



Inference on Reliability of Stress-Strength Model with Peng-Yan Extended Weibull Distributions

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Abstract. In this paper we estimate $R = P\{X < Y\}$ when X and Y are independent random variables following the Peng-Yan extended Weibull distribution. We find maximum likelihood estimator of R and its asymptotic distribution. This asymptotic distribution is used to construct asymptotic confidence intervals. In the case of equal shape parameters, we derive the exact confidence intervals, too. A procedure for deriving bootstrap-p confidence intervals is presented. The UMVUE of R and the UMVUE of its variance are derived and also the Bayes point and interval estimator of R for conjugate priors are obtained. Finally, we perform a simulation study in order to compare these estimators and provide a real data example.

1. Introduction

In reliability theory stress-strength models have been studied for many years, mainly due to their applicability in engineering, meteorology, climatology, quality control, medicine, etc. In the most common stress-strength model the system fails if the applied stress X is greater than strength Y , so the reliability parameter R , where $R = P\{X < Y\}$, is a measure of system performance. For example, if X represents the maximum pressure caused by flooding and Y represents the strength of the leg of a bridge, then R is a measure of the bridge solidity. In a broader interpretation, R can be viewed as a measure of difference between two populations. For example, in medicine, if Y represents the response of a treatment group, and X refers to a control group, R is a measure of the effect of the treatment.

An exhaustive bibliography on the estimation of R is available in [17]. Some recent papers on this topic include [22], [27], [14], [18], [7], [16], [28], [33] and [2–4].

As far as the Weibull distribution is concerned, various point and interval estimators of R were studied in [25], [20], [21], [19], [1], [26], [13] and [6]. Different cases were considered including two and three parameter distributions, with equal and unequal shape parameters based on complete and censored samples.

Since the Weibull distribution is not flexible enough to model all lifetime data, many of its extensions and modifications have been proposed. A review of such distributions can be found in [30], and some other

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papers on this topic include [5], [8] and [31]. Inferences on stress-strength reability of such extensions are studied in, for example, [32] and [23].

Recently, in [29], a new extended Weibull distribution was introduced. This distribution has one more shape parameter than standard two-parameter Weibull distribution and it can have both positive and negative skewness, as well as both increasing and upside-down bathtub shaped hazard rate function. As suggested by Peng and Yan, this distribution is very flexible and suitable for modeling data with upside-down bathtub shaped hazard rate, which commonly appears in reliability analysis, for example, the dynamic component data of the commercial vehicle engines [15], the maximum flood levels [24], the guinea pigs data [11]. Moreover, it has a closed expression for the distribution function, which may make it a preferred choice in practical applications in comparison to some other modifications of the Weibull distribution.

Its distribution and density functions are

$$\begin{aligned}
 F(x; \alpha, \beta, \lambda) &= 1 - e^{-\alpha x^\beta e^{-\frac{\lambda}{x}}}, \quad x > 0, \\
 f(x; \alpha, \beta, \lambda) &= \alpha(\lambda + \beta x)x^{\beta-2}e^{-\frac{\lambda}{x}-\alpha x^\beta e^{-\frac{\lambda}{x}}}, \quad x > 0,
 \end{aligned}
 \tag{1}$$

where $\alpha, \lambda > 0$, is the scale parameter and β and $\lambda, \beta > 0, \lambda \geq 0$, are shape parameters.

Let $X : \text{PYEW}(\alpha_1, \beta_1, \lambda_1)$ and $Y : \text{PYEW}(\alpha_2, \beta_2, \lambda_2)$ be independent random variables with Peng-Yan extended Weibull (PYEW) distribution. The reliability parameter R is

$$\begin{aligned}
 R = P\{X < Y\} &= \int_0^\infty \int_x^\infty \alpha_1(\lambda_1 + \beta_1 x)x^{\beta_1-2}e^{-\frac{\lambda_1}{x}-\alpha_1 x^{\beta_1} e^{-\frac{\lambda_1}{x}}} \alpha_2(\lambda_2 + \beta_2 y)y^{\beta_2-2}e^{-\frac{\lambda_2}{y}-\alpha_2 y^{\beta_2} e^{-\frac{\lambda_2}{y}}} dy dx \\
 &= \int_0^\infty \alpha_1(\lambda_1 + \beta_1 x)x^{\beta_1-2}e^{-\frac{\lambda_1}{x}-\alpha_1 x^{\beta_1} e^{-\frac{\lambda_1}{x}}} -\alpha_2 x^{\beta_2} e^{-\frac{\lambda_2}{x}} dx.
 \end{aligned}
 \tag{2}$$

In the case $\beta_1 = \beta_2$ and $\lambda_1 = \lambda_2$, using change of variables $s = -\alpha_1 x^{\beta_1} e^{-\frac{\lambda_1}{x}}$ the equation above simplifies to

$$R = \frac{\alpha_1}{\alpha_1 + \alpha_2}.
 \tag{3}$$

The rest of the paper is organized as follows. In section 2 the maximum likelihood estimator (MLE) of R and its asymptotic distribution are derived, and, based on it, the asymptotic, exact and bootstrap-p confidence intervals are constructed. The uniformly minimum variance unbiased estimator (UMVUE) of R and the UMVUE of its variance are obtained in section 3. Bayes estimator of R with respect to mean square error as well as the credible interval is derived in section 4. In section 5 we perform a simulation study and compare the obtained estimators, while in section 6 we present a real data example.

2. MLE of R and its Asymptotics

Let $\mathbf{X} = (X_1, X_2, \dots, X_n)$ and $\mathbf{Y} = (Y_1, Y_2, \dots, Y_m)$ be the samples from the distributions of random variables X and Y . The log-likelihood function of the combined sample $\ln L = \ln L(\alpha_1, \alpha_2, \beta_1, \beta_2, \lambda_1, \lambda_2; \mathbf{x}, \mathbf{y})$ is

$$\begin{aligned}
 \ln L &= n \ln \alpha_1 + \sum_{i=1}^n \ln(\lambda_1 + \beta_1 x_i) + (\beta_1 - 2) \sum_{i=1}^n \ln x_i - \lambda_1 \sum_{i=1}^n \frac{1}{x_i} - \alpha_1 \sum_{i=1}^n x_i^{\beta_1} e^{-\frac{\lambda_1}{x_i}} \\
 &+ m \ln \alpha_2 + \sum_{j=1}^m \ln(\lambda_2 + \beta_2 y_j) + (\beta_2 - 2) \sum_{j=1}^m \ln y_j - \lambda_2 \sum_{j=1}^m \frac{1}{y_j} - \alpha_2 \sum_{j=1}^m y_j^{\beta_2} e^{-\frac{\lambda_2}{y_j}}.
 \end{aligned}$$

The MLE's of the parameters are obtained from the system of likelihood equations:

$$\frac{\partial \ln L}{\partial \alpha_1} = \frac{n}{\alpha_1} - \sum_{i=1}^n x_i^{\beta_1} e^{-\frac{\lambda_1}{x_i}} = 0,$$

$$\begin{aligned}
 \frac{\partial \ln L}{\partial \alpha_2} &= \frac{m}{\alpha_2} - \sum_{j=1}^m y_j^{\beta_2} e^{-\frac{\lambda_2}{y_j}} = 0, \\
 \frac{\partial \ln L}{\partial \beta_1} &= \sum_{i=1}^n \frac{x_i}{\lambda_1 + \beta_1 x_i} + \sum_{i=1}^n \ln x_i - \alpha_1 \sum_{i=1}^n x_i^{\beta_1} e^{-\frac{\lambda_1}{x_i}} \ln x_i = 0, \\
 \frac{\partial \ln L}{\partial \beta_2} &= \sum_{j=1}^m \frac{y_j}{\lambda_2 + \beta_2 y_j} + \sum_{j=1}^m \ln y_j - \alpha_2 \sum_{j=1}^m y_j^{\beta_2} e^{-\frac{\lambda_2}{y_j}} \ln y_j = 0, \\
 \frac{\partial \ln L}{\partial \lambda_1} &= \sum_{i=1}^n \frac{1}{\lambda_1 + \beta_1 x_i} - \sum_{i=1}^n \frac{1}{x_i} + \alpha_1 \sum_{i=1}^n x_i^{\beta_1-1} e^{-\frac{\lambda_1}{x_i}} = 0, \\
 \frac{\partial \ln L}{\partial \lambda_2} &= \sum_{j=1}^m \frac{1}{\lambda_2 + \beta_2 y_j} - \sum_{j=1}^m \frac{1}{y_j} + \alpha_2 \sum_{j=1}^m y_j^{\beta_2-1} e^{-\frac{\lambda_2}{y_j}} = 0.
 \end{aligned}
 \tag{4}$$

In general case there is no analytical solution for this system and it can be solved by numerical methods, for example by the quasi-Newton algorithm. However, when $\beta_1, \beta_2, \lambda_1$ and λ_2 are known, the MLE's for α_1 and α_2 can be explicitly derived from the first two equations as

$$\tilde{\alpha}_1 = \frac{n}{\sum_{i=1}^n X_i^{\beta_1} e^{-\frac{\lambda_1}{X_i}}}, \quad \tilde{\alpha}_2 = \frac{m}{\sum_{j=1}^m Y_j^{\beta_2} e^{-\frac{\lambda_2}{Y_j}}}.
 \tag{5}$$

Then, using the invariance property of MLE, from equation (2) we get the MLE of R

$$\tilde{R} = \int_0^\infty \tilde{\alpha}_1 (\lambda_1 + \beta_1 x) x^{\beta_1-2} e^{-\frac{\lambda_1}{x} - \tilde{\alpha}_1 x^{\beta_1} e^{-\frac{\lambda_1}{x}} - \tilde{\alpha}_2 x^{\beta_2} e^{-\frac{\lambda_2}{x}}} dx.
 \tag{6}$$

When the corresponding known shape parameters are equal, using equation (3) we get

$$\tilde{R} = \frac{\tilde{\alpha}_1}{\tilde{\alpha}_1 + \tilde{\alpha}_2}.
 \tag{7}$$

2.1. Asymptotic Distribution

The standard regularity conditions [12] for the asymptotic normality of MLE's are satisfied (see Appendix for details). Hence, it holds

$$\sqrt{n}(\tilde{\alpha}_1 - \alpha_1, \tilde{\beta}_1 - \beta_1, \tilde{\lambda}_1 - \lambda_1) \xrightarrow{d} \mathcal{N}_3(\mathbf{0}, I^{-1}(\alpha_1, \beta_1, \lambda_1))
 \tag{8}$$

when $n \rightarrow \infty$, where $I(\alpha_1, \beta_1, \lambda_1)$ is Fisher information matrix

$$I(\alpha_1, \beta_1, \lambda_1) = - \begin{bmatrix} E\left(\frac{\partial^2 \ln f(X)}{\partial \alpha_1^2}\right) & E\left(\frac{\partial^2 \ln f(X)}{\partial \alpha_1 \partial \beta_1}\right) & E\left(\frac{\partial^2 \ln f(X)}{\partial \alpha_1 \partial \lambda_1}\right) \\ E\left(\frac{\partial^2 \ln f(X)}{\partial \beta_1 \partial \alpha_1}\right) & E\left(\frac{\partial^2 \ln f(X)}{\partial \beta_1^2}\right) & E\left(\frac{\partial^2 \ln f(X)}{\partial \beta_1 \partial \lambda_1}\right) \\ E\left(\frac{\partial^2 \ln f(X)}{\partial \lambda_1 \partial \alpha_1}\right) & E\left(\frac{\partial^2 \ln f(X)}{\partial \lambda_1 \partial \beta_1}\right) & E\left(\frac{\partial^2 \ln f(X)}{\partial \lambda_1^2}\right) \end{bmatrix},$$

$$f(x) = f(x; \alpha_1, \beta_1, \lambda_1).$$

It could be calculated using equalities

$$\frac{\partial^2 \ln f(x)}{\partial \alpha_1^2} = -\frac{1}{\alpha_1^2},$$

$$\begin{aligned} \frac{\partial^2 \ln f(x)}{\partial \alpha_1 \partial \beta_1} &= -x^{\beta_1} e^{-\frac{\lambda_1}{x}} \ln x, \\ \frac{\partial^2 \ln f(x)}{\partial \alpha_1 \partial \lambda_1} &= x^{\beta_1-1} e^{-\frac{\lambda_1}{x}}, \\ \frac{\partial^2 \ln f(x)}{\partial \beta_1^2} &= -\frac{x^2}{(\lambda_1 + \beta_1 x)^2} - \alpha_1 x^{\beta_1} e^{-\frac{\lambda_1}{x}} \ln^2 x, \\ \frac{\partial^2 \ln f(x)}{\partial \beta_1 \partial \lambda_1} &= -\frac{x}{(\lambda_1 + \beta_1 x)^2} + \alpha_1 x^{\beta_1-1} e^{-\frac{\lambda_1}{x}} \ln x, \\ \frac{\partial^2 \ln f(x)}{\partial \lambda_1^2} &= -\frac{1}{(\lambda_1 + \beta_1 x)^2} - \alpha_1 x^{\beta_1-2} e^{-\frac{\lambda_1}{x}}. \end{aligned}$$

Similarly,

$$\sqrt{m}(\tilde{\alpha}_2 - \alpha_2, \tilde{\beta}_2 - \beta_2, \tilde{\lambda}_2 - \lambda_2) \xrightarrow{d} \mathcal{N}_3(\mathbf{0}, I^{-1}(\alpha_2, \beta_2, \lambda_2)), \tag{9}$$

when $m \rightarrow \infty$, where $I(\alpha_2, \beta_2, \lambda_2)$ could be analogously calculated.

From equation (2) it follows that

$$\frac{\partial R}{\partial \alpha_1} = \int_0^\infty (\lambda_1 + \beta_1 x) x^{\beta_1-2} e^{-\frac{\lambda_1}{x} - \alpha_1 x^{\beta_1} e^{-\frac{\lambda_1}{x}} - \alpha_2 x^{\beta_2} e^{-\frac{\lambda_2}{x}}} (1 - \alpha_1 x^{\beta_1} e^{-\frac{\lambda_1}{x}}) dx, \tag{10}$$

$$\frac{\partial R}{\partial \beta_1} = \int_0^\infty \alpha_1 x^{\beta_1-2} e^{-\frac{\lambda_1}{x} - \alpha_1 x^{\beta_1} e^{-\frac{\lambda_1}{x}} - \alpha_2 x^{\beta_2} e^{-\frac{\lambda_2}{x}}} (x + (\lambda_1 + \beta_1 x) \ln x (1 - \alpha_1 x^{\beta_1} e^{-\frac{\lambda_1}{x}})) dx,$$

$$\frac{\partial R}{\partial \lambda_1} = \int_0^\infty \alpha_1 x^{\beta_1-3} e^{-\frac{\lambda_1}{x} - \alpha_1 x^{\beta_1} e^{-\frac{\lambda_1}{x}} - \alpha_2 x^{\beta_2} e^{-\frac{\lambda_2}{x}}} (x - (\lambda_1 + \beta_1 x) (1 - \alpha_1 x^{\beta_1} e^{-\frac{\lambda_1}{x}})) dx,$$

$$\frac{\partial R}{\partial \alpha_2} = - \int_0^\infty \alpha_1 (\lambda_1 + \beta_1 x) x^{\beta_1+\beta_2-2} e^{-\frac{\lambda_1}{x} - \frac{\lambda_2}{x} - \alpha_1 x^{\beta_1} e^{-\frac{\lambda_1}{x}} - \alpha_2 x^{\beta_2} e^{-\frac{\lambda_2}{x}}} dx, \tag{11}$$

$$\frac{\partial R}{\partial \beta_2} = - \int_0^\infty \alpha_1 \alpha_2 (\lambda_1 + \beta_1 x) x^{\beta_1+\beta_2-2} \ln x e^{-\frac{\lambda_1}{x} - \frac{\lambda_2}{x} - \alpha_1 x^{\beta_1} e^{-\frac{\lambda_1}{x}} - \alpha_2 x^{\beta_2} e^{-\frac{\lambda_2}{x}}} dx,$$

$$\frac{\partial R}{\partial \lambda_2} = \int_0^\infty \alpha_1 \alpha_2 (\lambda_1 + \beta_1 x) x^{\beta_1+\beta_2-3} e^{-\frac{\lambda_1}{x} - \frac{\lambda_2}{x} - \alpha_1 x^{\beta_1} e^{-\frac{\lambda_1}{x}} - \alpha_2 x^{\beta_2} e^{-\frac{\lambda_2}{x}}} dx.$$

Using above formulae the asymptotic distribution of \tilde{R} can be determined.

Theorem 2.1. Let the ratio $\frac{n}{m}$ converge to a positive number s when both n and m tend to infinity. Then

$$\sqrt{n}(\tilde{R} - R) \xrightarrow{d} \mathcal{N}(0, V)$$

when $n \rightarrow \infty$, where $V = BJB^T$,

$$B = \left[\frac{\partial R}{\partial \alpha_1} \quad \frac{\partial R}{\partial \beta_1} \quad \frac{\partial R}{\partial \lambda_1} \quad \frac{\partial R}{\partial \alpha_2} \quad \frac{\partial R}{\partial \beta_2} \quad \frac{\partial R}{\partial \lambda_2} \right]$$

and

$$J = \begin{bmatrix} I^{-1}(\alpha_1, \beta_1, \lambda_1) & \mathbf{0} \\ \mathbf{0} & sI^{-1}(\alpha_2, \beta_2, \lambda_2) \end{bmatrix}.$$

Proof. From expression (9) it follows that

$$\sqrt{n}(\tilde{\alpha}_2 - \alpha_2, \tilde{\beta}_2 - \beta_2, \tilde{\lambda}_2 - \lambda_2) = \sqrt{\frac{n}{m}} \sqrt{m}(\tilde{\alpha}_2 - \alpha_2, \tilde{\beta}_2 - \beta_2, \tilde{\lambda}_2 - \lambda_2) \xrightarrow{d} \mathcal{N}_3(\mathbf{0}, sI^{-1}(\alpha_2, \beta_2, \lambda_2)),$$

when both n and m tend to infinity. Combining this with expression (8) we get that

$$\sqrt{n}(\tilde{\alpha}_1 - \alpha_1, \tilde{\beta}_1 - \beta_1, \tilde{\lambda}_1 - \lambda_1, \tilde{\alpha}_2 - \alpha_2, \tilde{\beta}_2 - \beta_2, \tilde{\lambda}_2 - \lambda_2) \xrightarrow{d} \mathcal{N}_3(\mathbf{0}, J),$$

when $n \rightarrow \infty$. Using the method from [12, Corollary 6.4.1.] we get the statement of the theorem. □

In special cases the situation is much simpler.

Corollary 2.2. *Let the ratio $\frac{n}{m}$ converge to a positive number s when both n and m tend to infinity and let the shape parameters $\beta_1, \beta_2, \lambda_1$ and λ_2 be known constants. Then*

$$\sqrt{n}(\tilde{R} - R) \xrightarrow{d} \mathcal{N}(0, V_1)$$

when $n \rightarrow \infty$, where

$$V_1 = \alpha_1^2 \left(\frac{\partial R}{\partial \alpha_1} \right)^2 + s \alpha_2^2 \left(\frac{\partial R}{\partial \alpha_2} \right)^2 \tag{12}$$

and $\frac{\partial R}{\partial \alpha_1}$ and $\frac{\partial R}{\partial \alpha_2}$ can be determined using equations (10) and (11).

Moreover, in the case when $\beta_1 = \beta_2$ and $\lambda_1 = \lambda_2$, it holds

$$\sqrt{n}(\tilde{R} - R) \xrightarrow{d} \mathcal{N}\left(0, (1+s) \frac{\alpha_1^2 \alpha_2^2}{(\alpha_1 + \alpha_2)^4}\right) \tag{13}$$

when $n \rightarrow \infty$.

Proof. In the case when the shape parameters $\beta_1, \beta_2, \lambda_1$ and λ_2 are known constants, expressions (8) and (9) become

$$\sqrt{n}(\tilde{\alpha}_1 - \alpha_1) \xrightarrow{d} \mathcal{N}(0, \alpha_1^2)$$

and

$$\sqrt{m}(\tilde{\alpha}_2 - \alpha_2) \xrightarrow{d} \mathcal{N}(0, \alpha_2^2),$$

ie

$$\sqrt{n}(\tilde{\alpha}_2 - \alpha_2) \xrightarrow{d} \mathcal{N}(0, s\alpha_2^2).$$

Then, $B = \begin{bmatrix} \frac{\partial R}{\partial \alpha_1} & \frac{\partial R}{\partial \alpha_2} \end{bmatrix}$ and $J = \begin{bmatrix} \alpha_1^2 & 0 \\ 0 & s\alpha_2^2 \end{bmatrix}$. Using previous theorem we get the statement of the corollary.

In the case when the shape parameters are known constants and $\beta_1 = \beta_2$ and $\lambda_1 = \lambda_2$, from equation (3) it is obtained that

$$\frac{\partial R}{\partial \alpha_1} = \frac{\alpha_2}{(\alpha_1 + \alpha_2)^2}, \quad \frac{\partial R}{\partial \alpha_2} = -\frac{\alpha_1}{(\alpha_1 + \alpha_2)^2}.$$

Changing that in equation (12), we get expression (13). □

2.2. Confidence Intervals Based on MLE

2.2.1. Asymptotic Confidence Interval

Using the asymptotic distribution of R the asymptotic confidence interval of R can be constructed. In what follows, z_α denotes the α th quantile of the standard normal distribution.

In general case, using Theorem 2.1 we get the asymptotic interval of confidence level $1 - \gamma$ for R , which is given by

$$I_R^{(ASYM)} = \left(\tilde{R} - z_{1-\frac{\gamma}{2}} \frac{\sqrt{\tilde{V}}}{\sqrt{n}}, \tilde{R} + z_{1-\frac{\gamma}{2}} \frac{\sqrt{\tilde{V}}}{\sqrt{n}} \right), \tag{14}$$

where \tilde{R} is the MLE of R given by equation (2) and $\tilde{V} = \tilde{B}\tilde{J}\tilde{B}^T$. Using the invariance property of MLEs, \tilde{B} is obtained by replacing each parameter with its MLE, in the expression for B defined in Theorem 2.1, while \tilde{J} can be obtained in the same way, or by inverting the observed Fisher information matrix.

When the shape parameters are known, using Corollary 2.2 we obtain the asymptotic interval of confidence level $1 - \gamma$ for R , which is given by

$$I_R^{(ASYM)} = \left(\tilde{R} - z_{1-\frac{\gamma}{2}} \frac{\sqrt{\tilde{V}_1}}{\sqrt{n}}, \tilde{R} + z_{1-\frac{\gamma}{2}} \frac{\sqrt{\tilde{V}_1}}{\sqrt{n}} \right), \tag{15}$$

where \tilde{R} is calculated using equations (5) and (6), and \tilde{V}_1 is obtained analogously to \tilde{V} by using equations (5), (10), (11) and (12) and invariance property.

In the case when the shape parameters are known and $\beta_1 = \beta_2$ and $\lambda_1 = \lambda_2$, from expression (13) follows that the asymptotic interval of confidence level $1 - \gamma$ for R is given by

$$I_R^{(ASYM)} = \left(\tilde{R} - z_{1-\frac{\gamma}{2}} \sqrt{\frac{(1+s)}{n} \frac{\tilde{\alpha}_1^2 \tilde{\alpha}_2^2}{(\tilde{\alpha}_1 + \tilde{\alpha}_2)^4}}, \tilde{R} + z_{1-\frac{\gamma}{2}} \sqrt{\frac{(1+s)}{n} \frac{\tilde{\alpha}_1^2 \tilde{\alpha}_2^2}{(\tilde{\alpha}_1 + \tilde{\alpha}_2)^4}} \right), \tag{16}$$

where \tilde{R} , $\tilde{\alpha}_1$ and $\tilde{\alpha}_2$ are obtained using equations (5) and (7).

Remark 2.3. In case the lower or upper bounds fall outside the interval $[0, 1]$, the confidence interval is appropriately truncated.

2.2.2. Exact Confidence Interval

It is easy to prove that if $X^\beta e^{-\frac{x}{\lambda}}$ has exponential $\mathcal{E}(\alpha)$ distribution, then X has PYEW(α, β, λ) distribution. In the case when the shape parameters are known and $\beta_1 = \beta_2$ and $\lambda_1 = \lambda_2$ from equation (5) follows that $n\tilde{\alpha}_1^{-1}$ and $m\tilde{\alpha}_2^{-1}$ have gamma $\Gamma(n, \alpha_1)$ and gamma $\Gamma(m, \alpha_2)$ distributions respectively. Then, $2\alpha_1 n\tilde{\alpha}_1^{-1}$ and $2\alpha_2 m\tilde{\alpha}_2^{-1}$ have chi-square χ_{2n}^2 and chi-square χ_{2m}^2 distributions respectively, so that $\frac{\alpha_2 \tilde{\alpha}_1}{\alpha_1 \tilde{\alpha}_2}$ has Fisher's $F_{2m, 2n}$ distribution.

Using this fact and equation (3) we get that the exact interval of confidence level $1 - \gamma$ for R is given by

$$I_R^{(EXACT)} = \left(\frac{1}{1 + F_{2m, 2n; 1-\frac{\gamma}{2}} \frac{\tilde{\alpha}_2}{\tilde{\alpha}_1}}, \frac{1}{1 + F_{2m, 2n; \frac{\gamma}{2}} \frac{\tilde{\alpha}_2}{\tilde{\alpha}_1}} \right), \tag{17}$$

where $F_{2m, 2n; \alpha}$ is α th quantile from Fisher's $F_{2m, 2n}$ distribution and $\tilde{\alpha}_1$ and $\tilde{\alpha}_2$ are obtained using equations (5).

This fact also enables us to create a test for testing $H_0 : R \leq R_0$ against $H_1 : R > R_0$ for some fixed value R_0 . Since $\frac{\alpha_2}{\alpha_1} = \frac{1-R}{R}$, the test statistic

$$T = \frac{1 - R_0}{R_0} \frac{\tilde{\alpha}_1}{\tilde{\alpha}_2},$$

under H_0 , has Fisher's $F_{2m, 2n}$ distribution. The critical region is $\{T \geq C\}$, where C is the appropriate quantile of Fisher's distribution, and the p -value is $1 - F_T(T_0)$, where T_0 is the sample value of T .

2.2.3. Bootstrap-p Confidence Interval

The confidence intervals based on the asymptotic distribution do not perform very well for small sample sizes. Therefore, in the case when the shape parameters are known, we propose a construction of the confidence interval based on parametric bootstrap-p method. The algorithm is illustrated below.

- Step 1: From initial samples $\mathbf{x} = (x_1, x_2, \dots, x_n)$ and $\mathbf{y} = (y_1, y_2, \dots, y_m)$ calculate MLEs $\tilde{\alpha}_1$ and $\tilde{\alpha}_2$ using equations (5).
- Step 2: Use those estimates to generate bootstrap sample \mathbf{x}_i^* from PYEW($\tilde{\alpha}_1, \beta_1, \lambda_1$) distribution and bootstrap sample \mathbf{y}_i^* from PYEW($\tilde{\alpha}_2, \beta_2, \lambda_2$) distribution. Based on these bootstrap samples compute estimates $\tilde{\alpha}_{1i}^*$ and $\tilde{\alpha}_{2i}^*$ using equation (5), and \tilde{R}_i^* of R using equation (6), or, in the case of equal shape parameters, equation (7).
- Step 3: Repeat step 2, N boot times.
- Step 4: Let $\tilde{R}_{(\alpha)}^*$ be the α th empirical quantile of the \tilde{R}_i^* values obtained in step 3, that is, the $N\alpha$ th value in the ordered list of the N replications of \tilde{R}_i^* . If $N\alpha$ is not an integer, assuming $\alpha \leq 0.5$, the largest integer less or equal $(N + 1)\alpha$ should be used. The bootstrap-p interval of confidence level $1 - \gamma$ for R is given by

$$I_R^{(BOOTp)} = \left(\tilde{R}_{(\frac{\gamma}{2})}^*, \tilde{R}_{(1-\frac{\gamma}{2})}^* \right). \tag{18}$$

3. UMVUE of R

In this section we find the UMVUE of R , denoted by \widehat{R} , and the UMVUE of the variance of \widehat{R} .

When β and λ are unknown there is no complete sufficient statistics for parameters of Peng-Yan extended Weibull distribution. Therefore we consider only the case when shape parameters are known. Then,

$$T_X = \sum_{i=1}^n X_i^{\beta_1} e^{-\frac{\lambda_1}{X_i}} \text{ and } T_Y = \sum_{j=1}^m Y_j^{\beta_2} e^{-\frac{\lambda_2}{Y_j}}$$

are complete and sufficient statistics for α_1 and α_2 . These statistics, as a sum of independent identically distributed random variables with exponential distribution, are both gamma distributed with parameters (n, α_1) and (m, α_2) respectively.

Theorem 3.1. For \widehat{R} and \widehat{R}^2 hold

$$\widehat{R} = \frac{n-1}{t_X^{n-1} t_Y^{m-1}} \int_0^\infty (t_X - x^{\beta_1} e^{-\frac{\lambda_1}{x}})^{n-2} (t_Y - x^{\beta_2} e^{-\frac{\lambda_2}{x}})^{m-1} (\lambda_1 + \beta_1 x) x^{\beta_1-2} e^{-\frac{\lambda_1}{x}} I\{t_X \geq x^{\beta_1} e^{-\frac{\lambda_1}{x}}\} I\{t_Y \geq x^{\beta_2} e^{-\frac{\lambda_2}{x}}\} dx \tag{19}$$

and

$$\begin{aligned} \widehat{R}^2 = & \frac{(n-1)(n-2)}{t_X^{n-1} t_Y^{m-1}} \int_0^\infty \int_0^\infty ((\lambda_1 + \beta_1 x_1) x_1^{\beta_1-2} e^{-\frac{\lambda_1}{x_1}} (\lambda_1 + \beta_1 x_2) x_2^{\beta_1-2} e^{-\frac{\lambda_1}{x_2}} (t_X - x_1^{\beta_1} e^{-\frac{\lambda_1}{x_1}} - x_2^{\beta_1} e^{-\frac{\lambda_1}{x_2}})^{n-3} \\ & \cdot (t_Y - x_1^{\beta_2} e^{-\frac{\lambda_2}{x_1}} - x_2^{\beta_2} e^{-\frac{\lambda_2}{x_2}})^{m-1} I\{t_X \geq x_1^{\beta_1} e^{-\frac{\lambda_1}{x_1}} + x_2^{\beta_1} e^{-\frac{\lambda_1}{x_2}}\} I\{t_Y \geq x_1^{\beta_2} e^{-\frac{\lambda_2}{x_1}} + x_2^{\beta_2} e^{-\frac{\lambda_2}{x_2}}\}) dx_2 dx_1. \end{aligned} \tag{20}$$

The UMVUE of the variance of \widehat{R} is given by

$$\widehat{Var}(\widehat{R}) = (\widehat{R})^2 - \widehat{R}^2. \tag{21}$$

Proof. To calculate the UMVUE of R we shall use the procedure from [17, subsection 2.2.2]. Using the transformation of independent random variables $(X_1, T_X - X_1^{\beta_1} e^{-\frac{\lambda_1}{X_1}})$ to random variables (X_1, T_X) we get

$$g_{T_X, T_X}(x, t_X; \alpha_1, \beta_1, \lambda_1) = \alpha_1(\lambda_1 + \beta_1 x)x^{\beta_1-2}e^{-\frac{\lambda_1}{x} - \alpha_1 x^{\beta_1} e^{-\frac{\lambda_1}{x}}} (t_X - x^{\beta_1} e^{-\frac{\lambda_1}{x}})^{n-2} \cdot \frac{e^{-\alpha_1(t_X - x^{\beta_1} e^{-\frac{\lambda_1}{x}})} \alpha_1^{n-1}}{\Gamma(n-1)} I\{t_X \geq x^{\beta_1} e^{-\frac{\lambda_1}{x}}\}$$

and

$$g_{T_X|X_1=x}(t_X; \alpha_1, \beta_1, \lambda_1) = \frac{(t_X - x^{\beta_1} e^{-\frac{\lambda_1}{x}})^{n-2} e^{-\alpha_1(t_X - x^{\beta_1} e^{-\frac{\lambda_1}{x}})} \alpha_1^{n-1}}{\Gamma(n-1)} I\{t_X \geq x^{\beta_1} e^{-\frac{\lambda_1}{x}}\}$$

Putting 1 as arbitrary value for α_1 we obtain

$$\begin{aligned} \widehat{f}_X(x) &= \frac{g_{T_X|X_1=x}(t_X; 1, \beta_1, \lambda_1)}{g_{T_X}(t_X; 1, \beta_1, \lambda_1)} f(x; 1, \beta_1, \lambda_1) \\ &= \frac{(t_X - x^{\beta_1} e^{-\frac{\lambda_1}{x}})^{n-2} e^{-\alpha_1(t_X - x^{\beta_1} e^{-\frac{\lambda_1}{x}})}}{\frac{\Gamma(n-1)}{t_X^{n-1} e^{-t_X}} \Gamma(n)} I\{t_X \geq x^{\beta_1} e^{-\frac{\lambda_1}{x}}\} (\lambda_1 + \beta_1 x)x^{\beta_1-2} e^{-\frac{\lambda_1}{x} - x^{\beta_1} e^{-\frac{\lambda_1}{x}}} \\ &= \frac{n-1}{t_X^{n-1}} (t_X - x^{\beta_1} e^{-\frac{\lambda_1}{x}})^{n-2} (\lambda_1 + \beta_1 x)x^{\beta_1-2} e^{-\frac{\lambda_1}{x}} I\{t_X \geq x^{\beta_1} e^{-\frac{\lambda_1}{x}}\}, \end{aligned}$$

and analogously

$$\widehat{f}_Y(y) = \frac{m-1}{t_Y^{m-1}} (t_Y - y^{\beta_2} e^{-\frac{\lambda_2}{y}})^{m-2} (\lambda_2 + \beta_2 y)y^{\beta_2-2} e^{-\frac{\lambda_2}{y}} I\{t_Y \geq y^{\beta_2} e^{-\frac{\lambda_2}{y}}\}.$$

Using Theorem 2.5 from [17], and change of variable $s = t_Y - y^{\beta_2} e^{-\frac{\lambda_2}{y}}$, we get

$$\begin{aligned} \widehat{R} &= \iint I(x < y) \widehat{f}_X(x) \widehat{f}_Y(y) dx dy \\ &= \int_0^\infty \int_x^\infty \left(\frac{n-1}{t_X^{n-1}} (t_X - x^{\beta_1} e^{-\frac{\lambda_1}{x}})^{n-2} (\lambda_1 + \beta_1 x)x^{\beta_1-2} e^{-\frac{\lambda_1}{x}} I\{t_X \geq x^{\beta_1} e^{-\frac{\lambda_1}{x}}\} \right) \\ &\quad \cdot \left(\frac{m-1}{t_Y^{m-1}} (t_Y - y^{\beta_2} e^{-\frac{\lambda_2}{y}})^{m-2} (\lambda_2 + \beta_2 y)y^{\beta_2-2} e^{-\frac{\lambda_2}{y}} I\{t_Y \geq y^{\beta_2} e^{-\frac{\lambda_2}{y}}\} \right) dy dx \\ &= \frac{n-1}{t_X^{n-1} t_Y^{m-1}} \int_0^\infty (t_X - x^{\beta_1} e^{-\frac{\lambda_1}{x}})^{n-2} (t_Y - x^{\beta_2} e^{-\frac{\lambda_2}{x}})^{m-1} (\lambda_1 + \beta_1 x)x^{\beta_1-2} e^{-\frac{\lambda_1}{x}} I\{t_X \geq x^{\beta_1} e^{-\frac{\lambda_1}{x}}\} I\{t_Y \geq x^{\beta_2} e^{-\frac{\lambda_2}{x}}\} dx. \end{aligned}$$

In order to calculate the UMVUE of R^2 , we apply the identical procedure as above. Putting $\alpha_1 = 1$ we obtain

$$\begin{aligned} \widehat{f}_X(x_1, x_2) &= \frac{g_{T_X|X_1=x_1, X_2=x_2}(t_X; 1, \beta_1, \lambda_1)}{g_{T_X}(t_X; 1, \beta_1, \lambda_1)} \cdot \prod_{i=1}^2 f(x_i; 1, \beta_1, \lambda_1) \\ &= \frac{(n-1)(n-2)}{t_X^{n-1}} (t_X - x_1^{\beta_1} e^{-\frac{\lambda_1}{x_1}} - x_2^{\beta_1} e^{-\frac{\lambda_1}{x_2}})^{n-3} (\lambda_1 + \beta_1 x_1)x_1^{\beta_1-2} e^{-\frac{\lambda_1}{x_1}} \\ &\quad \cdot (\lambda_1 + \beta_1 x_2)x_2^{\beta_1-2} e^{-\frac{\lambda_1}{x_2}} I\{t_X \geq x_1^{\beta_1} e^{-\frac{\lambda_1}{x_1}} + x_2^{\beta_1} e^{-\frac{\lambda_1}{x_2}}\}, \end{aligned}$$

and analogously

$$\widehat{f}_Y(y_1, y_2) = \frac{(m-1)(m-2)}{t_Y^{m-1}} (t_Y - y_1^{\beta_2} e^{-\frac{\lambda_2}{y_1}} - y_2^{\beta_2} e^{-\frac{\lambda_2}{y_2}})^{m-3} (\lambda_2 + \beta_2 y_1) y_1^{\beta_2-2} e^{-\frac{\lambda_2}{y_1}} \cdot (\lambda_2 + \beta_2 y_2) y_2^{\beta_2-2} e^{-\frac{\lambda_2}{y_2}} I\{t_Y \geq y_1^{\beta_2} e^{-\frac{\lambda_2}{y_1}} + y_2^{\beta_2} e^{-\frac{\lambda_2}{y_2}}\}.$$

Using Theorem 2.5 from [17] we get

$$\begin{aligned} \widehat{R}^2 &= \iiint\limits_{x_1 < y_1} \iint\limits_{x_2 < y_2} \widehat{f}_X(x_1, x_2) \widehat{f}_Y(y_1, y_2) dx_1 dx_2 dy_1 dy_2 \\ &= \int_0^\infty \int_0^\infty \int_{x_1}^\infty \int_{x_2}^\infty \left(\frac{(n-1)(n-2)}{t_X^{n-1}} (t_X - x_1^{\beta_1} e^{-\frac{\lambda_1}{x_1}} - x_2^{\beta_1} e^{-\frac{\lambda_1}{x_2}})^{n-3} (\lambda_1 + \beta_1 x_1) x_1^{\beta_1-2} e^{-\frac{\lambda_1}{x_1}} (\lambda_1 + \beta_1 x_2) x_2^{\beta_1-2} e^{-\frac{\lambda_1}{x_2}} \right. \\ &\quad \cdot I\{t_X \geq x_1^{\beta_1} e^{-\frac{\lambda_1}{x_1}} + x_2^{\beta_1} e^{-\frac{\lambda_1}{x_2}}\} \left. \frac{(m-1)(m-2)}{t_Y^{m-1}} (t_Y - y_1^{\beta_2} e^{-\frac{\lambda_2}{y_1}} - y_2^{\beta_2} e^{-\frac{\lambda_2}{y_2}})^{m-3} (\lambda_2 + \beta_2 y_1) y_1^{\beta_2-2} e^{-\frac{\lambda_2}{y_1}} \right. \\ &\quad \cdot (\lambda_2 + \beta_2 y_2) y_2^{\beta_2-2} e^{-\frac{\lambda_2}{y_2}} I\{t_Y \geq y_1^{\beta_2} e^{-\frac{\lambda_2}{y_1}} + y_2^{\beta_2} e^{-\frac{\lambda_2}{y_2}}\} \Big) dy_2 dy_1 dx_2 dx_1 \\ &= \frac{(n-1)(n-2)}{t_X^{n-1} t_Y^{m-1}} \int_0^\infty \int_0^\infty \left((\lambda_1 + \beta_1 x_1) x_1^{\beta_1-2} e^{-\frac{\lambda_1}{x_1}} (\lambda_1 + \beta_1 x_2) x_2^{\beta_1-2} e^{-\frac{\lambda_1}{x_2}} (t_X - x_1^{\beta_1} e^{-\frac{\lambda_1}{x_1}} - x_2^{\beta_1} e^{-\frac{\lambda_1}{x_2}})^{n-3} \right. \\ &\quad \cdot (t_Y - x_1^{\beta_2} e^{-\frac{\lambda_2}{x_1}} - x_2^{\beta_2} e^{-\frac{\lambda_2}{x_2}})^{m-1} I\{t_X \geq x_1^{\beta_1} e^{-\frac{\lambda_1}{x_1}} + x_2^{\beta_1} e^{-\frac{\lambda_1}{x_2}}\} I\{t_Y \geq x_1^{\beta_2} e^{-\frac{\lambda_2}{x_1}} + x_2^{\beta_2} e^{-\frac{\lambda_2}{x_2}}\} \Big) dx_2 dx_1. \end{aligned}$$

The statement (21) follows from Theorem 2.6 from [17]. □

In special case the situation is simpler.

Corollary 3.2. *If the shape parameters are known and $\beta_1 = \beta_2 = \beta$ and $\lambda_1 = \lambda_2 = \lambda$, then expressions for \widehat{R} and \widehat{R}^2 simplify to*

$$\widehat{R} = \frac{n-1}{t_X^{n-1} t_Y^{m-1}} \sum_{k=0}^{m-1} \frac{\binom{m-1}{k} (t_Y - t_X)^{m-1-k}}{n+k-1} \left(t_X^{n+k-1} - (\max\{0, t_X - t_Y\})^{n+k-1} \right)$$

and

$$\widehat{R}^2 = \frac{(n-1)(n-2)}{t_X^{n-1} t_Y^{m-1}} \sum_{k=0}^{m-1} \frac{\binom{m-1}{k} (t_Y - t_X)^{m-1-k}}{n+k-2} \left(\frac{t_X^{n+k-1} - (\max\{0, t_X - t_Y\})^{n+k-1}}{n+k-1} - \min\{t_X, t_Y\} (\max\{0, t_X - t_Y\})^{n+k-2} \right).$$

Proof. Putting β instead β_1 and β_2 and λ instead λ_1 and λ_2 in the equations (19) and (20), after some calculations we get

$$\begin{aligned} \widehat{R} &= \frac{n-1}{t_X^{n-1} t_Y^{m-1}} \int_0^\infty (t_X - x^\beta e^{-\frac{\lambda}{x}})^{n-2} (t_Y - x^\beta e^{-\frac{\lambda}{x}})^{m-1} (\lambda + \beta x) x^{\beta-2} e^{-\frac{\lambda}{x}} I\{t_X \geq x^\beta e^{-\frac{\lambda}{x}}\} I\{t_Y \geq x^\beta e^{-\frac{\lambda}{x}}\} dx \\ &= \frac{n-1}{t_X^{n-1} t_Y^{m-1}} \int_{-\infty}^{t_X} s^{n-2} (t_Y - t_X + s)^{m-1} I\{s \geq \max\{0, t_X - t_Y\}\} ds \\ &= \frac{n-1}{t_X^{n-1} t_Y^{m-1}} \sum_{k=0}^{m-1} \frac{\binom{m-1}{k} (t_Y - t_X)^{m-1-k}}{n+k-1} \left(t_X^{n+k-1} - (\max\{0, t_X - t_Y\})^{n+k-1} \right) \end{aligned}$$

and

$$\widehat{R}^2 = \frac{(n-1)(n-2)}{t_X^{n-1} t_Y^{m-1}} \int_0^\infty \int_0^\infty \left((\lambda + \beta x_1) x_1^{\beta-2} e^{-\frac{\lambda}{x_1}} (\lambda + \beta x_2) x_2^{\beta-2} e^{-\frac{\lambda}{x_2}} (t_X - x_1^\beta e^{-\frac{\lambda}{x_1}} - x_2^\beta e^{-\frac{\lambda}{x_2}})^{n-3} \right.$$

$$\begin{aligned}
 & \cdot (t_Y - x_1^\beta e^{-\frac{\lambda}{x_1}} - x_2^\beta e^{-\frac{\lambda}{x_2}})^{m-1} I\{t_X \geq x_1^\beta e^{-\frac{\lambda}{x_1}} + x_2^\beta e^{-\frac{\lambda}{x_2}}\} I\{t_Y \geq x_1^\beta e^{-\frac{\lambda}{x_1}} + x_2^\beta e^{-\frac{\lambda}{x_2}}\} dx_2 dx_1 \\
 &= \frac{(n-1)(n-2)}{t_X^{n-1} t_Y^{m-1}} \int_0^\infty (\lambda + \beta x_1) x_1^{\beta-2} e^{-\frac{\lambda}{x_1}} dx_1 \int_{-\infty}^{t_X - x_1^\beta e^{-\frac{\lambda}{x_1}}} s^{n-3} (t_Y - t_X + s)^{m-1} I\{s \geq \max\{0, t_X - t_Y\}\} ds \\
 &= \frac{(n-1)(n-2)}{t_X^{n-1} t_Y^{m-1}} \sum_{k=0}^{m-1} \frac{\binom{m-1}{k} (t_Y - t_X)^{m-1-k}}{n+k-2} \int_0^\infty ((\lambda + \beta x_1) x_1^{\beta-2} e^{-\frac{\lambda}{x_1}} \\
 & \cdot ((t_X - x_1^\beta e^{-\frac{\lambda}{x_1}})^{n+k-2} - (\max\{0, t_X - t_Y\})^{n+k-2}) I\{t_X - x_1^\beta e^{-\frac{\lambda}{x_1}} \geq \max\{0, t_X - t_Y\}\} dx_1 \\
 &= \frac{(n-1)(n-2)}{t_X^{n-1} t_Y^{m-1}} \sum_{k=0}^{m-1} \frac{\binom{m-1}{k} (t_Y - t_X)^{m-1-k}}{n+k-2} \int_{-\infty}^{t_X} (s^{n+k-2} - (\max\{0, t_X - t_Y\})^{n+k-2}) I\{s \geq \max\{0, t_X - t_Y\}\} ds \\
 &= \frac{(n-1)(n-2)}{t_X^{n-1} t_Y^{m-1}} \sum_{k=0}^{m-1} \frac{\binom{m-1}{k} (t_Y - t_X)^{m-1-k}}{n+k-2} \left(\frac{t_X^{n+k-1} - (\max\{0, t_X - t_Y\})^{n+k-1}}{n+k-1} - \min\{t_X, t_Y\} (\max\{0, t_X - t_Y\})^{n+k-2} \right). \square
 \end{aligned}$$

4. Bayes Estimator of R

In this section we deal with Bayes estimator of R , denoted by \check{R} .

Since all our parameters are defined on $(0, \infty)$, a popular prior for such parameters is the gamma prior. Let, in the case when all parameters are unknown, the priors be $\alpha_i : \Gamma(a_i, b_i)$, $\beta_i : \Gamma(c_i, d_i)$, $\lambda_i : \Gamma(e_i, f_i)$, $i \in \{1, 2\}$. Let, for brevity, $\vartheta_1 = (\alpha_1, \beta_1, \lambda_1)$ and $\vartheta_2 = (\alpha_2, \beta_2, \lambda_2)$. The joint posterior distribution of random vector ϑ_1 is

$$\pi(\vartheta_1 | \mathbf{x}) = \frac{1}{K_1} \alpha_1^{a_1+n-1} e^{-\alpha_1 (b_1 + \sum_{i=1}^n x_i^{\beta_1} e^{-\frac{\lambda_1}{x_i}})} \left(\prod_{i=1}^n (\lambda_1 + \beta_1 x_i) x_i^{\beta_1-2} \right) \beta_1^{c_1-1} e^{-d_1 \beta_1} \lambda_1^{e_1-1} e^{-\lambda_1 (f_1 + \sum_{i=1}^n \frac{1}{x_i})},$$

where

$$K_1 = \int_{(0, \infty)^3} \alpha_1^{a_1+n-1} e^{-\alpha_1 (b_1 + \sum_{i=1}^n x_i^{\beta_1} e^{-\frac{\lambda_1}{x_i}})} \left(\prod_{i=1}^n (\lambda_1 + \beta_1 x_i) x_i^{\beta_1-2} \right) \beta_1^{c_1-1} e^{-d_1 \beta_1} \lambda_1^{e_1-1} e^{-\lambda_1 (f_1 + \sum_{i=1}^n \frac{1}{x_i})} d\vartheta_1,$$

with $d\vartheta_1 = d\alpha_1 d\beta_1 d\lambda_1$, is the proportionality constant. The analogous formula holds for $\pi(\vartheta_2 | \mathbf{y})$.

The Bayes estimator of R for mean square loss function is the posterior mean. Here it can be obtained as

$$\check{R} = E(R | \mathbf{x}, \mathbf{y}) = \int_{(0, \infty)^6} R(\vartheta_1, \vartheta_2) \pi(\vartheta_1 | \mathbf{x}) \pi(\vartheta_2 | \mathbf{y}) d\vartheta_1 d\vartheta_2, \tag{22}$$

where $R = R(\vartheta_1, \vartheta_2)$ is given by equation (2), or, in the case of equal shape parameters, by equation (3). The integral (22) can be in general case solved using Markov chain Monte Carlo algorithm and numerical integration.

In the case when the shape parameters are known, gamma priors are conjugate priors for rate parameters α_i , so let $\alpha_1 : \Gamma(a_1, b_1)$ and $\alpha_2 : \Gamma(a_2, b_2)$. We obtain the following posterior distributions

$$\alpha_1 | \mathbf{x} : \Gamma\left(a_1 + n, b_1 + \sum_{i=1}^n x_i^{\beta_1} e^{-\frac{\lambda_1}{x_i}}\right), \tag{23}$$

$$\alpha_2 | \mathbf{y} : \Gamma\left(a_2 + m, b_2 + \sum_{j=1}^m y_j^{\beta_2} e^{-\frac{\lambda_2}{y_j}}\right). \tag{24}$$

Then, the Bayes estimator of R , in the case of unequal shape parameters, is

$$\check{R} = E(R|\mathbf{x}, \mathbf{y}) = \int_0^\infty \int_0^\infty R(\alpha_1, \alpha_2)\pi(\alpha_1|\mathbf{x})\pi(\alpha_2|\mathbf{y})d\alpha_1d\alpha_2,$$

where $R = R(\alpha_1, \alpha_2)$ is given by equation (2).

When the shape parameters are known and $\beta_1 = \beta_2 = \beta$ and $\lambda_1 = \lambda_2 = \lambda$, the closed form expression for R , from equation (3), enables us to obtain the posterior density for R and Bayes estimator of R . Calculation is similar as in [9]. Denote, for simplicity, $a^* = a_1 + n$, $b^* = b_1 + \sum_{i=1}^n x_i^\beta e^{-\frac{\lambda}{x_i}}$, $c^* = a_2 + m$ and $d^* = b_2 + \sum_{j=1}^m y_j^\beta e^{-\frac{\lambda}{y_j}}$.

Then, the joint posterior distribution is

$$\pi(\alpha_1, \alpha_2|\mathbf{x}, \mathbf{y}) = \frac{(b^*)^{a^*} (d^*)^{c^*}}{\Gamma(a^*)\Gamma(c^*)} \alpha_1^{a^*-1} e^{-b^*\alpha_1} \alpha_2^{c^*-1} e^{-d^*\alpha_2}.$$

Transformation of variables $R = \frac{\alpha_1}{\alpha_1 + \alpha_2}$ and $W = \alpha_1 + \alpha_2$ gives us

$$h(r, w|\mathbf{x}, \mathbf{y}) = \frac{(b^*)^{a^*} (d^*)^{c^*}}{\Gamma(a^*)\Gamma(c^*)} r^{a^*-1} (1-r)^{c^*-1} w^{a^*+c^*-1} e^{-wd^*(1-cr)},$$

$0 < r < 1$, $w > 0$, where $c = \frac{d^*-b^*}{d^*}$ and $c < 1$. The posterior density for R is

$$f_{R|\mathbf{x}, \mathbf{y}}(r) = \int_0^\infty h(r, w|\mathbf{x}, \mathbf{y}) dw = \frac{1}{B(a^*, c^*)} \left(\frac{b^*}{d^*}\right)^{a^*} \frac{r^{a^*-1} (1-r)^{c^*-1}}{(1-cr)^{a^*+c^*}},$$

$0 < r < 1$. Finally, using equation 3.197.3 from [10] we get

$$\begin{aligned} \check{R} = E(R|\mathbf{x}, \mathbf{y}) &= \int_0^1 r f_{R|\mathbf{x}, \mathbf{y}}(r) dr \\ &= \begin{cases} \frac{a^*}{a^*+c^*} \left(\frac{b^*}{d^*}\right)^{a^*} {}_2F_1(a^*+c^*, a^*+1; a^*+c^*+1; c), & \text{for } |c| < 1, \\ \frac{a^*}{a^*+c^*} \left(\frac{d^*}{b^*}\right)^{c^*} {}_2F_1(a^*+c^*, c^*; a^*+c^*+1; \frac{c}{c-1}), & \text{for } c \leq -1, \end{cases} \end{aligned}$$

where ${}_2F_1$ is hypergeometric function.

When the shape parameters are known and $\beta_1 = \beta_2 = \beta$ and $\lambda_1 = \lambda_2 = \lambda$ a credible interval can be obtained. From relations (23) and (24) follow that, for integer-valued a_1 and a_2 , $2b^*\alpha_1|\mathbf{x}$ and $2d^*\alpha_2|\mathbf{y}$ have chi-square $\chi_{2a^*}^2$ and chi-square $\chi_{2c^*}^2$ distributions respectively, so the ratio

$$\frac{a^* d^* \alpha_2 | \mathbf{y}}{b^* c^* \alpha_1 | \mathbf{x}}$$

has Fisher's $F_{2c^*, 2a^*}$ distribution.

Using this fact and equation (3) we get that the credible interval of level $1 - \gamma$ for R is given by

$$I_R^{(BAYES)} = \left(\frac{1}{1 + \frac{b^*c^*}{a^*d^*} F_{2c^*, 2a^*; 1-\frac{\gamma}{2}}}, \frac{1}{1 + \frac{b^*c^*}{a^*d^*} F_{2c^*, 2a^*; \frac{\gamma}{2}}} \right).$$

Remark 4.1. The special case of Bayes estimators above, when all the hyperparameters are equal to 0, corresponds to the Jeffreys non-informative priors.

5. Simulation Study

In this section we conduct a simulation study for different sample sizes and different values of known shape parameters and unknown scale parameters. We choose various parameter combinations. The

inference about values of R larger than 0.5 is the same as the inference about $1 - R$ with interchanged parameters of X and Y , so we consider only the cases with R ranging from 0 to 0.5. For fixed values of sample sizes and distribution parameters, we do the following procedure. We choose a sample and calculate the MLE, using (6) and (7), as well as Bayes estimate using both non-informative Jeffreys priors (see Remark 4.1) and informative gamma priors with hyperparameters 5 and $\frac{5}{\alpha_i}$ (means of these prior distributions are equal to true parameter values), where we find the estimate from the posterior distribution for R using Monte Carlo method with 5000 replicates. For smaller sample sizes, $n, m \leq 20$, we also calculate the UMVUE, using Theorem 3.1 and Corollary 3.2, while in rest of the cases we are unable to do it due to the computational limitations. For all these point estimates we calculate their standard errors.

We also calculate the 95% asymptotic confidence interval, using (15) and (16), and 95% bootstrap-p confidence interval, using (18) with $N = 1000$ boot times, as well as 95% Bayes credible intervals based on the Monte Carlo method mentioned above. In the case of equal shape parameters, we also calculate the exact confidence interval, using (17).

This procedure is repeated for 500 samples and the averages for each estimate are calculated.

From Tables 1 and 3 one can notice that the standard errors increase when R gets closer to 0.5. For smaller values of R , the UMVUE, having the smallest bias, seems like the best estimator. For $R = 0.5$, the biases are almost equal, so Bayes noninformative and MLE become slightly better, since they have a bit smaller standard error. As expected, due to additional information, the informative Bayes estimate has the smallest standard error.

From Tables 2 and 4, we observe that all interval estimators, with the exception of the asymptotic confidence for smaller sample sizes interval, have good coverages. In the case of equal shape parameters, the exact confidence interval and the Bayes noninformative credible interval have the best performance, while in the case of unequal shape parameters, the Bayes noninformative credible interval is the best, while the bootstrap-p interval is better than the asymptotic one. It is also noticeable that the Bayes informative credible interval is too conservative for smaller sample sizes.

6. Real Data Application

In this section we compare daily wind speeds in two Atlantic coast cities, A Coruña (Spain) and Bergen (Norway), from January 1st 2010 till December 31st 2019, taken from the database of the European Climate Assessment & Dataset project available at <https://www.ecad.eu/dailydata/predefinedseries.php>.

Most of our formulae are obtained for the case when shape parameters are known. In practice, the parameters are usually fixed using some historical data or previous knowledge. Here, the values of the shape parameters are obtained from the whole populations. They are $\beta_1 = 1.2$, $\lambda_1 = 45$, and $\beta_2 = 1.2$, $\lambda_2 = 30$.

For illustrative purposes random samples of the size 30 are taken:

X (A Coruña): 81, 33, 39, 78, 28, 22, 53, 25, 25, 28,
 17, 44, 31, 28, 39, 22, 22, 42, 39, 31,
 36, 39, 44, 33, 36, 25, 33, 36, 28, 44;
 Y (Bergen): 36, 57, 26, 52, 29, 93, 50, 72, 53, 11,
 31, 27, 37, 15, 28, 38, 28, 26, 48, 17,
 34, 28, 35, 30, 45, 22, 100, 126, 21, 39.

To check that the $\text{PYEW}(\alpha_1, 1.2, 45)$ and $\text{PYEW}(\alpha_2, 1.2, 30)$ distributions fit the data, we used a Lilliefors-type modification of the Kolmogorov-Smirnov test with estimated parameter α . The null distribution of the test statistic is obtained by simulation, exploiting the fact that it does not depend on scale parameter. We were unable to reject the hypothesis that the data follow the PYEW distribution with shape parameters given above (see Table 5).

The point estimates with their estimated standard errors, as well as interval estimates of R are given in Table 6. The standard error of \tilde{R} is estimated as $\sqrt{\frac{\tilde{V}}{n}}$ (see Section 2). The UMVUE of R and estimate of its

Table 1: Point estimates for R and their standard errors for case $\beta_1 = \beta_2 = \beta$ and $\lambda_1 = \lambda_2 = \lambda$

α_1	α_2	β	λ	R	n	m	MLE		UMVUE		Bayes noninf.		Bayes inf.	
							\bar{R}	$SE(\bar{R})$	\hat{R}	$SE(\hat{R})$	\check{R}	$SE(\check{R})$	\check{R}	$SE(\check{R})$
0.5	1.5	0.40	10	0.25	10	10	0.26	0.08	0.25	0.08	0.27	0.08	0.26	0.07
					10	20	0.26	0.07	0.25	0.07	0.27	0.07	0.26	0.06
					20	10	0.26	0.07	0.25	0.07	0.27	0.07	0.26	0.06
					20	20	0.26	0.06	0.25	0.06	0.26	0.06	0.26	0.05
					100	100	0.25	0.03			0.25	0.03	0.25	0.03
0.5	1	0.40	10	0.33	10	10	0.33	0.10	0.33	0.10	0.34	0.09	0.34	0.08
					10	20	0.34	0.09	0.33	0.09	0.34	0.09	0.34	0.07
					20	10	0.33	0.09	0.33	0.09	0.34	0.09	0.34	0.07
					20	20	0.34	0.07	0.34	0.07	0.34	0.07	0.34	0.06
					100	100	0.33	0.03			0.33	0.03	0.33	0.03
1	1	0.40	10	0.5	10	10	0.50	0.11	0.50	0.11	0.50	0.10	0.50	0.09
					10	20	0.51	0.09	0.50	0.10	0.50	0.09	0.50	0.08
					20	10	0.48	0.09	0.49	0.10	0.49	0.09	0.50	0.08
					20	20	0.50	0.08	0.50	0.08	0.50	0.08	0.50	0.07
					100	100	0.50	0.04			0.50	0.04	0.50	0.03
1	1	1	10	0.5	10	10	0.50	0.11	0.50	0.12	0.50	0.11	0.50	0.09
					10	20	0.51	0.10	0.50	0.10	0.50	0.10	0.50	0.08
					20	10	0.49	0.10	0.50	0.10	0.50	0.09	0.50	0.08
					20	20	0.50	0.08	0.50	0.08	0.50	0.07	0.50	0.07
					100	100	0.50	0.04			0.50	0.04	0.50	0.03
1	1	3	50	0.5	10	10	0.50	0.10	0.50	0.11	0.50	0.10	0.50	0.09
					10	20	0.50	0.09	0.50	0.09	0.50	0.09	0.50	0.08
					20	10	0.49	0.09	0.50	0.10	0.50	0.09	0.50	0.08
					20	20	0.49	0.07	0.49	0.08	0.49	0.07	0.49	0.07
					100	100	0.50	0.04			0.50	0.04	0.50	0.03

Table 2: Confidence intervals for R and their coverage for case $\beta_1 = \beta_2 = \beta$ and $\lambda_1 = \lambda_2 = \lambda$

α_1	α_2	β	λ	R	n	m	exact			asymptotic			bootstrap-p			Bayes noninf.			Bayes inf.		
							l	u	cov	l	u	cov	l	u	cov	l	u	cov	l	u	cov
0.5	1.5	0.40	10	0.25	10	10	0.13	0.45	0.96	0.10	0.42	0.94	0.13	0.45	0.96	0.13	0.45	0.96	0.14	0.41	0.99
					10	20	0.15	0.44	0.97	0.12	0.40	0.94	0.15	0.44	0.94	0.14	0.41	0.96	0.15	0.39	0.98
					20	10	0.13	0.41	0.96	0.12	0.40	0.94	0.13	0.41	0.94	0.14	0.43	0.96	0.16	0.40	0.98
					20	20	0.15	0.38	0.95	0.14	0.37	0.94	0.15	0.38	0.96	0.15	0.38	0.94	0.17	0.37	0.98
					100	100	0.20	0.31	0.94	0.20	0.30	0.95	0.20	0.31	0.95	0.20	0.31	0.94	0.20	0.31	0.95
0.5	1	0.40	10	0.33	10	10	0.18	0.55	0.97	0.15	0.53	0.93	0.18	0.55	0.97	0.18	0.55	0.97	0.20	0.51	0.98
					10	20	0.21	0.54	0.93	0.18	0.51	0.91	0.21	0.54	0.91	0.19	0.52	0.94	0.21	0.49	0.98
					20	10	0.18	0.50	0.95	0.17	0.49	0.93	0.18	0.50	0.93	0.20	0.52	0.95	0.22	0.50	0.98
					20	20	0.21	0.47	0.96	0.19	0.46	0.94	0.21	0.47	0.96	0.21	0.47	0.96	0.23	0.47	0.98
					100	100	0.27	0.40	0.94	0.27	0.39	0.94	0.27	0.40	0.94	0.27	0.40	0.94	0.28	0.40	0.95
1	1	0.40	10	0.5	10	10	0.30	0.70	0.94	0.29	0.71	0.91	0.30	0.70	0.93	0.30	0.70	0.94	0.33	0.67	0.99
					10	20	0.34	0.70	0.92	0.33	0.69	0.90	0.34	0.70	0.90	0.32	0.68	0.94	0.34	0.65	0.99
					20	10	0.30	0.65	0.96	0.30	0.67	0.94	0.30	0.65	0.94	0.32	0.68	0.96	0.35	0.66	0.98
					20	20	0.35	0.65	0.94	0.35	0.65	0.92	0.35	0.65	0.94	0.35	0.65	0.94	0.37	0.64	0.98
					100	100	0.43	0.57	0.94	0.43	0.57	0.94	0.43	0.57	0.94	0.43	0.57	0.94	0.43	0.57	0.96
1	1	1	10	0.5	10	10	0.29	0.70	0.95	0.28	0.70	0.91	0.29	0.70	0.94	0.29	0.70	0.95	0.33	0.67	0.98
					10	20	0.33	0.69	0.94	0.32	0.69	0.91	0.34	0.69	0.91	0.31	0.67	0.94	0.35	0.66	0.97
					20	10	0.30	0.66	0.95	0.31	0.68	0.93	0.30	0.66	0.93	0.33	0.68	0.96	0.35	0.66	0.98
					20	20	0.35	0.65	0.94	0.35	0.65	0.93	0.35	0.65	0.94	0.35	0.65	0.95	0.37	0.63	0.98
					100	100	0.43	0.57	0.94	0.43	0.57	0.94	0.43	0.57	0.94	0.43	0.57	0.94	0.43	0.57	0.95
1	1	3	50	0.5	10	10	0.30	0.71	0.95	0.30	0.71	0.91	0.30	0.71	0.94	0.30	0.71	0.95	0.33	0.67	0.98
					10	20	0.34	0.70	0.94	0.33	0.70	0.92	0.35	0.70	0.92	0.32	0.68	0.95	0.34	0.65	0.98
					20	10	0.30	0.66	0.94	0.31	0.68	0.92	0.30	0.66	0.92	0.33	0.68	0.94	0.35	0.66	0.98
					20	20	0.34	0.64	0.94	0.34	0.64	0.90	0.34	0.64	0.93	0.34	0.64	0.93	0.36	0.63	0.97
					100	100	0.43	0.57	0.96	0.43	0.57	0.96	0.43	0.57	0.96	0.43	0.57	0.96	0.43	0.57	0.97

Table 3: Point estimates for R and their standard errors for case $\beta_1 \neq \beta_2$ or $\lambda_1 \neq \lambda_2$

α_1	β_1	λ_1	α_2	β_2	λ_2	R	n	m	MLE		UMVUE		Bayes noninf.		Bayes inf.	
									\bar{R}	$SE(\bar{R})$	\hat{R}	$SE(\hat{R})$	\check{R}	$SE(\check{R})$	\check{R}	$SE(\check{R})$
1	1	10	1	3	10	0.09	10	10	0.10	0.04	0.09	0.04	0.11	0.04	0.10	0.03
							10	20	0.10	0.03	0.09	0.03	0.10	0.03	0.10	0.03
							20	10	0.09	0.03	0.09	0.03	0.10	0.03	0.10	0.02
							20	20	0.10	0.02	0.09	0.03	0.10	0.02	0.10	0.02
							100	100	0.09	0.01			0.09	0.01	0.09	0.01
0.5	3	10	1.5	1	2	0.10	10	10	0.11	0.07	0.10	0.07	0.13	0.07	0.11	0.06
							10	20	0.11	0.05	0.11	0.05	0.12	0.05	0.12	0.04
							20	10	0.11	0.06	0.10	0.07	0.13	0.07	0.12	0.06
							20	20	0.11	0.05	0.10	0.05	0.12	0.05	0.12	0.04
							100	100	0.10	0.02			0.11	0.02	0.11	0.02
1	0.4	50	1	0.4	10	0.10	10	10	0.11	0.06	0.09	0.06	0.12	0.07	0.11	0.05
							10	20	0.10	0.05	0.09	0.05	0.11	0.05	0.11	0.04
							20	10	0.10	0.06	0.10	0.06	0.12	0.07	0.11	0.05
							20	20	0.10	0.04	0.09	0.04	0.11	0.05	0.10	0.04
							100	100	0.10	0.02			0.10	0.02	0.10	0.02
1.5	1	10	1	1	2	0.10	10	10	0.11	0.06	0.11	0.06	0.13	0.07	0.12	0.05
							10	20	0.11	0.04	0.10	0.05	0.11	0.05	0.11	0.04
							20	10	0.10	0.06	0.11	0.06	0.12	0.06	0.12	0.05
							20	20	0.10	0.04	0.11	0.04	0.11	0.04	0.11	0.04
							100	100	0.11	0.02			0.11	0.02	0.10	0.02
1	0.4	50	1.5	1	50	0.11	10	10	0.11	0.04	0.11	0.05	0.12	0.04	0.11	0.04
							10	20	0.11	0.04	0.11	0.04	0.12	0.04	0.11	0.03
							20	10	0.11	0.03	0.11	0.03	0.12	0.04	0.11	0.03
							20	20	0.11	0.03	0.11	0.03	0.11	0.03	0.11	0.03
							100	100	0.11	0.01			0.11	0.01	0.11	0.01
0.5	1	10	1.5	0.4	2	0.12	10	10	0.13	0.07	0.12	0.08	0.15	0.08	0.14	0.06
							10	20	0.13	0.05	0.12	0.06	0.14	0.06	0.13	0.05
							20	10	0.13	0.07	0.12	0.07	0.15	0.08	0.14	0.06
							20	20	0.13	0.05	0.12	0.06	0.14	0.05	0.13	0.05
							100	100	0.12	0.02			0.13	0.02	0.13	0.02
1	0.4	2	1	3	2	0.24	10	10	0.26	0.07	0.24	0.08	0.26	0.07	0.25	0.06
							10	20	0.26	0.07	0.24	0.07	0.26	0.07	0.25	0.06
							20	10	0.24	0.05	0.24	0.06	0.25	0.05	0.25	0.05
							20	20	0.25	0.05	0.24	0.05	0.25	0.05	0.25	0.04
							100	100	0.24	0.02			0.24	0.02	0.24	0.02
1.5	1	50	1	0.4	10	0.25	10	10	0.25	0.10	0.25	0.11	0.26	0.10	0.26	0.08
							10	20	0.25	0.07	0.25	0.08	0.26	0.07	0.26	0.07
							20	10	0.25	0.10	0.26	0.10	0.27	0.10	0.26	0.08
							20	20	0.25	0.07	0.25	0.08	0.26	0.07	0.25	0.07
							100	100	0.25	0.03			0.25	0.03	0.25	0.03

Table 3: Continuation

α_1	β_1	λ_1	α_2	β_2	λ_2	R	n	m	MLE		UMVUE		Bayes noninf.		Bayes inf.	
									\tilde{R}	$SE(\tilde{R})$	\hat{R}	$SE(\hat{R})$	\check{R}	$SE(\check{R})$	\check{R}	$SE(\check{R})$
1	3	10	1	1	2	0.27	10	10	0.26	0.10	0.27	0.11	0.28	0.10	0.28	0.08
							10	20	0.27	0.08	0.27	0.08	0.27	0.08	0.27	0.07
							20	10	0.26	0.10	0.25	0.10	0.28	0.10	0.27	0.08
							20	20	0.26	0.07	0.27	0.08	0.27	0.07	0.27	0.07
							100	100	0.27	0.03			0.27	0.03	0.26	0.03
1.5	0.4	50	0.5	0.4	10	0.32	10	10	0.32	0.11	0.32	0.11	0.33	0.11	0.33	0.09
							10	20	0.33	0.08	0.31	0.08	0.33	0.08	0.32	0.07
							20	10	0.31	0.10	0.32	0.11	0.32	0.10	0.33	0.09
							20	20	0.32	0.08	0.32	0.08	0.32	0.08	0.33	0.07
							100	100	0.32	0.04			0.32	0.04	0.31	0.03
1.5	0.4	2	1	3	2	0.33	10	10	0.33	0.09	0.33	0.08	0.36	0.09	0.34	0.07
							10	20	0.35	0.09	0.35	0.09	0.35	0.08	0.34	0.07
							20	10	0.34	0.07	0.34	0.07	0.34	0.07	0.34	0.06
							20	20	0.34	0.06	0.33	0.06	0.34	0.06	0.34	0.06
							100	100	0.33	0.03			0.33	0.03	0.33	0.03
1.5	0.4	10	1	1	10	0.34	10	10	0.35	0.09	0.34	0.10	0.35	0.09	0.34	0.08
							10	20	0.36	0.09	0.34	0.09	0.36	0.09	0.35	0.07
							20	10	0.34	0.08	0.34	0.08	0.35	0.08	0.35	0.07
							20	20	0.35	0.07	0.34	0.07	0.35	0.07	0.35	0.06
							100	100	0.34	0.03			0.34	0.03	0.34	0.03
1	3	10	1	0.4	2	0.48	10	10	0.46	0.11	0.47	0.11	0.47	0.11	0.48	0.09
							10	20	0.47	0.08	0.48	0.08	0.48	0.08	0.48	0.07
							20	10	0.46	0.11	0.48	0.11	0.47	0.10	0.48	0.09
							20	20	0.48	0.08	0.48	0.08	0.48	0.08	0.48	0.07
							100	100	0.48	0.04			0.48	0.03	0.48	0.03
1	0.4	10	0.5	3	50	0.49	10	10	0.52	0.11	0.50	0.11	0.51	0.10	0.49	0.09
							10	20	0.52	0.11	0.50	0.11	0.51	0.10	0.49	0.09
							20	10	0.51	0.08	0.49	0.08	0.50	0.08	0.49	0.07
							20	20	0.50	0.08	0.50	0.08	0.50	0.08	0.49	0.07
							100	100	0.49	0.03			0.49	0.03	0.49	0.03
0.5	3	2	1.5	1	2	0.49	10	10	0.49	0.11	0.50	0.11	0.49	0.10	0.49	0.09
							10	20	0.49	0.08	0.49	0.09	0.49	0.08	0.49	0.07
							20	10	0.49	0.10	0.50	0.11	0.50	0.10	0.49	0.08
							20	20	0.48	0.08	0.49	0.08	0.49	0.08	0.49	0.07
							100	100	0.49	0.03			0.49	0.03	0.50	0.03

Table 4: Confidence intervals for R and their coverage for case $\beta_1 \neq \beta_2$ or $\lambda_1 \neq \lambda_2$

α_1	β_1	λ_1	α_2	β_2	λ_2	R	n	m	asymptotic			bootstrap-p			Bayes noninf.			Bayes inf.		
									l	u	cov	l	u	cov	l	u	cov	l	u	cov
1	1	10	1	3	10	0.09	10	10	0.03	0.17	0.94	0.05	0.21	0.95	0.05	0.19	0.96	0.05	0.16	0.99
							10	20	0.04	0.16	0.94	0.05	0.19	0.95	0.05	0.17	0.96	0.05	0.15	0.98
							20	10	0.04	0.15	0.93	0.05	0.16	0.94	0.05	0.17	0.95	0.06	0.15	0.97
							20	20	0.05	0.14	0.94	0.06	0.16	0.95	0.06	0.15	0.95	0.06	0.14	0.98
							100	100	0.07	0.11	0.94	0.07	0.11	0.96	0.07	0.11	0.95	0.07	0.11	0.93
0.5	3	10	1.5	1	2	0.10	10	10	0.01	0.24	0.88	0.02	0.26	0.94	0.03	0.31	0.95	0.04	0.26	0.98
							10	20	0.02	0.20	0.92	0.04	0.22	0.96	0.04	0.24	0.96	0.04	0.22	0.97
							20	10	0.00	0.23	0.87	0.02	0.25	0.92	0.03	0.31	0.96	0.04	0.26	0.99
							20	20	0.02	0.20	0.90	0.04	0.21	0.94	0.04	0.24	0.94	0.05	0.22	0.99
							100	100	0.06	0.15	0.93	0.07	0.15	0.94	0.07	0.15	0.94	0.07	0.16	0.96
1	0.4	50	1	0.4	10	0.10	10	10	0.00	0.23	0.90	0.02	0.25	0.94	0.03	0.29	0.94	0.03	0.24	0.99
							10	20	0.01	0.19	0.92	0.03	0.21	0.95	0.04	0.22	0.96	0.04	0.20	0.98
							20	10	0.00	0.21	0.87	0.02	0.23	0.92	0.03	0.28	0.95	0.04	0.24	0.99
							20	20	0.02	0.18	0.90	0.03	0.20	0.93	0.04	0.22	0.95	0.04	0.20	0.98
							100	100	0.06	0.13	0.94	0.06	0.14	0.94	0.06	0.14	0.94	0.06	0.14	0.96
1.5	1	10	1	1	2	0.10	10	10	0.01	0.23	0.88	0.03	0.26	0.93	0.04	0.3	0.95	0.04	0.25	0.99
							10	20	0.02	0.19	0.91	0.04	0.22	0.95	0.04	0.23	0.95	0.05	0.21	0.98
							20	10	0.00	0.21	0.87	0.03	0.23	0.93	0.04	0.28	0.95	0.04	0.25	0.99
							20	20	0.02	0.18	0.90	0.04	0.20	0.94	0.04	0.22	0.95	0.05	0.20	0.97
							100	100	0.07	0.14	0.95	0.07	0.15	0.96	0.07	0.15	0.96	0.07	0.14	0.95
1	0.4	50	1.5	1	50	0.11	10	10	0.03	0.20	0.95	0.05	0.23	0.95	0.05	0.23	0.95	0.06	0.19	0.98
							10	20	0.04	0.19	0.95	0.06	0.22	0.95	0.05	0.20	0.96	0.06	0.18	0.98
							20	10	0.04	0.18	0.93	0.06	0.20	0.95	0.06	0.21	0.96	0.06	0.19	0.98
							20	20	0.05	0.17	0.96	0.07	0.19	0.95	0.06	0.18	0.95	0.07	0.17	0.96
							100	100	0.08	0.13	0.95	0.09	0.14	0.95	0.09	0.14	0.96	0.09	0.14	0.95
0.5	1	10	1.5	0.4	2	0.12	10	10	0.01	0.26	0.86	0.03	0.28	0.91	0.04	0.33	0.94	0.04	0.29	0.99
							10	20	0.03	0.24	0.93	0.05	0.25	0.96	0.05	0.27	0.96	0.06	0.25	0.97
							20	10	0.01	0.26	0.88	0.03	0.27	0.93	0.04	0.33	0.95	0.05	0.29	0.99
							20	20	0.03	0.23	0.91	0.04	0.24	0.94	0.05	0.26	0.94	0.06	0.25	0.96
							100	100	0.08	0.17	0.96	0.08	0.17	0.96	0.08	0.18	0.97	0.09	0.18	0.95
1	0.4	2	1	3	2	0.24	10	10	0.12	0.40	0.94	0.15	0.45	0.94	0.13	0.41	0.95	0.15	0.37	0.98
							10	20	0.12	0.39	0.93	0.16	0.45	0.92	0.13	0.40	0.94	0.15	0.36	0.98
							20	10	0.14	0.35	0.94	0.16	0.37	0.96	0.15	0.36	0.96	0.16	0.35	0.99
							20	20	0.15	0.35	0.95	0.17	0.37	0.96	0.16	0.35	0.96	0.16	0.34	0.96
							100	100	0.20	0.29	0.95	0.20	0.29	0.94	0.20	0.29	0.95	0.20	0.28	0.95
1.5	1	50	1	0.4	10	0.25	10	10	0.06	0.44	0.90	0.07	0.43	0.93	0.10	0.49	0.95	0.12	0.45	0.98
							10	20	0.11	0.40	0.92	0.12	0.40	0.94	0.13	0.42	0.95	0.14	0.40	0.97
							20	10	0.06	0.45	0.90	0.08	0.43	0.93	0.11	0.50	0.97	0.12	0.45	0.99
							20	20	0.11	0.39	0.92	0.12	0.39	0.94	0.14	0.42	0.95	0.14	0.40	0.97
							100	100	0.19	0.32	0.96	0.19	0.32	0.96	0.19	0.32	0.97	0.19	0.32	0.95

Table 4: Continuation

α_1	β_1	λ_1	α_2	β_2	λ_2	R	n	m	asymptotic			bootstrap-p			Bayes noninf.			Bayes inf.		
									l	u	cov	l	u	cov	l	u	cov	l	u	cov
1	3	10	1	1	2	0.27	10	10	0.07	0.46	0.89	0.08	0.45	0.92	0.11	0.50	0.94	0.13	0.46	0.98
							10	20	0.12	0.42	0.93	0.13	0.41	0.94	0.14	0.44	0.95	0.15	0.42	0.97
							20	10	0.07	0.46	0.90	0.08	0.44	0.92	0.12	0.50	0.95	0.16	0.45	0.99
							20	20	0.12	0.41	0.94	0.12	0.40	0.95	0.14	0.43	0.95	0.15	0.41	0.98
							100	100	0.20	0.33	0.95	0.20	0.33	0.95	0.21	0.34	0.96	0.20	0.33	0.96
1.5	0.4	50	0.5	0.4	10	0.32	10	10	0.11	0.53	0.91	0.12	0.51	0.93	0.15	0.56	0.94	0.17	0.51	0.99
							10	20	0.17	0.49	0.95	0.17	0.48	0.96	0.18	0.50	0.97	0.19	0.47	0.98
							20	10	0.11	0.51	0.89	0.11	0.49	0.91	0.15	0.54	0.95	0.18	0.51	0.99
							20	20	0.17	0.47	0.93	0.17	0.46	0.95	0.19	0.48	0.94	0.20	0.47	0.98
							100	100	0.25	0.39	0.94	0.25	0.38	0.94	0.25	0.39	0.94	0.25	0.38	0.94
1.5	0.4	2	1	3	2	0.33	10	10	0.18	0.53	0.93	0.22	0.58	0.94	0.19	0.54	0.95	0.21	0.49	0.99
							10	20	0.18	0.52	0.94	0.22	0.57	0.94	0.19	0.52	0.95	0.21	0.48	0.99
							20	10	0.21	0.47	0.95	0.23	0.49	0.96	0.22	0.48	0.96	0.23	0.46	0.98
							20	20	0.21	0.46	0.95	0.23	0.48	0.96	0.22	0.46	0.96	0.22	0.45	0.97
							100	100	0.28	0.39	0.94	0.28	0.39	0.94	0.28	0.39	0.94	0.28	0.38	0.96
1.5	0.4	10	1	1	10	0.34	10	10	0.17	0.54	0.92	0.20	0.56	0.95	0.19	0.55	0.95	0.20	0.50	0.98
							10	20	0.19	0.53	0.94	0.22	0.56	0.94	0.20	0.53	0.95	0.21	0.49	0.98
							20	10	0.19	0.50	0.91	0.20	0.51	0.93	0.21	0.52	0.94	0.22	0.49	0.97
							20	20	0.21	0.48	0.93	0.23	0.49	0.94	0.22	0.49	0.95	0.24	0.48	0.98
							100	100	0.28	0.40	0.96	0.29	0.41	0.96	0.29	0.40	0.96	0.28	0.40	0.94
1	3	10	1	0.4	2	0.48	10	10	0.25	0.67	0.90	0.22	0.63	0.93	0.27	0.68	0.93	0.31	0.65	0.99
							10	20	0.32	0.63	0.95	0.30	0.61	0.95	0.33	0.63	0.96	0.34	0.61	0.98
							20	10	0.25	0.67	0.91	0.22	0.63	0.92	0.28	0.68	0.94	0.31	0.65	0.98
							20	20	0.32	0.63	0.95	0.31	0.61	0.95	0.34	0.63	0.95	0.34	0.61	0.98
							100	100	0.41	0.55	0.95	0.41	0.54	0.94	0.41	0.55	0.96	0.41	0.54	0.95
1	0.4	10	0.5	3	50	0.49	10	10	0.31	0.72	0.94	0.35	0.76	0.94	0.30	0.70	0.96	0.32	0.66	0.99
							10	20	0.31	0.73	0.92	0.35	0.76	0.91	0.30	0.70	0.95	0.33	0.66	0.99
							20	10	0.35	0.66	0.94	0.37	0.68	0.94	0.35	0.65	0.96	0.36	0.63	0.97
							20	20	0.35	0.65	0.93	0.38	0.67	0.92	0.35	0.64	0.95	0.36	0.62	0.97
							100	100	0.43	0.56	0.96	0.43	0.57	0.96	0.43	0.56	0.96	0.43	0.56	0.96
0.5	3	2	1.5	1	2	0.49	10	10	0.28	0.70	0.91	0.26	0.67	0.94	0.29	0.70	0.95	0.32	0.66	0.98
							10	20	0.32	0.65	0.92	0.32	0.64	0.94	0.32	0.65	0.93	0.34	0.63	0.97
							20	10	0.29	0.69	0.93	0.27	0.66	0.94	0.31	0.70	0.96	0.33	0.66	0.98
							20	20	0.33	0.63	0.91	0.32	0.62	0.92	0.34	0.64	0.92	0.36	0.62	0.97
							100	100	0.42	0.56	0.95	0.42	0.56	0.94	0.42	0.56	0.95	0.44	0.57	0.95

Table 5: Results of KS tests

sample	$\bar{\alpha}$	KS stat	p-value
X	0.041	0.277	0.17
Y	0.020	0.199	0.35

standard error are calculated using Theorem 3.1, while the Bayes estimates are obtained from the sample of size 5000 from the posterior distribution with Jeffreys prior.

From the Table 6 we can see that the estimated probability is relatively close to 0.5. In addition, all interval estimators contain 0.5, from which one may deduce that the daily wind speed in these two rather distant Atlantic coast cities can be considered equal.

Table 6: Real data estimates

\tilde{R}	$SE(\tilde{R})$	\hat{R}	$SE(\hat{R})$	\check{R}	$SE(\check{R})$	asymptotic CI	bootstrap-p CI	Bayes CI
0.57	0.04	0.57	0.05	0.57	0.06	(0.49,0.65)	(0.45,0.69)	(0.45,0.68)

7. Conclusion

In this paper we considered the estimation of the probability $P\{X < Y\}$ when X and Y are two independent random variables following the PYEW distribution. Various point and interval estimators were constructed.

A simulation study was performed and the obtained point estimates were compared. The UMVUE seems the best for smaller (and larger) values of R , while the informative Bayes estimate is the best for R closer to 0.5. A comparison of interval estimates was also done, and we concluded that exact and Bayes noninformative interval estimates are the best, with bootstrap-p interval being close to them. For larger sample size and equal shape parameters, the asymptotic confidence interval is also acceptable.

Appendix

Most of the conditions from [12, chapter 6 and appendix A] are obviously satisfied. Some further explanations might be needed for conditions marked as (R7) and (R9). These conditions are essentially interchanging of differentiation and integration in some neighborhood of the true parameter values, which reduces to the convergence of appropriate integrals. For a density $f(x; \theta_1, \dots, \theta_p)$ these integrals are:

$$\int \frac{\partial f(x; \theta_1, \dots, \theta_p)}{\partial \theta_i} dx, \text{ for } i \in \{1, \dots, p\};$$

$$\int \frac{\partial^2 f(x; \theta_1, \dots, \theta_p)}{\partial \theta_i \partial \theta_j} dx, \text{ for } i, j \in \{1, \dots, p\};$$

$$\int \frac{\partial^3 \ln f(x; \theta_1, \dots, \theta_p)}{\partial \theta_i \partial \theta_j \partial \theta_k} f(x; \theta_1, \dots, \theta_p) dx, \text{ for } i, j, k \in \{1, \dots, p\}.$$

In our case all these integrals (19 in total) have very similar pattern. We shall prove convergence of one of them, and the rest can be shown analogously. First we prove the following lemma.

Lemma 7.1. For $m > 0$ it holds

$$\int_0^\infty x^m f(x; \alpha, \beta, \lambda) dx < +\infty,$$

where $f(x; \alpha, \beta, \lambda)$ is defined in equation (1).

Proof.

$$\int_0^\infty x^m f(x; \alpha, \beta, \lambda) dx = \int_0^1 x^m f(x; \alpha, \beta, \lambda) dx + \int_1^\infty x^m f(x; \alpha, \beta, \lambda) dx \leq 1 + \int_1^\infty x^m f(x; \alpha, \beta, \lambda) dx.$$

For $x \geq 1$ it holds that $e^{-\lambda} \leq e^{-\frac{\lambda}{x}} \leq 1$. Using this fact and the change of variables $s = \alpha x^\beta e^{-\lambda}$, we obtain that

$$\begin{aligned} \int_1^\infty x^m f(x; \alpha, \beta, \lambda) dx &\leq \lambda \int_1^\infty x^{m-1} \alpha x^{\beta-1} e^{-\alpha x^\beta e^{-\lambda}} dx + \int_1^\infty x^m \alpha \beta x^{\beta-1} e^{-\alpha x^\beta e^{-\lambda}} dx \\ &= \frac{\lambda e^\lambda}{\beta} \int_{\alpha e^{-\lambda}}^\infty \left(\frac{e^\lambda s}{\alpha}\right)^{\frac{m-1}{\beta}} e^{-s} ds + e^\lambda \int_{\alpha e^{-\lambda}}^\infty \left(\frac{e^\lambda s}{\alpha}\right)^{\frac{m}{\beta}} e^{-s} ds \\ &\leq \frac{\lambda e^\lambda}{\beta} \left(\frac{e^\lambda}{\alpha}\right)^{\frac{m-1}{\beta}} \Gamma\left(\frac{m-1}{\beta} + 1\right) + e^\lambda \left(\frac{e^\lambda}{\alpha}\right)^{\frac{m}{\beta}} \Gamma\left(\frac{m}{\beta} + 1\right), \end{aligned}$$

where Γ is the gamma function. Hence the lemma is proven. □

Without essential loss of generality we may consider interchanging of integration and differentiation along β .

$$\begin{aligned} \int_0^\infty \frac{\partial f(x; \alpha, \beta, \lambda)}{\partial \beta} dx &= \int_0^\infty \left(\frac{x}{\lambda + \beta x} + \ln x - \alpha e^{-\frac{\lambda}{x}} x^\beta \ln x\right) f(x; \alpha, \beta, \lambda) dx \\ &= \int_0^\infty \frac{x}{\lambda + \beta x} f(x; \alpha, \beta, \lambda) dx + \int_0^\infty \ln x f(x; \alpha, \beta, \lambda) dx - \int_0^\infty \alpha e^{-\frac{\lambda}{x}} x^\beta \ln x f(x; \alpha, \beta, \lambda) dx. \end{aligned}$$

It is clear that

$$\int_0^\infty \frac{x}{\lambda + \beta x} f(x; \alpha, \beta, \lambda) dx \leq \frac{1}{\lambda} \int_0^\infty x f(x; \alpha, \beta, \lambda) dx. \tag{25}$$

$$\int_0^\infty \ln x f(x; \alpha, \beta, \lambda) dx = \int_0^1 \ln x f(x; \alpha, \beta, \lambda) dx + \int_1^\infty \ln x f(x; \alpha, \beta, \lambda) dx \tag{26}$$

The density function f is bounded on $[0, 1]$, so there exists some positive constant C such that $f(x; \alpha, \beta, \lambda) \leq C$. It follows that

$$\left| \int_0^1 \ln x f(x; \alpha, \beta, \lambda) dx \right| \leq \int_0^1 |\ln x| f(x; \alpha, \beta, \lambda) dx = - \int_0^1 \ln x f(x; \alpha, \beta, \lambda) dx \leq -C \int_0^1 \ln x dx = C. \tag{27}$$

Using the inequality $\ln x < x$, for $x > 1$, we further obtain

$$\int_1^\infty \ln x f(x; \alpha, \beta, \lambda) dx < \int_1^\infty x f(x; \alpha, \beta, \lambda) dx < \int_0^\infty x f(x; \alpha, \beta, \lambda) dx. \tag{28}$$

Since $|x \ln x| \leq \frac{1}{e}$, for $0 \leq x \leq 1$, it follows that

$$\begin{aligned} \left| \int_0^\infty \alpha e^{-\frac{\lambda}{x}} x^\beta \ln x f(x; \alpha, \beta, \lambda) dx \right| &\leq \alpha \int_0^\infty x^\beta |\ln x| f(x; \alpha, \beta, \lambda) dx \\ &= \alpha \int_0^1 x^\beta |\ln x| f(x; \alpha, \beta, \lambda) dx + \alpha \int_1^\infty x^\beta \ln x f(x; \alpha, \beta, \lambda) dx \\ &< \frac{\alpha}{e} \int_0^1 x^{\beta-1} f(x; \alpha, \beta, \lambda) dx + \alpha \int_1^\infty x^{\beta+1} f(x; \alpha, \beta, \lambda) dx \end{aligned}$$

$$< \frac{\alpha}{e} \int_0^{\infty} x^{\beta-1} f(x; \alpha, \beta, \lambda) dx + \alpha \int_0^{\infty} x^{\beta+1} f(x; \alpha, \beta, \lambda) dx. \quad (29)$$

From the relations (25), (26), (27), (28), (29) and Lemma 7.1 it follows that the integral $\int_0^{\infty} \frac{\partial f(x; \alpha, \beta, \lambda)}{\partial \beta} dx$ converges.

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