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Transitive Maps in Bitopological Dynamical Systems

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Abstract. This paper introduces fundamental ideas of bitopological dynamical systems. Here, notions of bitopological transitivity, point transitivity, pairwise iterated compactness, weakly bitopological transitivity, etc. are introduced. Later, it is shown that under pairwise homeomorphism, weakly point transitivity implies weakly bitopological transitivity. Moreover, under pairwise homeomorphism; pairwise compactness and pairwise iterated compactness are found to be equivalent. Later, we apply our results in the development process of a human embryo from the zygote until birth. During the process of biological application, we disprove conjecture 1 of Nada and Zohny [S. I. Nada, H. Zohny, An application of relative topology in biology. Chaos, Solitons and Fractals, 42 (2009) 202–204].

1. Introduction

Dynamical systems deal with systems that evolve with time. In topological dynamical system, we are concerned with a continuous self map or a homeomorphism on a non-empty topological space. There are many aspects of topological dynamical system, which have been extensively studied by many researchers since last century. But, because of the involvement of only one topology, there is a limitation of topological dynamical system. It cannot represent a system having two physical states at the same time. For example, in the development of an organism from zygote, the brain together with the central nervous system and the other body parts grow parallelly since different stem cell layers generate them. Thus, they can be represented by two topologies. This motivates us to generalize the notion of topological dynamical system. We consider bitopological space to generalize the notion of topological dynamical system. In a bitopological space, we get two topologies on the same set that may represent two physical states related to an object at the same time. In this matter, we must not be confused with the notion "states" of quantum mechanics.

Kelly [16] introduced the concept of bitopological space. Later, bitopological space attracted the attention of many researchers of various branches. Fletcher et al. [14] introduced the concept of pairwise compactness in a bitopological space. Pervin [20] extended the concept of continuity and connectedness in a bitopological space. For recent theoretical works in bitopological space, one may refer to Acharjee and Tripathy [5], Acharjee et al. [3], Acharjee et al. [4] and many others. Recently, bitopological space has been applied in many areas of science and social science. One may find its applications in medical science [25], economics ([6, 11]), computer science [10], etc.

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Topological transitivity is one of the most studied notions in the theory of topological dynamical system. For the motivation of the notion of topological transitivity, one may think of a real physical system, where a state is never given or measured accurately, but contains certain errors [17]. According to [17], one should study open subsets of the phase space instead of points and describe how open subsets move in that space. It is also one of the most important components of chaos.

Kolyada [17], Akin et al. [7] and many others studied transitive map in compact metric spaces. Recently, transitive map was generalized to G-spaces by Garg and Das [15]. Also, Akin and Carlson [8], Mai and Sun [18] generalized transitive map in general topological spaces. In this paper, we introduce transitive map in a bitopological space and study some of its fundamental properties. As per our information; this paper is the first one to introduce bitopological dynamical system.

The paper is divided into three sections. In the preliminary section, we recall some existing definitions of bitopological space and topological dynamical system. In the next section, we introduce bitopological dynamical system. We also introduce bitopological transitivity, point transitivity, pairwise iterated compactness, etc. and establish some relationships among them. As an application of our theory, we show that the growth process of a human baby from the zygote till its birth can be represented by a bitopological dynamical system. Moreover, we give evidence from medical literature that our reasons to choose two topologies in the growth process of a human baby have advantage over choosing one topology as suggested by Nada and Zohny [19]. We also disprove conjecture 1 of Nada and Zohny [19] by providing suitable mathematical theory with the help of some results which are derived in this paper. Recently, Acharjee et al. [2] disproved the conjecture 2 of Nada and Zohny [19]. Moreover, conjecture 3 of Nada and Zohny [19] was disproved recently by Acharjee et al. [1].

2. Preliminary Definitions

This section consists of some existing definitions of topological dynamical system and bitopological space.

Definition 2.1. ([16]) A quasi-pseudo-metric on a set X is a non-negative real-valued function $p(,)$ on the product $X \times X$ such that:
(i) $p(x, x) = 0$, where $x \in X$;

(ii) $p(x, z) \leq p(x, y) + p(y, z)$, where $x, y, z \in X$.

Definition 2.2. ([16]) Let $p(,)$ be a quasi-pseudo-metric on X, and let $q(,)$ be defined by $q(x, y) = p(y, x)$, where $x, y \in X$. Then, $q(,)$ is also a quasi-pseudo metric on X. We say that $p(,)$ and $q(,)$ are conjugate, and denote the set X with the structure by (X, p, q).

If $p(,)$ is a quasi-pseudo-metric on a set X, then the open p-sphere with centre x and radius $\varepsilon > 0$ is the set $S_p(x, \varepsilon) = \{y : p(x, y) < \varepsilon\}$. The collection of all open p-spheres forms a base for a topology. Similarly, $q(,)$ determines a topology for X. We shall denote the topology determined by $p(,)$ by τ_1 and that of $q(,)$ by τ_2.

Definition 2.3. ([16]) A space X on which are defined two (arbitrary) topologies τ_1 and τ_2 is called a bitopological space and denoted by (X, τ_1, τ_2).

Definition 2.4. ([16]) A bitopological space (X, τ_1, τ_2) is quasi-pseudo-metrizable if there is a pair $p(,)$ and $q(,)$ of conjugate quasi-pseudo metrices such that τ_1 and τ_2 are determined by $p(,)$ and $q(,)$ respectively.

Definition 2.5. ([21]) A function f from a bitopological space (X, τ_1, τ_2) into a bitopological space (Y, ψ_1, ψ_2) is said to be pairwise continuous (respectively, a pairwise homeomorphism) iff the induced functions $f : (X, \tau_1) \to (Y, \psi_1)$ and $f : (X, \tau_2) \to (Y, \psi_2)$ are continuous (respectively, homeomorphisms).

Pervin [20] called this a continuous map. However, we shall call this as pairwise continuous map, due to Reilly [21].

Definition 2.6. ([12]) A subset A of a bitopological space (X, τ_1, τ_2) is called bidense in X if A is dense in both (X, τ_1) and (X, τ_2).

Definition 2.7. ([14]) A cover $\mathcal{U}$ of a bitopological space (X, τ_1, τ_2) is pairwise open if $\mathcal{U} \subset \tau_1 \cup \tau_2$, $\mathcal{U} \cap \tau_1$ contains a non-empty set, and $\mathcal{U} \cap \tau_2$ contains a non-empty set.

Definition 2.8. ([14]) A bitopological space (X, τ_1, τ_2) is pairwise compact provided every pairwise open cover of X has a finite subcover.

Definition 2.9. ([9]) Let X be a topological space. A continuous map $f : X \to X$ is said to be a topological dynamical system with discrete time or simply a topological dynamical system. When f is a homeomorphism (that is, a bijective continuous map with continuous inverse), we also say that it is an invertible topological dynamical system.

Let $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{N}$ denote the set of integers, the set of real numbers and the set of non-negative integers respectively.

The forward orbit [13] of a point $x \in X$ under f is defined as $O_+(x) = \{f^n(x) : n \in \mathbb{N}\}$, where f^n denotes the n^{th} iteration of the map f i.e. f^0 is the identity map on X and $f^n = f \circ f^{n-1}$, the composition of f and f^{n-1}. If f is a homeomorphism, then the backward orbit of x is the set $O_-(x) = \{f^{-n}(x) : n \in \mathbb{N}\}$ and the full orbit of x (or simply orbit of x) is the set $O(x) = \{f^n(x) : n \in \mathbb{Z}\}$.

Definition 2.10. ([8]) A set $A \subset X$ is +invariant when $f(A) \subset A$ and A is −invariant when $A \subset f(A)$. A is called invariant when $f(A) = A$.

We now recall the following notions from Akin and Carlson [8]:

A dynamical system (X, f), where X is a topological space and $f : X \to X$ is a continuous map, is:

(i) (TT) if for every pair U, V of non-empty open subsets of X, the set $N(U, V) = \{k \in \mathbb{Z} : f^k(U) \cap V \neq \emptyset\} = \{k \in \mathbb{Z} : U \cap f^{-k}(V) \neq \emptyset\}$ is non-empty. We call this as $k-$transitivity.

(ii) (DO_+) if there exists a point $x \in X$ with forward orbit of x, $O(x) = \{f^n(x) : n \in \mathbb{N}\}$ is dense in X. In this case, (X, f) is said to be point transitive and x as a transitive point.

The set of transitive points of X is labeled as $Trans_f$. When $Trans_f = X$, then the system (X, f) is called minimal.

3. Main Results

In this section, we define some fundamental notions of bitopological dynamical system. We give suitable examples and study their various interrelated properties.

Definition 3.1. Let (X, τ_1, τ_2) be a bitopological space. A bitopological dynamical system is a pair (X, f), where (X, τ_1, τ_2) is a bitopological space and $f : X \to X$ is a pairwise continuous map. The dynamics is obtained by iterating the map.

The forward orbit of a point $x \in X$ under f is defined as $O_+(x) = \{f^n(x) : n \in \mathbb{N}\}$, where f^n denotes the n^{th} iteration of the map f. If f is a homeomorphism, then the backward orbit of x is the set $O_-(x) = \{f^{-n}(x) : n \in \mathbb{N}\}$ and the full orbit of x (or simply orbit of x) is the set $O(x) = \{f^n(x) : n \in \mathbb{Z}\}$.

Here, homeomorphism of f indicates homeomorphism of the function $f : (X, \tau_i) \to (X, \tau_i)$ separately for all $i \in \{1, 2\}$ as it is clear in terms of bitopological space. Thus, we can consider pairwise homeomorphism equivalently.

Definition 3.2. Let (X, f) be a bitopological dynamical system. A point $x \in X$ is called a fixed point for f if $f(x) = x$. The point x is a periodic point of period n if $f^n(x) = x$, where $n \in \mathbb{N}$. The least positive integer n for which $f^n(x) = x$ is called the prime period of x.

Definition 3.3. Let (X, f) be a bitopological dynamical system. For $U \in \tau_1$ and $V \in \tau_2$, we define the following:

$$N(U, V) = \{(m, n) : m, n \in \mathbb{N}, \quad f^m(U) \cap V \neq \emptyset \quad and \quad U \cap f^n(V) \neq \emptyset\}.$$

The map f is called bitopologically transitive (or $(m, n)-$transitive) if for any pair of non-empty sets $U \in \tau_1$ and $V \in \tau_2$, the set $N(U, V)$ is non-empty.

The following definition is the generalization of the above definition. The definition given below is based on both forward iteration and backward iteration.

Definition 3.4. Let (X, f) be a bitopological dynamical system. For $U \in \tau_1$ and $V \in \tau_2$, we define:

$$Z(U, V) = \{(m, n) : m, n \in \mathbb{Z}, \quad f^m(U) \cap V \neq \emptyset \quad and \quad U \cap f^n(V) \neq \emptyset\}.$$

The map f is called weakly bitopologically transitive (or $(m, n)-$weakly transitive) if for any pair of non-empty sets $U \in \tau_1$ and $V \in \tau_2$, the set $Z(U, V)$ is non-empty.

Now, we have the following theorem which establishes the relationship between $(m, n)-$transitivity and iteration of the function on two topologies.

Theorem 3.1. *Let (X, f) be a bitopological dynamical system. If the map f is pairwise continuous, then the following conditions are equivalent:*
(i) f is bitopologically transitive (or $(m, n)-$transitive).
(ii) for every non-empty set $U \in \tau_1$, $\bigcup_{m \in \mathbb{N}} f^m(U)$ is dense in (X, τ_2) and for every non-empty set $V \in \tau_2$, $\bigcup_{n \in \mathbb{N}} f^n(V)$ is dense in (X, τ_1).

Proof. (i) $\Longrightarrow$ (ii): Let $U \in \tau_1$ be arbitrary. The bitopological transitivity of the map f implies that for each non-empty set $V \in \tau_2$, the set $N(U, V)$ is non-empty i.e. $f^m(U) \cap V \neq \emptyset$, where $m \in \mathbb{N}$ depends on $V \in \tau_2$. This gives $\bigcup_{m \in \mathbb{N}} f^m(U) \cap V \neq \emptyset$, for each non-empty set $V \in \tau_2$. Thus, $\bigcup_{m \in \mathbb{N}} f^m(U)$ is dense in (X, τ_2).

Again, let $V \in \tau_2$ be arbitrary. We get for each non-empty set $U \in \tau_1$, $U \cap f^n(V) \neq \emptyset$, where $n \in \mathbb{N}$ depends on $U \in \tau_1$ i.e. $\bigcup_{n \in \mathbb{N}} f^n(V) \cap U \neq \emptyset$, for each non-empty set $U \in \tau_1$. Thus, $\bigcup_{n \in \mathbb{N}} f^n(V)$ is dense in (X, τ_1).

(ii) $\Longrightarrow$ (i): Let $U \in \tau_1$ and $V \in \tau_2$ be two arbitrary non-empty sets. Since $\bigcup_{m \in \mathbb{N}} f^m(U)$ is dense in (X, τ_2), so $\bigcup_{m \in \mathbb{N}} f^m(U) \cap V \neq \emptyset$. This implies that there exists $m \in \mathbb{N}$ such that $f^m(U) \cap V \neq \emptyset$. Similarly, there exists $n \in \mathbb{N}$ such that $U \cap f^n(V) \neq \emptyset$. Thus, f is bitopologically transitive. □

Similarly, we can get equivalent conditions for weakly bitopologically transitive map. The theorem given below establishes the relation between forward iteration and backward iteration under weakly transitive map irrespective of $\mathbb{N}$, as we had in Theorem 3.1. Here, we are free to consider backward iterations.

Theorem 3.2. *Let (X, f) be a bitopological dynamical system. If the map f is pairwise continuous, then the following conditions are equivalent:*
(i) f is weakly bitopologically transitive (or $(m, n)-$weakly transitive).
(ii) for every non-empty set $U \in \tau_1$, $\bigcup_{m \in \mathbb{Z}} f^m(U)$ is dense in (X, τ_2) and for every non-empty set $V \in \tau_2$, $\bigcup_{n \in \mathbb{Z}} f^n(V)$ is dense in (X, τ_1).
(iii) f is $(-n, -m)-$weakly transitive.

Proof. Proceeding as in theorem 3.1., we can prove (i)$\Longleftrightarrow$(ii).

Now, we prove (i)$\Longleftrightarrow$(iii).

(i)$\Longleftrightarrow$(iii): Since f is $(m, n)-$weakly transitive, so for any pair of non-empty sets $U \in \tau_1$ and $V \in \tau_2$, we have $f^m(U) \cap V \neq \emptyset$ and $U \cap f^n(V) \neq \emptyset$, where $m, n \in \mathbb{Z}$. But, $f^m(U) \cap V \neq \emptyset \Longleftrightarrow U \cap f^{-m}(V) \neq \emptyset$ as both say that there exist $x \in U$ and $y \in V$ such that $y = f^m(x)$. Also, $U \cap f^n(V) \neq \emptyset \Longleftrightarrow f^{-n}(U) \cap V \neq \emptyset$. Hence, $f^{-n}(U) \cap V \neq \emptyset$ and $U \cap f^{-m}(V) \neq \emptyset$, where $m, n \in \mathbb{Z}$. Thus, f is $(-n, -m)-$weakly transitive. □

Since; pairwise homeomorphism is a very useful tool to study various properties of bitopological space, hence our intention is to find behaviour of bitopological dynamical system under pairwise homeomorphism. Thus, we have the following theorem.

Theorem 3.3. *Let f be $(m,n)-$weakly transitive and pairwise homeomorphism in a bitopological space (X,τ_1,τ_2). If every non-empty set $U \in \tau_1$ is $-$invariant and every non-empty set $V \in \tau_2$ is $+$invariant, then for any pair of non-empty sets $U \in \tau_1$ and $V \in \tau_2$, the set $Z(U,V)$ is infinite.*

Proof. For any pair of non-empty sets $U \in \tau_1$ and $V \in \tau_2$, either $U \cap V \neq \emptyset$ or $U \cap V = \emptyset$.

Case I: Let $U \cap V \neq \emptyset$. So, we consider $x \in U \cap V$. To prove the set $Z(U,V)$ is infinite, it is sufficient to prove that for any natural number n, $f^n(U) \cap V \neq \emptyset$ and $U \cap f^{-n}(V) \neq \emptyset$. We shall use the principle of mathematical induction to prove this.

Now, $f(x) \in f(U \cap V) = f(U) \cap f(V) \subset f(U) \cap V$, as V is +invariant. Hence, $f(x) \in f(U) \cap V$. Let for some positive integer k, $f^k(x) \in f^k(U) \cap V$. Then, $f(f^k(x)) \in f(f^k(U) \cap V) = f^{k+1}(U) \cap f(V) \subset f^{k+1}(U) \cap V$ i.e. $f^{k+1}(x) \in f^{k+1}(U) \cap V$. Thus, due to the principle of mathematical induction, $f^n(x) \in f^n(U) \cap V$ for any natural number n. In other words, $f^n(U) \cap V \neq \emptyset$ for any natural number n.

Again, $f^{-1}(x) \in f^{-1}(U \cap V) = f^{-1}(U) \cap f^{-1}(V) \subset U \cap f^{-1}(V)$, as U is $-$invariant i.e. $f^{-1}(x) \in U \cap f^{-1}(V)$. Let for some positive integer k, $f^{-k}(x) \in U \cap f^{-k}(V)$. Then, $f^{-1}(f^{-k}(x)) \in f^{-1}(U \cap f^{-k}(V)) = f^{-1}(U) \cap f^{-(k+1)}(V) \subset U \cap f^{-(k+1)}(V)$ i.e. $f^{-(k+1)}(x) \in U \cap f^{-(k+1)}(V)$. Thus, due to the principle of mathematical induction, $f^{-n}(x) \in U \cap f^{-n}(V)$ for any natural number n. In other words, $U \cap f^{-n}(V) \neq \emptyset$ for any natural number n. Thus, $f^n(U) \cap V \neq \emptyset$ and $U \cap f^{-n}(V) \neq \emptyset$ for any natural number n.

Case II: Let $U \cap V = \emptyset$. Then, $(m,n)-$weakly transitiveness of f implies that there exists an integer m such that $f^m(x) = y \in f^m(U) \cap V$, where $x \in U$ and $y \in V$. Since, f is a pairwise homeomorphism, so $f^m(U) = U_1$ is a non-empty τ_1-open set. Thus, we get $y \in U_1 \cap V$. Hence, $U_1 \cap V \neq \emptyset$. Proceeding as in case I, we get that, $f^{n_0}(U_1) \cap V \neq \emptyset$ for any natural number n_0 i.e. $f^{n_0}(f^m(U)) \cap V \neq \emptyset$. Thus, $f^{m+n_0}(U) \cap V \neq \emptyset$ for any natural number n_0.

Again, due to $(m,n)-$weakly transitiveness of f, there exists an integer n such that $f^n(y) = x \in U \cap f^n(V)$, where $x \in U$ and $y \in V$. Since, f is pairwise homeomorphism, so $f^n(V) = V_1$ is a non-empty τ_2-open set. Thus, we get $y \in U \cap V_1$, which yields $U \cap V_1 \neq \emptyset$. Proceeding as in case I, we get that, $U \cap f^{-n_0}(V_1) \neq \emptyset$ for any natural number n_0. Hence, $U \cap f^{-n_0}(f^n(V)) \neq \emptyset$. Thus, $U \cap f^{n-n_0}(V) \neq \emptyset$, for any natural number n_0. Hence, $f^{m+n_0}(U) \cap V \neq \emptyset$ and $U \cap f^{n-n_0}(V) \neq \emptyset$ for any natural number n_0. Thus, for any pair of non-empty sets $U \in \tau_1$ and $V \in \tau_2$, the set $Z(U,V)$ is infinite. □

Definition 3.5. Let (X,f) be a bitopological dynamical system. The map f is called point transitive if there exists a point $x \in X$ with $O_+(x)$ is bidense in X. Here, the point x is called a transitive point of f.

A point $x \in X$ which is not a transitive point is called an intransitive point. The set of all transitive points (intransitive points) of f is denoted by $tr(f)$ ($intr(f)$) respectively. If $tr(f) = X$, then the system (X,f) is called minimal.

Definition 3.6. Let (X,f) be a bitopological dynamical system, where f is pairwise homeomorphism. The map f is called weakly point transitive if there exists a point $x \in X$ with $O(x)$ is bidense in X. Here, the point x is called a weak transitive point of f.

A point $x \in X$ which is not a weak transitive point is called a weak intransitive point. The set of all weak transitive points (weak intransitive points) of f is denoted by $wtr(f)$ ($wintr(f)$) respectively. If $wtr(f) = X$, then the system (X,f) is called weakly minimal.

The relation between weakly point transitive map and weakly bitopologically transitive map is given below.

Theorem 3.4. *Let (X,f) be a bitopological dynamical system, where f is pairwise homeomorphism. If f is weakly point transitive, then f is weakly bitopologically transitive.*

Proof. Since f is weakly point transitive, so there exists a point $x \in X$ with $O(x)$ is bidense in X. Thus, for any pair of non-empty sets $U \in \tau_1$ and $V \in \tau_2$, we have $O(x) \cap U \neq \emptyset$ and $O(x) \cap V \neq \emptyset$. This implies there exist $m, n \in \mathbb{Z}$ such that $f^m(x) \in U$ and $f^n(x) \in V$. Then, the following cases may arise:

Case I: Let $m = n$. Then, $f^m(x) = y \in U \cap V$. Hence, $f^0(U) \cap V \neq \emptyset$ and $U \cap f^0(V) \neq \emptyset$. Therefore, f is weakly bitopologically transitive in this case.

Case II: Let $m < n$ and $n = m+r$. Then $f^r(f^m(x)) \in f^r(U)$ and $f^{-r}(f^n(x)) \in f^{-r}(V)$. This gives $f^{m+r}(x) \in f^r(U)$ and $f^{n-r}(x) \in f^{-r}(V)$. So, we have $f^n(x) \in f^r(U)$ and $f^m(x) \in f^{-r}(V)$. We get $f^n(x) \in f^r(U) \cap V$ and $f^m(x) \in U \cap f^{-r}(V)$. Hence, $f^r(U) \cap V \neq \emptyset$ and $U \cap f^{-r}(V) \neq \emptyset$, which means that f is weakly bitopologically transitive.

Case III: Let $m > n$ and $m = n + s$. Proceeding as in case II, we get $f^{-s}(U) \cap V \neq \emptyset$ and $U \cap f^s(V) \neq \emptyset$, which says that f is weakly bitopologically transitive. Thus, f is weakly bitopologically transitive. □

It is to be noted that the similar relation is not true between point transitive map and bitopological transitive map. We provide Example 3.3. to prove our claim. Now, we discuss some examples in support of the above notions.

Example 3.1. Consider the set $X = \{1, \frac{1}{2}, \frac{1}{3}\}$.We define $p : X \times X \to [0, \infty)$ by $p(x, y) = x$, if $x \neq y$ and $p(x, x) = 0, \forall x, y \in X$. Then, clearly p is a quasi-pseudo metric on X. The conjugate of p, i.e. $q(x, y) = p(y, x)\ \forall x, y \in X$ is also a quasi-pseudo metric on X. An open p-sphere with centre x and radius $\varepsilon > 0$ is of the form

$$S_p(x, \varepsilon) = \begin{cases} \{x\}, & \text{if } x \geq \varepsilon \\ X, & \text{if } x < \varepsilon \end{cases}$$

Thus, p induces discrete topology on X. Similarly, q also induces discrete topology on X. Now, we define $f : X \to X$ by $f(1) = \frac{1}{2}, f(\frac{1}{2}) = \frac{1}{3}, f(\frac{1}{3}) = 1$. Then, f is a bitopologically transitive and weakly bitopologically transitive. Also, $O_+(1) = \{1, \frac{1}{2}, \frac{1}{3}\} = O_+(\frac{1}{2}) = O_+(\frac{1}{3}) = X$. Hence, $tr(f) = X$. So, the system (X, f) is minimal. Moreover, $O(x) = X,\ \forall x \in X$. Thus, the system (X, f) is weakly minimal.

Example 3.2. ([22]) Let us consider the set $\mathbb{R}$ of real numbers. We define $p : \mathbb{R} \times \mathbb{R} \to [0, \infty)$ by $p(x, y) = max\{y-x, 0\}, \forall x, y \in X$. Then, p is a quasi-pseudo metric on $\mathbb{R}$. The conjugate of p i.e. $q(x, y) = p(y, x)\ \forall x, y \in \mathbb{R}$ is also a quasi-pseudo metric on $\mathbb{R}$. Then, an open p-sphere with centre x and radius $\varepsilon > 0$ is of the form

$$\begin{aligned} S_p(x, \varepsilon) &= \{y : p(x, y) < \varepsilon\} \\ &= \{y : y - x < \varepsilon\} \\ &= \{y : y < x + \varepsilon\} \\ &= (-\infty, x + \varepsilon) \end{aligned}$$

Similarly, $S_q(x, \varepsilon) = (x - \varepsilon, \infty)$. If τ_1 and τ_2 are the topologies generated by p-spheres and q-spheres respectively, then $(\mathbb{R}, \tau_1, \tau_2)$ is a bitopological space. We define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = x + 1$. Clearly, f is invertible and $f^{-1}(x) = x - 1$. Now, for any set $U_1 = (-\infty, a) \in \tau_1$ and $a \in \mathbb{R}$, we have

$f(U_1) = \{f(x) : f(x) < f(a)\} = \{y : f(x) = y,\ y < a + 1\} = (-\infty, a + 1)$ and $f^{-1}(U_1) = (-\infty, a - 1)$,

i.e. inverse image under f of each τ_1-open set is τ_1-open. Therefore, $f : (\mathbb{R}, \tau_1) \to (\mathbb{R}, \tau_1)$ is a homeomorphism. Similarly, $f : (\mathbb{R}, \tau_2) \to (\mathbb{R}, \tau_2)$ is a homeomorphism. Hence, f is a pairwise homeomorphism. Also, $O(1) = \{..., f^{-3}(1), f^{-2}(1), f^{-1}(1), 1, f(1), f(2), f(3), ...\} = \{..., -2, -1, 0, 1, 2, 3, 4, ...\}$, which is bidense in $(\mathbb{R}, \tau_1, \tau_2)$. Thus, f is weakly point transitive. Hence, by theorem 3.4., f is weakly bitopologically transitive.

Example 3.3. Let $X = \{\frac{1}{n} : n \in \mathbb{N} - \{0\}\}$. We define $p : X \times X \to [0, \infty)$ by $p(x, y) = x$, if $x \neq y$ and $p(x, x) = 0, \forall x, y \in X$. Then, p is a quasi-pseudo metric on X. Both p and its conjugate q i.e. $q(x, y) = p(y, x) \forall x, y \in X$, induces discrete topology on X. Now, we define $f : X \to X$ by $f(\frac{1}{n}) = \frac{1}{n+1}$, where $n \in \mathbb{N} - \{0\}$. Then, f is pairwise continuous, and one-one but not onto. Here, $O_+(1) = X$, i.e. $O_+(1)$ is bidense in X. Therefore, f is point transitive but f is not bitopologically transitive as there is not any $n \in \mathbb{N}$ such that $f^n(\{\frac{1}{2}\}) \cap \{1\} \neq \emptyset$.

In general, pairwise compactness and iterations are hardly discussed together. Thus, our motivation is to define pairwise iterated compactness and to establish relation with pairwise compactness.

Definition 3.7. Let (X, τ_1, τ_2) be a bitopological space and f be a pairwise continuous map. For $m_i, n_j \in \mathbb{N}$, a collection $\mathcal{U}_f = \{f^{m_i}(U_i), f^{n_j}(V_j) : i, j \in \Delta\}$ is called an iterated cover of X if $X = \{\bigcup_{i\in\Delta} f^{m_i}(U_i)\} \cup \{\bigcup_{j\in\Delta} f^{n_j}(V_j)\}$, where $U_i \in \tau_1$ and $V_j \in \tau_2$ are non-empty sets and Δ is an index set. In addition, if $\mathcal{U}_f$ contains the iteration of atleast one non-empty element of τ_1 and atleast one non-empty element of τ_2, then it is called pairwise iterated cover.

Definition 3.8. Let (X, τ_1, τ_2) be a bitopological space and f be a pairwise continuous map. If every iterated cover $\mathcal{U}_f$ of X has a finite subcover, then the space (X, τ_1, τ_2) is called iterated compact. Also, X is called pairwise iterated compact if every pairwise iterated cover $\mathcal{U}_f$ of X has a finite subcover.

Now, the following theorem establishes the relationship between pairwise compactness and pairwise iterated compactness.

Theorem 3.5. *Let (X, τ_1, τ_2) be a bitopological space and f be pairwise homeomorphism. Then, X is pairwise compact if and only if X is pairwise iterated compact.*

Proof. Let X be pairwise compact and let $\mathcal{U}_f = \{f^{m_i}(U_i), f^{n_j}(V_j) : i, j \in \Delta\}$ be a pairwise iterated cover of X, where Δ is an index set. Since f is pairwise homeomorphism, $f^{m_i}(U_i) = A_i$, which is a non-empty τ_1-open set and $f^{n_j}(V_j) = B_j$, which is a non-empty τ_2-open set. Thus, $\mathcal{U}_f$ is a pairwise open cover of X. Since X is pairwise compact, thus $\mathcal{U}_f$ has a finite subcover. Hence, X is pairwise iterated compact.

Conversely, let X be pairwise iterated compact and let $\mathcal{U} = \{U_i, V_j : i, j \in \Delta\}$ be a pairwise open cover of X, where atleast one $U_i \in \tau_1$ and $V_j \in \tau_2$ and Δ is an index set. Then, $\mathcal{U}_f = \{\{f^0(U_i), f^0(V_j)\} : i, j \in \Delta\}$ is a pairwise iterated cover of X. The pairwise iterated compactness of X implies that $\mathcal{U}_f$ has a finite subcover, which is the required finite subcover of $\mathcal{U}$. Hence, X is pairwise compact. □

Example 3.4. Let us consider the bitopological dynamical system as defined in Example 3.2. Then, X is pairwise iterated compact since X is pairwise compact.

Thus, we can conclude this section since we established some important results to introduce the theory of bitopological dynamical system. Now, we try to find application of our theoretical work from the scenario of reality.

4. Birth of a Child from the Zygote as a Bitopological Dynamical System

In this section, we study the growth of a human embryo from the viewpoint of bitopological dynamical system. At first, we recall some biological terms related to the development of a newborn from the zygote.

Definition 4.1. ([26]) Embryonic day is the number of days post conception, e.g. E25 i.e. embryonic day 25.

Definition 4.2. ([23]) Gastrulation is the process of forming the three primary germ layers from the epiblast involving movement of cells through the primitive streak to form endoderm and mesoderm.

Definition 4.3. ([27]) Anencephaly is a congenital absence of a major portion of the brain, skull, and scalp, with its genesis in the first month of gestation.

Definition 4.4. ([23]) Congenital malformation is synonymous with the term birth defect, it refers to any structural, behavioral, functional, or metabolic disorder present at birth.

According to Stiles and Jernigan [26], when the fertilization takes place, a zygote say Z, is formed and as time goes on, it differentiates to form different cells. At the end of second week after the conception, the embryo becomes a two layered structure and then gastrulation begins (on embryonic day 13, or E13). After gastrulation, the embryo becomes a three layered structure: endodermal stem cell layer, mesodermal stem cell layer and ectodermal stem cell layer. The structures of the gut and the respiratory tract are developed from the endodermal stem cell layer, while the mesodermal stem cell layer gives rise to structures such as muscle, bone, cartilage and the vascular system. Ectodermal layer stem cells are of two types: epidermal ectodermal stem cells and neuroectodermal stem cells. The epidermal ectodermal stem cells develop into skin, nails, and sweat glands, while neuroectodermal stem cells give rise to the brain and the central nervous system. The neuroectodermal stem cells are called the neural progenitor cells.

Thus, after gastrulation, the development process of the brain together with the central nervous system and the other body parts splits since different stem cell layers generate them. We show this by figure 1.

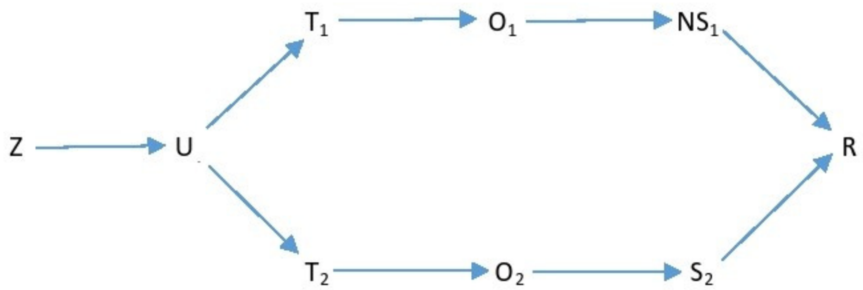

Figure 1: Here, Z- the zygote, U- development of the zygote just before gastrulation, T_1- Neural tissues, T_2- Non-neural tissues, O_1- Neural organs, O_2- Non-neural organs, NS_1- Neural organ systems, S_2- Non-neural organ systems and R- the baby at the time of birth.

Hence, we can define two topologies as given below:
$\tau_1 = \{(\phi, \tau_1(t_0)), (U_1, \tau_1(t_1)), (U_2, \tau_1(t_2)), ..., (U_m, \tau_1(t_m)), (R, \tau_1(T))\}$ and
$\tau_2 = \{(\phi, \tau_2(t_0)), (V_1, \tau_2(t_1)), (V_2, \tau_2(t_2)), ..., (V_n, \tau_2(t_n)), (R, \tau_2(T))\}$,
where $\phi = Z = U_0 = V_0$, since initially there is only the zygote from which the whole organism develops.

$U_n = X$ is the brain together with the central nervous system of the whole organism and $V_n = Y$ is the other body parts of the whole organism except the brain and the central nervous system. Also, U_1, U_2,... represent different development stages of the brain and the central nervous system; while V_1, V_2,... represent different development stages of the other body parts except the brain and the central nervous system. Here, t_0 is the time of fertilization and T is the time of birth without indicating the stages of the growth. In notional sense; $(U_i, \tau_j(t_i))$ indicates that to reach the stage U_i at the time of growth under topology τ_j; the required time is $\tau_j(t_i)$ where $j \in \{1, 2\}$. It is important to note that before gastrulation $U_i = V_i$. Here, X and Y together forms the whole organism, the baby say R, i.e. $X \cup Y = R$. Clearly, $(X, \tau_1|_R)$ and $(Y, \tau_2|_R)$ form dynamical relative topological spaces (in the sense of Nada and Zohny [19]) individually with respect to (R, τ_1, τ_2) as the growth rate of the organism depends on time. Here, $\tau_1|_R$ and $\tau_2|_R$ are two relative topologies on X. But, (R, τ_1, τ_2) is a dynamical bitopological space (in our sense bitopological dynamical system) which contains both the growth of the brain with the central nervous system and growth of other body parts, which all together yields the baby R.

Here, $Z = U_0 \subset U_1 \subset U_2 \subset ... \subset U_m \subset R$ and $Z = V_0 \subset V_1 \subset V_2 \subset ... \subset V_n \subset R$. Also, $\tau_1(t_0) < \tau_1(t_1) < \tau_1(t_2) < ... < \tau_1(t_m) < \tau_1(T)$ and $\tau_2(t_0) < \tau_2(t_1) < \tau_2(t_2) < ... < \tau_2(t_n) < \tau_2(T)$.

Now, let $f_i : R \to R$ (the unfolding map) be defined as

$$f_i(U_k) = \begin{cases} U_{k+1} & if\ i = k \\ U_k & if\ i \neq k \end{cases}$$

and $f_i(R) = R$, where $i, k \in \{0, 1, ..., m-1\}$. Also, we consider $f_i(V_k) = \emptyset$, $\forall\ V_k \subset R$

Also, let $g_j : R \to R$ be defined by

$$g_j(V_k) = \begin{cases} V_{k+1} & if\ j = k \\ V_k & if\ j \neq k \end{cases}$$

and $g_j(R) = R$, where $j, k \in \{0, 1, ..., n-1\}$. Also, we consider $g_j(U_k) = \emptyset$, $\forall\ U_k \subset R$

Let $F = f_{m-1} \circ f_{m-2} \circ ... \circ f_0$ and $G = g_{n-1} \circ g_{n-2} \circ ... \circ g_0$.

Now, we define the map $H : R \to R$ by

$$H(A) = \begin{cases} R & if\ A = R \\ F(A) & if\ A \in \tau_1 \setminus \tau_2, A \neq R \\ G(A) & if\ A \in \tau_2 \setminus \tau_1, A \neq R \\ F(A) \cup G(A) & if A \in \tau_1 \cap \tau_2, A \neq R \end{cases}$$

Then, (R, H) forms a bitopological dynamical system.

Also, the forward orbit of the zygote Z is

$$\begin{aligned} O_+(Z) &= \bigcup_{n \in \mathbb{N}} H^n(Z) \\ &= H(Z) \cup H^2(Z) \cup ... \\ &= \{F(Z) \cup G(Z)\} \cup H^2(Z) \cup ... \\ &= \{(f_{m-1} \circ f_{m-2} \circ ... \circ f_0)(Z) \cup (g_{n-1} \circ g_{n-2} \circ ... \circ g_0)(Z)\} \cup H^2(Z) \cup ... \\ &= \{(f_{m-1} \circ f_{m-2} \circ ... \circ f_1)(f_0(Z)) \cup (g_{n-1} \circ g_{n-2} \circ ... \circ g_1)(g_0(Z))\} \cup H^2(Z) \cup ... \\ &= \{(f_{m-1} \circ f_{m-2} \circ ... \circ f_2)(f_1(U_1)) \cup (g_{n-1} \circ g_{n-2} \circ ... \circ g_2)(g_1(V_1)\} \cup H^2(Z) \cup ... \end{aligned}$$

After some steps of calculation, we get the following result

$$\begin{aligned} O_+(Z) &= (U_m \cup V_n) \cup H^2(Z) \cup ... \\ &= (X \cup Y) \cup H(H(Z)) \cup ... \\ &= R \cup H(R) \cup ... \\ &= R \cup R \cup ... \\ &= R \end{aligned}$$

Thus, $O_+(Z) = R$, which is bidense in (R, τ_1, τ_2). Hence, H is a point transitive map. Similarly, for any $A \in \tau_1 \cap \tau_2$, $O_+(A)$ is bidense in (R, τ_1, τ_2).

Hence, we have shown that the growth of human baby can be represented by a bitopological dynamical system. Medically [23] or from the view point of literature [26], we may conclude that our mathematical ideas are predicting the growth of a baby from the zygote more accurately than the topological dynamical system as conjectured by Nada and Zohny [19] in conjecture 1. Thus, it should be our next step to investigate more accurately the growth process of a baby from the zygote based on growth of the brain together with the central nervous system and growth of the other body parts from the perspective of bitopological dynamical system.

The advantage of choosing two topologies in this paper over one topology as considered by Nada and Zohny [19] is that bitopological dynamical system may be able to find the hidden theoretical causes of problems like the infants with anencephaly has congenital malformation of other body parts ([27], [24]), viz, agenesis of kidney, heart anomaly, etc. On the basis of discussions made in medical literature ([23], [24], [26]), our later observations and findings [1, 2], we firmly believe that bitopological dynamical system is a better tool in comparison to topological dynamical system as proposed by Nada and Zohny [19]. Hence, we disprove the conjecture 1 of Nada and Zohny [19] with our results in this section.

5. Open Questions

In this section, we formulate some questions which are still open. More sophisticated theoretical results in bitopological dynamical system will be needed to handle these. The questions are given below:

Q.1. Is there any connection between the growth process of a cell to form an organ of the baby and pairwise iterated compactness?

Q.2. Does the growth process of nervous system follow theoretical ideas related to pairwise Hausdorffness and related structures?

We are expecting that answers to these questions may be helpful for the treatments of various forms of cancer along with brain disorders.

6. Conclusion

In this paper, we introduced the theory of bitopological dynamical system and defined bitopological transitivity, point transitivity, pairwise iterated compactness and established some relationships between them. Later, we showed that the development of a human embryo from the zygote until birth can be represented by a bitopological dynamical system. Also, medical evidences were provided to justify our theoretical claims. We firmly believe that if the theoretical relations between the development process of the brain together with central nervous system and the development of the other body parts may be found in future, then the medical treatments may be given to stop any congenital malformation just by studying brain development or body development in prenatal stage. To make this a reality, we need a deep research in human embryo development as well as in bitopological dynamical system. At the end, we disproved the conjecture 1 of Nada and Zohny [19].

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