + T)\hat{k}_\lambda, \hat{k}_\lambda\rangle}{2}\right)^m \right.\\
&\quad \left. - (\langle X^*T^{1/2}f^2(T^{-1/2}ST^{-1/2})T^{1/2}X\hat{k}_\lambda, \hat{k}_\lambda\rangle\langle T\hat{k}_\lambda, \hat{k}_\lambda\rangle)^{m/2}\right\}.
\end{aligned}$$

Taking supremum over $\lambda \in \Omega$, we get the desired result. □

Putting $m = 1 = r$ in Theorem 2.1 and using the fact that $ber(T) \leq \|T\|$, we get the following result as follows.

Corollary 2.2. *Let $T, S, X \in \mathcal{L}(\mathcal{H})$ be such that T, S are positive invertible. Then*

$$ber((T\sigma_f S)X) \leq \frac{1}{2}ber(X^*T^{1/2}f^2(T^{-1/2}ST^{-1/2})T^{1/2}X + T) - \inf_{\lambda\in\Omega}\xi(\hat{k}_\lambda),$$

where

$$\xi(\hat{k}_\lambda) = \left\langle \frac{X^*T^{1/2}f^2(T^{-1/2}ST^{-1/2})T^{1/2}X + T)}{2}\hat{k}_\lambda, \hat{k}_\lambda \right\rangle - (\langle X^*T^{1/2}f^2(T^{-1/2}ST^{-1/2})T^{1/2}X\hat{k}_\lambda, \hat{k}_\lambda\rangle\langle T\hat{k}_\lambda, \hat{k}_\lambda\rangle)^{1/2}.$$

Taking $f(t) = t^{1/2}$ and $m = 1 = r$ in Theorem 2.1, obtain the following result.

Corollary 2.3. *Let $T, S, X \in \mathcal{L}(\mathcal{H})$ be such that T, S are positive invertible. Then*

$$ber((T \sharp S)X) \leq \frac{1}{2} ber(X^*SX + T) - \inf_{\lambda \in \Omega} \xi(\hat{k}_\lambda),$$

where

$$\xi(\hat{k}_\lambda) = \left\langle \frac{(X^*SX + T)}{2}\hat{k}_\lambda, \hat{k}_\lambda \right\rangle - (\langle (X^*SX)\hat{k}_\lambda, \hat{k}_\lambda\rangle\langle T\hat{k}_\lambda, \hat{k}_\lambda\rangle)^{1/2}.$$

Taking $X = I$ in Theorem 2.1 we get the following.

Corollary 2.4. *Let $T, S \in \mathcal{L}(\mathcal{H})$ be positive invertible. Then for $m \in \mathbb{N}, r \geq 1$*

$$\|T\sigma_f S\|_{ber}^m \leq 2^{-m/r} \|(T^{1/2}f^2(T^{-1/2}ST^{-1/2})T^{1/2})^r + T^r\|_{ber}^{m/r} - \inf \xi(\hat{k}_\lambda),$$

where

$$\xi(\hat{k}_\lambda) = \left\langle \frac{(T^{1/2}f^2(T^{-1/2}ST^{-1/2})T^{1/2} + T)}{2}\hat{k}_\lambda, \hat{k}_\lambda \right\rangle^m - (\langle T^{1/2}f^2(T^{-1/2}ST^{-1/2})T^{1/2}\hat{k}_\lambda, \hat{k}_\lambda\rangle\langle T\hat{k}_\lambda, \hat{k}_\lambda\rangle)^{m/2}.$$

Taking $r = 1$, $X = I$ and $f(t) = t^{1/2}$, we have the following simplified form.

Corollary 2.5. *Let $T, S \in \mathcal{L}(\mathcal{H})$ be such that T, S are positive invertible. Then for $m \in \mathbb{N}$,*

$$\|T \sharp S\|_{ber}^m \leq 2^{-m} \|S + T\|_{ber}^m - \inf_{\lambda \in \Omega} \xi(\hat{k}_\lambda),$$

where

$$\xi(\hat{k}_\lambda) = \left\langle \frac{S+T}{2}\hat{k}_\lambda, \hat{k}_\lambda \right\rangle^m - \langle S\hat{k}_\lambda, \hat{k}_\lambda\rangle^{m/2}\langle T\hat{k}_\lambda, \hat{k}_\lambda\rangle^{m/2}.$$

Proposition 2.6. *Let $T, S, X \in \mathcal{L}(\mathcal{H})$ such that $T, S > 0$ and $r > 1, m \in \mathbb{N}$. Then*

$$\|(T\sigma_f S)X\|_{ber}^m \leq 2^{-m/r}(\|X^*T^{1/2}f^2(T^{-1/2}ST^{-1/2})T^{1/2}X\|_{ber}^r + \|T\|_{ber}^r)^{m/r} - \inf_{\lambda,\ \mu \in \Omega} \xi(\hat{k}_\lambda, \hat{k}_\mu),$$

where

$$\xi(\hat{k}_\lambda, \hat{k}_\mu) = \left(\frac{\langle X^*T^{1/2}f^2(T^{-1/2}ST^{-1/2})T^{1/2}\hat{k}_\lambda\hat{k}_\lambda, \hat{k}_\lambda\rangle + \langle T\hat{k}_\mu, \hat{k}_\mu\rangle}{2} \right)^m - \left(\langle X^*T^{1/2}f^2(T^{-1/2}ST^{-1/2})T^{1/2}X\hat{k}_\lambda, \hat{k}_\lambda\rangle\langle T\hat{k}_\mu, \hat{k}_\mu\rangle\right)^{m/2}.$$

Proof. Let $\hat{k}_\lambda, \hat{k}_\mu \in \mathcal{H}(\Omega)$, then

$$\begin{aligned}
|\langle (T\sigma_f S)X\hat{k}_\lambda, \hat{k}_\mu\rangle|^m &= |\langle T^{1/2}f(T^{-1/2}ST^{-1/2})T^{1/2}X\hat{k}_\lambda, \hat{k}_\mu\rangle|^m \\
&= |\langle f(T^{-1/2}ST^{-1/2})T^{1/2}X\hat{k}_\lambda, T^{1/2}\hat{k}_\mu\rangle|^m \\
&\leq \langle X^*T^{1/2}f^2(T^{-1/2}ST^{-1/2})T^{1/2}X\hat{k}_\lambda, \hat{k}_\lambda\rangle^{m/2}\langle T\hat{k}_\mu, \hat{k}_\mu\rangle^{m/2}.
\end{aligned}$$

Using similar technique as in Theorem 2.1, we get

$$\langle (T\sigma_f S)X\hat{k}_\lambda, \hat{k}_\mu\rangle^m \le 2^{-m/r}(\langle X^*T^{1/2}f^2(T^{-1/2}ST^{-1/2})T^{1/2}X\hat{k}_\lambda, \hat{k}_\lambda\rangle^r + \langle T\hat{k}_\mu, \hat{k}_\mu\rangle^r)^{m/r} - \left\{\left(\frac{\langle X^*T^{1/2}f^2(T^{-1/2}ST^{-1/2})T^{1/2}X\hat{k}_\lambda, \hat{k}_\lambda\rangle + \langle T\hat{k}_\mu, \hat{k}_\mu\rangle}{2}\right)^m - (\langle X^*T^{1/2}f^2(T^{-1/2}ST^{-1/2})T^{1/2}X\hat{k}_\lambda, \hat{k}_\lambda\rangle\langle T\hat{k}_\mu, \hat{k}_\mu\rangle)^{m/2}\right\}$$

Taking supremum over $\lambda, \mu \in \Omega$, we have

$$\|(T\sigma_f S)X\|^m \le 2^{-m/r}(\|X^*T^{1/2}f^2(T^{-1/2}ST^{-1/2})T^{1/2}X\|^r + \|T\|^r)^{m/r} - \inf_{\lambda,\ \mu\in\Omega} \xi(\hat{k}_\lambda, \hat{k}_\mu),$$

where

$$\xi(\hat{k}_\lambda, \hat{k}_\mu) = \left(\frac{\langle X^*T^{1/2}f^2(T^{-1/2}ST^{-1/2})T^{1/2}X\hat{k}_\lambda, \hat{k}_\lambda\rangle + \langle T\hat{k}_\mu, \hat{k}_\mu\rangle}{2}\right)^m - \left(\langle X^*T^{1/2}f^2(T^{-1/2}ST^{-1/2})T^{1/2}X\hat{k}_\lambda, \hat{k}_\lambda\rangle\langle T\hat{k}_\mu, \hat{k}_\mu\rangle\right)^{m/2}.$$

□

In particular, letting $f(t) = t^{1/2}$, $m = 1, 2, ...$ we have the following simplified form.

Corollary 2.7. *Let $T, S, X \in \mathcal{L}(\mathcal{H})$ such that $T, S > 0$ and let $r > 1$. Then*

$$\|(T\sharp S)X\|_{ber}^m \le 2^{-m/r}(\|X^*SX\|_{ber}^r + \|T\|_{ber}^r)^{m/r} - \inf_{\lambda,\ \mu\ \in\Omega}\left\{\left(\frac{\langle X^*SX\hat{k}_\lambda, \hat{k}_\lambda\rangle + \langle T\hat{k}_\mu, \hat{k}_\mu\rangle}{2}\right)^m - (\langle X^*SX\hat{k}_\lambda, \hat{k}_\lambda\rangle\langle T\hat{k}_\mu, \hat{k}_\mu\rangle)^{m/2}\right\}.$$

2.2. Generalized Euclidean Berezin number inequalities.

In this subsection, we show our main results; starting with the generalized Euclidean Berezin number. Our first result is a generalized refinement of [6, Theorem 9].

Theorem 2.8. *Let $\mathcal{H} = \mathcal{H}(\Omega)$ be a reproducing kernel Hilbert space on Ω and $A_i, B_i, S_i \in \mathcal{L}(\mathcal{H})$ $(i = 1, 2, \ldots, n)$ and let f and g be non negative continuous functions on $[0, \infty)$ such that $f(t)g(t) = t$ for all $t \in [0, \infty)$. Then for $m = 1, 2, \ldots$ and $r, p \ge m$,*

$$ber_p^p(A_1^*S_1B_1, \ldots, A_n^*S_nB_n) \le \frac{n^{1-\frac{m}{r}}}{2^{\frac{m}{r}}} ber^{\frac{m}{r}}\Big(\sum_{i=1}^n([B_i^*f^2(|S_i|)B_i]^{\frac{pr}{m}} + [A_i^*g^2(|S_i^*|)A_i]^{\frac{pr}{m}})\Big) - \inf_{\lambda\in\Omega}\xi(\hat{k}_\lambda),$$

where

$$\xi(\hat{k}_\lambda) = \sum_{i=1}^n\left[\frac{1}{2^m}\big(\langle([B_i^*f^2(|S_i|)B_i]^{\frac{p}{m}} + [A_i^*g^2(|S_i^*|)A_i]^{\frac{p}{m}})\hat{k}_\lambda, \hat{k}_\lambda\rangle\big)^m - \big(\langle[B_i^*f^2(|S_i|)B_i]^{\frac{p}{m}}\hat{k}_\lambda, \hat{k}_\lambda\rangle\langle[A_i^*g^2(|S_i^*|)A_i]^{\frac{p}{m}}\hat{k}_\lambda, \hat{k}_\lambda\rangle\big)^{\frac{m}{2}}\right].$$

Proof. Let $\hat{k}_\lambda$ is the normalized reproducing kernel of $\mathcal{H}(\Omega)$, then

$$\begin{aligned}
&\sum_{i=1}^{n} |\langle A_i^* S_i B_i \hat{k}_\lambda, \hat{k}_\lambda\rangle|^p \\
&= \sum_{i=1}^{n} |\langle S_i B_i \hat{k}_\lambda, A_i \hat{k}_\lambda\rangle|^p \\
&\le \sum_{i=1}^{n} \|f(|S_i|) B_i \hat{k}_\lambda\|^p \|g(|S_i^*|) A_i \hat{k}_\lambda\|^p \quad \text{(by Lemma 1.1)} \\
&= \sum_{i=1}^{n} \langle f(|S_i|) B_i \hat{k}_\lambda, f(|S_i|) B_i \hat{k}_\lambda\rangle^{\frac{p}{2}} \langle g(|S_i^*|) A_i \hat{k}_\lambda, g(|S_i^*|) A_i \hat{k}_\lambda\rangle^{\frac{p}{2}} \\
&= \sum_{i=1}^{n} \left(\langle B_i^* f^2(|S_i|) B_i \hat{k}_\lambda, \hat{k}_\lambda\rangle^{\frac{p}{2m}} \langle A_i^* g^2(|S_i^*|) A_i \hat{k}_\lambda, \hat{k}_\lambda\rangle^{\frac{p}{2m}} \right)^m \\
&\le \sum_{i=1}^{n} \left(\langle [B_i^* f^2(|S_i|) B_i]^{\frac{p}{m}} \hat{k}_\lambda, \hat{k}_\lambda\rangle^{1/2} \langle [A_i^* g^2(|S_i^*|) A_i]^{\frac{p}{m}} \hat{k}_\lambda, \hat{k}_\lambda\rangle^{1/2} \right)^m \quad \text{(by Lemma 1.2)} \\
&\le \sum_{i=1}^{n} \left[\frac{1}{2} \left(\langle [B_i^* f^2(|S_i|) B_i]^{\frac{p}{m}} \hat{k}_\lambda, \hat{k}_\lambda\rangle^r + \langle [A_i^* g^2(|S_i^*|) A_i]^{\frac{p}{m}} \hat{k}_\lambda, \hat{k}_\lambda\rangle^r \right) \right]^{\frac{m}{r}} \\
&\quad - \sum_{i=1}^{n} \Bigg[\left(\frac{1}{2} \left(\langle [B_i^* f^2(|S_i|) B_i]^{\frac{p}{m}} \hat{k}_\lambda, \hat{k}_\lambda\rangle + \langle [A_i^* g^2(|S_i^*|) A_i]^{\frac{p}{m}} \hat{k}_\lambda, \hat{k}_\lambda\rangle \right) \right)^m \\
&\quad - \left(\langle [B_i^* f^2(|S_i|) B_i]^{\frac{p}{m}} \hat{k}_\lambda, \hat{k}_\lambda\rangle \langle [A_i^* g^2(|S_i^*|) A_i]^{\frac{p}{m}} \hat{k}_\lambda, \hat{k}_\lambda\rangle \right)^{m/2} \Bigg] \quad \text{(by (9))} \\
&\le \frac{n^{1-m/r}}{2^{m/r}} \left\langle \sum_{i=1}^{n} \left([B_i^* f^2(|S_i|) B_i]^{\frac{pr}{m}} + [A_i^* g^2(|S_i^*|) A_i]^{\frac{pr}{m}} \right) \hat{k}_\lambda, \hat{k}_\lambda \right\rangle^{\frac{m}{r}} \\
&\quad - \sum_{i=1}^{n} \Bigg[\langle \frac{1}{2} \left([B_i^* f^2(|S_i|) B_i]^{\frac{p}{m}} + [A_i^* g^2(|S_i^*|) A_i]^{\frac{p}{m}} \right) \hat{k}_\lambda, \hat{k}_\lambda\rangle^m \\
&\quad - \left(\langle [B_i^* f^2(|S_i|) B_i]^{\frac{p}{m}} \hat{k}_\lambda, \hat{k}_\lambda\rangle \langle [A_i^* g^2(|S_i^*|) A_i]^{\frac{p}{m}} \hat{k}_\lambda, \hat{k}_\lambda\rangle \right)^{\frac{m}{2}} \Bigg],
\end{aligned}$$

where the last inequality follows from (11), noting concavity of the mapping $t \mapsto t^{\frac{m}{r}}$, as we have $m \le r$. Taking the supremum over $\lambda \in \Omega$, we get the desired inequality. □

Letting $m = 1$ in Theorem 2.8, we get [6, Theorem 9].

Corollary 2.9. *Let $A_i, B_i, S_i \in \mathcal{L}(\mathcal{H})$ $(i = 1, \dots, n)$ and let f and g be non negative continuous functions on $[0, \infty)$ such that $f(t)g(t) = t$ for all $t \in [0, \infty)$. Then for all $r, p \ge 1$,*

$$ber_p^p(A_1^* S_1 B_1, \dots, A_n^* S_n B_n) \le \frac{n^{1-\frac{1}{r}}}{2^{1/r}} ber^{\frac{1}{r}} \left(\sum_{i=1}^{n} \left([B_i^* f^2(|S_i|) B_i]^{pr} + [A_i^* g^2(|S_i^*|) A_i]^{pr} \right) \right) - \inf_{\lambda \in \Omega} \xi(\hat{k}_\lambda),$$

where

$$\xi(\hat{k}_\lambda) = \sum_{i=1}^{n} \Bigg[\langle \frac{1}{2} \left([B_i^* f^2(|S_i|) B_i]^p + [A_i^* g^2(|S_i^*|) A_i]^p \right) \hat{k}_\lambda, \hat{k}_\lambda\rangle - \left(\langle [B_i^* f^2(|S_i|) B_i]^p \hat{k}_\lambda, \hat{k}_\lambda\rangle \langle [A_i^* g^2(|S_i^*|) A_i]^p \hat{k}_\lambda, \hat{k}_\lambda\rangle \right)^{\frac{1}{2}} \Bigg].$$

Choosing $f(t) = g(t) = t^{\frac{1}{2}}$ and $S_i = I$ for $i = 1, 2, ..., n$ in Theorem 2.8 we obtain the following simpler form.

Corollary 2.10. *Let $A_i, B_i \in \mathcal{L}(\mathcal{H})$ $(i = 1, 2, \ldots, n)$ and let f and g be non negative continuous functions on $[0, \infty)$ such that $f(t)g(t) = t$ for all $t \in [0, \infty)$. Then for $m = 1, 2, \ldots,$ and $r, p \geq m$,*

$$ber_p^p(A_1^*B_1, \ldots, A_n^*B_n) \leq \frac{n^{1-\frac{m}{r}}}{2^{\frac{m}{r}}} ber^{\frac{m}{r}}\left(\sum_{i=1}^n \left(|B_i|^{\frac{2pr}{m}} + |A_i|^{\frac{2pr}{m}}\right)\right) - \inf_{\lambda\in\Omega} \xi(\hat{k}_\lambda),$$

where

$$\xi(\hat{k}_\lambda) = \sum_{i=1}^n \left[\left\langle \frac{1}{2}\left(|B_i|^{\frac{2p}{m}} + |A_i|^{\frac{2p}{m}}\right)\hat{k}_\lambda, \hat{k}_\lambda\right\rangle^m - \left(\langle |B_i|^{\frac{2p}{m}}\hat{k}_\lambda, \hat{k}_\lambda\rangle\langle |A_i|^{\frac{2p}{m}}\hat{k}_\lambda, \hat{k}_\lambda\rangle\right)^{\frac{m}{2}}\right].$$

Following theorem of this article, we present an upper bound for the generalized Berezin number.

Theorem 2.11. *Let $S_i \in \mathcal{L}(\mathcal{H})$ $(1 \leq i \leq n)$. Then for $0 \leq \beta \leq 1, m \in \mathbb{N}$ and $p \geq 2m$,*

$$ber_p^p(S_1, \ldots, S_n) \leq ber\left(\sum_{i=1}^n \left(\beta|S_i|^{\frac{p}{m}} + (1-\beta)|S_i^*|^{\frac{p}{m}}\right)^m\right) - \inf_{\lambda\in\Omega} \xi(\hat{k}_\lambda),$$

where

$$\xi(\hat{k}_\lambda) = (2\ \min\{\beta, 1-\beta\})^m \sum_{i=1}^n \left(\left\langle \frac{|S_i|^{\frac{p}{m}} + |S_i^*|^{\frac{p}{m}}}{2}\hat{k}_\lambda, \hat{k}_\lambda\right\rangle^m - (\langle |S_i|^{\frac{p}{m}}\hat{k}_\lambda, \hat{k}_\lambda\rangle\langle |S_i^*|^{\frac{p}{m}}\hat{k}_\lambda, \hat{k}_\lambda\rangle)^{\frac{m}{2}}\right).$$

Proof. Let $\hat{k}_\lambda$ is the normalized reproducing kernel of $\mathcal{H}(\Omega)$, then

$$\begin{aligned}
&\sum_{i=1}^n |\langle S_i\hat{k}_\lambda, \hat{k}_\lambda\rangle|^p \\
&= \sum_{i=1}^n (\langle |S_i|^{2\beta}\hat{k}_\lambda, \hat{k}_\lambda\rangle^{\frac{1}{2}}\langle |S_i^*|^{2(1-\beta)}\hat{k}_\lambda, \hat{k}_\lambda\rangle^{\frac{1}{2}})^p \quad \text{(by (10))} \\
&= \sum_{i=1}^n (\langle |S_i|^{2\beta}\hat{k}_\lambda, \hat{k}_\lambda\rangle\langle |S_i^*|^{2(1-\beta)}\hat{k}_\lambda, \hat{k}_\lambda\rangle)^{\frac{p}{2}} \\
&\leq \sum_{i=1}^n \left(\langle |S_i|^{\frac{p}{m}}\hat{k}_\lambda, \hat{k}_\lambda\rangle^\beta\langle |S_i^*|^{\frac{p}{m}}\hat{k}_\lambda, \hat{k}_\lambda\rangle^{1-\beta}\right)^m \quad \text{(by Lemma 1.2)} \\
&\leq \sum_{i=1}^n \left(\beta\langle |S_i|^{\frac{p}{m}}\hat{k}_\lambda, \hat{k}_\lambda\rangle + (1-\beta)\langle |S_i^*|^{\frac{p}{m}}\hat{k}_\lambda, \hat{k}_\lambda\rangle\right)^m - \sum_{i=1}^n (2\min\{\beta, 1-\beta\})^m \\
&\quad \times \left(\left(\frac{\langle |S_i|^{\frac{p}{m}}\hat{k}_\lambda, \hat{k}_\lambda\rangle + \langle |S_i^*|^{\frac{p}{m}}\hat{k}_\lambda, \hat{k}_\lambda\rangle}{2}\right)^m - (\langle |S_i|^{\frac{p}{m}}\hat{k}_\lambda, \hat{k}_\lambda\rangle\langle |S_i^*|^{\frac{p}{m}}\hat{k}_\lambda, \hat{k}_\lambda\rangle)^{\frac{m}{2}}\right) \quad \text{(by (5))} \\
&= \sum_{i=1}^n \left\langle (\beta|S_i|^{\frac{p}{m}} + (1-\beta)|S_i^*|^{\frac{p}{m}})\hat{k}_\lambda, \hat{k}_\lambda\right\rangle^m - \sum_{i=1}^n (2\min\{\beta, 1-\beta\})^m \\
&\quad \times \left(\left\langle \frac{|S_i|^{\frac{p}{m}} + |S_i^*|^{\frac{p}{m}}}{2}\hat{k}_\lambda, \hat{k}_\lambda\right\rangle^m - (\langle |S_i|^{\frac{p}{m}}\hat{k}_\lambda, \hat{k}_\lambda\rangle\langle |S_i^*|^{\frac{p}{m}}\hat{k}_\lambda, \hat{k}_\lambda\rangle)^{\frac{m}{2}}\right) \\
&\leq \sum_{i=1}^n \left\langle (\beta|S_i|^{\frac{p}{m}} + (1-\beta)|S_i^*|^{\frac{p}{m}})^m\hat{k}_\lambda, \hat{k}_\lambda\right\rangle - \sum_{i=1}^n (2\min\{\beta, 1-\beta\})^m \\
&\quad \times \left(\left\langle \frac{|S_i|^{\frac{p}{m}} + |S_i^*|^{\frac{p}{m}}}{2}\hat{k}_\lambda, \hat{k}_\lambda\right\rangle^m - (\langle |S_i|^{\frac{p}{m}}\hat{k}_\lambda, \hat{k}_\lambda\rangle\langle |S_i^*|^{\frac{p}{m}}\hat{k}_\lambda, \hat{k}_\lambda\rangle)^{\frac{m}{2}}\right) \quad \text{(by Lemma 1.2)} \\
&= \left\langle \sum_{i=1}^n (\beta|S_i|^{\frac{p}{m}} + (1-\beta)|S_i^*|^{\frac{p}{m}})^m\hat{k}_\lambda, \hat{k}_\lambda\right\rangle - \sum_{i=1}^n (2\min\{\beta, 1-\beta\})^m \\
&\quad \times \left(\left\langle \frac{|S_i|^{\frac{p}{m}} + |S_i^*|^{\frac{p}{m}}}{2}\hat{k}_\lambda, \hat{k}_\lambda\right\rangle^m - (\langle |S_i|^{\frac{p}{m}}\hat{k}_\lambda, \hat{k}_\lambda\rangle\langle |S_i^*|^{\frac{p}{m}}\hat{k}_\lambda, \hat{k}_\lambda\rangle)^{\frac{m}{2}}\right).
\end{aligned}$$

Taking supremum over $\lambda \in \Omega$, we get

$$ber_p^p(S_1,\dots,S_n) \le ber\left(\sum_{i=1}^{n}\left(\beta|S_i|^{\frac{p}{m}} + (1-\beta)|S_i^*|^{\frac{p}{m}}\right)^m\right) - \inf_{\lambda\in\Omega}\xi(\hat{k}_\lambda),$$

where

$$\xi(\hat{k}_\lambda) = (2\min\{\beta,1-\beta\})^m \sum_{i=1}^{n}\left(\left\langle \frac{|S_i|^{\frac{p}{m}} + |S_i^*|^{\frac{p}{m}}}{2}\hat{k}_\lambda,\hat{k}_\lambda\right\rangle^m - (\langle |S_i|^{\frac{p}{m}}\hat{k}_\lambda,\hat{k}_\lambda\rangle\langle |S_i^*|^{\frac{p}{m}}\hat{k}_\lambda,\hat{k}_\lambda\rangle)^{\frac{m}{2}}\right).$$

□

The following simpler form follows from Theorem 2.11 by letting $m = 1$.

Corollary 2.12. *Let $S_i \in \mathcal{L}(\mathcal{H})$ $(1 \le i \le n)$. Then for $0 \le \beta \le 1$ and $p \ge 2$,*

$$ber_p^p(S_1,\dots,S_n) \le ber\left(\sum_{i=1}^{n}\beta|S_i|^p + (1-\beta)|S_i^*|^p\right) - \inf_{\lambda\in\Omega}\xi(\hat{k}_\lambda),$$

where

$$\xi(\hat{k}_\lambda) = 2\min\{\beta,1-\beta\}\sum_{i=1}^{n}\left(\left\langle\left(\frac{|S_i|^p + |S_i^*|^p}{2}\right)\hat{k}_\lambda,\hat{k}_\lambda\right\rangle - (\langle |S_i|^p\hat{k}_\lambda,\hat{k}_\lambda\rangle\langle |S_i^*|^p\hat{k}_\lambda,\hat{k}_\lambda\rangle)^{1/2}\right).$$

Letting $\beta = \frac{1}{2}$ and $m = 1$ in Theorem 2.11, we obtain the following corollary.

Corollary 2.13. *Let $A, B \in \mathcal{L}(\mathcal{H})$. Then for $p \ge 2$,*

$$ber_p^p(A,B) \le \frac{1}{2}ber(|A|^p + |A^*|^p + |B|^p + |B^*|^p) - \inf_{\lambda\in\Omega}\xi(\hat{k}_\lambda),$$

where

$$\xi(\hat{k}_\lambda) = \left\langle\left(\frac{|A|^p + |A^*|^p}{2}\right)\hat{k}_\lambda,\hat{k}_\lambda\right\rangle + \left\langle\left(\frac{|B|^p + |B^*|^p}{2}\right)\hat{k}_\lambda,\hat{k}_\lambda\right\rangle - \langle |A|^p\hat{k}_\lambda,\hat{k}_\lambda\rangle^{1/2}\langle |A^*|\hat{k}_\lambda,\hat{k}_\lambda\rangle^{1/2}$$
$$- \langle |B|^p\hat{k}_\lambda,\hat{k}_\lambda\rangle^{1/2}\langle |B^*|\hat{k}_\lambda,\hat{k}_\lambda\rangle^{1/2}.$$

In particular, we have

$$\begin{aligned} ber^2(A) &\le \frac{1}{2}ber(|A|^2 + |A^*|^2) \\ &\quad - \inf_{\lambda\in\Omega}\left\{\left\langle\left(\frac{|A|^2 + |A^*|^2}{2}\right)\hat{k}_\lambda,\hat{k}_\lambda\right\rangle - \langle |A|^2\hat{k}_\lambda,\hat{k}_\lambda\rangle^{1/2}\langle |A^*|^2\hat{k}_\lambda,\hat{k}_\lambda\rangle^{1/2}\right\} \\ &= \frac{1}{2}ber(A^*A + AA^*) - \inf_{\lambda\in\Omega}\left\{\left\langle\left(\frac{A^*A + AA^*}{2}\right)\hat{k}_\lambda,\hat{k}_\lambda\right\rangle - \|A\hat{k}_\lambda\|\|A^*\hat{k}_\lambda\|\right\} \\ &= \frac{1}{2}ber(A^*A + AA^*) - \inf_{\lambda\in\Omega}\left\{\frac{1}{2}(\|A\hat{k}_\lambda\|^2 + \|A^*\hat{k}_\lambda\|^2) - \|A\hat{k}_\lambda\|\|A^*\hat{k}_\lambda\|\right\}; \end{aligned}$$

which is a refinement of the inequality [5, Corollary 3.2].

$$ber^2(A) \le \frac{1}{2}ber(A^*A + AA^*).$$

Once we finish studying the Euclidean Berezin number, we show a Berezin number inequality. Hajmohamadi et al. [13] established that

$$ber^r(A^\beta XB^{1-\beta}) \le \|X\|^r\left(ber(\beta A^r + (1-\beta)B^r) - \inf_{\|\hat{k}_\lambda\|=1}\xi(\hat{k}_\lambda)\right), \tag{13}$$

where $A, B, X \in \mathcal{L}(\mathcal{H})$, with $A, B \ge 0$, $r \ge 2$, $\xi(\hat{k}_\lambda) = r_0(\langle A^r\hat{k}_\lambda,\hat{k}_\lambda\rangle^{\frac{1}{2}} - \langle B^r\hat{k}_\lambda,\hat{k}_\lambda\rangle^{\frac{1}{2}})^2$, $r_0 = \min\{\beta, 1-\beta\}$ and $0 \le \beta \le 1$.

The following result is the generalized improvement of (13).

Theorem 2.14. *Let $A, B, X \in \mathcal{L}(\mathcal{H})$ such that A, B are positive. Then*

$$ber^r(A^\beta X B^{1-\beta}) \leq \|X\|^r \left[ber(\beta A^{\frac{r}{2m}} + (1-\beta) B^{\frac{r}{2m}}) - \inf_{\lambda \in \Omega} \xi(\hat{k}_\lambda) \right],$$

where

$$\xi(\hat{k}_\lambda) = (2r_0)^m \left(\left(\frac{\langle A^{\frac{r}{2m}} \hat{k}_\lambda, \hat{k}_\lambda \rangle + \langle B^{\frac{r}{2m}} \hat{k}_\lambda, \hat{k}_\lambda \rangle}{2} \right)^m - (\langle A^{\frac{r}{2m}} \hat{k}_\lambda, \hat{k}_\lambda \rangle \langle B^{\frac{r}{2m}} \hat{k}_\lambda, \hat{k}_\lambda \rangle)^{\frac{m}{2}} \right),$$

where, $r_0 = \min\{\beta, 1-\beta\}$, $r \geq 2m$ and $0 \leq \beta \leq 1$.

Proof. Let $\hat{k}_\lambda$ is the normalized reproducing kernel of $\mathcal{H}(\Omega)$, then we have

$$\begin{aligned}
&|\langle A^\beta X B^{1-\beta} \hat{k}_\lambda, \hat{k}_\lambda \rangle|^r \\
&= |\langle X B^{1-\beta} \hat{k}_\lambda, A^\beta \hat{k}_\lambda \rangle|^r \\
&\leq \|X\|^r \|B^{1-\beta} \hat{k}_\lambda\|^r \|A^\beta \hat{k}_\lambda\|^r \quad \text{(by the Cauchy Schwartz inequality)} \\
&= \|X\|^r \left(\langle B^{2(1-\beta)} \hat{k}_\lambda, \hat{k}_\lambda \rangle^{\frac{r}{2m}} \langle A^{2\beta} \hat{k}_\lambda, \hat{k}_\lambda \rangle^{\frac{r}{2m}} \right)^m \\
&\leq \|X\|^r \left(\langle A^{\frac{r}{m}} \hat{k}_\lambda, \hat{k}_\lambda \rangle^\beta \langle B^{\frac{r}{m}} \hat{k}_\lambda, \hat{k}_\lambda \rangle^{1-\beta} \right)^m \quad \text{(by Lemma 1.2)} \\
&\leq \|X\|^r \Bigg[\left(\beta \langle A^{\frac{r}{m}} \hat{k}_\lambda, \hat{k}_\lambda \rangle + (1-\beta) \langle B^{\frac{r}{m}} \hat{k}_\lambda, \hat{k}_\lambda \rangle \right)^m \\
&\quad - (2r_0)^m \left(\left(\frac{\langle A^{\frac{r}{2m}} \hat{k}_\lambda, \hat{k}_\lambda \rangle + \langle B^{\frac{r}{2m}} \hat{k}_\lambda, \hat{k}_\lambda \rangle}{2} \right)^m - (\langle A^{\frac{r}{2m}} \hat{k}_\lambda, \hat{k}_\lambda \rangle \langle B^{\frac{r}{2m}} \hat{k}_\lambda, \hat{k}_\lambda \rangle)^{\frac{m}{2}} \right) \Bigg],
\end{aligned}$$

where the last inequality follows from (5). Taking supremum over $\lambda \in \Omega$, we deduce the desired inequality. □

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ORCID

Satyajit Sahoo http://orcid.org/0000-0002-1363-0103
Mojtaba Bakherad https://orcid.org/0000-0003-0323-6310

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