Filomat 35:6 (2021), 1963–1975
https://doi.org/10.2298/FIL2106963G

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Some Notes on Metallic Kähler Manifolds

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Abstract. The present paper deals with metallic Kähler manifolds. Firstly, we define a tensor H which can be written in terms of the $(0,4)-$Riemannian curvature tensor and the fundamental 2−form of a metallic Kähler manifold and study its properties and some hybrid tensors. Secondly, we obtain the conditions under which a metallic Hermitian manifold is conformal to a metallic Kähler manifold. Thirdly, we prove that the conformal recurrency of a metallic Kähler manifold implies its recurrency and also obtain the Riemannian curvature tensor form of a conformally recurrent metallic Kähler manifold with non-zero scalar curvature. Finally, we present a result related to the notion of Z recurrent form on a metallic Kähler manifold.

1. Introduction

It is known that the number $\eta = \frac{1+\sqrt{5}}{2} \approx 1,61803398874989...$, which is the positive root of the equation $x^2 - x - 1 = 0$, is called the golden mean. As a literature review related to golden mean has been done, we see that two well-known generalizations have existed. First of them is called the golden $p-$proportions of golden mean and defined as positive root of the equation $x^{p+1} - x^p - 1 = 0$, $(p = 0, 1, 2, 3, ...)$ in [13]. The second generalization named metallic means family (or metallic proportions) was introduced by V. W. de Spinadel in [9],[10],[11],[12]. For two positive integers p and q, the positive solutions of the equation $x^2 - px - q = 0$ are named as members of the metallic means family. All the members of the metallic means family are positive quadratic irrational numbers $\sigma_{p,q} = \frac{p+\sqrt{p^2+4q}}{2}$ and these numbers $\sigma_{p,q}$ are also called $(p,q)-$metallic numbers. In [6], authors have defined a metallic structure as a $(1,1)$ tensor field $\mathcal{J}$ satisfying $\mathcal{J}^2 - p\mathcal{J} - qI = 0$ on a Riemannian manifold and studied some properties. Now, we take into account a new equation $x^2 - px + \frac{3}{2}q = 0$. If we want to ensure that the new equation has complex roots, then we should give these two conditions $q > 0$ and $-\sqrt{6q} < p < \sqrt{6q}$. So, the numbers $\sigma^c_{p,q} = \frac{p\pm\sqrt{p^2-6q}}{2}$ are obtained as the complex roots of this equation. $\sigma^c_{p,q} = \frac{p+\sqrt{p^2-6q}}{2}$ that is one of the roots of this equation will be called as members of metallic complex means family. If $p = 1$ and $q = 1$, then the complex metallic means family $\sigma^c_{p,q} = \frac{p+\sqrt{p^2-6q}}{2}$ reduces to the complex golden mean: $\sigma^c_{1,1} = \frac{1+\sqrt{5}i}{2}, i^2 = -1$ which is a complex analog of well-known golden mean [2]. Any $(1,1)-$tensor field $\mathcal{J}$ satisfying $\mathcal{J}^2 - \mathcal{J} + \frac{3}{2}I = 0$ is called

2010 *Mathematics Subject Classification*. Primary 53C15; Secondary 53C55
Keywords. Conformal recurrency, conformal transformation, metallic Kähler manifold, hybrid tensor, $Z-$form.
Received: 19 May 2020; Revised: 08 September 2020; Accepted: 16 September 2020
Communicated by Mića Stancović
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almost complex golden structure. Almost complex golden structures were defined in [2]. Note that there is a bijection between almost complex structures and almost complex golden structure, as it is shown in [1, 3]. If a $(1,1)-$tensor field $\mathcal{J}$ provides the equation $x^2 - px + \frac{3}{2}q = 0$, then we call it an almost complex metallic structure. Precisely, an almost complex metallic structure is a polynomial structure as defined in [4, 5], with the structural polynomial $Q(\mathcal{J}) = \mathcal{J}^2 - p\mathcal{J} + \frac{3}{2}qI$.

Different kinds of geometric structures (such as almost product, almost contact, almost paracontact etc.) allow to get rich results while studying on Riemannian manifolds. Recently, Riemannian manifolds with almost metallic structures are defined and studied in [14]. In this paper, it has been considered metallic Kähler manifolds with an almost complex metallic structure and a Riemannian metric. As our main goal, it is examined how we can get results when we transfer the known geometric properties on Kähler manifolds to our own space. Many different techniques have been used in the process due to the different characteristics of an almost complex metallic structure and some results are obtained. An almost complex metallic structure allows to reinterpret plenty of structures on Riemannian manifolds. Thanks to the structure tensor which is introduced in this article, we hope to have results in the studies done with the structure tensor in the long run.

2. Preliminaries

In this section, some definitions that are relevant to the whole paper are made. It is stated that all geometric expressions mentioned in this paper are assumed to be class of C^∞. Let M_n be an $n-$dimensional diffrentiable manifold. For being one to one correspondence between almost complex structures and almost complex metallic structures (see [14]), the dimension n must be even. An almost complex metallic structure on M_{2k} satisfies the following equation

$$\mathcal{J}_i{}^k \mathcal{J}_k{}^j = p\mathcal{J}_i{}^j - \frac{3}{2}q\delta_i{}^j.$$

Consider a $(1,1)-$tensor field $\widehat{\mathcal{J}}$ defined by

$$\widehat{\mathcal{J}}_j{}^k = p\delta_j{}^k - \mathcal{J}_j{}^k.$$

It is easy to demonstrate that $\widehat{\mathcal{J}}$ also is an almost complex metallic structure. We call it conjugate almost metallic structure and it satisfies

$$\widehat{\mathcal{J}}_j{}^k \mathcal{J}_k{}^i = \frac{3}{2}q\delta_j{}^i.$$

If M_{2k} has an almost complex metallic structures $\mathcal{J}$, the pair $(M_{2k}, \mathcal{J})$ is an almost complex metallic manifold. If the almost complex metallic structure $\mathcal{J}$ is integrable, i.e., its Nijenhuis tensor vanishes, then it is called a complex metallic structure and then the pair $(M_{2k}, \mathcal{J})$ is called a complex metallic manifold. A Riemannian metric which satisfies

$$g_{ij}\mathcal{J}_k{}^i = -g_{ki}\mathcal{J}_j{}^i$$

is called a hybrid metric. We will say that an almost complex metallic manifold equipped with a hybrid metric g is an almost metallic Hermitian manifold. In addition, by the conditions $N_{\mathcal{J}} = 0$ and $d\omega = 0$ (equivalently, $\nabla\mathcal{J} = 0$), the triple $(M_{2k}, g, \mathcal{J})$ is a metallic Kähler manifold [14]. Here, ∇ is the Levi-Civita connection and $\omega_{ij} = g_{ih}\mathcal{J}_j^h$ is the fundamental $2-$form. Note that if the triple $(M_{2k}, g, \mathcal{J})$ is a metallic Kähler manifold, then $\left(M_{2k}, g, \widehat{\mathcal{J}}\right)$ is so.

Example 2.1. *Let us consider the* $\mathbb{R}^{2k}$ *endowed with the Euclidean metric* g *, i.e.*

$$g = \begin{pmatrix} \delta_j^i & 0 \\ 0 & \delta_{\bar{j}}^{\bar{i}} \end{pmatrix}, \; i, j = 1, ..., k, \quad \bar{i}, \bar{j} = k+1, ..., 2k.$$

Two complex metallic structures on $\mathbb{R}^{2k}$ *are given by*

$$\mathcal{J}_{\pm}=\begin{pmatrix} \frac{p}{2}\delta^{i}_{j} & \pm(\frac{2\sigma^{c}_{p,q}-p}{2})\delta^{i}_{\bar{j}} \\ \mp(\frac{2\sigma^{c}_{p,q}-p}{2})\delta^{\bar{i}}_{j} & \frac{p}{2}\delta^{\bar{i}}_{\bar{j}} \end{pmatrix},\ i,j=1,...,k,\quad \bar{i},\bar{j}=k+1,...,2k.$$

It is easy to see that the triple $(\mathbb{R}^{2k},\mathcal{J}_{\pm},g)$ *are two metallic Kähler Euclidean spaces.*

Example 2.2. *Let* M *be a k-dimensional differentiable Riemannian manifold of class* C^{∞} *and with a Riemannian metric* g, $T(M)$ *its tangent bundle, and* π *the natural projection* $T(M)\rightarrow M$. *Let* ∇ *be the Levi-Civita connection on* M *and denote by* ${}^{V}X$ *and* ${}^{H}X$ *the vertical and horizontal lift respectively to the tangent bundle* $T(M)$ *([16]) of a vector field* X *on* M.

The Sasaki metric ${}^{S}g$ *is defined on* $T(M)$ *by the three equations [16]:*

$$\begin{aligned} {}^{S}g\left({}^{V}X,{}^{V}Y\right) &= {}^{V}\left(g\left(X,Y\right)\right),\\ {}^{S}g\left({}^{V}X,{}^{H}Y\right) &= 0,\\ {}^{S}g\left({}^{H}X,{}^{H}Y\right) &= {}^{V}\left(g\left(X,Y\right)\right). \end{aligned}$$

We can define two almost complex metallic structures $\widehat{\mathcal{J}_{\pm}}$ *on* $T(M)$ *by*

$$\begin{cases} \widehat{\mathcal{J}_{\pm}}({}^{H}X)=\frac{p}{2}{}^{H}X\mp(\frac{2\sigma^{c}_{p,q}-p}{2})^{V}X,\\ \widehat{\mathcal{J}_{\pm}}({}^{V}X)=\frac{p}{2}{}^{V}X\pm(\frac{2\sigma^{c}_{p,q}-p}{2})^{H}X. \end{cases}$$

Also note that ${}^{S}g$ *is hybrid with respect to* $\widehat{\mathcal{J}_{\pm}}$. *Then we can say that the triple* $\left(T\left(M\right),\widehat{\mathcal{J}_{\pm}},{}^{S}g\right)$ *are two almost metallic Hermitian manifolds.*

The Levi-Civita connection ${}^{S}\nabla$ *of* ${}^{S}g$ *satisfies [7]:*

$$\begin{aligned} &(i)\quad {}^{S}\nabla_{{}^{H}X}{}^{H}Y={}^{H}\left(\nabla_{X}Y\right)-\frac{1}{2}{}^{V}\left(\mathcal{R}\left(X,Y\right)u\right),\\ &(ii)\quad {}^{S}\nabla_{{}^{V}X}{}^{H}Y=\frac{1}{2}{}^{H}\left(\mathcal{R}\left(u,X\right)Y\right),\\ &(iii)\quad {}^{S}\nabla_{{}^{H}X}{}^{V}Y={}^{V}\left(\nabla_{X}Y\right)+\frac{1}{2}{}^{H}\left(\mathcal{R}\left(u,Y\right)X\right),\\ &(iv)\quad {}^{S}\nabla_{{}^{V}X}{}^{V}Y=0. \end{aligned}$$

We now consider the covariant derivative of $\widehat{\mathcal{J}_{\pm}}$. *We obtain*

$$\left({}^{S}\nabla_{{}^{V}X}\widehat{\mathcal{J}_{\pm}}\right)\left({}^{V}Y\right)=\left(\frac{2\sigma^{c}_{p,q}-p}{4}\right)^{H}\left(\mathcal{R}\left(u,X\right)Y\right)$$

from which it follows that $\left({}^{S}\nabla_{{}^{V}X}\widehat{\mathcal{J}_{\pm}}\right)\left({}^{V}Y\right)=0$ *if and only if the Riemannian curvature tensor* $\mathcal{R}$ *is zero. In the case, the Levi-Civita connection reduces to*

$$\begin{aligned} &(i)\quad {}^{S}\nabla_{{}^{H}X}{}^{H}Y={}^{H}\left(\nabla_{X}Y\right),\\ &(ii)\quad {}^{S}\nabla_{{}^{V}X}{}^{H}Y=0,\\ &(iii)\quad {}^{S}\nabla_{{}^{H}X}{}^{V}Y={}^{V}\left(\nabla_{X}Y\right),\\ &(iv)\quad {}^{S}\nabla_{{}^{V}X}{}^{V}Y=0. \end{aligned}$$

Hence, it is easy to see

$$\left({}^{S}\nabla_{{}^{V}X}\widehat{\mathcal{J}_{\pm}}\right)\left({}^{H}Y\right)=\left({}^{S}\nabla_{{}^{H}X}\widehat{\mathcal{J}_{\pm}}\right)\left({}^{V}Y\right)=\left({}^{S}\nabla_{{}^{V}X}\widehat{\mathcal{J}_{\pm}}\right)\left({}^{V}Y\right)=0.$$

From all of the above, we can say that the triple $\left(T\left(M\right),\widehat{\mathcal{J}_{\pm}},{}^{S}g\right)$ *are two metallic Kähler manifolds if and only if the base manifold is flat.*

3. Properties of some tensors on metallic Kähler manifolds

This section can be divided into three parts. In the first part, it is examined some properties of the curvature tensor on metallic Kähler manifold. In the second part, we define a tensor H_{kj} and study relations between the curvature tensor and the tensor H_{kj}. In the last part, the harmonicity of the tensor H_{kj} is examined.

From the Ricci identity, we write

$$\nabla_k\nabla_j \mathcal{J}_i^{\ h} - \nabla_j\nabla_k \mathcal{J}_i^{\ h} = \mathcal{R}_{kjs}^{\ \ \ h}\, \mathcal{J}_i^{\ s} - \mathcal{R}_{kji}^{\ \ \ s}\, \mathcal{J}_s^{\ h},$$

where $\mathcal{R}_{kjs}^{\ \ \ h}$ are the components of the Riemannian curvature tensor field $\mathcal{R}$. In a metallic Kähler manifold, we immediately find

$$\mathcal{R}_{kjs}^{\ \ \ h}\, \mathcal{J}_i^{\ s} = \mathcal{R}_{kji}^{\ \ \ s}\, \mathcal{J}_s^{\ h}. \tag{1}$$

On multiplying (1) by $\widehat{\mathcal{J}}_r^{\ i} g_{hm}$, we obtain

$$\mathcal{R}_{kjrm} = -\frac{2}{3q}\, \mathcal{R}_{kjis}\, \mathcal{J}_m^{\ s}\, \widehat{\mathcal{J}}_r^{\ i}.$$

If we transvect (1) by g^{ji}, we infer that

$$\begin{aligned} \mathcal{R}_{kjs}^{\ \ \ h}\, \mathcal{J}_i^{\ s} g^{ji} &= \mathcal{R}_{kji}^{\ \ \ s}\, \mathcal{J}_s^{\ h} g^{ji} \\ \mathcal{R}_{kjs}^{\ \ \ h}\, \widetilde{\omega}^{js} &= \mathcal{R}_k^{\ s} \mathcal{J}_s^{\ h}, \end{aligned} \tag{2}$$

where $\mathcal{R}_k^{\ h} = \mathcal{R}_{kjs}^{\ \ \ h}\, g^{js}$ and $\widetilde{\omega}^{js}$ denotes the contravariant components of ω in M_{2k}. By using left side of (2), we write down

$$\begin{aligned} \widetilde{\omega}^{js}\mathcal{R}_{kjs}^{\ \ \ h} &= \frac{1}{2}\widetilde{\omega}^{js}\left(\mathcal{R}_{kjs}^{\ \ \ h} + \mathcal{R}_{kjs}^{\ \ \ h}\right) \\ \widetilde{\omega}^{js}\mathcal{R}_{kjs}^{\ \ \ h} &= \frac{1}{2}\widetilde{\omega}^{js}\left(\mathcal{R}_{kjs}^{\ \ \ h} - \mathcal{R}_{jks}^{\ \ \ h}\right) \\ \widetilde{\omega}^{js}\mathcal{R}_{kjs}^{\ \ \ h} &= \frac{1}{2}\widetilde{\omega}^{js}\mathcal{R}_{kjs}^{\ \ \ h} - \frac{1}{2}\widetilde{\omega}^{js}\mathcal{R}_{jks}^{\ \ \ h} \\ \widetilde{\omega}^{js}\mathcal{R}_{kjs}^{\ \ \ h} &= \frac{1}{2}\widetilde{\omega}^{js}\mathcal{R}_{kjs}^{\ \ \ h} - \frac{1}{2}\widetilde{\omega}^{sj}\mathcal{R}_{skj}^{\ \ \ h} \\ \widetilde{\omega}^{js}\mathcal{R}_{kjs}^{\ \ \ h} &= \frac{1}{2}\widetilde{\omega}^{js}\mathcal{R}_{kjs}^{\ \ \ h} + \frac{1}{2}\widetilde{\omega}^{js}\mathcal{R}_{skj}^{\ \ \ h} \\ \widetilde{\omega}^{js}\mathcal{R}_{kjs}^{\ \ \ h} &= \frac{1}{2}\widetilde{\omega}^{js}\left(\mathcal{R}_{kjs}^{\ \ \ h} + \mathcal{R}_{skj}^{\ \ \ h}\right) \\ \widetilde{\omega}^{js}\mathcal{R}_{kjs}^{\ \ \ h} &= -\frac{1}{2}\widetilde{\omega}^{\ \ js}\mathcal{R}_{jsk}^{\ \ \ h}. \end{aligned} \tag{3}$$

If we substitute (3) into (2), we get

$$-\frac{1}{2}\,\widetilde{\omega}^{js}\mathcal{R}_{jsk}^{\ \ \ h} = \mathcal{R}_k^{\ s}\, \mathcal{J}_s^{\ h} \tag{4}$$

which gives

$$\mathcal{R}_k^{\ s}\, \mathcal{J}_s^{\ h} = -\frac{1}{2}\,\widetilde{\omega}^{js}\left(\frac{2}{3q}\mathcal{R}_{jsm}^{\ \ \ n}\, \mathcal{J}_n^{\ h}\, \widehat{\mathcal{J}}_k^{\ m}\right). \tag{5}$$

After transvecting (5) by $\widehat{\mathcal{J}}_h^r$, we have

$$\begin{aligned}
\mathcal{R}_k^{\ s}\,\mathcal{J}_s^{\ h}\widehat{\mathcal{J}}_h^r &= -\frac{1}{2}\,\widetilde{\omega}^{js}\frac{2}{3q}\mathcal{R}_{jsm}^{\ \ \ n}\,\mathcal{J}_n^{\ h}\,\widehat{\mathcal{J}}_k^{\ m}\widehat{\mathcal{J}}_h^r\\
\mathcal{R}_k^{\ s}\,\frac{3q}{2}\delta_s^{\ r} &= -\frac{1}{2}\,\widetilde{\omega}^{js}\frac{2}{3q}\mathcal{R}_{jsm}^{\ \ \ n}\,\mathcal{J}_n^{\ h}\,\widehat{\mathcal{J}}_k^{\ m}\widehat{\mathcal{J}}_h^{\ r}\\
\frac{3q}{2}\mathcal{R}_k^{\ r} &= -\frac{1}{2}\,\widetilde{\omega}^{js}\frac{2}{3q}\mathcal{R}_{jsm}^{\ \ \ n}\frac{3q}{2}\delta_n^{\ r}\widehat{\mathcal{J}}_k^{\ m}\\
\frac{3q}{2}\mathcal{R}_k^{\ r} &= -\frac{1}{2}\,\widetilde{\omega}^{js}\,\mathcal{R}_{jsm}^{\ \ \ r}\,\widehat{\mathcal{J}}_k^{\ m}\\
\frac{3q}{2}\mathcal{R}_k^{\ r} &= \mathcal{R}_m^{\ s}\mathcal{J}_s^{\ r}\,\widehat{\mathcal{J}}_k^{\ m}\\
\mathcal{R}_k^{\ r} &= \frac{2}{3q}\,\mathcal{R}_m^{\ s}\,\mathcal{J}_s^{\ r}\,\widehat{\mathcal{J}}_k^{\ m}.
\end{aligned}$$

So, we can state the following result:

Theorem 3.1. *The tensor $\mathcal{R}_k^{\ r}$ in a metallic Kähler manifold $(M_{2k}, g, \mathcal{J})$ satisfies the following equation*

$$\mathcal{R}_k^{\ r} = \frac{2}{3q}\,\mathcal{R}_m^{\ s}\,\mathcal{J}_s^{\ r}\,\widehat{\mathcal{J}}_k^{\ m}. \tag{6}$$

By transvecting (6) with $\mathcal{J}_l^{\ k}$, we find

$$\begin{aligned}
\mathcal{R}_k^{\ r}\,\mathcal{J}_l^{\ k} &= \frac{2}{3q}\,\mathcal{R}_m^{\ s}\,\mathcal{J}_s^{\ r}\,\widehat{\mathcal{J}}_k^{\ m}\mathcal{J}_l^{\ k}\\
\mathcal{R}_k^{\ r}\,\mathcal{J}_l^{\ k} &= \frac{2}{3q}\,\mathcal{R}_m^{\ s}\,\mathcal{J}_s^{\ r}\,\frac{3q}{2}\,\delta_l^{\ m}\\
\mathcal{R}_k^{\ r}\,\mathcal{J}_l^{\ k} &= \mathcal{R}_l^{\ s}\mathcal{J}_s^{\ r}\\
\mathcal{R}_k^{\ r}\,\mathcal{J}_l^{\ k}g_{rm} &= \mathcal{R}_l^{\ s}\mathcal{J}_s^{\ r}g_{rm}\\
\mathcal{R}_{km}\,\mathcal{J}_l^{\ k} &= \mathcal{R}_l^{\ s}\,\omega_{sm}\\
\mathcal{R}_{km}\,\mathcal{J}_l^{\ k} &= -\mathcal{R}_l^{\ s}\,\omega_{ms}\\
\mathcal{R}_{km}\,\mathcal{J}_l^{\ k} &= -\mathcal{R}_{lk}\,\mathcal{J}_m^{\ k}.
\end{aligned} \tag{7}$$

Hence, we obtain the following result:

Corollary 3.2. *In a metallic Kähler manifold $(M_{2k}, \mathcal{J}, g)$, the tensor $\mathcal{R}_{km}$ is hybrid with respect to $\mathcal{J}$.*

A tensor H_{kj} in a metallic Kähler manifold is defined by (for Kähler manifold, see [15])

$$H_{kj} = \mathcal{R}_{kjsr}\,\widetilde{\omega}^{sr}.$$

It follows from (4) that

$$\begin{aligned}
-\frac{1}{2}\widetilde{\omega}^{sr}\mathcal{R}_{srk}^{\ \ \ h} &= \mathcal{R}_k^{\ s}\,\mathcal{J}_s^{\ h}\\
-\frac{1}{2}\widetilde{\omega}^{sr}\mathcal{R}_{srk}^{\ \ \ h} &= \mathcal{R}_{ks}\,\widetilde{\omega}^{sh}\\
-\frac{1}{2}\mathcal{R}_{srkm}\,g^{mh}\widetilde{\omega}^{sr} &= \mathcal{R}_{ks}\,\widetilde{\omega}^{sh}\\
\frac{1}{2}\,\mathcal{R}_{srkm}\,g^{mh}\widetilde{\omega}^{sr} &= \mathcal{R}_{ks}\,\widetilde{\omega}^{hs}\\
\frac{1}{2}\mathcal{R}_{kmsr}\,g^{mh}\widetilde{\omega}^{sr} &= \mathcal{R}_{ks}\,\mathcal{J}_j^{\ s}g^{jh}\\
H_{km}\,g^{mh} &= \mathcal{R}_{ks}\,\mathcal{J}_j^{\ s}g^{jh}\\
H_{kj}\,g^{jh} &= \mathcal{R}_{ks}\,\mathcal{J}_j^{\ s}g^{jh}\\
H_{kj} &= \mathcal{R}_{ks}\,\mathcal{J}_j^{\ s}\,.
\end{aligned}\tag{8}$$

If we transvect (8) with $\widehat{\mathcal{J}}_l^{\ j}$, we get

$$\begin{aligned}
H_{kj}\,\widehat{\mathcal{J}}_l^{\ j} &= \mathcal{R}_{ks}\,\mathcal{J}_j^{\ s}\widehat{\mathcal{J}}_l^{\ j}\\
H_{kj}\,\widehat{\mathcal{J}}_l^{\ j} &= \mathcal{R}_{ks}\frac{3q}{2}\delta_l^{\ s}\\
\mathcal{R}_{kl} &= \frac{2}{3q}H_{kj}\,\widehat{\mathcal{J}}_l^{\ j}.
\end{aligned}\tag{9}$$

After multiplying (8) by $\widehat{\omega}^{kj} = \widehat{\mathcal{J}}_m^{\ j}g^{mk}$, we obtain

$$\begin{aligned}
H_{kj}\,\widehat{\omega}^{kj} &= \mathcal{R}_{ks}\,\mathcal{J}_j^{\ s}\widehat{\omega}^{kj}\\
H_{kj}\,\widehat{\omega}^{kj} &= \mathcal{R}_{ks}\,\mathcal{J}_j^{\ s}\widehat{\omega}^{kj}\\
H_{kj}\,\widehat{\omega}^{kj} &= \mathcal{R}_{ks}\,\mathcal{J}_j^{\ s}\widehat{\mathcal{J}}_m^{\ j}\,g^{mk}\\
H_{kj}\,\widehat{\omega}^{kj} &= \mathcal{R}_{ks}\frac{3q}{2}\delta_m^{\ s}\,g^{mk}\\
\frac{2}{3q}H_{kj}\,\widehat{\omega}^{kj} &= \mathcal{R}_{km}\,g^{mk}\\
\mathcal{R} &= \frac{2}{3q}H_{kj}\,\widehat{\omega}^{kj}.
\end{aligned}\tag{10}$$

Thus, we yield:

Theorem 3.3. *In a metallic Kähler manifold $(M_{2k}, \mathcal{J}, g)$, the following equations are satisfied*

$$\begin{aligned}
H_{kj} &= \mathcal{R}_{ks}\,\mathcal{J}_j^{\ s}\,,\\
\mathcal{R}_{kl} &= \frac{2}{3q}H_{kj}\,\widehat{\mathcal{J}}_l^{\ j},\\
\mathcal{R} &= \frac{2}{3q}H_{kj}\,\widehat{\omega}^{kj}.
\end{aligned}$$

As a direct result of the above theorem we have the following theorem:

Theorem 3.4. *A metallic Kähler manifold $(M_{2k}, \mathcal{J}, g)$ is an Einstein manifold if and only if the tensor H_{kj} is proportional to the fundamental 2–form $\widehat{\omega}_{kj}$.*

As we take into account the fundamental 2−form $\omega_{kj} = g_{mj}\mathcal{J}_k^{\ m}$, in a metallic Kähler manifold we easily see that $\nabla\omega = 0$. Let us apply the derivative operator $\nabla^k = g^{kh}\nabla_h$ to the both of sides of (8). Then we obtain

$$\nabla^k H_{kj} = (\nabla^k \mathcal{R}_{ks})\,\mathcal{J}_j^{\ s}.$$

From (10), standard calculations give

$$\nabla_m H_j^{\ m} = -(\nabla_m \mathcal{R})\; J_j^{\ m}. \tag{11}$$

On the other hand, from the Bianchi identity we write down

$$\nabla_l \mathcal{R}_{kjih} + \nabla_k \mathcal{R}_{jlih} + \nabla_j \mathcal{R}_{lkih} = 0. \tag{12}$$

If we transvect (12) by $\widetilde{\omega}^{ih}g^{lk}$, we get

$$\begin{aligned}
\nabla_l(\mathcal{R}_{kjih}\,\widetilde{\omega}^{ih}g^{lk}) + \nabla_k(\mathcal{R}_{jlih}\,\widetilde{\omega}^{ih}g^{lk}) + \nabla_j(\mathcal{R}_{lkih}\,\widetilde{\omega}^{ih}g^{lk}) &= 0\\
g^{lk}\nabla_l H_{kj} + \nabla_k\,(H_{jl}g^{lk}) + \nabla_j\,(H_{lk}g^{lk}) &= 0\\
\nabla^k H_{kj} + \nabla_k\, H_j^{\ k} + \nabla_j\, H_l^{\ l} &= 0\\
\nabla^k H_{kj} + \nabla_k H_j^{\ k} &= 0
\end{aligned}$$

from which, by taking into account (11), we obtain

$$\nabla^k\, H_{kj} - (\nabla_k\, \mathcal{R})\;\mathcal{J}_j^{\ k} = 0. \tag{13}$$

From this, we obtain immediately the following theorem:

Theorem 3.5. *The tensor H_{kj} in a metallic Kähler manifold $(M_{2k}, \mathcal{J}, g)$ is harmonic if and only if the scalar curvature tensor of the metallic Kähler manifold is constant.*

4. Conformal transformation

In this section, we are going to find out the conditions under which a metallic Hermitian manifold is conformal to a metallic Kähler manifold.

By a conformal transformation, a hybrid metric g_{ij} is transformed into $\overline{g}_{ij}$, the structure tensor $\mathcal{J}_j^{\ k}$ remains unchanged and the fundamental 2−form ω_{ij} is transformed into $\overline{\omega}_{ij}$:

$$\begin{aligned}
\overline{\omega}_{ij} &= \varphi^2\omega_{ij}\,,\\
\overline{g}_{ij} &= \varphi^2 g_{ij}.
\end{aligned}$$

If we suppose that the transformed metallic Hermitian manifold is a metallic Kähler manifold, then

$$(d\overline{\omega})_{kji} = \partial_k\overline{\omega}_{ji} + \partial_j\overline{\omega}_{ik} + \partial_i\overline{\omega}_{kj} = 0$$

$$(d\overline{\omega})_{kji} = \partial_k\left(\varphi^2\omega_{ji}\right) + \partial_j\left(\varphi^2\omega_{ik}\right) + \partial_i\left(\varphi^2\omega_{kj}\right) = 0$$

$$2\varphi\varphi_k\omega_{ji} + \varphi^2\left(\partial_k\omega_{ji}\right) + 2\varphi\varphi_j\omega_{ik} + \varphi^2\left(\partial_j\omega_{ik}\right) + 2\varphi\varphi_i\omega_{kj} + \varphi^2\left(\partial_i\omega_{kj}\right) = 0$$

$$2\varphi\varphi_k\omega_{ji} + 2\varphi\varphi_j\omega_{ik} + 2\varphi\varphi_i\omega_{kj} + \varphi^2\,(d\omega)_{kji} = 0. \tag{14}$$

Dividing both sides of (14) with $2\varphi^2$, we get

$$\begin{aligned}
\frac{\varphi_k}{\varphi}\omega_{ji} + \frac{\varphi_j}{\varphi}\omega_{ik} + \frac{\varphi_i}{\varphi}\omega_{kj} + \frac{1}{2}(d\omega)_{kji} &= 0\\
\varphi_k\omega_{ji} + \varphi_j\omega_{ik} + \varphi_i\omega_{kj} + \frac{1}{2}(d\omega)_{kji} &= 0,
\end{aligned} \tag{15}$$

where $\varphi_k = \partial_k ln\varphi$, transvecting (15) with $\widehat{\omega}^{ji}$, we obtain

$$\begin{aligned}
\varphi_k\omega_{ji}\widehat{\omega}^{ji} + \varphi_j\omega_{ik}\widehat{\omega}^{ji} + \varphi_i\omega_{kj}\widehat{\omega}^{ji} + \frac{1}{2}(d\omega)_{kji}\widehat{\omega}^{ji} &= 0 \\
-3qn\,\varphi_k + \frac{3q}{2}\,\varphi_k + \frac{3q}{2}\,\varphi_k + \omega_k &= 0 \\
3q\,(-n+1)\,\varphi_k + \omega_k &= 0 \\
\frac{\omega_k}{3q\,(n-1)} &= \varphi_k. \qquad (16)
\end{aligned}$$

If we substitute (16) into (15), we infer

$$\begin{aligned}
\frac{\omega_k}{3q\,(n-1)}\omega_{ji} + \frac{\omega_j}{3q\,(n-1)}\omega_{ik} + \frac{\omega_i}{3q\,(n-1)}\omega_{kj} + \frac{1}{2}(d\omega)_{kji} &= 0 \\
(d\omega)_{kji} + \frac{2}{3q\,(n-1)}\left(\omega_k\omega_{ji} + \omega_j\omega_{ik} + \omega_i\omega_{kj}\right) &= 0. \qquad (17)
\end{aligned}$$

From (17), we obtain

$$\partial_l\omega_{kji} - \partial_k\omega_{lji} - \partial_j\omega_{kli} - \partial_i\omega_{kjl} = 0$$

$$\frac{2}{3q\,(n-1)}((\partial_l\omega_k - \partial_k\omega_l)\,\omega_{ji} - \left(\partial_l\omega_j - \partial_j\omega_l\right)\omega_{ki} - (\partial_l\omega_i - \partial_i\omega_l)\,\omega_{jk}$$

$$-\left(\partial_j\omega_i - \partial_i\omega_j\right)\omega_{lk} - (\partial_k\omega_i - \partial_i\omega_k)\,\omega_{lj} - \left(\partial_j\omega_k - \partial_k\omega_j\right)\omega_{li} = 0$$

from which, by transvecting with $\widehat{\omega}^{ji}$, we find

$$\frac{2}{3q\,(n-1)}(-3qn\,(\partial_l\omega_k - \partial_k\omega_l) + \frac{3q}{2}\delta^j_k\left(\partial_l\omega_j - \partial_j\omega_l\right) + \frac{3q}{2}\delta^i_k\,(\partial_l\omega_i - \partial_i\omega_l)$$

$$-\left(\partial_j\omega_i - \partial_i\omega_j\right)\omega_{lk}\widehat{\omega}^{ji} - \frac{3q}{2}\delta^i_l\,(\partial_k\omega_i - \partial_i\omega_k) + \frac{3q}{2}\delta^j_l\left(\partial_j\omega_k - \partial_k\omega_j\right)) = 0$$

$$\frac{2}{3q\,(n-1)}(-3qn\,(\partial_l\omega_k - \partial_k\omega_l) + \frac{3q}{2}\,(\partial_l\omega_k - \partial_k\omega_l) + \frac{3q}{2}\,(\partial_l\omega_k - \partial_k\omega_l)$$

$$-\left(\partial_j\omega_i - \partial_i\omega_j\right)\omega_{lk}\widehat{\omega}^{ji} - \frac{3q}{2}\,(\partial_k\omega_l - \partial_l\omega_k) + \frac{3q}{2}\,(\partial_l\omega_k - \partial_k\omega_l)) = 0$$

$$\frac{2}{3q\,(n-1)}(-3qn\,(\partial_l\omega_k - \partial_k\omega_l) + \frac{3q}{2}\,(\partial_l\omega_k - \partial_k\omega_l) + \frac{3q}{2}\,(\partial_l\omega_k - \partial_k\omega_l)$$

$$-\left(\partial_j\omega_i - \partial_i\omega_j\right)\omega_{lk}\widehat{\omega}^{ji} + \frac{3q}{2}\,(\partial_l\omega_k - \partial_k\omega_l) + \frac{3q}{2}\,(\partial_l\omega_k - \partial_k\omega_l)$$

$$\frac{2}{3q\,(n-1)}\left((-3qn+6q)\,(\partial_l\omega_k - \partial_k\omega_l) - \left(\partial_j\omega_i - \partial_i\omega_j\right)\omega_{lk}\widehat{\omega}^{ji}\right)$$

$$-\frac{2}{(n-1)}\left((n-2)\,(\partial_l\omega_k - \partial_k\omega_l) - \left(\partial_j\omega_i - \partial_i\omega_j\right)\omega_{lk}\widehat{\omega}^{ji}\right) = 0. \qquad (18)$$

After multiplying (18) by $\widehat{\omega}^{lk}$, we have

$$\begin{aligned}
-\frac{2\ (n-2)}{(n-1)}\left(\left(\partial_l\omega_k-\partial_k\omega_l\right)\widehat{\omega}^{lk}-\left(\partial_j\omega_i-\partial_i\omega_j\right)\omega_{lk}\widehat{\omega}^{ji}\widehat{\omega}^{lk}\right) &= 0\\
-\frac{2\ (n-2)}{(n-1)}\left(\left(\partial_l\omega_k-\partial_k\omega_l\right)\widehat{\omega}^{lk}+\left(\partial_j\omega_i-\partial_i\omega_j\right)3qn\ \widehat{\omega}^{ji}\right) &= 0\\
-\frac{2\ (n-2)}{(n-1)}\left(\left(\partial_l\omega_k-\partial_k\omega_l\right)\widehat{\omega}^{lk}+\left(\partial_l\omega_k-\partial_k\omega_l\right)3qn\ \widehat{\omega}^{lk}\right) &= 0\\
\left(-\frac{2\ (n-2)}{(n-1)}+3qn\right)\left(\partial_l\omega_k-\partial_k\omega_l\right)\widehat{\omega}^{lk} &= 0\\
\frac{3qn^2-(3q+2)\,n+4}{n-1}\left(\partial_l\omega_k-\partial_k\omega_l\right)\widehat{\omega}^{lk} &= 0.
\end{aligned}$$

We can see that $\left(\partial_l\omega_k-\partial_k\omega_l\right)\widehat{\omega}^{lk}=0$ if $3qn^2-(3q+2)\,n+4\neq 0$. When we use this equation in (18), we get

$$-\frac{2\,(n-2)}{(n-1)}\left(\left(\partial_l\omega_k-\partial_k\omega_l\right)\right)=0.$$

For $n\neq 2$, we obtain

$$\partial_l\omega_k-\partial_k\omega_l=0.$$

Theorem 4.1. *A necessary and sufficient condition for a metallic Hermitian manifold to be conformal to a metallic Kähler manifold is that, for* $2n>4$

$$(d\omega)_{kji}+\frac{2}{3q\,(n-1)}\left(\omega_k\omega_{ji}+\omega_j\omega_{ik}+\omega_i\omega_{kj}\right)=0$$

and, for $n\neq 2$ *and* $3qn^2-(3q+2)\,n+4\neq 0$

$$\partial_l\omega_k-\partial_k\omega_l=0.$$

5. Conformally recurrent metallic Kähler manifolds

Let M_n be an $n-$dimensional metallic Kähler manifold ($n=2k$, $k\neq 1,2$) and $C_{kji}^{\ \ h}$ be its conformal curvature tensor. If the following condition is satisfied

$$\nabla_l C_{kjih}=\lambda_l C_{kjih}, \tag{19}$$

where $C_{kjih}=C_{kji}^{\ \ h}g_{rh}$, then the metallic Kähler manifold is called a conformally recurrent metallic Kähler manifold. Here

$$C_{kjih}=\mathcal{R}_{kjih}+g_{kh}\mathcal{L}_{ji}-g_{jh}\mathcal{L}_{ki}+g_{ji}\mathcal{L}_{kh}-g_{ki}\mathcal{L}_{jh}, \tag{20}$$

$$\mathcal{L}_{ji}=-\frac{1}{2n-2}\mathcal{R}_{ji}+\frac{1}{2\,(2n-1)\,(2n-2)}\mathcal{R}g_{ji}. \tag{21}$$

Therefore, we can write (19) as

$$\nabla_l\mathcal{R}_{kjih}+\nabla_l g_{kh}L_{ji}-\nabla_l g_{jh}L_{ki}+\nabla_l g_{ji}L_{kh}-\nabla_l g_{ki}L_{jh}=$$

$$\lambda_l\left(\mathcal{R}_{kjih}+g_{kh}\mathcal{L}_{ji}-g_{jh}\mathcal{L}_{ki}+g_{ji}\mathcal{L}_{kh}-g_{ki}\mathcal{L}_{jh}\right). \tag{22}$$

Transvecting (22) with $\widetilde{\omega}^{ih}$, we get

$$2\nabla_l H_{kj} + \frac{4}{2n-2}\nabla_l H_{jk} + \frac{4}{2(2n-1)(2n-2)}\omega_{jk}\nabla_l \mathcal{R} =$$

$$\lambda_l\left(\frac{4}{2n-2}H_{jk} + \frac{4}{2(2n-1)(2n-2)}\omega_{jk}\mathcal{R}\right). \tag{23}$$

By using (10), after transvecting (23) with $\widehat{\omega}^{kj}$, we infer

$$(\nabla_l \mathcal{R} - \lambda_l \mathcal{R})\left(\frac{6qn(2n-1) - 2\omega_{kj}\widehat{\omega}^{kj}}{(2n-1)(2n-2)}\right) = 0$$

which gives

$$\nabla_l \mathcal{R} = \lambda_l \mathcal{R}. \tag{24}$$

Substituting (24) into (23), we obtain

$$\nabla_l H_{kj} = \lambda_l H_{kj}$$

from which

$$\begin{aligned}
\nabla_l H_{kj} &= \lambda_l H_{kj} \\
\nabla_l H_{kj} &= \lambda_l \mathcal{R}_{kj}\mathcal{J}_j^{\ s} \\
\nabla_l H_{kj}\widehat{\mathcal{J}}_j^{\ m} &= \mathcal{J}_j^{\ s}\widehat{\mathcal{J}}_j^{\ m}\lambda_l \mathcal{R}_{ks} \\
\frac{3q}{2}\nabla_l \mathcal{R}_{km} &= \frac{3q}{2}\lambda_l \mathcal{R}_{km} \\
\nabla_l \mathcal{R}_{km} &= \lambda_l \mathcal{R}_{km}.
\end{aligned} \tag{25}$$

From (21) we immediately see that

$$\nabla_l \mathcal{L}_{ji} = \lambda_l \mathcal{L}_{ji}.$$

Therefore (22) reduces to

$$\nabla_l \mathcal{R}_{kjih} = \lambda_l \mathcal{R}_{kjih} \tag{26}$$

which implies that $\mathcal{R}_{kjih}$ is recurrent with recurrence vector λ_l. Thus we can sate the following theorem:

Theorem 5.1. *A metallic Kähler manifold is conformally recurrent if and only if it is recurrent.*

The vector $u_i = \lambda_s \mathcal{J}_i^{\ s}$ is called associated vector of λ_i with respect to $\mathcal{J}$. It is easy to see that

$$u^l \lambda_l = 0 \tag{27}$$

$$u^l u_l = \lambda^l \lambda_s. \tag{28}$$

Since $\lambda^l \lambda_l = \theta \neq 0$. we easily say that λ_l and u_l are non-null orthogonal vectors with equal lengths.

From (13) , we have

$$\nabla^k H_{kj} = \mathcal{R} u_j.$$

By transvecting (12) by $\widetilde{\omega}^{ih}$, we get

$$\nabla_l H_{kj} + \nabla_k H_{jl} + \nabla_j H_{lk} = 0$$

from which standard calculations give

$$\theta H_{jk} = -\mathcal{R}\left(\lambda_k u_j - \lambda_j u_k\right) \tag{29}$$

whence

$$\mathcal{R}_{jm} = \frac{2}{3q\theta}\mathcal{R}\left(p\frac{3q}{2}\lambda_j\lambda_m + u_m u_j - p\lambda_m u_j\right). \tag{30}$$

By (26), we can write Bianchi's identity as

$$\begin{aligned}
\nabla_l R_{kjih} + \nabla_k R_{jlih} + \nabla_j \mathcal{R}_{lkih} &= 0 \\
\lambda_l \mathcal{R}_{kjih} + \lambda_k \mathcal{R}_{jlih} + \lambda_j \mathcal{R}_{lkih} &= 0 \qquad (31)\\
\lambda_l\lambda^l \mathcal{R}_{kjih} + \lambda_k\lambda^l \mathcal{R}_{jlih} + \lambda_j\lambda^l \mathcal{R}_{lkih} &= 0 \\
\theta\mathcal{R}_{kjih} + \lambda_k\lambda^l \mathcal{R}_{ihjl} - \lambda_j\lambda^l \mathcal{R}_{ihkl} &= 0. \qquad (32)
\end{aligned}$$

Transvecting (31) with g^{kh}, we get

$$\begin{aligned}
\lambda_l \mathcal{R}_{kjih}g^{kh} + \lambda_k \mathcal{R}_{jlih}g^{kh} + \lambda_j \mathcal{R}_{lkih}g^{kh} &= 0 \\
\lambda_l \mathcal{R}_{jikh}g^{kh} + \lambda_k \mathcal{R}_{jlih}g^{kh} - \lambda_j \mathcal{R}_{likh}g^{kh} &= 0 \\
\lambda_l \mathcal{R}_{ji} + \lambda_k \mathcal{R}_{jlih}g^{kh} - \lambda_j \mathcal{R}_{li} &= 0 \\
\lambda_k \mathcal{R}_{jlih}g^{kh} &= \lambda_j \mathcal{R}_{li} - \lambda_l \mathcal{R}_{ji}. \qquad (33)
\end{aligned}$$

Using the equation (30) in (33), we get

$$\lambda^h \mathcal{R}_{jlih} = \frac{2}{3q\theta}\mathcal{R}\left[\left(u_i - p\lambda_i\right)\left(\lambda_j u_l - \lambda_l u_j\right)\right]. \tag{34}$$

By using (34) in (32), we obtain

$$\mathcal{R}_{kjih} = \frac{2}{3q\theta^2}\mathcal{R}\left(\lambda_i u_h - \lambda_h u_i\right)\left(\lambda_j u_k - \lambda_k u_j\right).$$

Substituting (29) into the last equation, we have

$$\mathcal{R}_{kjih} = -\frac{2}{3q\mathcal{R}}H_{kj}H_{ih}.$$

Hence we have the following theorem:

Theorem 5.2. *In a conformally recurrent metallic Kähler manifold of non-zero scalar curvature, the Riemannian curvature tensor has the following form*

$$\mathcal{R}_{kjih} = -\frac{2}{3q\mathcal{R}}H_{kj}H_{ih}.$$

6. Recurrent Z-forms on metallic Kähler manifolds

The last chapter is devoted to $Z-$forms on metallic Kähler manifolds.
A tensor Z_{kl} on a Riemannian manifold M is defined by

$$Z_{kl} = \mathcal{R}_{kl} + \phi g_{kl}, \tag{35}$$

where ϕ is a scalar function [8]. From (35), we can write the following equations

$$Z_{kl}\mathcal{J}_i^{\ k} = -Z_{ik}\mathcal{J}_l^{\ k} \tag{36}$$

$$Z_{ml} = -\frac{2}{3q}Z_{ik}\mathcal{J}_l^{\ k}\widehat{\mathcal{J}}_m^{\ i}.$$

The $Z-$form on M is recurrent if and only if [8]

$$\nabla_k Z_{ij} - \nabla_j Z_{ik} = \omega_k Z_{ij} - \omega_j Z_{ik}. \tag{37}$$

Transvecting (37) with $-\frac{2}{3q}\mathcal{J}_h^{\ j}\widehat{\mathcal{J}}_l^{\ i}$, we get

$$\nabla_k Z_{ij} + \frac{2}{3q}\nabla_\beta\left(Z_{\alpha k}\mathcal{J}_j^{\ \beta}\widehat{\mathcal{J}}_i^{\ \alpha}\right) = \omega_k Z_{ij} + \frac{2}{3q}\omega_j\left(Z_{ik}\mathcal{J}_j^{\ \beta}\widehat{\mathcal{J}}_i^{\ \alpha}\right). \tag{38}$$

Exchanging roles of the indices k and i in equation (38), we get

$$\nabla_i Z_{kj} + \frac{2}{3q}\nabla_\beta\left(Z_{\alpha i}\mathcal{J}_j^{\ \beta}\widehat{\mathcal{J}}_k^{\ \alpha}\right) = \omega_i Z_{kj} + \frac{2}{3q}\omega_\beta\left(Z_{\alpha i}\mathcal{J}_j^{\ \beta}\widehat{\mathcal{J}}_k^{\ \alpha}\right). \tag{39}$$

When the equations (38) and (39) are added, we have

$$\nabla_k Z_{ij} + \nabla_i Z_{kj} + \frac{2}{3q}\nabla_\beta\left(pZ_{\alpha k}\mathcal{J}_j^{\ \beta}\mathcal{J}_i^{\ \alpha} - Z_{\alpha k}\mathcal{J}_j^{\ \beta}\mathcal{J}_i^{\ \alpha} + pZ_{\alpha i}\mathcal{J}_j^{\ \beta}\mathcal{J}_k^{\ \alpha} - Z_{\alpha i}\mathcal{J}_j^{\ \beta}\mathcal{J}_k^{\ \alpha}\right) =$$

$$\omega_k Z_{ij} + \omega_i Z_{kj} + \frac{2}{3q}\omega_\beta\left(pZ_{\alpha k}\mathcal{J}_j^{\ \beta}\mathcal{J}_i^{\ \alpha} - Z_{\alpha k}\mathcal{J}_j^{\ \beta}\mathcal{J}_i^{\ \alpha} + pZ_{\alpha i}\mathcal{J}_j^{\ \beta}\mathcal{J}_k^{\ \alpha} - Z_{\alpha i}\mathcal{J}_j^{\ \beta}\mathcal{J}_k^{\ \alpha}\right)$$

from which, by using (36), we get

$$\nabla_k Z_{ij} + \nabla_i Z_{kj} = \omega_k Z_{ij} + \omega_i Z_{kj}. \tag{40}$$

By exchanging again roles of the indices in (40) properly, we have

$$\nabla_k Z_{ij} + \nabla_j Z_{ki} = \omega_k Z_{ij} + \omega_j Z_{ki}. \tag{41}$$

If the equations (40) and (41) are added, then we obtain

$$\begin{aligned}\nabla_k Z_{ij} - \nabla_j Z_{ik} + \nabla_k Z_{ij} + \nabla_j Z_{ki} &= \omega_k Z_{ij} - \omega_j Z_{ik} + \omega_k Z_{ij} + \omega_j Z_{ki}\\ 2\nabla_k Z_{ij} &= 2\omega_k Z_{ij}\\ \nabla_k Z_{ij} &= \omega_k Z_{ij}.\end{aligned}$$

So, we are able to say the following theorem:

Theorem 6.1. *On a metallic Kähler manifold, the notion of Z recurrent form is equivalent to the ordinary reccurency of the Z tensor.*

Acknowledgement: *We would like to thank the reviewer for his/her time spent on reviewing our manuscript and his/her comments helping us improving the article.*

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