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Estimates for Initial Coefficients of Certain Bi–Univalent Functions

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Abstract. Estimates are obtained for the initial coefficients of a normalized analytic function f in the unit disk $\mathbb{D}$ such that f and the analytic extension of f^{-1} to $\mathbb{D}$ belong to certain subclasses of univalent functions. The bounds obtained improve some existing known bounds.

1. Introduction and Preliminaries

Let $\mathcal{A}$ be the class of analytic functions defined on the unit disk $\mathbb{D} := \{z : |z| < 1\}$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1}$$

Suppose that $\mathcal{S}$ is the subclass of $\mathcal{A}$ consisting of univalent functions. Being univalent, the functions in class $\mathcal{S}$ are invertible; however, the inverse need not be defined on the entire unit disk. The Koebe one-quarter theorem ensures that the image of the unit disk under every univalent function contains a disk of radius 1/4. Thus, a function $f \in \mathcal{S}$ has an inverse defined on a disk containing disk $|z| < 1/4$. It can be easily seen that

$$f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + \cdots$$

in some disk of radius at least 1/4. A function $f \in \mathcal{A}$ is said to be bi–univalent in $\mathbb{D}$ if both f and analytic extension of f^{-1} to $\mathbb{D}$ are univalent in $\mathbb{D}$. The class of bi–univalent functions, denoted by σ, was introduced by Lewin [15] in 1967, who also showed that the second coefficient of a bi–univalent function satisfies the inequality $|a_2| \leq 1.51$. Let σ_1 be the class of the functions $f = \phi \circ \psi^{-1}$, where ϕ and ψ are univalent analytic functions mapping $\mathbb{D}$ onto a domain containing $\mathbb{D}$ and satisfy $\phi'(0) = \psi'(0)$. Clearly, $\sigma_1 \subset \sigma$, though $\sigma_1 \neq \sigma$ (see [8]). In 1969, Suffridge [25] gave a function in class σ_1 with $a_2 = 4/3$ and conjectured that $|a_2| \leq 4/3$ for the functions in the class σ. Netanyahu [20] proved the conjecture for the subclass σ_1. The conjecture was later disproved by Styler and Wright [24] in 1981, who showed that $a_2 > 4/3$ for some function in σ.

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Brannan and Clunie [6] conjectured that $|a_2| \leq \sqrt{2}$ for $f \in \sigma$. Kedzierawski [11] proved this for a special case when the functions f and f^{-1} are starlike functions.

For analytic functions f and g in $\mathbb{D}$, the function f is *subordinate* to the function g, written as $f(z) \prec g(z)$, if there is a Schwarz function w such that $f = g \circ w$. If g is univalent, then $f(z) \prec g(z)$ if and only if $f(0) = g(0)$ and $f(\mathbb{D}) \subseteq g(\mathbb{D})$. The method of subordination is quite useful for establishing relations in terms of inequalities in the complex plane. Padmanabhan and Parvatham [21] gave a unified representation of various classes of starlike and convex functions using convolution with the function $z/(1-z)^\alpha$, for $\alpha \in \mathbb{R}$. Later in 1989, for a convex function h and a fixed function g, Shanmugam [22] introduced a class $\mathcal{S}^*_g(h)$ which consists of functions $f \in \mathcal{A}$ satisfying $z(f*g)'(z)/(f*g)(z) \prec h(z)$. Further, if $g(z) = z/(1-z)$ and $h = \varphi$ is an analytic function with positive real part in $\mathbb{D}$ such that $\varphi(0) = 1$, $\varphi'(0) > 0$ and $\varphi(\mathbb{D})$ is symmetric about the real axis and starlike with respect to 1, then the class $\mathcal{S}^*_g(h)$ reduces to the class $\mathcal{S}^*(\varphi)$ which was introduced by Ma and Minda [16]. The growth, distortion and covering theorems for the class $\mathcal{S}^*(\varphi)$ are also proved in [16]. For particular choices of φ, we have the following subclasses of the univalent functions. If $\varphi(z) = (1+Az)/(1+Bz)$, where $-1 \leq B < A \leq 1$, then the class $\mathcal{S}^*(\varphi)$ is termed as the class of *Janowski starlike functions* [10], denoted by $\mathcal{S}^*[A,B]$. For $0 \leq \beta < 1$, the class $\mathcal{S}^*[1-2\beta,-1] =: \mathcal{S}^*(\beta)$ is the class of *starlike functions of order* β and for $\beta = 0$, the class $\mathcal{S}^* := \mathcal{S}^*(0)$ is simply the class of *starlike functions*. If $0 < \alpha \leq 1$, then the class $\mathcal{SS}^*(\alpha) := \mathcal{S}^*(((1+z)/(1-z))^\alpha)$ is the class of *strongly starlike functions of order* α. Similarly, the class $\mathcal{K}(\varphi)$ of convex functions consists of the univalent functions satisfying $1 + zf''(z)/f'(z) \prec \varphi(z)$. Let $\mathcal{R}(\varphi)$ be the class of univalent functions satisfying $f'(z) \prec \varphi(z)$. For $b \in \mathbb{C} \setminus \{0\}$ and $p \in \mathbb{N}$, the classes $\mathcal{R}_{b,p}(\varphi)$ and $\mathcal{S}^*_{b,p}(\varphi)$ consist of the functions of the form $f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n$ satisfying

$$1 + \frac{1}{b}\left(\frac{f'(z)}{pz^{p-1}} - 1\right) \prec \varphi(z) \quad \text{and} \quad 1 + \frac{1}{b}\left(\frac{1}{p}\frac{zf'(z)}{f(z)} - 1\right) \prec \varphi(z),$$

respectively. Ali *et al.* [4] obtained Fekete-Szegö inequalities and bound on the coefficient a_{p+3} for the functions in these classes. On coefficient estimates for the functions that belong to certain subclasses of univalent functions, one can refer [12, 14].

Analogous to the class of starlike (and convex) functions of order β (with $0 \leq \beta < 1$), the class of *bi–starlike* (and *bi-convex*) *functions of order* β, denoted by $\mathcal{S}^*_\sigma(\beta)$ (and $\mathcal{K}_\sigma(\beta)$), is the class of bi–univalent functions f such that f and analytic extension of f^{-1} to $\mathbb{D}$ are both starlike (and convex) of order β in $\mathbb{D}$. For $0 < \alpha \leq 1$, a bi–univalent function f is in class $\mathcal{S}^*_\sigma[\alpha]$ of *strongly bi–starlike functions of order* α if f and analytic extension of f^{-1} to $\mathbb{D}$ are strongly starlike functions of order α in $\mathbb{D}$. Brannan and Taha [7] introduced these classes and gave bound on initial coefficients of the functions in these classes. Also, for a function $f \in \mathcal{K}_\sigma(0)$ given by (1), they showed $|a_2| \leq 1$ and $|a_3| \leq 1$ with extremal function given by $z/(1-z)$ and its rotations. Particularly if $\beta = 0$, then the class $\mathcal{S}^*_\sigma(\beta)$ reduces to the class of bi–starlike functions. Kedzierawski [11] proved that for a bi–starlike function f of the form (1), $|a_2| \leq \sqrt{2}$. Further, [17] and [18] improved the estimates for coefficients a_2 and a_3 and also found estimates for the fourth coefficient for the functions in classes $\mathcal{S}^*_\sigma(\beta)$ and $\mathcal{S}^*_\sigma[\alpha]$. For coefficient estimates for the functions in some particular subclasses of bi–univalent functions, one may see [3, 9, 13, 19, 23, 26–28].

Let the function φ be an analytic function in $\mathbb{D}$ of the form

$$\varphi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots, \tag{2}$$

where $B_1 > 0$. For the function φ and $\lambda \geq 0$, Kumar *et al.* [13] introduced the following subclass $\mathcal{R}_\sigma(\lambda,\varphi)$ of bi–univalent functions.

Definition 1.1. *Let* $\lambda \geq 0$*. A bi–univalent function* f *given by* (1) *is in class* $\mathcal{R}_\sigma(\lambda,\varphi)$*, if it satisfies*

$$(1-\lambda)\frac{f(z)}{z} + \lambda f'(z) \prec \varphi(z) \quad \text{and} \quad (1-\lambda)\frac{g(w)}{w} + \lambda g'(w) \prec \varphi(w),$$

where g *denotes the univalent extension of* f^{-1} *to the unit disk.*

With the particular values of λ and φ, the class $\mathcal{R}_\sigma(\lambda,\varphi)$ reduces to many earlier classes as mentioned below:

(i) $\mathcal{R}_\sigma(\lambda, (1+(1-2\beta)z)/(1-z)) = \mathcal{R}_\sigma(\lambda, \beta) \quad (\lambda \geq 1; 0 \leq \beta < 1)$ [9, Definition 3.1].
(ii) $\mathcal{R}_\sigma(\lambda, ((1+z)/(1-z))^\alpha) = \mathcal{R}_{\sigma,\alpha}(\lambda) \quad (\lambda \geq 1; 0 < \alpha \leq 1)$ [9, Definition 2.1].
(iii) $\mathcal{R}_\sigma(1, \varphi) = \mathcal{R}_\sigma(\varphi)$ [3, p. 345].
(iv) $\mathcal{R}_\sigma(1, (1+(1-2\beta)z)/(1-z)) = \mathcal{R}_\sigma(\beta) \quad (0 \leq \beta < 1)$ [23, Definition 2].
(v) $\mathcal{R}_\sigma(1, ((1+z)/(1-z))^\alpha) = \mathcal{R}_{\sigma,\alpha} \quad (0 < \alpha \leq 1)$ [23, Definition 1].

The class of bi–starlike functions of Ma-Minda type was given by Ali *et al.* [3].

Definition 1.2. *A function $f \in \sigma$ of the form* (1), *is said to be in the class of* Ma-Minda bi–starlike functions, *denoted by $\mathcal{S}^*_\sigma(\varphi)$, if the following subordinations hold:*

$$\frac{zf'(z)}{f(z)} \prec \varphi(z) \quad \text{and} \quad \frac{wg'(w)}{g(w)} \prec \varphi(w),$$

where g denotes the univalent extension of f^{-1} to $\mathbb{D}$ and φ is the function of the form (2) *satisfying the conditions as in the definition of the class $\mathcal{S}^*(\varphi)$ as mentioned earlier.*

The class $\mathcal{S}^*_\sigma(\varphi)$ includes some well-known classes of the bi–univalent functions. For example:

(i) $\mathcal{S}^*_\sigma((1+(1-2\beta)z)/(1-z)) =: \mathcal{S}^*_\sigma(\beta), 0 \leq \beta < 1$.
(ii) $\mathcal{S}^*_\sigma(((1+z)/(1-z))^\alpha) =: \mathcal{S}^*_\sigma[\alpha], 0 < \alpha \leq 1$.

In this paper, using the Fekete-Szegö inequalities and principles of subordination, the estimates for the coefficients a_2 and a_3 of the functions of the form (1) in the classes $\mathcal{R}_\sigma(\lambda, \varphi)$ and $\mathcal{S}^*_\sigma(\varphi)$ have been obtained. The estimates so obtained are observed to be an improvement over the ones derived in [3, 5, 13]. For some particular choices of λ and φ, the bounds determined are smaller than those mentioned in [7, 9, 17, 18, 23].

More precisely, the following theorem derives the estimates for the coefficients a_2 and a_3 for the functions given by (1) that belong to the class $\mathcal{R}_\sigma(\lambda, \varphi)$.

Theorem 1.3. *Let φ be an analytic function given by the series* (2) *such that $B_2 \in \mathbb{R}$. For $\lambda \geq 0$, let the function $f \in \mathcal{R}_\sigma(\lambda, \varphi)$ and $\tau := ((1+\lambda)^2)/(1+2\lambda)$.*

(a) *If $\tau B_2 \leq B_1^2$, then*

$$|a_2| \leq \frac{B_1\sqrt{B_1}}{\sqrt{(1+2\lambda)(B_1^2 - \tau B_2 + \tau B_1)}} \quad \text{and} \quad |a_3| \leq \frac{B_1}{1+2\lambda} \max\left\{\frac{B_1^2}{B_1^2 - \tau B_2 + \tau B_1}, 1\right\}.$$

(b) *If $\tau B_2 \geq B_1^2$, then*

$$|a_2| \leq \frac{B_1\sqrt{B_1}}{\sqrt{(1+2\lambda)(\tau B_2 + \tau B_1 - B_1^2)}} \quad \text{and} \quad |a_3| \leq \frac{B_1}{1+2\lambda} \max\left\{\frac{B_1^2}{\tau B_2 + \tau B_1 - B_1^2}, 1\right\}.$$

Remark 1.4. *For an analytic function φ of the form* (2), *Kumar* et al. *[13, Theorem 2.2] obtained a bound on the coefficient a_2 of the function $f \in \mathcal{R}_\sigma(\lambda, \varphi)$ for $\lambda \geq 0$. In addition, if $B_2 \in \mathbb{R}$, by the means of the following comparisons, it may be noted that Theorem 1.3 gives an estimate for a_2 which is smaller than the one given by [13, Theorem 2.2]. We can see that if $\tau B_2 \leq B_1^2$ and $B_2 \leq B_1$, then*

$$\frac{2B_1 - B_2}{B_1} - \frac{B_1^2}{B_1^2 - \tau B_2 + \tau B_1} = \frac{(B_1 - B_2)(2\tau B_1 + B_1^2 - \tau B_2)}{B_1(B_1^2 - \tau B_2 + \tau B_1)} \geq 0.$$

Therefore,

$$\min\left\{\frac{2B_1 - B_2}{B_1}, \frac{B_1^2}{B_1^2 - \tau B_2 + \tau B_1}\right\} = \frac{B_1^2}{B_1^2 - \tau B_2 + \tau B_1}$$

which implies that the estimate obtained for a_2 using Theorem 1.3 for this case is less than $\sqrt{(2B_1 - B_2)/(1+2\lambda)}$. *Similarly, if $\tau B_2 \leq B_1^2$ and $B_2 \geq B_1$, then*

$$\frac{B_2}{B_1} - \frac{B_1^2}{B_1^2 - \tau B_2 + \tau B_1} = \frac{(B_1^2 - \tau B_2)(B_2 - B_1)}{B_1(B_1^2 - \tau B_2 + \tau B_1)} \geq 0.$$

Next, in the case when the conditions $\tau B_2 \geq B_1^2$ and $B_2 \leq B_1$ hold, we have

$$\frac{2B_1 - B_2}{B_1} - \frac{B_1^2}{\tau B_2 + \tau B_1 - B_1^2} = \frac{\tau B_1 B_2 - 3B_1^3 + 2B_1^2\tau - \tau B_2^2 + B_2 B_1^2}{B_1(\tau B_2 + \tau B_1 - B_1^2)}$$

Using the conditions $\tau B_2 \geq B_1^2$ and $B_2 \leq B_1$, we get

$$\frac{2B_1 - B_2}{B_1} - \frac{B_1^2}{\tau B_2 + \tau B_1 - B_1^2} \geq \frac{B_1^3 - 3B_1^3 + 2B_1^2\tau - \tau B_1^2 + B_1^4/\tau}{B_1(\tau B_2 + \tau B_1 - B_1^2)} = \frac{B_1(\tau - B_1)^2}{\tau(\tau B_2 + \tau B_1 - B_1^2)} \geq 0$$

and further, if $\tau B_2 \geq B_1^2$ and $B_2 \geq B_1$, then the inequality

$$\frac{B_2}{B_1} - \frac{B_1^2}{\tau B_2 + \tau B_1 - B_1^2} = \frac{(\tau B_2 - B_1^2)(B_2 + B_1)}{B_1(\tau B_2 + \tau B_1 - B_1^2)} \geq 0$$

holds.

Now let us consider the class $\mathcal{R}_{\sigma,\alpha}(\lambda) := \mathcal{R}_\sigma(\lambda, ((1+z)/(1-z))^\alpha)$ for $0 < \alpha \leq 1$. Clearly, $B_1 = 2\alpha$ and $B_2 = 2\alpha^2$. For a function f given by (1) *in the class $\mathcal{R}_{\sigma,\alpha}(\lambda)$, Theorem 1.3 yields*

$$|a_2| \leq \begin{cases} \frac{2\alpha}{\sqrt{(1+\lambda)^2 + \alpha(1-\lambda^2+2\lambda)}} & \text{if} \quad 1 \leq \lambda \leq 1+\sqrt{2}, \\ \frac{2\alpha}{\sqrt{(1+\lambda)^2 - \alpha(1-\lambda^2+2\lambda)}} & \text{if} \quad \lambda \geq 1+\sqrt{2}. \end{cases}$$

It can be verified that if $1 \leq \lambda \leq 1+\sqrt{2}$, then the bound derived for a_2 coincides with that obtained by Frasin and Aouf [9, Theorem 2.2], whereas the estimate obtained for a_2 for the part $\lambda \geq 1+\sqrt{2}$ is smaller than that in [9, Theorem 2.2]. Likewise, using Theorem 1.3, we can see that $|a_3| \leq 2\alpha/(1+2\lambda)$ which is less than the bound for a_3 derived in [9, Theorem 2.2].

Similarly, let $\varphi(z) = (1+(1-2\beta)z)/(1-z)$ for $0 \leq \beta < 1$. As a result of Theorem 1.3, the functions in the class $\mathcal{R}_\sigma(\lambda,\beta)$ satisfy

$$|a_2| \leq \begin{cases} \sqrt{\frac{2(1-\beta)}{1+2\lambda}} & \text{if} \quad 0 \leq \beta \leq \frac{1-\lambda^2+2\lambda}{2(1+2\lambda)}, \\ (1-\beta)\sqrt{\frac{2}{\lambda^2+\beta(1+2\lambda)}} & \text{if} \quad \frac{1-\lambda^2+2\lambda}{2(1+2\lambda)} \leq \beta < 1 \end{cases}$$

and $|a_3| \leq 2(1-\beta)/(1+2\lambda)$. Again, the estimate for a_2 so determined for the part when $0 \leq \beta \leq (1-\lambda^2+2\lambda)/(2(1+2\lambda))$ is same as that obtained by Frasin and Aouf [9, Theorem 3.2]. For $(1-\lambda^2+2\lambda)/(2(1+2\lambda)) \leq \beta < 1$, the estimate for a_2, derived using Theorem 1.3, is refined in comparison with [9, Theorem 3.2]. The estimate for the coefficient a_3 obtained using Theorem 1.3 is smaller than the one in [9, Theorem 3.2]. Moreover, the coefficient estimates derived above for the functions in classes $\mathcal{R}_{\sigma,\alpha}(\lambda)$ and $\mathcal{R}_\sigma(\lambda,\beta)$ are valid for $\lambda \geq 0$.

Also, Ali et al. *[3, Theorem 2.1] derived bound on the coefficients a_2 and a_3 of a function $f \in \mathcal{R}_\sigma(\varphi)$ of the form* (1). *It may be noted that the estimates for the coefficients a_2 and a_3 of the function $f \in \mathcal{R}_\sigma(\varphi)$ given using Theorem 1.3 improve the estimates given in [3, Theorem 2.1] provided $\varphi''(0) \in \mathbb{R}$.*

Furthermore, the coefficient estimates for the functions in the classes $\mathcal{R}_{\sigma,\alpha}$ and $\mathcal{R}_\sigma(\beta)$ determined in [23, Theorem 1] and [23, Theorem 2], respectively are particular cases for the above-mentioned estimates.

The next theorem determines the estimates for the initial coefficients for a function in the class $\mathcal{S}^*_\sigma(\varphi)$.

Theorem 1.5. *Let $f \in \mathcal{S}^*_\sigma(\varphi)$, where $\varphi''(0) \in \mathbb{R}$.*

(a) *If $B_2 \le B_1^2$, then*

$$|a_2| \le \frac{B_1\sqrt{B_1}}{\sqrt{B_1^2 + B_1 - B_2}} \quad \text{and} \quad |a_3| \le \max\left\{\frac{B_1^3}{B_1^2 - B_2 + B_1}, \frac{B_1}{2}\right\}.$$

(b) *If $B_2 \ge B_1^2$, then*

$$|a_2| \le \frac{B_1\sqrt{B_1}}{\sqrt{B_2 + B_1 - B_1^2}} \quad \text{and} \quad |a_3| \le \max\left\{\frac{B_1^3}{B_2 + B_1 - B_1^2}, \frac{B_1}{2}\right\}.$$

Remark 1.6. *Ali* et al. *[3, Corollary 2.1], Bohra* et al. *[5, Corollary 2.3] and Kumar* et al. *[13, Theorem 2.5] gave estimates on the coefficients a_2 and a_3 of the functions in the class $\mathcal{S}^*_\sigma(\varphi)$. In addition, let us assume that $B_2 \in \mathbb{R}$. By means of inequalities similar to those in Remark 1.4, we can see that the estimates for the coefficients a_2 and a_3 of a function in the class $\mathcal{S}^*_\sigma(\varphi)$ which are obtained using Theorem 1.5 improve those derived in the above references.*

Particularly if $\varphi(z) = ((1+z)/(1-z))^\alpha$, $(0 < \alpha \le 1)$, the Theorem 1.5 readily yields that for a function $f \in \mathcal{S}^*_\sigma[\alpha]$ of the form (1)*, we have $|a_2| \le 2\alpha/(\sqrt{\alpha+1})$, while*

$$|a_3| \le \alpha \quad \text{if} \quad 0 < \alpha \le 1/3, \quad \text{and} \quad |a_3| \le 4\alpha^2/(\alpha+1) \quad \text{if} \quad 1/3 \le \alpha \le 1$$

which coincides with the estimates for a_3 as mentioned in [18, Theorem 2.1]. For a function $f \in \mathcal{S}^*_\sigma(\beta)$ $(0 \le \beta < 1)$, a bi–starlike function of order β, using Theorem 1.5, we may solve to get

$$|a_2| \le \sqrt{2(1-\beta)} \quad \text{if} \quad 0 \le \beta \le 1/2, \quad \text{whereas} \quad |a_2| \le (1-\beta)\sqrt{2/\beta} \quad \text{if} \quad 1/2 \le \beta < 1.$$

Further,

$$|a_3| \le \begin{cases} 2(1-\beta) & \text{if} \quad 0 \le \beta \le 1/2, \\ 2(1-\beta)^2/\beta & \text{if} \quad 1/2 \le \beta \le 2/3, \\ 1-\beta & \text{if} \quad 2/3 \le \beta < 1. \end{cases}$$

The bounds for a_2 and a_3 obtained above are smaller than those given by [17]. Also it can be seen that the bounds obtained as a result of Theorem 1.5 are an improvement over the ones given by Brannan and Taha [7].

2. Proofs of the main results

We now prove following lemmas which are useful to prove the main result.

Lemma 2.1. *Let $\xi \in \mathbb{R}$, $\eta > 0$. Let the function $G\colon \mathbb{R} \to [0,\infty)$ be defined by*

$$G(x) := \max\{1, |\eta x - \xi|\}.$$

Then

$$\inf_{x,y\in\mathbb{R}} \frac{G(x)+G(y)}{|2-x-y|} = \begin{cases} \frac{1}{1-\gamma} & \text{if} \quad \xi \le \eta, \\ \frac{1}{\rho-1} & \text{if} \quad \xi \ge \eta, \end{cases}$$

where $\gamma := (\xi-1)/\eta$ and $\rho := (\xi+1)/\eta$.

Proof. The function $G(x)$ can be simplified to

$$G(x) = \begin{cases} \xi - \eta x & \text{if} \quad x \le \gamma, \\ 1 & \text{if} \quad \gamma \le x \le \rho, \\ \eta x - \xi & \text{if} \quad x \ge \rho. \end{cases}$$

Let H be a function on $\mathbb{R}^2 \setminus \{(x, y) : x + y = 2\}$ defined by

$$H(x, y) := \frac{G(x) + G(y)}{|2 - x - y|}.$$

Being a non-negative real-valued function, H has a non-negative infimum in $\mathbb{R}$. To find the infimum of the function H, we consider the possibilities as x and y vary in the domain of definition. Therefore, the procedure is divided into the following nine cases.

1. $x \le \gamma$ and $y \le \gamma$.
2. $x \le \gamma$ and $\gamma \le y \le \rho$.
3. $x \le \gamma$ and $y \ge \rho$.
4. $\gamma \le x \le \rho$ and $\gamma \le y \le \rho$.
5. $x \ge \rho$ and $\gamma \le y \le \rho$.
6. $x \ge \rho$ and $y \ge \rho$.
7. $\gamma \le x \le \rho$ and $y \le \gamma$.
8. $\gamma \le x \le \rho$ and $y \ge \rho$.
9. $x \ge \rho$ and $y \le \gamma$.

Since H is a symmetric function in the variables x and y, the infimum of the function H in Case 7 coincides with that in Case 2. Similarly, the conditions in Case 9 and Case 8 result into the same infimum of the function H as that from Case 3 and Case 5, respectively. Therefore, the process of determining the infimum of the function H reduces to finding infimum of H in the six cases:

Case 1: $x \le \gamma$ and $y \le \gamma$.

The function H becomes

$$H(x, y) = \frac{1}{|2 - x - y|}(2\xi - (x + y)\eta).$$

This case may be divided into three subcases *viz.* $\xi \le \eta$, $\eta \le \xi \le \eta + 1$ and $\xi \ge \eta + 1$. If $\xi \le \eta$, it is clear that $\gamma < 1$. Hence, the function H becomes

$$H(x, y) = \frac{2\xi - (x + y)\eta}{2 - x - y} = \eta + \frac{2(\xi - \eta)}{2 - x - y}. \tag{3}$$

Since $\xi \le \eta$, the function H satisfies $H(x, y) \le \eta$. We intend to find the infimum of the function H. In view of the conditions $x \le \gamma$ and $y \le \gamma$, we have $x + y \le 2\gamma$ and hence

$$H(x, y) \ge \eta + \frac{2(\xi - \eta)}{2 - \gamma - \gamma} = H(\gamma, \gamma).$$

Therefore, $H(x, y)$ attains its minimum at the point (γ, γ) and

$$\min H(x, y) = H(\gamma, \gamma) = \frac{1}{1 - \gamma}.$$

Let us now assume that $\eta \le \xi \le \eta + 1$ which implies $\gamma \le 1$. In this case, the function H has the form (3) and satisfies $H(x, y) \ge \eta$ for $x, y \le \gamma$. Therefore, in this case, $\inf H(x, y) = \eta$.

Suppose $\xi > \eta + 1$ which means $\gamma > 1$. Whenever $x + y < 2$, by virtue of the subcase $\eta \leq \xi \leq \eta + 1$, we observe that $\inf H(x, y) = \eta$, whereas if $x + y > 2$, then the function H is given by

$$H(x, y) = \frac{2\xi - (x + y)\eta}{x + y - 2} = -\eta + \frac{2(\xi - \eta)}{x + y - 2}.$$

It is known that $x, y \leq \gamma$. We observe that $H(x, y) \geq H(\gamma, \gamma) = 1/(\gamma - 1)$. On choosing the least amongst the values η and $1/(\gamma - 1)$, we infer that

$$\inf H(x, y) = \begin{cases} \eta & \text{if} \quad \eta + 1 < \xi \leq \eta + 2, \\ 1/(\gamma - 1) & \text{otherwise} \end{cases}$$

provided $\xi > \eta + 1$. In view of the observations made above in each of the subcases, on selecting the least of the corresponding infimum, it follows that

$$\inf_{x \leq \gamma, y \leq \gamma} H(x, y) = \begin{cases} \eta & \text{if} \quad \eta \leq \xi \leq \eta + 2, \\ 1/|1 - \gamma| & \text{otherwise.} \end{cases}$$

Case 2: $x \leq \gamma$ and $\gamma \leq y \leq \rho$.
In this case, the function H becomes

$$H(x, y) = \frac{1}{|2 - x - y|}(\xi + 1 - x\eta) = \frac{\eta}{|2 - x - y|}(\rho - x).$$

We may divide this case into the subcases $\xi \leq \eta$, $\eta \leq \xi \leq \eta + 1$ and $\xi \geq \eta + 1$. Whenever $\xi \leq \eta$, we have $\gamma + \rho \leq 2$. Since $x \leq \gamma$ and $\gamma \leq y \leq \rho$, the function H reduces to

$$H(x, y) = \frac{\xi - \eta x + 1}{2 - x - y} = \frac{\eta(\rho - x)}{2 - x - y}.$$

Since $x \leq \gamma < \rho$, using the condition $y \geq \gamma$. we get that

$$H(x, y) \geq \frac{\eta(\rho - x)}{2 - x - \gamma} = H(x, \gamma).$$

Being a decreasing function of x, the function $H(x, \gamma)$ attains its minimum at the point (γ, γ) and in this situation, $\min H(x, y) = 1/(1 - \gamma)$.

Let us assume that $\eta < \xi \leq \eta + 1$ which is same as $\gamma + \rho > 2$ and $\gamma \leq 1$. If $x + y < 2$, it is computed above that $H(x, y) \geq H(x, \gamma)$. In this case, since the function $H(x, \gamma)$ is increasing in x, we have $\inf H(x, y) = \lim_{x \to -\infty} H(x, \gamma) = \eta$. On the other hand, if $x + y > 2$, performing similar computations, we get

$$H(x, y) \geq H(x, \rho) \geq H(\gamma, \rho) = \frac{2}{\gamma + \rho - 2}.$$

Whenever $\xi \geq \eta + 1$, as observed above, we get that $\inf H(x, y) = \lim_{x \to -\infty} H(x, \gamma) = \eta$ provided $x + y < 2$, while the condition $x + y > 2$ results into

$$\min H(x, y) = H(\gamma, \rho) = \frac{2}{\gamma + \rho - 2}.$$

Choosing the least of all the values so obtained, we deduce that

$$\inf_{x \leq \gamma, \gamma \leq y \leq \rho} H(x, y) = \begin{cases} 1/(1 - \gamma) & \text{if} \quad \xi \leq \eta, \\ \eta & \text{if} \quad \eta \leq \xi \leq \eta + 1, \\ 2/(\gamma + \rho - 2) & \text{if} \quad \xi \geq \eta + 1. \end{cases}$$

Working on similar lines, infimum for the function H for each case may be observed as follows:

Case 3: $x \leq \gamma$ and $y \geq \rho$.

The Case 3 can be viewed in three subcases as given by $\xi \leq \eta - 1$, $\eta - 1 \leq \xi \leq \eta + 1$ and $\xi \geq \eta + 1$. For $\xi \leq \eta - 1$, we can see that the minimum of the function H is attained at the point (γ, ρ) with the minimum value $2/(2 - \gamma - \rho)$, whereas if $\eta - 1 \leq \xi \leq \eta + 1$, then

$$\inf H(x, y) = \lim_{x \to -\infty} H(x, y) = \lim_{y \to \infty} = H(x, y) = \eta.$$

Solving for the part $\xi \geq \eta + 1$ similarly and selecting the least value, we get

$$\inf_{x \leq \gamma, y \geq \rho} H(x, y) = \begin{cases} \eta & \text{if} \quad \eta - 1 \leq \xi \leq \eta + 1, \\ 2/(|2 - \gamma - \rho|) & \text{otherwise.} \end{cases}$$

Case 4: $\gamma \leq x \leq \rho$ and $\gamma \leq y \leq \rho$.

The function H can be written as

$$H(x, y) = \frac{2}{|2 - x - y|}.$$

Whenever $x + y < 2$, the function H attains its minimum at the point (γ, γ) and $\min H(x, y) = 1/(1 - \gamma)$. On the other hand, if $x + y > 2$, then the function H has the minimum value $1/(\rho - 1)$. Calculating the least of the two values, we get that

$$\min_{\gamma \leq x \leq \rho, \gamma \leq y \leq \rho} H(x, y) = \begin{cases} 1/(1 - \gamma) & \text{if} \quad \xi \leq \eta, \\ 1/(\rho - 1) & \text{if} \quad \xi \geq \eta. \end{cases}$$

Case 5: $x \geq \rho$ and $\gamma \leq y \leq \rho$.

Let $\xi \leq \eta - 1$ which signifies the condition $\rho \leq 1$. In this case, $\min H(x, y) = H(\rho, \gamma) = 2/(2 - \gamma - \rho)$. Suppose that $\eta - 1 \leq \xi \leq \eta$ that is $\gamma + \rho \leq 2$ and $\rho \geq 1$ for which we have $\inf H(x, y) = \lim_{x \to \infty} H(x, y) = \eta$. Given that $\xi \geq \eta$ that is $\gamma + \rho \geq 2$, $\min H(x, y) = H(\rho, \rho) = 1/(\rho - 1)$. Consequently, we have

$$\inf_{x \geq \rho, \gamma \leq y \leq \rho} H(x, y) = \begin{cases} 2/(2 - \gamma - \rho) & \text{if} \quad \xi \leq \eta - 1, \\ \eta & \text{if} \quad \eta - 1 \leq \xi \leq \eta, \\ 1/(\rho - 1) & \text{if} \quad \eta \leq \xi. \end{cases}$$

Case 6: $x \geq \rho$ and $y \geq \rho$.

Again, the case may be divided into the subcases given by $\xi \leq \eta - 1$, $\eta - 1 \leq \xi \leq \eta$ and $\xi \geq \eta$. On the similar lines as followed in the Case 1, it can be verified that

$$\inf_{x \geq \rho, y \geq \rho} H(x, y) = \begin{cases} \eta & \text{if} \quad \eta - 2 \leq \xi \leq \eta, \\ 1/|\rho - 1| & \text{otherwise.} \end{cases}$$

In view of the six cases, we use some simple computations to select the least value amongst all the values of infimum obtained above. Therefore, it can be concluded that

$$\inf_{x, y \in \mathbb{R}} H(x, y) = \begin{cases} 1/(1 - \gamma) & \text{if} \quad \xi \leq \eta, \\ 1/(\rho - 1) & \text{if} \quad \xi \geq \eta \end{cases}$$

and the lemma holds. □

Lemma 2.2. *Let $\xi \in \mathbb{R}$ and $\eta > 0$. Let the function $G\colon \mathbb{R} \to [0, \infty)$ be defined as in Lemma 2.1. Then*

$$\inf_{x, y \in \mathbb{R}} \frac{|2 - y|G(x) + |x|G(y)}{|2 - x - y|} = \begin{cases} \frac{1}{1-\gamma} & \text{if} \quad 1 \leq \xi \leq \eta, \\ \frac{1}{\rho-1} & \text{if} \quad \eta \leq \xi \leq 2\eta - 1, \\ 1 & \text{otherwise,} \end{cases}$$

where $\gamma := (\xi - 1)/\eta$ and $\rho := (\xi + 1)/\eta$.

Proof. Let H be a function on $\mathbb{R}^2 \setminus \{(x,y) : x+y=2\}$ defined by

$$H(x,y) := \frac{|2-y|G(x)+|x|G(y)}{|2-x-y|}.$$

Being a non-negative real-valued function, H has a non-negative infimum in $\mathbb{R}$. In order to obtain the infimum of the function H, we consider the following cases:

Case 1: $x \le \gamma$ and $y \le \gamma$.
The function H becomes

$$H(x,y) = \frac{1}{|2-x-y|}\left(|2-y|(\xi-\eta x)+|x|(\xi-\eta y)\right).$$

To minimize the function H, this case is divided into the subcases *viz.* $\xi \le 1$, $1 \le \xi \le \eta+1$, $\eta+1 \le \xi \le 2\eta+1$ and $\xi \ge 2\eta+1$. If $\xi \le 1$ which yields $\gamma \le 0$, the function H simplifies to

$$H(x,y) = \xi - \frac{2\eta x(1-y)}{2-x-y}. \tag{4}$$

Since the function H has no critical points in the set $(-\infty,\gamma)\times(-\infty,\gamma)$, it does not acquire a minimum value in this region. Thus, the function H has infimum as x or y approach $-\infty$ or has minimum along the edges $x=\gamma$ or $y=\gamma$. It can easily be verified that along the edge $y=\gamma$, the function $H(x,\gamma)$ is a decreasing function of x, hence attains its minimum at the point (γ,γ) with the minimum value $H(\gamma,\gamma)=1$ and so does the function $H(\gamma,y)$. Also, we may note that the values $\lim_{x\to-\infty} H(x,y) = \xi - 2\eta(1-y)$ and $\lim_{y\to-\infty} H(x,y) = \xi - 2\eta x$ exceed 1. This implies $\inf H(x,y) = \min H(x,y) = 1$ whenever $\xi \le 1$.

Now if $1 \le \xi \le \eta+1$ which means $0 \le \gamma \le 1$, it is clear that $y < 2$. So, the case can be split into parts when $x \le 0$ and when $x \ge 0$. If $x \le 0$, then the function $H(x,y)$ is given by equation (4). Since $y \le \gamma \le 1$ and $x \le 0$, we have $H(x,y) \ge \xi$. Further, if $x \ge 0$, then the function H satisfies

$$H(x,y) = \xi + \frac{2x(\xi-\eta)}{2-x-y} \ge \xi$$

provided $\xi \ge \eta$. Let $\xi \le \eta$. It is known that $y \le \gamma$. Thus, with $x \ge 0$, we can observe that

$$H(x,y) \ge \xi + \frac{2x(\xi-\eta)}{2-x-\gamma} = H(x,\gamma).$$

The function $H(x,\gamma)$ being a decreasing function of x attains its minimum at the point (γ,γ) with the minimum value $\min H(\gamma,\gamma) = 1/(1-\gamma)$. Consequently, for $1 \le \xi \le \eta+1$, simplifying the values so obtained, it can be seen that if $\eta \le 1$, then $\inf H(x,y) = \xi$, otherwise we have

$$\inf H(x,y) = \begin{cases} 1/(1-\gamma) & \text{if } \ 1 \le \xi \le \eta, \\ \xi & \text{if } \ \eta \le \xi \le \eta+1. \end{cases}$$

The subcase when we have $\eta+1 < \xi \le 2\eta+1$ which results into $1 < \gamma \le 2$ is further divided in the parts as $x \le 0$, $x \ge 0$ and $x+y<2$, and $x \ge 0$ and $x+y>2$. Let $x \le 0$. It can be verified that H is a decreasing function of y and hence

$$H(x,y) \ge H(x,\gamma) = \xi - \frac{2\eta x(1-\gamma)}{2-x-\gamma}.$$

The function $H(x,\gamma) \ge \lim_{x\to-\infty} H(x,\gamma) = 2\eta - \xi + 2$. Suppose that $x \ge 0$ and $x+y<2$. It is known that $\xi > \eta+1$. Hence, the function H satisfies

$$H(x,y) = \xi + \frac{2x(\xi-\eta)}{2-x-y} \ge \xi.$$

Thus, infimum of the function $H(x, y)$, in this case, is ξ. The part when $x \geq 0$ and $x + y > 2$, the minimum value of the function $H(x, y)$ occurs at the point (γ, γ) with $H(\gamma, \gamma) = 1/(\gamma - 1)$. Choosing minimum amongst the values so obtained, for $\eta + 1 < \xi \leq 2\eta + 1$, we observe that if $\eta \leq 1$, then $\min H(x, y) = 2\eta - \xi + 2$. If $\eta \geq 1$, then

$$\inf H(x,y) = \begin{cases} 2\eta - \xi + 2 & \text{if} \quad \eta + 1 < \xi \leq \eta + 2, \\ 1/(\gamma - 1) & \text{if} \quad \eta + 2 \leq \xi \leq 2\eta + 1. \end{cases}$$

The case $\xi \geq 2\eta + 1$ which is equivalent to $\gamma \geq 2$ may be subdivided into six parts depending upon the signs of $2 - y$, x and $2 - x - y$. For the part $x \leq 0$ and $y \leq 2$, being a decreasing function of y, the function H has its infimum as $y = 2$ and hence, $H(x, y) \geq H(x, 2) = \xi - 2\eta$. The case when $x \geq 0$ and $x + y < 2$, it is clear that $y < 2$ and the function H satisfies

$$H(x,y) = \xi + \frac{2x(\xi - \eta)}{2 - x - y} \geq \xi.$$

Similarly, for $x \geq 0$, $y \leq 2$ and $x + y > 2$, we get that the function $H(x, y)$ satisfies

$$H(x,y) \geq -\xi + \frac{2x(\xi - \eta)}{x} = \xi - 2\eta.$$

Likewise, the minimum of the function $H(x, y)$ for the case $x \geq 0$ and $y \geq 2$ is 1. Further, for the part $x \leq 0$, $y \geq 2$ with $x + y > 2$, the function $H(x, \gamma)$ has minimum given by $H(0, \gamma) = \xi$. If $x \leq 0$, $y \geq 2$ with $x + y < 2$, then $\inf H(x, y) = \xi - 2\eta$. The infimum of the function $H(x, y)$ is the least of the values for infimum obtained in different parts above. In this way, we see that the $\min H(x, y) = 1$ whenever $\xi \geq 2\eta + 1$.

Briefly, we observe that if $\eta \leq 1$, then

$$\inf_{x\leq\gamma, y\leq\gamma} H(x,y) = \begin{cases} \xi & \text{if} \quad 1 \leq \xi \leq \eta + 1, \\ 2\eta - \xi + 2 & \text{if} \quad \eta + 1 \leq \xi \leq 2\eta + 1, \\ 1 & \text{otherwise.} \end{cases}$$

But if $\eta \geq 1$, then

$$\inf_{x\leq\gamma, y\leq\gamma} H(x,y) = \begin{cases} 1/(1 - \gamma) & \text{if} \quad 1 \leq \xi \leq \eta, \\ \xi & \text{if} \quad \eta \leq \xi \leq \eta + 1, \\ 2\eta - \xi + 2 & \text{if} \quad \eta + 1 \leq \xi \leq \eta + 2, \\ 1/(\gamma - 1) & \text{if} \quad \eta + 2 \leq \xi \leq 2\eta + 1, \\ 1 & \text{otherwise.} \end{cases}$$

For the rest of the cases, we follow a similar trend by dividing the subcases into parts in accordance with the sign of $2 - y$, x and $2 - x - y$. The analysis of their subcases is done as in the Case 1. In order to avoid the length of the proof of the lemma, the estimates are given. The details are left for the reader. In this way, we obtain the infimum of the function H due to the cases that follow.

Case 2: $x \leq \gamma$ and $\gamma \leq y \leq \rho$.
The case if $\eta \leq 1$, we can solve to get

$$\inf_{x\leq\gamma, \gamma\leq y\leq\rho} H(x,y) = 1.$$

Let $1 \leq \eta \leq 2$. Then

$$\inf_{x\leq\gamma, \gamma\leq y\leq\rho} H(x,y) = \begin{cases} 1/(1 - \gamma) & \text{if} \quad 1 \leq \xi \leq \eta, \\ 2\eta - \xi & \text{if} \quad \eta \leq \xi \leq 2\eta - 1, \\ 1 & \text{otherwise} \end{cases}$$

and the case when $\eta \geq 2$, we have

$$\inf_{x\leq\gamma,\gamma\leq y\leq\rho} H(x,y) = \begin{cases} 1/(1-\gamma) & \text{if} \quad 1 \leq \xi \leq \eta, \\ 2\eta - \xi & \text{if} \quad \eta \leq \xi \leq \eta+1, \\ (2-\rho+\gamma)/(\gamma+\rho-2) & \text{if} \quad \eta+1 \leq \xi \leq 2\eta-1, \\ 1 & \text{otherwise.} \end{cases}$$

Case 3: $x \leq \gamma$ and $y \geq \rho$.
For $\eta \leq 1$, we have that the function $H(x,y)$ has infimum given by

$$\inf_{x\leq\gamma,y\geq\rho} H(x,y) = \begin{cases} \xi - 2\eta + 2 & \text{if} \quad 2\eta-1 \leq \xi \leq \eta, \\ 2-\xi & \text{if} \quad \eta \leq \xi \leq 1, \\ 1 & \text{otherwise.} \end{cases}$$

If we have the condition $1 \leq \eta \leq 2$, then

$$\inf_{x\leq\gamma,y\geq\rho} H(x,y) = \begin{cases} \xi & \text{if} \quad 1 \leq \xi \leq \eta, \\ 2\eta-\xi & \text{if} \quad \eta \leq \xi \leq 2\eta-1, \\ 1 & \text{otherwise.} \end{cases}$$

The possibility when $\eta \geq 2$, then it may noted that

$$\inf_{x\leq\gamma,y\geq\rho} H(x,y) = \begin{cases} (2-\rho+\gamma)/(2-\gamma-\rho) & \text{if} \quad 1 \leq \xi \leq \eta-1, \\ \xi & \text{if} \quad \eta-1 \leq \xi \leq \eta, \\ 2\eta-\xi & \text{if} \quad \eta \leq \xi \leq \eta+1, \\ (2-\rho+\gamma)/(\gamma+\rho-2) & \text{if} \quad \eta+1 \leq \xi \leq 2\eta-1, \\ 1 & \text{otherwise.} \end{cases}$$

Case 4: $\gamma \leq x \leq \rho$ and $y \leq \gamma$.
As in the cases above, we have infimum of the function $H(x,y)$ to be ξ whenever $1 \leq \xi \leq \eta+1$; $(2+\rho-\gamma)/(\gamma+\rho-2)$ if $\eta+1 \leq \xi \leq 2\eta+1$ and 1 elsewhere provided $\eta \leq 1$. In case $\eta \geq 1$, then

$$\inf_{\gamma\leq x\leq\rho,y\leq\gamma} H(x,y) = \begin{cases} 1/(1-\gamma) & \text{if} \quad 1 \leq \xi \leq \eta, \\ \xi & \text{if} \quad \eta \leq \xi \leq \eta+1, \\ (2+\rho-\gamma)/(\gamma+\rho-2) & \text{if} \quad \eta+1 \leq \xi \leq 2\eta+1, \\ 1 & \text{otherwise.} \end{cases}$$

Case 5: $\gamma \leq x \leq \rho$ and $\gamma \leq y \leq \rho$.
We compute that for the case if $\eta \leq 1$, $\inf H(x,y) = 1$ as ξ ranges over the real line. For the part when $\eta \geq 1$, we have

$$\inf_{\gamma\leq x\leq\rho,\gamma\leq y\leq\rho} H(x,y) = \begin{cases} 1/(1-\gamma) & \text{if} \quad 1 \leq \xi \leq \eta, \\ 1/(\rho-1) & \text{if} \quad \eta \leq \xi \leq 2\eta-1, \\ 1 & \text{otherwise.} \end{cases}$$

Case 6: $\gamma \leq x \leq \rho$ and $y \geq \rho$.
Again with condition $\eta \leq 1$, we note that $\inf H(x,y) = 1$ for $\xi \in \mathbb{R}$ and the case when $1 \leq \eta \leq 2$, then $\inf H(x,y) = \xi$ whenever $1 \leq \xi \leq \eta$ and $\inf H(x,y) = 1/(\rho-1)$ provided $\eta \leq \xi \leq 2\eta-1$ and the infimum is 1 elsewhere. On the other hand, if $\eta \geq 2$, then

$$\inf_{\gamma\leq x\leq\rho,y\geq\rho} H(x,y) = \begin{cases} (2-\rho+\gamma)/(2-\gamma-\rho) & \text{if} \quad 1 \leq \xi \leq \eta-1, \\ \xi & \text{if} \quad \eta-1 \leq \xi \leq \eta, \\ 1/(\rho-1) & \text{if} \quad \eta \leq \xi \leq 2\eta-1, \\ 1 & \text{otherwise.} \end{cases}$$

Case 7: $x \geq \rho$ and $y \leq \gamma$.
In this case, we may see that

$$\inf_{x\geq\rho, y\leq\gamma} H(x,y) = \begin{cases} (2+\rho-\gamma)/(2-\gamma-\rho) & \text{if} \quad -1 \leq \xi \leq \eta - 1, \\ \xi + 2 & \text{if} \quad \eta - 1 \leq \xi \leq \eta, \\ 2\eta - \xi + 2 & \text{if} \quad \eta \leq \xi \leq \eta + 1, \\ (2+\rho-\gamma)/(\gamma+\rho-2) & \text{if} \quad \eta + 1 \leq \xi \leq 2\eta + 1, \\ 1 & \text{otherwise.} \end{cases}$$

Case 8: $x \geq \rho$ and $\gamma \leq y \leq \rho$.
This case is also partitioned as $\eta \leq 1$ and $\eta \geq 1$. Let $\eta \leq 1$. Then the function H has infimum given by $(2+\rho-\gamma)/(2-\gamma-\rho)$, whenever $-1 \leq \xi \leq \eta - 1$ and is given by $2\eta - \xi$ if $\eta - 1 \leq \xi \leq 2\eta - 1$ and is otherwise 1. If $\eta \geq 1$, then it can be seen that

$$\inf_{x\geq\rho, \gamma\leq y\leq\rho} H(x,y) = \begin{cases} (2+\rho-\gamma)/(2-\gamma-\rho) & \text{if} \quad -1 \leq \xi \leq \eta - 1, \\ 2\eta - \xi & \text{if} \quad \eta - 1 \leq \xi \leq \eta, \\ 1/(\rho-1) & \text{if} \quad \eta \leq \xi \leq 2\eta - 1, \\ 1 & \text{otherwise.} \end{cases}$$

Case 9: $x \geq \rho$ and $y \geq \rho$.
For this case, if $\eta \leq 1$, then infimum of H happens to be $\xi + 2$ if $-1 \leq \xi \leq \eta - 1$; $2\eta - \xi$ if $\eta - 1 \leq \xi \leq 2\eta - 1$ and 1 elsewhere. For the part $\eta \geq 1$, we observe

$$\inf_{x\geq\rho, y\geq\rho} H(x,y) = \begin{cases} 1/(1-\rho) & \text{if} \quad -1 \leq \xi \leq \eta - 2, \\ \xi + 2 & \text{if} \quad \eta - 2 \leq \xi \leq \eta - 1, \\ 2\eta - \xi & \text{if} \quad \eta - 1 \leq \xi \leq \eta, \\ 1/(\rho-1) & \text{if} \quad \eta \leq \xi \leq 2\eta - 1, \\ 1 & \text{otherwise.} \end{cases}$$

Drawing the conclusion by choosing least of all the infimum values obtained above, the infimum of the function H is determined and is obtained to be

$$\inf_{x,y\in\mathbb{R}} H(x,y) = \begin{cases} \frac{1}{1-\gamma} & \text{if} \quad 1 \leq \xi \leq \eta, \\ \frac{1}{\rho-1} & \text{if} \quad \eta \leq \xi \leq 2\eta - 1, \\ 1 & \text{otherwise.} \end{cases}$$

Hence, the lemma holds. □

Using the above-mentioned lemmas, we now prove the theorems stated in Section 1.

Proof. [Proof of Theorem 1.3] Let $f \in \mathcal{R}_\sigma(\lambda, \varphi)$. Using Definition 1.1, we know that there exist two analytic functions $r, s \colon \mathbb{D} \to \mathbb{D}$ satisfying $r(0) = 0 = s(0)$ such that

$$(1-\lambda)\frac{f(z)}{z} + \lambda f'(z) = \varphi(r(z)) \quad \text{and} \quad (1-\lambda)\frac{g(w)}{w} + \lambda g'(w) = \varphi(s(w)). \tag{5}$$

Define the functions p and q by

$$p(z) := \frac{1+r(z)}{1-r(z)} = 1 + p_1 z + p_2 z^2 + \cdots \text{ and } q(w) := \frac{1+s(w)}{1-s(w)} = 1 + q_1 w + q_2 w^2 + \cdots. \tag{6}$$

It may be noted that the functions p and q are analytic with positive real part in $\mathbb{D}$ and $p(0) = 1 = q(0)$. Using equations (5) and (6), it is clear that

$$(1-\lambda)\frac{f(z)}{z} + \lambda f'(z) = \varphi\left(\frac{p(z)-1}{p(z)+1}\right) \tag{7}$$

and

$$(1-\lambda)\frac{g(w)}{w} + \lambda g'(w) = \varphi\left(\frac{q(w)-1}{q(w)+1}\right). \tag{8}$$

Comparing the coefficients on the both sides of equation (7), we have the relations

$$(1+\lambda)a_2 = \frac{B_1 p_1}{2} \quad \text{and} \quad (1+2\lambda)a_3 = \frac{B_1}{2}p_2 + \frac{p_1^2}{4}(B_2 - B_1). \tag{9}$$

Similarly, using equation (8), we get

$$(1+\lambda)a_2 = -\frac{B_1 q_1}{2} \quad \text{and} \quad (1+2\lambda)(2a_2^2 - a_3) = \frac{B_1}{2}q_2 + \frac{q_1^2}{4}(B_2 - B_1). \tag{10}$$

Some simple calculations in equation (9) yield

$$a_3 - xa_2^2 = \frac{B_1}{2(1+2\lambda)}\left(p_2 - \frac{\nu}{2}p_1^2\right), \tag{11}$$

where $\nu := x\frac{B_1}{\tau} - \frac{B_2}{B_1} + 1$ and $\tau := ((1+\lambda)^2)/(1+2\lambda)$. From [1, Lemma 2], we have

$$|p_2 - (\nu/2)p_1^2| \leq \max\{2, 2|\nu - 1|\}. \tag{12}$$

Using the inequality (12) in (11), we get

$$|a_3 - xa_2^2| \leq \frac{B_1}{1+2\lambda}\max\left\{1, \left|x\frac{B_1}{\tau} - \frac{B_2}{B_1}\right|\right\}. \tag{13}$$

A similar computation in relation (10) gives

$$|a_3 - (2-y)a_2^2| \leq \frac{B_1}{1+2\lambda}\max\left\{1, \left|y\frac{B_1}{\tau} - \frac{B_2}{B_1}\right|\right\}. \tag{14}$$

Using triangle's inequality with the inequalities (13) and (14), we arrive at

$$|(2-x-y)a_2^2| \leq |a_3 - xa_2^2| + |a_3 - (2-y)a_2^2| \leq \frac{B_1}{1+2\lambda}(G(x) + G(y)),$$

where

$$G(x) := \max\left\{1, \left|x\frac{B_1}{\tau} - \frac{B_2}{B_1}\right|\right\}.$$

Hence, if $x, y \in \mathbb{R}$, then we have

$$|a_2|^2 \leq \frac{B_1}{1+2\lambda}\inf_{x,y\in\mathbb{R}}\frac{G(x)+G(y)}{|2-x-y|}.$$

Since $B_2 \in \mathbb{R}$, in view of Lemma 2.1 by taking $\eta := B_1/\tau$ and $\xi := B_2/B_1$ which implies

$$\gamma = \frac{\tau}{B_1}\left(\frac{B_2}{B_1} - 1\right) \quad \text{and} \quad \rho = \frac{\tau}{B_1}\left(\frac{B_2}{B_1} + 1\right),$$

we have that $|a_2|$ is bounded by

$$|a_2| \leq \sqrt{\frac{B_1}{1+2\lambda}} \begin{cases} \frac{B_1}{\sqrt{B_1^2-\tau B_2+\tau B_1}} & \text{if } \frac{B_2}{B_1} \leq \frac{B_1}{\tau}, \\ \frac{B_1}{\sqrt{\tau B_2+\tau B_1-B_1^2}} & \text{if } \frac{B_2}{B_1} \geq \frac{B_1}{\tau}. \end{cases}$$

In order to obtain estimate for the coefficient a_3, we use triangle's inequality and get

$$|(2-x-y)a_3| = |(2-y)a_3 - (2-y)xa_2 - (xa_3 - (2-y)xa_2)| \leq |2-y||a_3 - xa_2| + |x||a_3 - (2-y)a_2|.$$

By means of relations (13) and (14), we obtain

$$|a_3| \leq \frac{B_1}{1+2\lambda} \inf_{x,y\in\mathbb{R}} \frac{|2-y|G(x)+|x|G(y)}{|2-x-y|}.$$

By Lemma 2.2 with $\eta := B_1/\tau$ and $\xi := B_2/B_1$, it can be seen that

$$|a_3| \leq \frac{B_1}{1+2\lambda} \begin{cases} \frac{B_1^2}{B_1^2-\tau B_2+\tau B_1} & \text{if } 1 \leq \frac{B_2}{B_1} \leq \frac{B_1}{\tau}, \\ \frac{B_1^2}{\tau B_2+\tau B_1-B_1^2} & \text{if } \frac{B_1}{\tau} \leq \frac{B_2}{B_1} \leq \frac{2B_1}{\tau} - 1, \\ 1 & \text{otherwise} \end{cases}$$

which completes the proof of the theorem. □

Illustration 2.3. *Let $\varphi(z) = (1+z)/(1-z)$ and $\lambda = 1$. For $\nu \geq \sqrt{2}$, it is asserted that the function*

$$f_\nu(z) := \frac{\nu z}{\nu - z} = z + \frac{1}{\nu}z^2 + \frac{1}{\nu^2}z^3 + \cdots \in \mathcal{R}_\sigma(1,\varphi).$$

The assertion can be justified as follows:
The analytic continuation of the inverse of the function f_ν to $\mathbb{D}$, denoted by g_ν, is given by $g_\nu(w) = \nu w/(\nu + w)$. Given that $\nu \geq \sqrt{2}$, we can see that f_ν and g_ν are univalent in $\mathbb{D}$. For f_ν to belong to the class $\mathcal{R}_\sigma(1,(1+z)/(1-z))$, it is required that for $z, w \in \mathbb{D}$, the subordinations

$$f_\nu'(z) = \frac{\nu^2}{(\nu-z)^2} \prec \frac{1+z}{1-z} \quad \text{and} \quad g_\nu'(w) = \frac{\nu^2}{(\nu+w)^2} \prec \frac{1+w}{1-w}$$

hold. These relations hold because the function $f_\nu'(z)$ maps unit disk $\mathbb{D}$ onto the domain

$$\left|\sqrt{z} - \frac{\nu^2}{\nu^2-1}\right| < \frac{\nu}{\nu^2-1}$$

which lies in the right-half plane if and only if $\nu \geq \sqrt{2}$. Similarly, $g_\nu'(w)$ maps $\mathbb{D}$ onto a domain which is contained in the right-half plane provided $\nu \geq \sqrt{2}$. Thus, we have $f_\nu \in \mathcal{R}_\sigma(1,(1+z)/(1-z))$.

Let $a_2 := 1/\nu$ and $a_3 := 1/\nu^2$. Since $\varphi(z) = (1+z)/(1-z)$, in view of equation (2), the coefficients $B_1 = B_2 = 2$ and $\tau = 4/3$ as $\lambda = 1$. Using Theorem 1.3, we obtain

$$|a_2| = \frac{1}{|\nu|} \leq \sqrt{\frac{2}{3}} \quad \text{and} \quad |a_3| = \frac{1}{|\nu^2|} \leq \frac{2}{3}.$$

From the hypothesis, we have $\nu \geq \sqrt{2}$ which implies

$$\frac{1}{\nu} \leq \frac{1}{\sqrt{2}} < \sqrt{\frac{2}{3}}.$$

Hence, the Theorem 1.3 is verified for $\lambda = 1$ and $\varphi(z) = (1+z)/(1-z)$.

Furthermore, the function $f_\nu(z)/z$ maps the unit disk onto the disk

$$\left| z - \frac{\nu^2}{\nu^2 - 1} \right| < \frac{\nu}{\nu^2 - 1} \tag{15}$$

which is contained in the right half plane provided $\nu \geq 1$. Similar is the case for the mapping $g_\nu(w)/w$. Therefore, for $\nu \geq 1$ and $z, w \in \mathbb{D}$, the subordinations

$$\frac{f_\nu(z)}{z} = \frac{\nu}{\nu - z} \prec \frac{1+z}{1-z} \quad \textit{and} \quad \frac{g_\nu(w)}{w} = \frac{\nu}{\nu + w} \prec \frac{1+w}{1-w}$$

hold. Hence, the function $f_\nu \in \mathcal{R}_\sigma(0, (1+z)/(1-z))$. Here, $\tau = 1$ and $B_1 = B_2 = 2$. Therefore, Theorem 1.3 implies

$$|a_2| = \frac{1}{|\nu|} \leq \sqrt{2} \quad \textit{and} \quad |a_3| = \frac{1}{|\nu|^2} \leq 2.$$

Since $\nu \geq 1$, we have $1/|\nu| \leq 1 < \sqrt{2}$ and hence the Theorem 1.3 holds true in this case.

Now for $\varphi(z) = \sqrt{1+z}$ and $\lambda = 0$, the function $f_\nu \in \mathcal{R}_\sigma(0, \sqrt{1+z})$ if the image of the unit disk under the mappings $f_\nu(z)/z$ and $g_\nu(w)/w$ lie in the region bounded by the right of lemniscate of Bernoulli given by $\{w : |w^2 - 1| = 1\}$. By means of [2, Lemma 2.2], it may be noted that if $\nu \geq \sqrt{2}(\sqrt{2}+1)$, then the disk (15) *is contained in the set $\{w : |w^2 - 1| < 1\}$. Thus, we infer that if $\nu \geq \sqrt{2}(\sqrt{2}+1)$, then $f_\nu \in \mathcal{R}_\sigma(0, \sqrt{1+z})$. It is known that $\nu \geq \sqrt{2}(\sqrt{2}+1)$. For $\varphi(z) = \sqrt{1+z}$, the coefficients $B_1 = 1/2$ and $B_2 = -1/8$. Substituting these values in Theorem 1.3, we gain*

$$|a_2| = \frac{1}{|\nu|} \leq \frac{1}{\sqrt{2}(\sqrt{2}+1)} < \frac{1}{\sqrt{7}} \quad \textit{and} \quad |a_3| = \frac{1}{|\nu|^2} \leq \frac{1}{2(\sqrt{2}+1)^2} < \frac{1}{2}$$

which justifies Theorem 1.3.

Analysing all the subcases as in Lemma 2.2, the following lemma holds.

Lemma 2.4. *Let $\xi \in \mathbb{R}$ and $\eta > 0$. Let the function $G \colon \mathbb{R} \to [0, \infty)$ be defined as in Lemma 2.1. Then*

$$\inf_{x,y \in \mathbb{R}} \frac{|3-y|G(x) + |x+1|G(y)}{|2-x-y|} = \begin{cases} \frac{2}{1-\gamma} & \textit{if} \quad 1-\eta \leq \xi \leq \eta, \\ \frac{2}{\rho-1} & \textit{if} \quad \eta \leq \xi \leq 3\eta - 1, \\ 1 & \textit{otherwise}, \end{cases}$$

where $\gamma := (\xi - 1)/\eta$ and $\rho := (\xi + 1)/\eta$.

Proof. [Proof of Theorem 1.5] Since the function $f \in \mathcal{S}^*_\sigma(\varphi)$, the Definition 1.2 states that there exist two Schwarz functions r and s such that

$$\frac{zf'(z)}{f(z)} = \varphi(r(z)) \quad \text{and} \quad \frac{wg'(w)}{g(w)} = \varphi(s(w)). \tag{16}$$

Let the functions p and q be defined by equation (6). Clearly, the functions p and q are analytic functions in $\mathbb{D}$ with positive real part and $p(0) = 1 = q(0)$. Therefore, equation (16) and (6) yield

$$\frac{zf'(z)}{f(z)} = \varphi\left(\frac{p(z)-1}{p(z)+1}\right) \quad \text{and} \quad \frac{wg'(w)}{g(w)} = \varphi\left(\frac{q(w)-1}{q(w)+1}\right).$$

Comparing the coefficients on each side of the above two relations, we get

$$a_2 = \frac{B_1 p_1}{2}, \quad 2a_3 - a_2^2 = \frac{B_2 p_1^2}{4} + \frac{B_1}{2}\left(p_2 - \frac{p_1^2}{2}\right),$$

$$a_2 = -\frac{B_1 q_1}{2}, \quad \text{and} \quad 3a_2^2 - 2a_3 = \frac{B_2 q_1^2}{4} + \frac{B_1}{2}\left(q_2 - \frac{q_1^2}{2}\right).$$

A similar computations as that in proof of Theorem 1.3 leads to the following inequalities:

$$|2a_3 - (x+1)a_2^2| \le B_1 G(x) \quad \text{and} \quad |2a_3 - (3-y)a_2^2| \le B_1 G(y), \tag{17}$$

where $G(x) := \max\{1, |xB_1 - B_2/B_1|\}$. On computing using triangle's inequality, it is easy to see that

$$|(2-x-y)a_2^2| \le |a_3 - xa_2^2| + |a_3 - (2-y)a_2^2| \le B_1(G(x) + G(y))$$

which implies

$$|a_2|^2 \le B_1 \inf_{x,y\in\mathbb{R}} \frac{G(x)+G(y)}{|2-x-y|}.$$

Since $B_2 \in \mathbb{R}$, upon taking $\xi = B_2/B_1$ and $\eta = B_1$, Lemma 2.1 gives

$$|a_2| \le \sqrt{\frac{B_1}{1-\gamma}} \quad \left(\text{if} \quad \frac{B_2}{B_1} \le B_1\right) \quad \text{and} \quad |a_2| \le \sqrt{\frac{B_1}{\rho - 1}} \quad \left(\text{if} \quad \frac{B_2}{B_1} \ge B_1,\right)$$

where $\gamma = \frac{1}{B_1}\left(\frac{B_2}{B_1} - 1\right)$ and $\rho = \frac{1}{B_1}\left(\frac{B_2}{B_1} + 1\right)$. Besides, keeping in view the relation (17), we may solve to get

$$|a_3| \le \frac{B_1}{2} \inf_{x,y\in\mathbb{R}} \frac{|3-y|G(x) + |x+1|G(y)}{|2-x-y|}.$$

By means of Lemma 2.4 with $\xi = B_2/B_1$ and $\eta = B_1$, on simplifying the above relations, we get the desired estimates for the third coefficient of a function in class $\mathcal{S}^*_\sigma(\varphi)$. □

Illustration 2.5. *Let $\varphi(z) = (1+z)/(1-z)$. It is asserted that for $\nu \ge 1$, the function $f_\nu(z) := \nu z/(\nu - z) \in \mathcal{S}^*_\sigma((1+z)/(1-z))$. The function f_ν and the analytic extension of its inverse to $\mathbb{D}$, denoted by g_ν, are univalent in $\mathbb{D}$ for $\nu \ge 1$. For f_ν to belong to the class $\mathcal{S}^*_\sigma((1+z)/(1-z))$, the following subordinations must hold:*

$$\frac{zf_\nu'(z)}{f_\nu(z)} = \frac{\nu}{\nu - z} \prec \frac{1+z}{1-z} \quad \text{and} \quad \frac{wg_\nu'(w)}{g_\nu(w)} = \frac{\nu}{\nu + w} \prec \frac{1+w}{1-w}.$$

As in Illustration 2.3, the functions $zf_\nu'(z)/f_\nu(z)$ and $wg_\nu'(w)/g_\nu(w)$ map the unit disk onto the region contained in the right-half plane if and only if $\nu \ge 1$. Hence, $f_\nu \in \mathcal{S}^*_\sigma((1+z)/(1-z))$ for $\nu \ge 1$. Furthermore

$$|a_2| = \frac{1}{|\nu|} \le 1 < \sqrt{2} \quad \text{and} \quad |a_3| = \frac{1}{|\nu|^2} \le 1 < 2.$$

Hence, Theorem 1.5 holds valid.

Assuming $\nu \ge \sqrt{2}(\sqrt{2}+1)$, using [2, Lemma 2.2], we can see that the mappings $zf_\nu'(z)/f_\nu(z)$ and $wg_\nu'(w)/g_\nu(w)$ map the unit disk onto the disks that are contained in the region $\{w : |w^2 - 1| < 1\}$. Hence, the function $f_\nu \in \mathcal{S}^*_\sigma(\sqrt{1+z})$. Since

$$|a_2| = \frac{1}{|\nu|} \le \frac{1}{\sqrt{2}(\sqrt{2}+1)} < \frac{1}{\sqrt{7}} \quad \text{and} \quad |a_3| = \frac{1}{|\nu|^2} \le \frac{1}{2(\sqrt{2}+1)^2} < \frac{1}{4},$$

the Theorem 1.5 gets verified.

Remark 2.6. *It may be noted that with $\varphi(z) = (1+z)/(1-z)$, whenever $\nu \ge 1$, the function $f_\nu := \nu z/(\nu - z) \in \mathcal{S}^*_\sigma(\varphi)$ and $f_\nu \in \mathcal{R}_\sigma(0,\varphi)$ but for $1 \le \nu < \sqrt{2}$, $f_\nu \notin \mathcal{R}_\sigma(1,\varphi)$.*

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