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Oscillation of Three-Dimensional Time Scale Systems with Fixed Point Theorems

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Abstract. Oscillation and nonoscillation theories play very important roles in gaining information about the long-time behavior of solutions of a system. Therefore, we investigate the asymptotic behavior of nonoscillatory solutions as well as the existence of such solutions so that one can determine the limit behavior. For the existence, we use some fixed point theorems such as Schauder's fixed point theorem and the Knaster fixed point theorem.

1. Introduction

We consider a three-dimensional(3D) system of first order dynamic equations on time scales

$$\begin{cases} x^{\Delta}(t) = a(t)|y(t)|^{\alpha} \operatorname{sgn} y(t) \\ y^{\Delta}(t) = b(t)|z(t)|^{\beta} \operatorname{sgn} z(t) \\ z^{\Delta}(t) = c(t)|x^{\sigma}(t)|^{\gamma} \operatorname{sgn} x^{\sigma}(t), \end{cases} \tag{1}$$

where $a, b \in C_{rd}\left([t_0,\infty)_{\mathbb{T}}, \mathbb{R}^+ \cup \{0\}\right)$, $c \in C_{rd}\left([t_0,\infty)_{\mathbb{T}}, \mathbb{R}^+\right)$, and α, β, γ are the ratio of odd positive integers. A time scale $\mathbb{T}$ is a nonempty closed subset of real numbers $\mathbb{R}$. Here C_{rd} stands for the set of *rd-continuous* functions, that is functions that are continuous at right dense points in $\mathbb{T}$ and its left-sided limits exist as a finite number at left-dense points in $\mathbb{T}$ (see [5]). The time scale-theory offers a mathematical structure that encompasses differential equations and difference equations synchronously. For example, when the domain is the set of real numbers $\mathbb{R}$, then the results are valid for ordinary differential equations and derivative is defined by $f'(t) = \lim\limits_{h\to 0} \frac{f(t+h)-f(t)}{h}$. On the other hand, when the domain is the set of integers $\mathbb{Z}$, then the results are valid for difference equations and the derivative is defined by $\Delta f(t) = f(t+1) - f(t)$, where Δ is called *forward difference operator*. For general time scales, we use the notation f^{Δ} and call it

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delta-derivative(Hilger derivative) of f. There are several things in real life which are not continuous, e.g., population phenomena. Therefore, besides the continuous and discrete cases, there are several time scales and their real-life applications to describe the natural developments in life.

This article is motivated by [2], where the authors consider a 3D time scale system

$$\begin{cases} x^{\Delta}(t) = a(t)f(y(t)) \\ y^{\Delta}(t) = b(t)g(z(t)) \\ z^{\Delta}(t) = \lambda c(t)h(x(t)), \end{cases} \tag{2}$$

where $f, g, h : \mathbb{R} \to \mathbb{R}$ are continuous functions such that $uf(u) > 0, ug(u) > 0$, and $uh(u) > 0$ for $u \neq 0$. They determine the oscillation criteria for $\lambda = \pm 1$. In our paper, on the other hand, we do not only determine the oscillation criteria but also we obtain the existence of nonoscillatory solutions by using well-known fixed point theorems. In addition, we find new α and β relations for the oscillation criteria. We also provide two examples that validate our theoretical claims.

When $\mathbb{T} = \mathbb{Z}$, the oscillatory and boundedness properties of solutions of the following system were considered in [19, 20]

$$\begin{cases} \Delta x_n = a_n f(y_{n-l}) \\ \Delta y_n = b_n g(z_{n-m}) \\ \Delta z_n = \delta c_n h(x_{n-k}), \end{cases}$$

where $\delta = \pm 1$ and l, m, k are positive integers. Continuous and time-scale versions of a system similar to system (1) in the 2D and 3D were considered in [8, 10, 13–16, 18].

In system (1), $\sigma(t) : \mathbb{T} \to \mathbb{T}$ is known as the forward jump operator and we define the function $f^{\sigma} : \mathbb{T} \to \mathbb{R}$ by $f^{\sigma}(t) = f(\sigma(t))$. When $\mathbb{T} = \mathbb{R}$, then $\sigma(t) = t$ while $\sigma(t) = t + 1$ when $\mathbb{T} = \mathbb{Z}$. Due to page limitations we cannot give the all details about time scale theory. However, we refer interested readers to the books by Bohner and Peterson [5, 6], that were published in 2001 and 2003, respectively. In addition to these books, Bohner and Peterson have made a huge contribution to the time scale theory with hundreds of articles.

In this paper, we only consider unbounded time scales and whenever we write $t \geq t_0$, we mean $t \in [t_0, \infty)_{\mathbb{T}} := [t_0, \infty) \cap \mathbb{T}$. For the sake of this article, we give the following definitions (see [2, 11]).

Definition 1.1. *By a solution of* (1)*, we mean a collection of functions* (x, y, z) *that are differentiable and whose derivatives are rd-continuous for* $T \geq t_0$ *and* (x, y, z) *satisfies system* (1) *for all large* $t \geq T$.

Definition 1.2. *A solution* (x, y, z) *of system* (1) *is called proper if*

$$\sup\{|x(s)|, |y(s)|, |z(s)| : s \in [t, \infty)_{\mathbb{T}}\} > 0, \quad t \geq t_0.$$

Definition 1.3. *A proper solution of system* (1) *is said to be nonoscillatory if all components* x, y, z *are nonoscillatory, i.e., either eventually positive or eventually negative. Otherwise, it is said to be oscillatory.*

Definition 1.4. *System* (1) *has Property B if every solution* (x, y, z) *of the system is either oscillatory or*

$$\lim_{t\to\infty} |x(t)| = \lim_{t\to\infty} |y(t)| = \lim_{t\to\infty} |z(t)| = \infty. \tag{3}$$

Throughout the paper, we assume

$$\int_{t_0}^{\infty} a(t)\Delta t = \int_{t_0}^{\infty} b(t)\Delta t = \infty.$$

For the existence of nonoscillatory solutions, we use the Knaster's fixed point theorem stated as follows, see [12].

Theorem 1.5. *If* $(Y, \leq)$ *is a complete lattice and* $F : Y \to Y$ *is order-preserving, then* F *has a fixed point. As a matter of fact, the set of fixed points of* F *is a complete lattice.*

The organization of the paper is as follows: In Section 2, we give the nonoscillation criteria for system (1), show the existence of such solutions and provide two examples. In Section 3, we introduce the Property B criteria of system (1) by using α, β and γ relations. In Section 4, we offer some open problems.

2. Nonoscillatory Solutions of System (1)

In this section, we give some results for nonoscillatory solutions of system (1) by using improper integrals.

2.1. Preliminaries

To show the nonoscillation results, we need the following preliminary lemmas whose proofs are similar to Lemmas 2.1, 2.2 and 2.3 in [2], respectively. These can be considered classification results.

Lemma 2.1. *For $t \in \mathbb{T}$, every nonoscillatory solution of system* (1) *is one of the following types*

$$N^+ := \text{Type(a)} : \operatorname{sgn} x(t) = \operatorname{sgn} y(t) = \operatorname{sgn} z(t)$$
$$N^- := \text{Type(c)} : \operatorname{sgn} x(t) = \operatorname{sgn} y(t) \neq \operatorname{sgn} z(t).$$

Lemma 2.2. *Let (x, y, z) be any* Type(a) *solution of system* (1)*. Then the component functions x and y have the following limit behaviors:*

$$\lim_{t\to\infty} |x(t)| = \infty \quad \textit{and} \quad \lim_{t\to\infty} |y(t)| = \infty.$$

Lemma 2.3. *The component function z of any* Type(c) *solution (x, y, z) of system* (1) *satisfies the asymptotic behavior:*

$$\lim_{t\to\infty} z(t) = 0.$$

Next, we show the possible limit behaviors of Type(a) and Type(c) solutions of system (1). Suppose that (x, y, z) is a Type(a) solution. Without loss of generality, assume that $x > 0$ eventually. Then by using the first equation of system (1), we have x is a positive increasing function. Therefore, we have $x(t) \to c_1$ or $x(t) \to \infty$ as $t \to \infty$, where $0 < c_1 < \infty$. Lemma 2.2 gives that x must tend to infinity. Using a similar idea for y and z, and Lemma 2.3, we have the following subclasses for N^+ and N^-:

$$N^-_{B,B,0} := \left\{(x, y, z) \in N^- : \lim_{t\to\infty} |x(t)| = c_1, \lim_{t\to\infty} |y(t)| = c_2, \lim_{t\to\infty} |z(t)| = 0\right\}$$
$$N^-_{B,0,0} := \left\{(x, y, z) \in N^- : \lim_{t\to\infty} |x(t)| = c_1 \lim_{t\to\infty} |y(t)| = \lim_{t\to\infty} |z(t)| = 0\right\}$$
$$N^-_{\infty,B,0} := \left\{(x, y, z) \in N^- : \lim_{t\to\infty} |x(t)| = \infty, \lim_{t\to\infty} |y(t)| = c_2, \lim_{t\to\infty} |z(t)| = 0\right\}$$
$$N^-_{\infty,0,0} := \left\{(x, y, z) \in N^- : \lim_{t\to\infty} |x(t)| = \infty, \lim_{t\to\infty} |y(t)| = \lim_{t\to\infty} |z(t)| = 0\right\}$$
$$N^+_{\infty,\infty,B} := \left\{(x, y, z) \in N^+ : \lim_{t\to\infty} |x(t)| = \lim_{t\to\infty} |y(t)| = \infty, \lim_{t\to\infty} |z(t)| = c_3\right\}$$
$$N^+_{\infty,\infty,\infty} := \left\{(x, y, z) \in N^+ : \lim_{t\to\infty} |x(t)| = \lim_{t\to\infty} |y(t)| = \lim_{t\to\infty} |z(t)| = \infty\right\},$$

where c_1, c_2, c_3 nonzero finite numbers.

We now consider the existence of Type(a) and Type(c) solutions of system (1) by using the following improper integrals:

$$J_1(t_0,\infty) = \int_{t_0}^{\infty} a(t)\left(k_1 + \int_t^{\infty} b(s)\left(k_2 \int_s^{\infty} c(u)\Delta u\right)^{\beta} \Delta s\right)^{\alpha} \Delta t,$$
$$J_2(t_0,\infty) = \int_{t_0}^{\infty} b(t)\left(\int_t^{\infty} c(s)\Delta s\right)^{\beta} \Delta t,$$
$$J_3(t_0,\infty) = \int_{t_0}^{\infty} b(t)\left(k_3 \int_t^{\infty} c(s)\left(k_4 \int_{t_0}^{\sigma(s)} a(u)\Delta u\right)^{\gamma} \Delta s\right)^{\beta} \Delta t,$$
$$J_4(t_0,\infty) = \int_{t_0}^{\infty} c(t)\left(k_5 \int_{t_0}^{\sigma(t)} a(s)\Delta s\right)^{\gamma} \Delta t,$$
$$J_5(t_0,\infty) = \int_{t_0}^{\infty} c(t)\left(k_6 + \int_{t_0}^{\sigma(t)} a(s)\left(k_7 + \int_{t_0}^{s} b(u)\Delta u\right)^{\alpha} \Delta s\right)^{\gamma} \Delta t,$$
$$A(t_0,t) = \int_{t_0}^{t} a(s)\Delta s, \quad B(t_0,t) = \int_{t_0}^{t} b(s)\Delta s, \quad C(t_0,t) = \int_{t_0}^{t} c(s)\Delta s,$$

where $k_i > 0, i = 1,2,\cdots,7$ and $\alpha,\ \beta,\ \gamma$ are the ratio of odd positive integers. A comment about notation; by $J_n(t_0,\infty) = \infty$ for $1 \le n \le 5$, we mean $J_n(t_0,\infty) = \lim_{T\to\infty} J_n(t_0,T)$ for $1 \le n \le 5$. Now, we provide a relationship between the improper integrals and we use these results in the proof of our main results.

Lemma 2.4. *We have the following relationships for all k_i, $i = 1,2,\cdots,5$:*

i.) If $J_2(t_0,\infty) < \infty$, then $C(t_0,\infty) < \infty$.
ii.) If $J_3(t_0,\infty) < \infty$, then $J_4(t_0,\infty) < \infty$.
iii.) If $J_4(t_0,\infty) < \infty$, then $C(t_0,\infty) < \infty$.

Proof. We only prove part (ii) since parts (i) and (iii) can be found in [4] or [17]. So, suppose that $J_3(t_0,\infty) < \infty$. Then

$$\begin{aligned}
&\int_{t_0}^{T} [b(t)]^{\frac{\beta}{\beta}} \left(k_3 \int_t^{T} c(s)\left(k_4 \int_{t_0}^{\sigma(s)} a(u)\Delta u\right)^{\gamma} \Delta s\right)^{\beta} \Delta t \\
&= \int_{t_0}^{T} (k_3)^{\beta} \left(\int_t^{T} b^{\frac{1}{\beta}}(t) c(s)\left(k_4 \int_{t_0}^{\sigma(s)} a(u)\Delta u\right)^{\gamma} \Delta s\right)^{\beta} \Delta t \\
&\ge \int_{t_0}^{t_1} (k_3)^{\beta} \left(b^{\frac{1}{\beta}}(t) \int_t^{T} c(s)\left(k_4 \int_{t_0}^{\sigma(s)} a(u)\Delta u\right)^{\gamma} \Delta s\right)^{\beta} \Delta t \\
&\ge (k_3)^{\beta} \int_{t_0}^{t_1} b(t)\Delta t \left(\int_{t_1}^{T} c(s)\left(k_4 \int_{t_0}^{\sigma(s)} a(u)\Delta u\right)^{\gamma} \Delta s\right)^{\beta}.
\end{aligned}$$

Therefore, as $T \to \infty$, the assertion follows. □

2.2. Existence in N^-

Next, we show the existence of nonoscillatory solutions of system (1) by using Theorem 1.5 and the integrals defined in Section 2.1.

Theorem 2.5. *If $J_1(t_0,\infty) = \infty$ and $J_3(t_0,\infty) < \infty$ for all $k_i, i = 1,...,4$, then $N^-_{\infty,B,0} \neq \emptyset$.*

Proof. Suppose that $J_1(t_0,\infty)=\infty$ and $J_3(t_0,\infty)<\infty$ for $k_1,k_2,k_3,k_4>0$. Then we can choose $t_1\geq t_0$ so large that $J_3(t_1,\infty)<1$ and $J_1(t_1,\infty)>1$ for $t\geq t_1$, where $k_3=1$ and $k_4=\left(\frac{3}{2}\right)^{\alpha}$.

Let X be the partially ordered Banach space of all real-valued continuous functions with the norm $\|x\|=\sup_{t\geq t_1}\dfrac{|x(t)|}{A(t_0,t)}$ and the usual pointwise ordering $\leq$. Let Φ be a subset of X such that

$$\Phi:=\left\{x\in X:\quad 1\leq x(t)\leq\left(\frac{3}{2}\right)^{\alpha}\int_{t_1}^{t}a(s)\Delta s,\quad t\geq t_1\right\}$$

and $Fx:X\to X$ be the operator given by

$$(Fx)(t)=\int_{t_1}^{t}a(s)\left(\frac{1}{2}+\int_s^\infty b(u)\left(\int_u^\infty c(\tau)(x^\sigma(\tau))^\gamma\Delta\tau\right)^\beta\Delta u\right)^\alpha\Delta s$$

First, note that $(\Phi,\leq)$ is a complete lattice, since $\inf\Omega\in\Phi$ and $\sup\Omega\in\Phi$ for any subset Ω of Φ.

$$1\leq(Fx)(t)\leq\left(\frac{3}{2}\right)^{\alpha}\int_{t_1}^{t}a(s)\Delta s.$$

Hence, it follows $Fx:\Phi\to\Phi$. Also, since $Fx_1\leq Fx_2$ for $x_1\leq x_2$, we have that F is an increasing mapping. Therefore, by Theorem 1.5, there exists $\bar{x}\in\Phi$ such that $\bar{x}=F\bar{x}$. So we have that $\bar{x}(t)>0$ for $t\geq t_1$ and diverges to ∞ as $t\to\infty$. Now, let us show that the component function y has a finite limit. Taking the derivative of $F\bar{x}$ and setting

$$\bar{y}(t)=\frac{1}{2}+\int_t^\infty b(u)\left(\int_u^\infty c(\tau)(\bar{x}^\sigma(\tau))^\gamma\Delta\tau\right)^\beta\Delta u$$

leads us $\bar{y}(t)\to\frac{1}{2}$ as $t\to\infty$ since

$$\begin{aligned}&\int_t^\infty b(u)\left(\int_u^\infty c(\tau)(\bar{x}^\sigma(\tau))^\gamma\Delta\tau\right)^\beta\Delta u\\ &\leq\int_t^\infty b(u)\left(\int_u^\infty c(\tau)\left(\frac{3}{2}\int_{t_1}^{\sigma(\tau)}a(s)\Delta s\right)^\gamma\Delta\tau\right)^\beta\Delta u<\infty,\quad t\geq t_1.\end{aligned}$$

Taking the derivative of $\bar{y}$ gives

$$\bar{y}^\Delta(t)=-b(t)\left(\int_t^\infty c(\tau)(\bar{x}^\sigma(\tau))^\gamma\Delta\tau\right)^\beta=b(t)|\bar{z}(t)|^\beta\operatorname{sgn}\bar{z}(t),$$

where

$$\bar{z}(t):=-\int_t^\infty c(\tau)(\bar{x}^\sigma(\tau))^\gamma\Delta\tau.$$

Then $\bar{z}^\Delta(t)=c(t)|\bar{x}^\sigma(t)|^\gamma\operatorname{sgn}\bar{x}^\sigma(t)$. Therefore, we end up with that $(\bar{x},\bar{y},\bar{z})$ is a solution of system (1). Finally, let us show the limit behavior of $\bar{z}$. Note that

$$-\int_t^\infty c(\tau)\left(\frac{3}{2}\int_{t_1}^{\sigma(\tau)}a(s)\Delta s\right)^\gamma\Delta\tau\leq\bar{z}(t)\leq-\left(\frac{1}{2}\right)^\gamma\int_t^\infty c(\tau)\Delta\tau$$

and that the left and right hand sides of the inequality are convergent because of Lemma 2.4 (ii) and (iii). Therefore $\bar{z}(t)\to 0$ as $t\to\infty$. It follows that $N^-_{\infty,B,0}\neq\emptyset$. □

With the same assumption as in the proof of Theorem 2.5, we can prove the following theorem. Therefore, its proof is left to the reader.

Theorem 2.6. *If $J_1(t_0,\infty)=\infty$ and $J_3(t_0,\infty)<\infty$ for all k_1,k_2,k_3 and k_4, then $N^-_{\infty,0,0}\neq\emptyset$.*

Proof. Suppose $J_1=\infty$ and $J_3<\infty$ for $k_1,k_2,k_3,k_4>0$. Then, by defining the operator F by

$$(Fx)(t)=\int_{t_1}^{t} a(s)\left(\int_s^\infty b(u)\left(\int_u^\infty c(\tau)(x^\sigma(\tau))^\gamma\Delta\tau\right)^\beta\Delta u\right)^\alpha\Delta s$$

and applying a similar procedure as in the proof of Theorem 2.5, one can easily show the existence of solution (x,y,z) of system (1) in $N^-_{\infty,0,0}$. □

We finish this section by the following example, which highlights Theorem 2.5.

Example 2.7. *Consider the time scale $\mathbb{T}=3^{\mathbb{N}_0}$, where $\mathbb{N}_0=\{0,1,2\cdots\}$. The derivative and integral are respectively defined by*

$$f^\Delta(t)=\frac{f(\sigma(t))-f(t)}{\mu(t)}$$

and

$$\int_a^\infty f(t)\Delta t=\lim_{b\to\infty}\int_a^b f(t)\Delta t=\sum_{t\in[a,b)_{3^{\mathbb{N}_0}}} f(t)\mu(t),$$

where $\sigma(t)=3t,\mu(t)=2t,a\in\mathbb{T}$ and f is rd-continuous on $[a,\infty)_\mathbb{T}$, (see [5, Definition 1.82]). Consider also the system

$$\begin{cases} x^\Delta(t)=\frac{\sqrt{3}-1}{2t^{\frac13}(t^{\frac12}+1)^{\frac13}}(y(t))^{\frac13}\\ y^\Delta(t)=-\frac{\sqrt{3}-1}{2\sqrt{3}t^{\frac{5}{22}}}(-z(t))^{\frac{7}{11}}\\ z^\Delta(t)=\frac{4}{9t^{\frac72}}x(3t).\end{cases}\tag{4}$$

First we need to show $A(1,\infty)=B(1,\infty)=\infty$. Indeed,

$$\int_1^T a(t)\Delta t=\sum_{t\in[1,T)_\mathbb{T}}\frac{\sqrt{3}-1}{2t^{\frac13}(t^{\frac12}+1)^{\frac13}}\cdot 2t.$$

So, as $T\to\infty$, we have that $A(1,\infty)=\infty$ by the divergence test. Similarly,

$$\int_1^T b(t)\Delta t=\sum_{t\in[1,T)_{3^{\mathbb{N}_0}}}\frac{\sqrt{3}-1}{2\sqrt{3}t^{\frac{5}{22}}}\cdot 2t$$

is divergent as $T\to\infty$. Now, we need to show $J_1(1,\infty)=\infty$ and $J_3(1,\infty)<\infty$ for $k_1=k_2=k_3=k_4=1$. First note that

$$\int_s^T c(u)\Delta u=\sum_{u\in[s,T)_\mathbb{T}}\frac{4}{9u^{\frac72}}\cdot 2u=\frac89\sum_{u\in[s,T)_\mathbb{T}}\frac{1}{u^{\frac52}}$$

Therefore, as $T\to\infty$, we have

$$\int_s^\infty c(u)\Delta u=\frac89\frac{3^{\frac52}}{3^{\frac52}-1}\frac{1}{s^{\frac52}}.\tag{5}$$

Now by (5), *we have*

$$\int_t^T b(s)\left(\int_s^\infty c(u)\Delta u\right)^\beta \Delta s = l_1 \sum_{s\in[t,T)_\mathbb{T}} \frac{1}{s^{\frac{5}{22}}}\left(\frac{1}{s^{\frac{5}{2}}}\right)^{\frac{7}{11}} \cdot s = l_1 \sum_{s\in[t,T)_\mathbb{T}} \frac{1}{s^{\frac{18}{22}}},$$

where $l_1 = \frac{\sqrt{3}-1}{\sqrt{3}}\left[\frac{8}{9}\frac{3^{\frac{5}{2}}}{3^{\frac{5}{2}}-1}\right]^{\frac{7}{11}}$. *So as* $T\to\infty$, *we obtain*

$$\int_t^\infty b(s)\left(\int_s^\infty c(u)\Delta u\right)^\beta \Delta s = l_2\frac{1}{t^{\frac{18}{22}}}, \tag{6}$$

where $l_2 = l_1\cdot\frac{3^{\frac{18}{22}}}{3^{\frac{18}{22}}-1}$. *Finally, substituting* (6) *in* $J_1(1,T)$ *yields us*

$$J_1(1,T) \geq \int_1^T a(t)\left(l_2\frac{1}{t^{\frac{18}{22}}}\right)^{\frac{1}{3}}\Delta t = l_3 \sum_{t\in[1,T)_\mathbb{T}} \frac{1}{t^{\frac{1}{3}}(t^{\frac{1}{2}}+1)^{\frac{1}{3}}}\frac{1}{t^{\frac{6}{22}}}\cdot t,$$

where $l_3 = l_2^{\frac{1}{3}}(\sqrt{3}-1)$. *Therefore, we conclude that* $J_1(1,\infty)=\infty$ *as* $T\to\infty$ *by the divergence test. It remains to show* $J_3(1,\infty)<\infty$. *First note that*

$$\begin{aligned}\int_1^{\sigma(s)} a(u)\Delta u = (\sqrt{3}-1)\sum_{u\in[1,\sigma(s))_\mathbb{T}} \frac{1}{u^{\frac{1}{3}}(u^{\frac{1}{2}}+1)^{\frac{1}{3}}}\cdot u &\leq \sum_{u\in[1,\sigma(s))_\mathbb{T}} u^{\frac{1}{2}}\\ &\leq \frac{\sqrt{3}}{\sqrt{3}-1}s^{\frac{1}{2}}.\end{aligned}$$

Then

$$\int_t^T c(s)\left(\int_1^{\sigma(s)} a(u)\Delta u\right)\Delta s \leq \int_t^T \frac{4\sqrt{3}}{9s^{\frac{7}{2}}}s^{\frac{1}{2}}\Delta s = l_4\sum_{s\in[t,T)_\mathbb{T}}\frac{1}{s^2},$$

where $l_4 = \frac{8\sqrt{3}}{9(\sqrt{3}-1)}$. *So as* $T\to\infty$, *we have that*

$$\int_t^\infty c(s)\left(\int_1^{\sigma(s)} a(u)\Delta u\right)\Delta s \leq l_5\frac{1}{t^2},$$

where $l_5 = \frac{\sqrt{3}}{\sqrt{3}-1}$. *It follows that*

$$J_3(1,T) \leq \int_1^T \frac{\sqrt{3}-1}{2\sqrt{3}t^{\frac{5}{22}}}\left(l_5\frac{1}{t^2}\right)^{\frac{7}{11}}\Delta t = l_6\sum_{t\in[1,T)_\mathbb{T}}\frac{1}{t^{\frac{33}{22}}}\cdot t = l_6\sum_{t\in[1,T)_\mathbb{T}}\frac{1}{t^{\frac{1}{2}}},$$

where $l_6 = \frac{(\sqrt{3}-1)^{\frac{4}{11}}}{3^{\frac{4}{22}}}$. *Letting* $t=3^n$ *and taking the limit as* $T\to\infty$ *yield us*

$$\sum_{n=0}^\infty \frac{1}{3^{\frac{n}{2}}}$$

is convergent by the ratio test. Therefore $J_3(1,\infty)<\infty$ *by the direct comparison test. Lastly, we can show that* $\left(t^{\frac{1}{2}}, 1+\frac{1}{t^{\frac{1}{2}}}, -\frac{1}{t^2}\right)$ *is a solution of system* (4) *such that* $x(t)\to\infty$, $y(t)\to 1$, *and* $z(t)\to 0$ *as* $t\to\infty$, *i.e.,* $N^-_{\infty,B,0}\neq\emptyset$ *by Theorem 2.5.*

2.3. Existence in N^+

This section presents the existence of nonoscillatory solutions of system (1) in N^+. Note that we have only two sub-classes in N^+ and the existence of nonoscillatory solutions in $N^+_{\infty,\infty,\infty}$ and $N^+_{\infty,\infty,B}$ is not easy to show. Hence, by assuming the existence of nonoscillatory solutions of system (1), we show that such solutions belong to $N^+_{\infty,\infty,\infty}$ or $N^+_{\infty,\infty,B}$. Therefore, we have the following theorems.

Theorem 2.8. *Let (x, y, z) a be nonoscillatory solution of system* (1) *in N^+ and $0 < \alpha\gamma\beta < 1$. Then such a solution belongs to $N^+_{\infty,\infty,B}$ if $J_5(t_0, \infty) < \infty$ for all k_6 and k_7.*

Proof. Suppose that $J_5(t_0, \infty) < \infty$ holds for the corresponding constants and (x, y, z) is a solution in N^+. Then since x, y and z are positive increasing functions, there exist $c_4, c_5, c_6 > 0$ and $t_1 \geq t_0$ such that $x(t) \geq c_4$, $y(t) \geq c_5$ and $z(t) \geq c_6$ for $t \geq t_1$. Integrating the first and second equations of system (1) from t_1 to t yield

$$x(t) = x(t_1) + \int_{t_1}^{t} a(s)y^{\alpha}(s)\Delta s \geq c_5^{\alpha} \int_{t_1}^{t} a(s)\Delta s \tag{7}$$

and

$$y(t) = y(t_1) + \int_{t_1}^{t} b(s)z^{\beta}(s)\Delta s \geq y(t_1) + c_6^{\beta} \int_{t_1}^{t} b(s)\Delta s. \tag{8}$$

So as $t \to \infty$, we have that $x(t), y(t) \to \infty$.

Next we show z has a finite limit. Indeed, by integrating the third equation of system (1) and using (7) and (8) as well as the increasing nature of z, we obtain

$$\begin{aligned} z(t) &= z(t_1) + \int_{t_1}^{t} c(s)\left(x^{\sigma}(s)\right)^{\gamma} \Delta s \\ &= z(t_1) + \int_{t_1}^{t} c(s)\left(x^{\sigma}(t_1) + \int_{t_1}^{\sigma(s)} a(u)\left(y(t_1) + \int_{t_1}^{u} b(v)z^{\beta}(v)\Delta v\right)^{\alpha} \Delta u\right)^{\gamma} \Delta s \\ &\leq z(t_1) + z^{\beta\alpha\gamma}(t) \int_{t_1}^{t} c(s)\left(\frac{x^{\sigma}(t_1)}{z^{\beta\alpha}(s)} + \int_{t_1}^{\sigma(s)} a(u)\left(\frac{y(t_1)}{z^{\beta}(u)} + \int_{t_1}^{u} b(v)\Delta v\right)^{\alpha} \Delta u\right)^{\gamma} \Delta s \end{aligned}$$

for $t \geq s \geq u \geq v$. Then dividing the both sides of the latter inequality by $z^{\beta\alpha\gamma}(t)$ and again using the monotonicty of z give

$$z^{1-\beta\alpha\gamma}(t) \leq c_7 + \int_{t_1}^{t} c(s)\left(c_8 + \int_{t_1}^{\sigma(s)} a(u)\left(c_9 + \int_{t_1}^{u} b(v)\Delta v\right)^{\alpha} \Delta u\right)^{\gamma} \Delta s,$$

where $c_7 = \frac{z(t_1)}{c_6^{\beta\alpha\gamma}}$, $c_8 = \frac{x^{\sigma}(t_1)}{c_6^{\beta\alpha}}$, and $c_9 = \frac{y(t_1)}{c_6^{\beta}}$. As $t \to \infty$, we have that the assertion follows. □

We give the following example on $\mathbb{T} = \mathbb{R}$ to highlight Theorem 2.8.

Example 2.9. *Consider system* (1) *for $a(t) = e^{\frac{7t}{9}}$, $b(t) = \frac{2e^{\frac{60}{29}t}}{(e^{2t}-1)^{\frac{1}{29}}}$, $c(t) = 2e^{\frac{-8t}{3}}$, $\alpha = \frac{1}{9}$, $\beta = \frac{1}{29}$ and $\gamma = \frac{1}{3}$ for $\mathbb{T} = \mathbb{R}$. Please note that $\alpha\gamma - \beta > 0$. Also it is easy to show $A(1, \infty) = B(1, \infty) = \infty$. Therefore, we only show $J_5(1, \infty) < \infty$ for $k_6 = \frac{261}{263}e^{\frac{263}{261}}$ and $k_7 = \frac{29}{30}e^{\frac{60}{29}}$. First,*

$$k_7 + \int_{1}^{s} b(u)du = k_7 + \int_{1}^{s} \frac{2e^{\frac{60}{29}u}}{(e^{2u}-1)^{\frac{1}{29}}}du \leq k_7 + \int_{1}^{s} 2e^{\frac{60}{29}u}du \leq e^{\frac{60}{29}s} \tag{9}$$

Then

$$k_6 + \int_{1}^{t} a(s)\left(k_7 + \int_{1}^{s} b(u)du\right)^{\alpha} ds \leq k_6 + \int_{1}^{t} e^{\frac{7s}{9}} e^{\frac{60s}{261}} ds = k_6 + \int_{1}^{t} e^{\frac{263s}{261}} ds = \frac{261}{263}e^{\frac{263t}{261}} \leq e^{\frac{263t}{261}} \tag{10}$$

Finally, by substituting (9) *and* (10) *in* $J_5(1,T)$*, we have*

$$J_5(1,T) \leq \int_1^T 2e^{\frac{-8t}{3}} e^{\frac{263t}{783}} dt < \infty$$

as $T \to \infty$*. One can also show that* $(e^t, e^{2t}, 1 - e^{-2t})$ *is a nonoscillatory solution of system* (1) *in* N^+ *such that* $x(t) \to \infty, y(t) \to \infty, z(t) \to 1$ *as* $t \to \infty$*, i.e.,* $N^+_{\infty,\infty,B} \neq \emptyset$*.*

The following theorem can be easily proved by integrating the system from t_0 to ∞ and by using the monotonicity of the component functions x, y and z. Therefore, we leave it to readers.

Theorem 2.10. *Let* (x, y, z) *be nonoscillatory solutions of system* (1) *in* N^+*. If* $C(t_0, \infty) = \infty$*, then all nonoscillatory solutions in* N^+ *belong to* $N^+_{\infty,\infty,\infty}$*.*

3. Property B Solutions of System (1)

In the previous section, we showed the existence of nonoscillatory solution of system (1) by using the fixed point theorems and improper integrals. This section presents the solutions of system (1) that have Property B.

The following theorem can be shown similar to Theorem 3.1 proved by Akın et al in [2], therefore, we omit it.

Theorem 3.1. *Suppose that* $C(t_0, \infty) = \infty$ *for* $t_0 \in \mathbb{T}$*. Then system* (1) *has Property B.*

For simplicity, set

$$J_6(t_0,\infty) = \int_{t_0}^{\infty} a(t)\left(\int_{t_0}^{t} b(u)\Delta u\right)^{\alpha}\left(\int_{\sigma(t)}^{\infty} c(\tau)\Delta\tau\right)^{\frac{1}{\gamma}} \Delta t,$$

$$J_7(t_0,\infty) = \int_{t_0}^{\infty} b(t)\left(\int_{t}^{\infty} c(s)\Delta s\right)^{\beta}\left(\int_{t_0}^{t} a(u)\Delta u\right)^{\frac{1}{\alpha}} \Delta t,$$

$$J_8(t_0,\infty) = \int_{t_0}^{\infty} c(t)\left(\int_{t_0}^{\sigma(t)} a(s)\left(\int_{t_0}^{s} b(u)\Delta u\right)^{\alpha}\Delta s\right)^{\gamma} \Delta t,$$

$$\Upsilon_{\lambda_1} = \int_{t_0}^{\infty} d(t)\left(\int_{t_0}^{t} e(s)\Delta s\right)^{\lambda_1} \Delta t,$$

$$Z_{\lambda_2} = \int_{t_0}^{\infty} e(t)\left(\int_{\sigma(t)}^{\infty} d(s)\Delta s\right)^{\frac{1}{\lambda_2}} \Delta t,$$

where $\lambda_1, \lambda_2 > 0$, $d, e \in C_{rd}([t_0,\infty)_{\mathbb{T}}, \mathbb{R}^+)$ and $\lim_{n\to\infty} J_n(t_0,T) = J_n(t_0,\infty)$ for $n = 6, 7, 8$.

Theorem 3.2. *If* $J_8(t_0,\infty)$ *is infinite for* $t_0 \in \mathbb{T}$*, then any* Type(a) *solution* (x, y, z) *of system* (1) *satisfies* (3).

Proof. Suppose that $J_8(t_0,\infty) = \infty$ for $t_0 \in \mathbb{T}$ and that is a Type(a) solution (x, y, z) of system (1). By Lemma 2.2, we have that both x and y are divergent. So it would be enough to show $z(t) \to \infty$ as $t \to \infty$. Since the component function z is positive and increasing, there exists a positive $t_1 \geq t_0$ such that $z(t) \geq c_1$ for $t \geq t_1$, where $0 < c_1 < \infty$. Integrating the second equation of (1) from t_1 to t gives

$$y(t) \geq c_1^{\beta} \int_{t_1}^{t} b(s)\Delta s. \tag{11}$$

Next, integrating the first equation of system (1) from t_1 to $\sigma(t)$ and using (11) yield us

$$x(\sigma(t)) \geq \int_{t_1}^{\sigma(t)} a(s)\left(c_1^{\beta}\int_{t_1}^{s} b(u)\Delta u\right)^{\alpha}\Delta s. \tag{12}$$

Finally, by integrating the last equation of system (1) from t_1 to t and using (12), we have:

$$z(t) \geq \int_{t_1}^{t} c(s)\left(\int_{t_1}^{\sigma(s)} a(u)\left(c_1^{\beta}\int_{t_1}^{u} b(\tau)\Delta\tau\right)^{\alpha}\Delta u\right)^{\gamma}\Delta s.$$

Therefore, as $t \to \infty$, the desired result is obtained and the proof is complete. □

The following lemma provides relationships between Y_{λ_1} and Z_{λ_2}, and we use them to prove some oscillatory properties of system (1); see [17] for the proofs.

Lemma 3.3. *The following hold:*

i.) Let $\lambda_1 = \lambda_2 \leq 1$. If $Z_{\lambda_1} = \infty$, then $Y_{\lambda_1} = \infty$.
ii.) Let $\lambda_1 = \lambda_2 \geq 1$. If $Y_{\lambda_1} = \infty$, then $Z_{\lambda_1} = \infty$.

Next, we obtain our results by using α, β and γ relationships between α, β and γ and Lemma 3.3.

Theorem 3.4. *Let $\alpha\beta\gamma > 1$ and $\alpha \leq 1$. If $J_7(t_0, \infty) = \infty$, then every nonoscillatory solution (x, y, z) of system* (1) *is of Type*(a)*.*

Proof. Suppose that $\alpha\beta\gamma > 1, \alpha \leq 1$ and $J_7(t_0, \infty) = \infty$. Set

$$d(t) = b(t)\left(\int_{t}^{\infty} c(u)\Delta u\right)^{\beta},$$

$e(t) = a(t)$, and $\lambda_1 = \lambda_2 = \frac{1}{\alpha}$ in Y_{λ_1}. It turns out that $J_7(t_0, \infty) = Y_{\lambda_1} = \infty$. Then, by Lemma 3.3(ii), we have

$$Z_{\lambda_1} = Z_{\frac{1}{\alpha}} = \int_{t_0}^{\infty} a(t)\left(\int_{\sigma(t)}^{\infty} b(s)\left(\int_{s}^{\infty} c(u)\Delta u\right)^{\beta}\Delta s\right)^{\alpha}\Delta t = \infty.$$

Then, by Theorem 3.3 in [2], the assertion follows. □

Our last result gives sufficient conditions for the system (1) to have Property B.

Theorem 3.5. *Suppose that $\alpha\beta\gamma > 1$, $\alpha \leq 1$ and $\gamma \leq 1$. Then if $J_6(t_0, \infty) = \infty$ and $J_7(t_0, \infty) = \infty$, then system* (1) *has Property B.*

Proof. Assume $\alpha\beta\gamma >$, $\alpha \leq 1$ and $\gamma \leq 1$, $J_6(t_0, \infty) = J_7(t_0, \infty) = \infty$. Set $\lambda_1 = \lambda_2 = \gamma$, $d(t) = c(t)$, $e(t) = a(t)\left(\int_{t_0}^{t} b(u)\Delta u\right)^{\alpha}$ for $t \geq t_0$ in Z_{λ_2}. Then

$$J_6(t_0, \infty) = Z_{\lambda_1} = \infty,$$

which implies Y_{λ_1} is also divergent by Lemma 3.3(i). Note that

$$Y_{\lambda_1} = \int_{t_0}^{\infty} c(t)\left(\int_{t_0}^{t} a(s)\left(\int_{t_0}^{s} b(u)\Delta u\right)^{\alpha}\Delta s\right)^{\gamma}\Delta t = \infty.$$

By Theorem 3.3 in [2], it follows that system (1) has Property B. □

4. Conclusion and Open Problems

This article shows us that it is possible to have nonoscillatory solutions in Type(a) and Type(c) under the convergence/divergence of improper integrals. In addition to the results in [2], we find new conditions and α, β, γ relationships to determine almost oscillation. Note that we could not show the existence of such solutions in $N^-_{B,B,0}$, $N^-_{B,0,0}$, $N^+_{\infty,\infty,B}$ and $N^+_{\infty,\infty,\infty}$. The main reason is the inability of setting up the operators required by the fixed point theorems. The presence of σ in the last equation of system (1) causes the problematic cases which prevent us from proving the existence. Therefore, it still remains as an open problem. Also, one can consider the following system:

$$\begin{cases} x^\Delta(t) = a(t)|y(t)|^\alpha \operatorname{sgn} y(t) \\ y^\Delta(t) = b(t)|z(t)|^\beta \operatorname{sgn} z(t) \\ z^\Delta(t) = -c(t)|x^\sigma(t)|^\gamma \operatorname{sgn} x^\sigma(t), \end{cases} \tag{13}$$

and find the oscillation/nonoscillation criteria. System (13) is known as a third order Emden-Fowler dynamical system and has several applications in astrophysics and fluid mechanics. Therefore, system (13) may be considered as a future work.

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