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Advanced Ordinary and Fractional Approximation by Positive Sublinear Operators

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Abstract. Here we consider the ordinary and fractional approximation of functions by sublinear positive operators with applications to generalized convolution type operators expressed by sublinear integrals such as of Choquet and Shilkret ones. The fractional approximation is under fractional differentiability of Caputo, Canavati and Iterated-Caputo types. We produce Jackson type inequalities under basic initial conditions. So our way is quantitative by producing inequalities with their right hand sides involving the modulus of continuity of ordinary and fractional derivatives of the function under approximation. We give also an application related to Picard singular integral operators.

1. Background - I

Here we follow [3], pp. 1-17.

Let $I \subset \mathbb{R}$ be a bounded or unbounded interval, $n \in \mathbb{N}$, and

$$CB_+^n(I) = \left\{ f : I \to \mathbb{R}_+ : f^{(i)} \text{ is continuous and bounded on } I, \text{ for both } i = 0, n \right\}. \tag{1}$$

We define for

$$f \in CB_+(I) = \{ f : I \to \mathbb{R}_+ : f \text{ is continuous and bounded on } I \},$$

the first modulus of continuity

$$\omega_1(f, \delta) = \sup_{\substack{x,y \in I: \\ |x-y| \leq \delta}} |f(x) - f(y)|, \tag{2}$$

where $0 < \delta \leq diameter(I)$, also defined the same for just uniformly continuous functions $f : I \to \mathbb{R}_+$.

Call $C_+(I) = \{ f : I \to \mathbb{R}_+ : f \text{ is continuous on } I \}$.

Let $L_N : C_+(I) \to CB_+(I)$, $n, N \in \mathbb{N}$ be a sequence of operators satisfying the following properties (see also [5], p. 17):

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(i) (positive homogeneous)

$$L_N(\alpha f) = \alpha L_N(f), \ \forall\, \alpha \geq 0, f \in C_+(I), \tag{3}$$

(ii) (Monotonicity)

$$\text{if } f, g \in C_+(I) \text{ satisfy } f \leq g, \text{ then } L_N(f) \leq L_N(g), \ \ \forall\, N \in \mathbb{N}, \tag{4}$$

and

(iii) (Subadditivity)

$$L_N(f+g) \leq L_N(f) + L_N(g), \ \ \forall\, f, g \in C_+(I). \tag{5}$$

We call L_N positive sublinear operators.

In particular we consider the restrictions $L_N|_{CB_+^n(I)} : CB_+^n(I) \to CB_+(I)$.

As in [5], p. 17, we get that for $f, g \in CB_+(I)$,

$$\left|L_N(f)(x) - L_N(g)(x)\right| \leq L_N\left(\left|f-g\right|\right)(x), \ \ \forall\, x \in I. \tag{6}$$

Furthermore, also from [5], p. 17, we have

$$\left|L_N(f)(x) - f(x)\right| \leq L_N\left(\left|f(\cdot) - f(x)\right|\right)(x) + \left|f(x)\right| \left|L_N(1)(x) - 1\right|, \ \ \forall\, x \in I. \tag{7}$$

Given that $L_N(1) = 1$, $\forall\, N \in \mathbb{N}$, we get

$$\left|L_N(f)(x) - f(x)\right| \leq L_N\left(\left|f(\cdot) - f(x)\right|\right)(x), \ \ \forall\, x \in I, \ \forall\, N \in \mathbb{N}. \tag{8}$$

We mention Hölder's inequality for positive sublinear operators

Theorem 1.1. *([3], p. 6) Let $L : C_+(I) \to CB_+(I)$, be a positive sublinear operator and $f, g \in C_+(I)$, furthermore let $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$. Assume that $L\left((f(\cdot))^p\right)(s_*), L\left((g(\cdot))^q\right)(s_*) > 0$ for some $s_* \in I$. Then*

$$L(f(\cdot)\, g(\cdot))(s_*) \leq \left(L\left((f(\cdot))^p\right)(s_*)\right)^{\frac{1}{p}} \left(L\left((g(\cdot))^q\right)(s_*)\right)^{\frac{1}{q}}. \tag{9}$$

By assuming $L_N\left(|\cdot - x|^{n+1}\right)(x) > 0$, (9) and $L_N(1) = 1$, we obtain

$$L_N\left(|\cdot - x|^n\right)(x) \leq \left(L_N\left(|\cdot - x|^{n+1}\right)(x)\right)^{\frac{n}{n+1}}, \tag{10}$$

in case of $n = 1$ we derive

$$L_N\left(|\cdot - x|\right)(x) \leq \sqrt{\left(L_N\left((\cdot - x)^2\right)(x)\right)}. \tag{11}$$

We mention also the following result.

Theorem 1.2. *([3], p. 7) Let $(L_N)_{N \in \mathbb{N}}$ be a sequence of positive sublinear operators from $C_+(I)$ into $CB_+(I)$, and $f \in CB_+^n(I)$, $f^{(n)}$ could be only uniformly continuous, where $n \in \mathbb{N}$ and $I \subset \mathbb{R}$ a bounded or unbounded interval. Assume $L_N(1) = 1$, $\forall\, N \in \mathbb{N}$, and $f^{(i)}(x) = 0$, $i = 1, ..., n$, for a fixed $x \in I$. Furthermore assume that $L_N\left(|\cdot - x|^{n+1}\right)(x) > 0$, $\forall\, N \in \mathbb{N}$.*

Then

$$\left|L_N(f)(x) - f(x)\right| \leq \frac{\omega_1\left(f^{(n)}, \left(L_N\left(|\cdot - x|^{n+1}\right)(x)\right)^{\frac{1}{n+1}}\right)}{n!} \cdot$$

$$\left[L_N\left(|\cdot - x|^n\right)(x) + \frac{\left(L_N\left(|\cdot - x|^{n+1}\right)(x)\right)^{\frac{n}{n+1}}}{(n+1)}\right], \ \ \forall\, N \in \mathbb{N}. \tag{12}$$

We mention ($n = 1$ case)

Corollary 1.3. *([3], p. 7) Let $(L_N)_{N\in\mathbb{N}}$ be a sequence of positive sublinear operators from $C_+(I)$ into $CB_+(I)$, and $f \in CB_+^1(I)$, f' could be only uniformly continuous, and $I \subset \mathbb{R}$ a bounded or unbounded interval. Assume $L_N(1) = 1$, $\forall\, N \in \mathbb{N}$, and $f'(x) = 0$, for a fixed $x \in I$. Furthermore assume that $L_N\left((\cdot - x)^2\right)(x) > 0$, $\forall\, N \in \mathbb{N}$.*

Then

$$\left|L_N(f)(x) - f(x)\right| \leq \omega_1\left(f', \sqrt{\left(L_N\left((\cdot - x)^2\right)(x)\right)}\right) \cdot$$

$$\left[L_N(|\cdot - x|)(x) + \frac{\sqrt{\left(L_N\left((\cdot - x)^2\right)(x)\right)}}{2}\right], \quad \forall\, N \in \mathbb{N}. \tag{13}$$

Remark 1.4. *([3], p. 7) (i) to Theorem 1.2: Assuming $L_N\left(|\cdot - x|^{n+1}\right)(x) \to 0$, as $N \to \infty$, using (10), we get that $(L_N(f))(x) \to f(x)$, as $N \to \infty$.*

(ii) to Corollary 1.3: Assuming $L_N\left((\cdot - x)^2\right)(x) \to 0$, as $N \to \infty$, using (11), we get that $(L_N(f))(x) \to f(x)$, as $N \to \infty$.

(iii) The right hand sides of (12), (13) are finite.

We also mention the basic result ($n = 0$ case).

Theorem 1.5. *([3], p. 8) Let $(L_N)_{N\in\mathbb{N}}$ be a sequence of positive sublinear operators from $C_+(I)$ into $CB_+(I)$, and $f \in CB_+(I)$, f could be only uniformly continuous, where $I \subset \mathbb{R}$ a bounded or unbounded interval. Assume that $L_N(|\cdot - x|)(x) > 0$, for some fixed $x \in I$, $\forall\, N \in \mathbb{N}$. Then*

1)

$$\left|L_N(f)(x) - f(x)\right| \leq f(x)\left|L_N(1)(x) - 1\right| +$$

$$\left[L_N(1)(x) + 1\right]\omega_1\left(f, L_N(|\cdot - x|)(x)\right), \quad \forall\, N \in \mathbb{N}, \tag{14}$$

2) when $L_N(1) = 1$, we get

$$\left|L_N(f)(x) - f(x)\right| \leq 2\omega_1\left(f, L_N(|\cdot - x|)(x)\right), \quad \forall\, N \in \mathbb{N}. \tag{15}$$

Remark 1.6. *([3], p. 8) (to Theorem 1.5) Here $x \in I$ is fixed.*

i) Assume $L_N(1)(x) \to 1$, as $N \to \infty$, and $L_N(|\cdot - x|)(x) \to 0$, as $N \to \infty$, given that f is uniformly continuous we get that $L_n(f)(x) \to f(x)$, as $N \to \infty$ (use of (14)). Notice here that $L_N(1)(x)$ is bounded.

ii) Assume that $L_N(1) = 1$, and $L_N(|\cdot - x|)(x) \to 0$, as $N \to \infty$, and f is uniformly continuous on I, then $L_n(f)(x) \to f(x)$, as $N \to \infty$ (use of (15)).

iii) The right hand sides of (14) and (15) are finite.

2. Background - II ([4])

Consider $\Omega \neq \varnothing$ and let $\mathcal{F}$ be a σ-algebra in Ω. Here μ is a set function $\mu : \mathcal{F} \to [0, +\infty)$ which is monotone, i.e. for $A, B \in \Omega : A \subset B$ we have $\mu(A) \leq \mu(B)$, furthermore it holds $\mu(\varnothing) = 0$.

Here $f, g : \Omega \to \mathbb{R}_+ = [0, +\infty)$ are $\mathcal{F}$-measurable, we write it as $f, g \in M(\Omega, \mathbb{R}_+)$.

We consider a functional denoted by the integral symbol $(SL)\int_A f d\mu$, $\forall\, A \in \mathcal{F}$, which is positive, i.e. $\int_A f d\mu \geq 0$.

We assume the following properties:

(i) (positive homogeneous)

$$(SL)\int_A \alpha f d\mu = \alpha\,(SL)\int_A f d\mu, \;\; \forall\, \alpha \geq 0, \forall\, f \in M(\Omega, \mathbb{R}_+).$$

(ii) (Monotonicity) if $f, g \in M(\Omega, \mathbb{R}_+)$ satisfy $f \leq g$, then $(SL)\int_A f d\mu \leq (SL)\int_A g d\mu$, $\forall A \in \mathcal{F}$.
And
(iii) (Subadditivity)

$$(SL)\int_A (f+g)\,d\mu \leq (SL)\int_A f d\mu + (SL)\int_A g d\mu,\ \forall A \in \mathcal{F}.$$

(iv)

$$(SL)\int_A 1 d\mu = \mu(A),\ \forall A \in \mathcal{F}.$$

(v) If $\Omega = \mathbb{R}^d$, $d \in \mathbb{N}$, we assume that μ is strictly positive, i.e. $\mu(A) > 0$, for any A compact subset of $\mathbb{R}^d$. Here $\mathcal{F} = \mathcal{B}$ the Borel σ-algebra.

We call $(SL)\int_A f d\mu$ a sublinear integral.

We notice the following:

$$f(x) = f(x) - g(x) + g(x) \leq \left|f(x) - g(x)\right| + g(x),$$

hence

$$(SL)\int_A f(x)\,d\mu(x) \leq (SL)\int_A \left(\left|f(x) - g(x)\right| + g(x)\right) d\mu(x) \leq$$

$$(SL)\int_A \left|f(x) - g(x)\right| d\mu(x) + (SL)\int_A g(x)\,d\mu(x),$$

i.e.

$$(SL)\int_A f(x)\,d\mu(x) - (SL)\int_A g(x)\,d\mu(x) \leq (SL)\int_A \left|f(x) - g(x)\right| d\mu(x).$$

Similarly, we get that

$$(SL)\int_A g(x)\,d\mu(x) - (SL)\int_A f(x)\,d\mu(x) \leq (SL)\int_A \left|f(x) - g(x)\right| d\mu(x).$$

In conclusion, it holds

$$\left|(SL)\int_A f(x)\,d\mu(x) - (SL)\int_A g(x)\,d\mu(x)\right| \leq (SL)\int_A \left|f(x) - g(x)\right| d\mu(x), \tag{16}$$

$\forall A \in \mathcal{F}$ and $\forall f, g \in M(\Omega, \mathbb{R}_+)$.

3. Background - III

About the Choquet integral:
We make

Definition 3.1. *Consider $\Omega \neq \varnothing$ and let $\mathcal{C}$ be a σ-algebra of subsets in Ω.*

(i) (see, e.g., [13], p. 63) The set function $\mu : \mathcal{C} \to [0, +\infty]$ is called a monotone set function (or capacity) if $\mu(\varnothing) = 0$ and $\mu(A) \leq \mu(B)$ for all $A, B \in \mathcal{C}$, with $A \subset B$. Also, μ is called submodular if

$$\mu(A \cup B) + \mu(A \cap B) \leq \mu(A) + \mu(B),\ \textit{for all } A, B \in \mathcal{C}. \tag{17}$$

μ is called bounded if $\mu(\Omega) < +\infty$ and normalized if $\mu(\Omega) = 1$.

(ii) (see, e.g., [13], p. 233, or [7]) If μ is a monotone set function on $\mathcal{C}$ and if $f:\Omega\to\mathbb{R}$ is $\mathcal{C}$-measurable (that is, for any Borel subset $B\subset\mathbb{R}$ it follows $f^{-1}(B)\in\mathcal{C}$), then for any $A\in\mathcal{C}$, the Choquet integral is defined by

$$(C)\int_A fd\mu=\int_0^{+\infty}\mu\left(F_\beta(f)\cap A\right)d\beta+\int_{-\infty}^0\left[\mu\left(F_\beta(f)\cap A\right)-\mu(A)\right]d\beta, \tag{18}$$

where we used the notation $F_\beta(f)=\{\omega\in\Omega:f(\omega)\geq\beta\}$. Notice that if $f\geq 0$ on A, then in the above formula we get $\int_{-\infty}^0=0$.

The integrals on the right-hand side are the usual Riemann integral.

The function f will be called Choquet integrable on A if $(C)\int_A fd\mu\in\mathbb{R}$.

Next we list some well known properties of the Choquet integral.

Remark 3.2. *If $\mu:\mathcal{C}\to[0,+\infty]$ is a monotone set function, then the following properties hold:*

(i) For all $a\geq 0$ we have $(C)\int_A afd\mu=a\cdot(C)\int_A fd\mu$ (if $f\geq 0$ then see, e.g., [13], Theorem 11.2, (5), p. 228 and if f is arbitrary sign, then see, e.g., [8], p. 64, Proposition 5.1, (ii)).

(ii) For all $c\in\mathbb{R}$ and f of arbitrary sign, we have (see, e.g., [13], pp. 232-233, or [8], p. 65) $(C)\int_A(f+c)d\mu=(C)\int_A fd\mu+c\cdot\mu(A)$.

If μ is submodular too, then for all f,g of arbitrary sign and lower bounded, we have (see, e.g., [8], p. 75, Theorem 6.3)

$$(C)\int_A(f+g)d\mu\leq(C)\int_A fd\mu+(C)\int_A gd\mu. \tag{19}$$

(iii) If $f\leq g$ on A then $(C)\int_A fd\mu\leq(C)\int_A gd\mu$ (see, e.g., [13], p. 228, Theorem 11.2, (3) if $f,g\geq 0$ and p. 232 if f,g are of arbitrary sign).

(iv) Let $f\geq 0$. If $A\subset B$ then $(C)\int_A fd\mu\leq(C)\int_B fd\mu$. In addition, if μ is finitely subadditive, then

$$(C)\int_{A\cup B}fd\mu\leq(C)\int_A fd\mu+(C)\int_B fd\mu. \tag{20}$$

(v) It is immediate that $(C)\int_A 1\cdot d\mu(t)=\mu(A)$.

(vi) If μ is a countably additive bounded measure, then the Choquet integral $(C)\int_A fd\mu$ reduces to the usual Lebesgue type integral (see, e.g., [8], p. 62, or [13], p. 226).

(vii) If $\Omega=\mathbb{R}^d$, $d\in\mathbb{N}$, we assume μ is strictly positive, i.e. $\mu(A)>0$, for every A compact subset of $\mathbb{R}^d$. Here $\mathcal{C}=\mathcal{B}$ the Borel σ-algebra.

Clearly here, for μ being submodular, we get

$$\left|(C)\int_A f(x)d\mu(x)-(C)\int_A g(x)d\mu(x)\right|\leq(C)\int_A\left|f(x)-g(x)\right|d\mu(x), \tag{21}$$

$\forall\ A\in\mathcal{C}$ and $\forall\ f,g\in M(\Omega,\mathbb{R}_+)$ (f,g are measurable with respect to $\mathcal{C}$ σ-algebra).

(viii) If $f\geq 0$, then $(C)\int_A fd\mu\geq 0$.

From now on in this article we assume that $\mu:\mathcal{C}\to[0,+\infty)$ and is submodular.

4. Background - IV

Here we follow [12].

Let $\mathcal{F}$ be a σ-field of subsets of an arbitrary set Ω. An extended non-negative real valued function μ on $\mathcal{F}$ is called maxitive if $\mu(\varnothing)=0$ and

$$\mu\left(\cup_{i\in I}E_i\right)=\sup_{i\in I}\mu(E_i), \tag{22}$$

where the set I is of cardinality at most countable, where $\{E_i\}_{i\in I}$ is a disjoint collection of sets from $\mathcal{F}$. We notice that μ is monotone and (22) is true even $\{E_i\}_{i\in I}$ are not disjoint. For more properties of μ see [12]. We also call μ a maxitive measure. Here f stands for a non-negative measurable function on Ω. In [12], Niel Shilkret developed his non-additive integral defined as follows:

$$(N^*)\int_D f d\mu := \sup_{y\in Y}\{y\cdot\mu(D\cap\{f\geq y\})\}, \tag{23}$$

where $Y=[0,m]$ or $Y=[0,m)$ with $0<m\leq\infty$, and $D\in\mathcal{F}$. Here we take $Y=[0,\infty)$.

It is easily proved that

$$(N^*)\int_D f d\mu = \sup_{y>0}\{y\cdot\mu(D\cap\{f>y\})\}. \tag{24}$$

The Shilkret integral takes values in $[0,\infty]$.

The Shilkret integral ([12]) has the following properties:

$$(N^*)\int_\Omega \chi_E d\mu = \mu(E), \tag{25}$$

where χ_E is the indicator function on $E\in\mathcal{F}$,

$$(N^*)\int_D cf d\mu = c(N^*)\int_D f d\mu,\ \ c\geq 0, \tag{26}$$

$$(N^*)\int_D \sup_{n\in\mathbb{N}} f_n d\mu = \sup_{n\in\mathbb{N}}(N^*)\int_D f_n d\mu, \tag{27}$$

where f_n, $n\in\mathbb{N}$, is an increasing sequence of elementary (countably valued) functions converging uniformly to f. Furthermore we have

$$(N^*)\int_D f d\mu \geq 0, \tag{28}$$

$$f\geq g \text{ implies } (N^*)\int_D f d\mu \geq (N^*)\int_D g d\mu, \tag{29}$$

where $f,g:\Omega\to[0,\infty]$ are measurable.

Let $a\leq f(\omega)\leq b$ for almost every $\omega\in E$, then

$$a\mu(E)\leq (N^*)\int_E f d\mu \leq b\mu(E); \tag{30}$$

$$(N^*)\int_E 1 d\mu = \mu(E); \tag{31}$$

$f>0$ almost everywhere and $(N^*)\int_E f d\mu = 0$ imply $\mu(E)=0$;

$(N^*)\int_\Omega f d\mu = 0$ if and only $f=0$ almost everywhere;

$(N^*)\int_\Omega f d\mu < \infty$ implies that

$$\overline{N}(f) := \{\omega\in\Omega | f(\omega)\neq 0\} \text{ has } \sigma\text{-finite measure;}$$

$$(N^*)\int_D (f+g) d\mu \leq (N^*)\int_D f d\mu + (N^*)\int_D g d\mu; \tag{32}$$

and

$$\left|(N^*)\int_D fd\mu-(N^*)\int_D gd\mu\right|\leq (N^*)\int_D\left|f-g\right|d\mu. \tag{33}$$

From now on in this article we assume that $\mu:\mathcal{F}\to[0,+\infty)$.

If $\Omega=\mathbb{R}^d$, $d\in\mathbb{N}$, we assume μ is strictly positive, i.e. $\mu(A)>0$, for every A compact subset of $\mathbb{R}^d$. Here $\mathcal{F}=\mathcal{B}$ the Borel σ-algebra.

Conclusion 4.1. *We observe that the Choquet integral $(C)\int_A fd\mu$ and Shilkret integral $(N^*)\int_A fd\mu$ are perfect examples of the sublinear integral $(SL)\int_A fd\mu$ of Section 2, fulfilling all properties and they have great applications in many areas of pure and applied mathematics and mathematical economics.*

Therefore, all the results presented in this article which are for the general integral $(SL)\int_A fd\mu$ are of course valid for the Choquet and Shilkret integrals.

5. Main Results

5.1. Ordinary Approximation

All terms, notations and assumptions here will be as in Backgrounds I-IV.

So here it is $f\in C^n(\mathbb{R},\mathbb{R}_+)$ with f and $f^{(n)}$ are being bounded, but $f^{(n)}$ could be uniformly continuous regardless if it is bounded or not, $n\in\mathbb{Z}_+$.

Let $\mathcal{B}$ be the Borel σ-algebra on $\mathbb{R}$ and μ_N $(N\in\mathbb{N})$ be a sequence of monotone set functions from $\mathcal{B}$ into $\mathbb{R}_+$, i.e. for $A,B\in\mathcal{B}:A\subset B$ we have $\mu_N(A)\leq\mu_N(B)$, and $\mu_N(\varnothing)=0$, with $\mu_N(\mathbb{R})=1$, $\forall\, N\in\mathbb{N}$.

We will study here the approximation properties of the following sequence of positive sublinear convolution type operators

$$P_N(f)(x)=(SL)\int_{\mathbb{R}} f(x+t)\,d\mu_N(t), \tag{34}$$

$\forall\, N\in\mathbb{N}$, to $f(x)$, where $x\in\mathbb{R}$ is fixed, pointwise and uniform in a quantitative way.

We would assume that $P_N(f)\in CB_+(\mathbb{R})$, $\forall\, N\in\mathbb{N}$. Clearly it holds $P_N(1)=1$. Notice here that

$$P_N\left(|\cdot-x|^{n+1}\right)(x)=(SL)\int_{\mathbb{R}}|t|^{n+1}\,d\mu_N(t), \tag{35}$$

$\forall\, N\in\mathbb{N}$, where $n\in\mathbb{Z}_+$.

Based on the above we present

Theorem 5.1. *Assume further that $f^{(i)}(x)=0$, $i=1,...,n$, for a fixed $x\in\mathbb{R}$ and that*

$$(SL)\int_{\mathbb{R}}|t|^{n+1}\,d\mu_N(t)>0,\ \ \forall\, N\in\mathbb{N}.$$

Then

$$\left|P_N(f)(x)-f(x)\right|=\left|(SL)\int_{\mathbb{R}} f(x+t)\,d\mu_N(t)-f(x)\right|\leq$$

$$\frac{(n+2)}{(n+1)!}\omega_1\left(f^{(n)},\left((SL)\int_{\mathbb{R}}|t|^{n+1}\,d\mu_N(t)\right)^{\frac{1}{n+1}}\right)$$

$$\left((SL)\int_{\mathbb{R}}|t|^{n+1}\,d\mu_N(t)\right)^{\frac{n}{n+1}},\ \ \forall\, N\in\mathbb{N}. \tag{36}$$

Proof. By Theorem 1.2 and (10). □

Remark 5.2. *If* $(SL)\int_{\mathbb{R}}|t|^{n+1}d\mu_N(t)\to 0$, *then* $P_N(f)(x)\to f(x)$, *as* $N\to+\infty$.

The $n=1$ case follows:

Corollary 5.3. *Assume further that* $f'(x)=0$, *for a fixed* $x\in\mathbb{R}$ *and that*

$$(SL)\int_{\mathbb{R}}t^2d\mu_N(t)>0,\ \ \forall\, N\in\mathbb{N}.$$

Then

$$\left|P_N(f)(x)-f(x)\right|\leq\frac{3}{2}\omega_1\left(f',\left((SL)\int_{\mathbb{R}}t^2d\mu_N(t)\right)^{\frac{1}{2}}\right)\left((SL)\int_{\mathbb{R}}t^2d\mu_N(t)\right)^{\frac{1}{2}},\tag{37}$$

$\forall\, N\in\mathbb{N}$.

If $(SL)\int_{\mathbb{R}}t^2d\mu_N(t)\to 0$, *then* $P_N(f)(x)\to f(x)$, *as* $N\to+\infty$.

Proof. By Corollary 1.3 and (11). □

The case $n=0$ comes next.

Theorem 5.4. *Assume that* $(SL)\int_{\mathbb{R}}|t|\,d\mu_N(t)>0,\ \forall\, N\in\mathbb{N}$. *Then*

$$\left\|P_N(f)-f\right\|_{\infty}\leq 2\omega_1\left(f,(SL)\int_{\mathbb{R}}|t|\,d\mu_N(t)\right),\tag{38}$$

$\forall\, N\in\mathbb{N}$.

Given that f *is uniformly continuous and not necessarily bounded, from* $\mathbb{R}$ *into* $\mathbb{R}_+$, *then (38) is again valid and if* $(SL)\int_{\mathbb{R}}t^2d\mu_N(t)\to 0$, *then* $P_N(f)\to f$, *uniformly, as* $N\to+\infty$.

Proof. By Theorem 1.5 (15). □

Application 5.5. *Consider the well-known Picard singular integral operators:*

$$P_{\xi}^{*}(f)(x):=\frac{1}{2\xi}\int_{-\infty}^{\infty}f(x+t)\,e^{-\frac{|t|}{\xi}}dt,\tag{39}$$

where $\xi>0$. *Here* f *is chosen so that* $P_{\xi}^{*}(f)(x)\in\mathbb{R}$, $\forall\, x\in\mathbb{R}$, *e.g.* f *is bounded. Also* $P_{\xi}^{*}(f)$ *is continuous when* f *is uniformly continuous.*

We notice that

$$\frac{1}{2\xi}\int_{-\infty}^{\infty}e^{-\frac{|t|}{\xi}}dt=1,\ \ \xi>0.\tag{40}$$

In [11] they obtained the degree of convergence of the operators P_{ξ}^{*} *to the unit operator* I *with rates over the class of Hölder-continuous functions as* $\xi\to 0$. *In [10] they derived some more refined convergence to* I *(as* $\xi\to 0$*), however only over the set of* $(C_{2\pi})$ 2π*-periodic continuous functions on* $\mathbb{R}$. *See also [2], pp. 127-129 for bounded and uniformly continuous functions over* $\mathbb{R}$*,we get uniform convergence of* $P_{\xi}^{*}\to I$.

We consider here only $\xi>0$ *such that* $\frac{1}{\xi}=N\in\mathbb{N}$, *as* $\xi\to 0$.

Thus, it holds

$$\frac{N}{2}\int_{-\infty}^{\infty}e^{-|t|N}dt=1,\ \ \forall\, N\in\mathbb{N}.\tag{41}$$

Clearly here according to our theory

$$d\mu_N(t) = \frac{N}{2} e^{-|t|N} dt, \ \ \forall\, N \in \mathbb{N},$$

and $P^*_{N^{-1}}(f)$ *is a special case of* $P_N(f)$.

We observe here that $(n \in \mathbb{Z}_+)$

$$(SL)\int_{\mathbb{R}} |t|^{n+1} d\mu_N(t) = \frac{N}{2}\int_{-\infty}^{\infty} |t|^{n+1} e^{-|t|N} dt = N\int_0^{\infty} t^{n+1} e^{-tN} dt \tag{42}$$

$$= \frac{N(n+1)!}{N^{n+2}} = \frac{(n+1)!}{N^{n+1}} \to 0, \ \ as\ N \to +\infty.$$

Therefore our special case of $(SL)\int_{\mathbb{R}} |t|^{n+1} d\mu_N(t)$ *generated from the Picard operators converges to zero as* $N \to +\infty$, *where* $n \in \mathbb{Z}_+$. *Furthermore our results apply to Picard operators convergence to the unit I.*

This application realizes our general convergence theory making possible our assumptions.

5.2. *Fractional Approximation*

We need

Definition 5.6. *Let* $\nu \geq 0$, $n = \lceil \nu \rceil$ ($\lceil \cdot \rceil$ *is the ceiling of the number*), $f \in AC^n([a,b])$ *(space of functions* f *with* $f^{(n-1)} \in AC([a,b])$*, absolutely continuous functions). We call left Caputo fractional derivative of order* $\nu > 0$ *(see [9], p. 49, [1], p. 394) the function*

$$D^{\nu}_{*a} f(x) = \frac{1}{\Gamma(n-\nu)} \int_a^x (x-t)^{n-\nu-1} f^{(n)}(t)\, dt, \ \ \forall\, x \in [a,b], \tag{43}$$

where Γ *is the gamma function* $\Gamma(v) = \int_0^{\infty} e^{-t} t^{v-1} dt$, $v > 0$.

We set $D^0_{*a} f(x) = f(x)$, $\forall\, x \in [a,b]$.

Exactly the same way one can define $D^{\nu}_{*x_0} f$ over $[x_0, +\infty)$, $x_0 \in \mathbb{R}$, for $f \in AC^n([x_0, b])$, $\forall\, b \in \mathbb{R}$, $b \geq x_0$.

We also need

Definition 5.7. *(see also [2], p. 336) Let* $f \in AC^r([a,b])$, $r = \lceil \alpha \rceil$, $\alpha \geq 0$. *We right Caputo fractional derivative of order* $\alpha > 0$ *is given by*

$$D^{\alpha}_{b-} f(x) = \frac{(-1)^r}{\Gamma(r-\alpha)} \int_x^b (\zeta - x)^{r-\alpha-1} f^{(r)}(\zeta)\, d\zeta, \ \ \forall\, x \in [a,b]. \tag{44}$$

We set $D^0_{b-} f(x) = f(x)$.

Exactly the same way one can define $D^{\alpha}_{x_0-} f$ over $(-\infty, x_0]$, $x_0 \in \mathbb{R}$, for $f \in AC^r([a, x_0])$, $\forall\, a \in \mathbb{R} : a \leq x_0$.

We make

Convention 5.8. *We assume that*

$$D^{a}_{*x_0} f(x) = 0, \textit{for } x < x_0, \tag{45}$$

and

$$D^{\alpha}_{x_0-} f(x) = 0, \textit{for } x > x_0, \tag{46}$$

for all $x, x_0 \in \mathbb{R}$.

We also make

Convention 5.9. *Let a real number $m > 0$, from now on we assume that $D^m_{x_0-}f$ is either bounded or uniformly continuous function on $(-\infty, x_0]$, similarly from now on we assume that $D^m_{*x_0}f$ is either bounded or uniformly continuous function on $[x_0, +\infty)$.*

We need

Definition 5.10. *Let $D^m_{x_0}f$ (real number $m > 0$) denote any of $D^m_{x_0-}f$, $D^m_{*x_0}f$ and $\delta > 0$. We set*

$$\omega_1\left(D^m_{x_0}f, \delta\right)_{\mathbb{R}} := \max\left\{\omega_1\left(D^m_{x_0-}f, \delta\right)_{(-\infty, x_0]}, \omega_1\left(D^m_{*x_0}f, \delta\right)_{[x_0, +\infty)}\right\}, \tag{47}$$

where $x_0 \in \mathbb{R}$. Notice that $\omega_1\left(D^m_{x_0}f, \delta\right)_{\mathbb{R}} < +\infty$.

We will use

Theorem 5.11. *([3], p. 89) Let the real number $m > 0$, $m \notin \mathbb{N}$, $\lambda = \lceil m \rceil$, $x_0 \in \mathbb{R}$, $f \in AC^{\lambda}([a, b], \mathbb{R}_+)$ (i.e. $f^{(\lambda-1)} \in AC[a, b]$, absolutely continuous functions on $[a, b]$), $\forall\ [a, b] \subset \mathbb{R}$, and $f^{(\lambda)} \in L_\infty(\mathbb{R})$. Furthermore we assume that $f^{(k)}(x_0) = 0$, $k = 1, ..., \lambda - 1$. The Convention 5.9 is imposed here. Then*

$$\left|f(x) - f(x_0)\right| \leq \frac{\omega_1\left(D^m_{x_0}f, \delta\right)_{\mathbb{R}}}{\Gamma(m+1)}\left[|x - x_0|^m + \frac{|x - x_0|^{m+1}}{(m+1)\delta}\right], \quad \delta > 0, \tag{48}$$

for all $x \in \mathbb{R}$, $\delta > 0$.

If $0 < m < 1$, then we do not need initial conditions.

We make

Remark 5.12. *Let L_N, $N \in \mathbb{N}$, be a sequence of positive sublinear operators from $C_+(\mathbb{R})$ into $CB_+(\mathbb{R})$. Here all are as in Theorem 5.11 for $x = x_0$ and we can rewrite (48) as follows:*

$$\left|f(\cdot) - f(x)\right| \leq \frac{\omega_1\left(D^m_x f, \delta\right)_{\mathbb{R}}}{\Gamma(m+1)}\left[|\cdot - x|^m + \frac{|\cdot - x|^{m+1}}{(m+1)\delta}\right], \tag{49}$$

valid over $\mathbb{R}$. Assume that $L_N(1) = 1$, $\forall\ N \in \mathbb{N}$.

By (8) we obtain

$$\left|L_N(f)(x) - f(x)\right| \leq$$

$$\frac{\omega_1\left(D^m_x f, \delta\right)_{\mathbb{R}}}{\Gamma(m+1)}\left[L_N\left(|\cdot - x|^m\right)(x) + \frac{L_N\left(|\cdot - x|^{m+1}\right)(x)}{(m+1)\delta}\right] =: (\xi). \tag{50}$$

We also assume that $L_N\left(|\cdot - x|^{m+1}\right)(x) > 0$, $\forall\ N \in \mathbb{N}$.

By (9) we get that

$$L_N\left(|\cdot - x|^m\right)(x) \leq \left(L_N\left(|\cdot - x|^{m+1}\right)(x)\right)^{\frac{m}{m+1}}. \tag{51}$$

Choose

$$\delta := \left(L_N\left(|\cdot - x|^{m+1}\right)(x)\right)^{\frac{1}{m+1}} > 0, \tag{52}$$

i.e.

$$\delta^{m+1} = L_N\left(|\cdot - x|^{m+1}\right)(x).$$

Therefore we have

$$(\xi) \leq \frac{\omega_1\left(D_x^m f, \left(L_N\left(|\cdot - x|^{m+1}\right)(x)\right)^{\frac{1}{m+1}}\right)_{\mathbb{R}}}{\Gamma(m+1)}\left[\delta^m + \frac{\delta^{m+1}}{(m+1)\delta}\right] = \tag{53}$$

$$\frac{(m+2)}{\Gamma(m+2)}\omega_1\left(D_x^m f, \left(L_N\left(|\cdot - x|^{m+1}\right)(x)\right)^{\frac{1}{m+1}}\right)_{\mathbb{R}}\left(L_N\left(|\cdot - x|^{m+1}\right)(x)\right)^{\frac{m}{m+1}},$$

$\forall\, N \in \mathbb{N}$.

We have proved

Theorem 5.13. *Let $m > 0$, $m \notin \mathbb{N}$, $\lambda = \lceil m \rceil$, $x \in \mathbb{R}$, $f \in AC^{\lambda}([a,b], \mathbb{R}_+)$, $\forall\, [a,b] \subset \mathbb{R}$, and $f^{(\lambda)} \in L_\infty(\mathbb{R})$. Furthermore we assume that $f^{(k)}(x) = 0$, $k = 1, ..., \lambda - 1$. We assume that $D_{x-}^m f$, $D_{*x}^m f$ are either bounded or uniformly continuous over $(-\infty, x]$, $[x, +\infty)$, respectively. Let L_N ($N \in \mathbb{N}$) be a sequence of positive sublinear operators from $C_+(\mathbb{R})$ into $CB_+(\mathbb{R})$. Assume that $L_N(1) = 1$, and $L_N\left(|\cdot - x|^{m+1}\right)(x) > 0$, $\forall\, N \in \mathbb{N}$. Then*

$$\left|L_N(f)(x) - f(x)\right| \leq \frac{(m+2)}{\Gamma(m+2)}\omega_1\left(D_x^m f, \left(L_N\left(|\cdot - x|^{m+1}\right)(x)\right)^{\frac{1}{m+1}}\right)_{\mathbb{R}}$$

$$\left(L_N\left(|\cdot - x|^{m+1}\right)(x)\right)^{\frac{m}{m+1}}, \quad \forall\, N \in \mathbb{N}. \tag{54}$$

If $L_N\left(|\cdot - x|^{m+1}\right)(x) \to 0$, then $L_N(f)(x) \to f(x)$, as $N \to +\infty$.

Next we specialize for $L_N = P_N$, $\forall\, N \in \mathbb{N}$, where P_N is as in (34), see Section 5.1 for the full description. We give

Corollary 5.14. *All are as in Theorem 5.13 regarding f.*
Assume that $(SL)\int_{\mathbb{R}} |t|^{m+1} d\mu_N(t) > 0$, and $P_N(f) \in CB_+(\mathbb{R})$, $\forall\, N \in \mathbb{N}$. Then

$$\left|P_N(f)(x) - f(x)\right| = \left|(SL)\int_{\mathbb{R}} f(x+t)\, d\mu_N(t) - f(x)\right| \leq$$

$$\frac{(m+2)}{\Gamma(m+2)}\omega_1\left(D_x^m f, \left((SL)\int_{\mathbb{R}} |t|^{m+1} d\mu_N(t)\right)^{\frac{1}{m+1}}\right)_{\mathbb{R}}$$

$$\left((SL)\int_{\mathbb{R}} |t|^{m+1} d\mu_N(t)\right)^{\frac{m}{m+1}}, \quad \forall\, N \in \mathbb{N}. \tag{55}$$

If $(SL)\int_{\mathbb{R}} |t|^{m+1} d\mu_N(t) \to 0$, then $P_N(f)(x) \to f(x)$, as $N \to +\infty$.

Proof. By Theorem 5.13. Notice also that
$P_N\left(|\cdot - x|^{m+1}\right)(x) = (SL)\int_{\mathbb{R}} |t|^{m+1} d\mu_N(t)$, $\forall\, N \in \mathbb{N}$. □

We need

Definition 5.15. *(see [1], p. 24, and [2], p. 334) Let $x, x_0 \in \mathbb{R}$ be such that $x \geq x_0$, $\nu > 0$, $\nu \notin \mathbb{N}$, such that $p = [\nu]$, $[\cdot]$ the integral part, $\alpha = \nu - p$ $(0 < \alpha < 1)$.*
Let $f \in C^p(\mathbb{R})$ and define

$$\left(J_\nu^{x_0} f\right)(x) := \frac{1}{\Gamma(\nu)}\int_{x_0}^{x} (x-t)^{\nu-1} f(t)\, dt, \quad x_0 \leq x < +\infty. \tag{56}$$

the left generalized Riemann-Liouville fractional integral.

Let $x, x_0 \in \mathbb{R}$ be such that $x \leq x_0$, $\nu > 0$, $\nu \notin \mathbb{N}$, such that $p = [\nu]$, $\alpha = \nu - p$ $(0 < \alpha < 1)$.

Let $f \in C^p(\mathbb{R})$ and define

$$\left(J^{\nu}_{x_0-}f\right)(x) := \frac{1}{\Gamma(\nu)}\int_x^{x_0}(z-x)^{\nu-1}f(z)\,dz, \quad -\infty < x \leq x_0. \tag{57}$$

the right generalized Riemann-Liouville fractional integral.

We need

Definition 5.16. *(see also [6]) Let $x, x_0 \in \mathbb{R}$, $x \geq x_0$, $\nu > 0$, $\nu \notin \mathbb{N}$, $p = [\nu]$, $\alpha = \nu - p$.*

Let $f \in C^p_b(\mathbb{R})$, i.e. $f \in C^p(\mathbb{R})$ with $\left\|f^{(p)}\right\|_\infty < +\infty$, where $\|\cdot\|_\infty$ is the supremum norm.

Here $(J^{x_0}_{\nu}f)(x)$ is defined via (56) over $[x_0, +\infty)$.

We define the subspace $C^{\nu}_{x_0+}(\mathbb{R})$ of $C^p_b(\mathbb{R})$:

$$C^{\nu}_{x_0+}(\mathbb{R}) := \left\{f \in C^p_b(\mathbb{R}) : J^{x_0}_{1-\alpha}f^{(p)} \in C^1([x_0, +\infty))\right\}.$$

For $f \in C^{\nu}_{x_0+}(\mathbb{R})$, we define the left generalized ν-fractional derivative of f over $[x_0, +\infty)$ as

$$D^{\nu}_{x_0+}f = \left(J^{x_0}_{1-\alpha}f^{(p)}\right)'. \tag{58}$$

We need

Definition 5.17. *(see also [2], p. 345) Let $x, x_0 \in \mathbb{R}$, $x \leq x_0$, $\nu > 0$, $\nu \notin \mathbb{N}$, $p = [\nu]$, $\alpha = \nu - p$. Let $f \in C^p_b(\mathbb{R})$. Here $\left(J^{\nu}_{x_0-}f\right)(x)$ is defined via (57) over $(-\infty, x_0]$.*

We define the subspace of $C^{\nu}_{x_0-}(\mathbb{R})$ of $C^p_b(\mathbb{R})$:

$$C^{\nu}_{x_0-}(\mathbb{R}) := \left\{f \in C^p_b(\mathbb{R}) : \left(J^{1-\alpha}_{x_0-}f^{(p)}\right) \in C^1((-\infty, x_0])\right\}.$$

For $f \in C^{\nu}_{x_0-}(\mathbb{R})$, we define the right generalized ν-fractional derivative of f over $(-\infty, x_0]$ as

$$\overline{D}^{\nu}_{x_0-}f = (-1)^{p-1}\left(J^{1-\alpha}_{x_0-}f^{(p)}\right)'. \tag{59}$$

We make

Convention 5.18. *Let a real number $m > 1$, from now on we assume that $\overline{D}^m_{x_0-}f$ is either bounded or uniformly continuous function on $(-\infty, x_0]$, similarly from now on we assume that $D^m_{x_0+}f$ is either bounded or uniformly continuous function on $[x_0, +\infty)$.*

We use

Definition 5.19. *Let $\overline{D}^m_{x_0}f$ (real number $m > 1$) denote any of $\overline{D}^m_{x_0-}f$, $D^m_{x_0+}f$ and $\delta > 0$. We set*

$$\omega_1\left(\overline{D}^m_{x_0}f, \delta\right)_{\mathbb{R}} := \max\left\{\omega_1\left(\overline{D}^m_{x_0-}f, \delta\right)_{(-\infty, x_0]}, \omega_1\left(D^m_{x_0+}f, \delta\right)_{[x_0, +\infty)}\right\}, \tag{60}$$

where $x_0 \in \mathbb{R}$. Notice that $\omega_1\left(\overline{D}^m_{x_0}f, \delta\right)_{\mathbb{R}} < +\infty$.

We give

Theorem 5.20. *([3], p. 113) Let $m > 1$, $m \notin \mathbb{N}$, $p = [m]$, $x_0 \in \mathbb{R}$, and $f \in C^m_{x_0+}(\mathbb{R}) \cap C^m_{x_0-}(\mathbb{R})$. Assume that $f^{(k)}(x_0) = 0$, $k = 1, ..., p-1$, and $\left(D^m_{x_0+} f\right)(x_0) = \left(\overline{D}^m_{x_0-} f\right)(x_0) = 0$. The Convention 5.18 is imposed. Then*

$$\left|f(x) - f(x_0)\right| \leq \frac{\omega_1\left(\overline{D}^m_{x_0} f, \delta\right)_{\mathbb{R}}}{\Gamma(m+1)}\left[|x - x_0|^m + \frac{|x - x_0|^{m+1}}{(m+1)\delta}\right], \ \ \delta > 0, \tag{61}$$

for all $x \in \mathbb{R}$.

We give

Theorem 5.21. *Let $m > 1$, $m \notin \mathbb{N}$, $p = \lceil m \rceil$, $x \in \mathbb{R}$, $f : \mathbb{R} \to \mathbb{R}_+$ with $f \in C^m_{x+}(\mathbb{R}) \cap C^m_{x-}(\mathbb{R})$. Assume that $f^{(k)}(x) = 0$, $k = 1, ..., p-1$, and $(D^m_{x+} f)(x) = \left(\overline{D}^m_{x-} f\right)(x) = 0$. We assume that $\overline{D}^m_{x-} f$, $D^m_{x+} f$ are either bounded or uniformly continuous over $(-\infty, x]$, $[x, +\infty)$, respectively.*

Let L_N ($N \in \mathbb{N}$) be a sequence of positive sublinear operators from $C_+(\mathbb{R})$ into $CB_+(\mathbb{R})$. Assume that $L_N(1) = 1$, and $L_N\left(|\cdot - x|^{m+1}\right)(x) > 0$, $\forall\, N \in \mathbb{N}$. Then

$$\left|L_N(f)(x) - f(x)\right| \leq \frac{(m+2)}{\Gamma(m+2)}\omega_1\left(\overline{D}^m_x f, \left(L_N\left(|\cdot - x|^{m+1}\right)(x)\right)^{\frac{1}{m+1}}\right)_{\mathbb{R}}$$

$$\left(L_N\left(|\cdot - x|^{m+1}\right)(x)\right)^{\frac{m}{m+1}}, \ \ \forall\, N \in \mathbb{N}. \tag{62}$$

If $L_N\left(|\cdot - x|^{m+1}\right)(x) \to 0$, then $L_N(f)(x) \to f(x)$, as $N \to +\infty$.

Proof. Similar to the proof of Theorem 5.13, by using (61). □

We have

Corollary 5.22. *All are as in Theorem 5.21 regarding f, with $L_N = P_N$, $\forall\, N \in \mathbb{N}$. Assume that $(SL)\int_{\mathbb{R}} |t|^{m+1} d\mu_N(t) > 0$, and $P_N(f) \in CB_+(\mathbb{R})$, $\forall\, N \in \mathbb{N}$. Then*

$$\left|P_N(f)(x) - f(x)\right| = \left|(SL)\int_{\mathbb{R}} f(x+t)\, d\mu_N(t) - f(x)\right| \leq$$

$$\frac{(m+2)}{\Gamma(m+2)}\omega_1\left(\overline{D}^m_x f, \left((SL)\int_{\mathbb{R}} |t|^{m+1} d\mu_N(t)\right)^{\frac{1}{m+1}}\right)_{\mathbb{R}}$$

$$\left((SL)\int_{\mathbb{R}} |t|^{m+1} d\mu_N(t)\right)^{\frac{m}{m+1}}, \ \ \forall\, N \in \mathbb{N}. \tag{63}$$

If $(SL)\int_{\mathbb{R}} |t|^{m+1} d\mu_N(t) \to 0$, then $P_N(f)(x) \to f(x)$, as $N \to +\infty$.

Proof. By Theorem 5.21. □

5.3. Iterated Fractional Approximation

Here $D^{\alpha}_{*x_0}, D^{\alpha}_{x_0-}$ stand for the Caputo left and right fractional derivatives, see (43), (44). For $n \in \mathbb{N}$, denote the iterated left and right fractional derivatives as $D^{n\alpha}_{*x_0} = D^{\alpha}_{*x_0} D^{\alpha}_{*x_0} ... D^{\alpha}_{*x_0}$ and $D^{n\alpha}_{x_0-} = D^{\alpha}_{x_0-} D^{\alpha}_{x_0-} ... D^{\alpha}_{x_0-}$ (n-times).

We make

Remark 5.23. *Let $f:\mathbb{R}\to\mathbb{R}$ such that $f'\in L_\infty(\mathbb{R})$, $x_0\in\mathbb{R}$, $0<\alpha<1$. The left Caputo fractional derivative $\left(D^{\alpha}_{*x_0}f\right)(x)$ is given for $x\geq x_0$. Clearly it holds $\left(D^{\alpha}_{*x_0}f\right)(x_0)=0$, and we define $\left(D^{\alpha}_{*x_0}f\right)(x)=0$, for $x<x_0$.*

Let us assume that $D^{k\alpha}_{*x_0}f\in C([x_0,+\infty))$, $k=0,1,...,n+1$; $n\in\mathbb{N}$.

The right Caputo fractional derivative $\left(D^{\alpha}_{x_0-}f\right)(x)$ is given for $x\leq x_0$. Clearly it holds $\left(D^{\alpha}_{x_0-}f\right)(x_0)=0$, and define $\left(D^{\alpha}_{x_0-}f\right)(x)=0$, for $x>x_0$.

Let us assume that $D^{k\alpha}_{x_0-}f\in C((-\infty,x_0])$, $k=0,1,...,n+1$.

Here we restrict ourselves to $\frac{1}{n+1}<\alpha<1$, that is $\overline{\lambda}:=(n+1)\alpha>1$. We denote $D^{\overline{\lambda}}_{*x_0}f:=D^{(n+1)\alpha}_{*x_0}f$, and $D^{\overline{\lambda}}_{x_0-}f:=D^{(n+1)\alpha}_{x_0-}f$.

We make

Convention 5.24. *We assume that $D^{\overline{\lambda}}_{x_0-}f$ is either bounded or uniformly continuous function on $(-\infty,x_0]$, similarly we assume that $D^{\overline{\lambda}}_{*x_0}f$ is either bounded or uniformly continuous function on $[x_0,+\infty)$.*

We need

Definition 5.25. *Let $D^{\overline{\lambda}}_{x_0}f$ denote any of $D^{\overline{\lambda}}_{x_0-}f$, $D^{\overline{\lambda}}_{*x_0}f$ and $\delta>0$. We set*

$$\omega_1\left(D^{\overline{\lambda}}_{x_0}f,\delta\right)_{\mathbb{R}}:=\max\left\{\omega_1\left(D^{\overline{\lambda}}_{x_0-}f,\delta\right)_{(-\infty,x_0]},\omega_1\left(D^{\overline{\lambda}}_{*x_0}f,\delta\right)_{[x_0,+\infty)}\right\}, \tag{64}$$

where $x_0\in\mathbb{R}$. Notice that $\omega_1\left(D^{\overline{\lambda}}_{x_0}f,\delta\right)_{\mathbb{R}}<+\infty$.

We mention

Theorem 5.26. *([3], p. 137) Let $\frac{1}{n+1}<\alpha<1$, $n\in\mathbb{N}$, $\overline{\lambda}:=(n+1)\alpha>1$, $f:\mathbb{R}\to\mathbb{R}$, $f'\in L_\infty(\mathbb{R})$, $x_0\in\mathbb{R}$. Assume that $D^{k\alpha}_{*x_0}f\in C([x_0,+\infty))$, $k=0,1,...,n+1$, and $\left(D^{i\alpha}_{*x_0}f\right)(x_0)=0$, $i=2,3,...,n+1$. Suppose that $D^{k\alpha}_{x_0-}f\in C((-\infty,x_0])$, for $k=0,1,...,n+1$, and $\left(D^{i\alpha}_{x_0-}f\right)(x_0)=0$, for $i=2,3,...,n+1$. Convention 5.24 is imposed. Then*

$$\left|f(x)-f(x_0)\right|\leq\frac{\omega_1\left(D^{\overline{\lambda}}_{x_0}f,\delta\right)_{\mathbb{R}}}{\Gamma\left(\overline{\lambda}+1\right)}\left[|x-x_0|^{\overline{\lambda}}+\frac{|x-x_0|^{\overline{\lambda}+1}}{\left(\overline{\lambda}+1\right)\delta}\right], \tag{65}$$

$\forall\, x\in\mathbb{R}$, $\delta>0$.

We present

Theorem 5.27. *Let $\frac{1}{n+1}<\alpha<1$, $n\in\mathbb{N}$, $\overline{\lambda}:=(n+1)\alpha>1$, $f:\mathbb{R}\to\mathbb{R}_+$, $f'\in L_\infty(\mathbb{R})$, $x_0\in\mathbb{R}$. Assume that $D^{k\alpha}_{*x}f\in C([x,+\infty))$, $k=0,1,...,n+1$, and $\left(D^{i\alpha}_{*x}f\right)(x)=0$, $i=2,3,...,n+1$. Suppose that $D^{k\alpha}_{x-}f\in C((-\infty,x])$, for $k=0,1,...,n+1$, and $\left(D^{i\alpha}_{x-}f\right)(x)=0$, for $i=2,3,...,n+1$. We assume that $D^{\overline{\lambda}}_{x-}f$, $D^{\overline{\lambda}}_{*x}f$ are either bounded or uniformly continuous over $(-\infty,x]$, $[x,+\infty)$, respectively. Let L_N ($N\in\mathbb{N}$) be a sequence of positive sublinear operators from $C_+(\mathbb{R})$ into $CB_+(\mathbb{R})$. Assume that $L_N(1)=1$, and $L_N\left(|\cdot-x|^{\overline{\lambda}+1}\right)(x)>0$, $\forall\, N\in\mathbb{N}$. Then*

$$\left|L_N(f)(x)-f(x)\right|\leq\frac{\left(\overline{\lambda}+2\right)}{\Gamma\left(\overline{\lambda}+2\right)}\omega_1\left(D^{\overline{\lambda}}_{x}f,\left(L_N\left(|\cdot-x|^{\overline{\lambda}+1}\right)(x)\right)^{\frac{1}{\overline{\lambda}+1}}\right)_{\mathbb{R}}$$

$$\left(L_N\left(|\cdot-x|^{\overline{\lambda}+1}\right)(x)\right)^{\frac{\overline{\lambda}}{\overline{\lambda}+1}},\ \ \forall\, N\in\mathbb{N}. \tag{66}$$

If $L_N\left(|\cdot-x|^{\overline{\lambda}+1}\right)(x)\to 0$, then $L_N(f)(x)\to f(x)$, as $N\to+\infty$.

Proof. Similar to the proof of Theorem 5.13, by using (65). $\square$

We finish with

Corollary 5.28. *All are as in Theorem 5.27 regarding f and $L_N = P_N$, $\forall N \in \mathbb{N}$. Assume that $(SL)\int_{\mathbb{R}} |t|^{\overline{\lambda}+1} d\mu_N(t) > 0$, and $P_N(f) \in CB_+(\mathbb{R})$, $\forall N \in \mathbb{N}$. Then*

$$\left|P_N(f)(x) - f(x)\right| \leq \frac{\left(\overline{\lambda}+2\right)}{\Gamma\left(\overline{\lambda}+2\right)}\omega_1\left(D_x^{\overline{\lambda}} f, \left((SL)\int_{\mathbb{R}} |t|^{\overline{\lambda}+1} d\mu_N(t)\right)^{\frac{1}{\overline{\lambda}+1}}\right)_{\mathbb{R}}$$

$$\left((SL)\int_{\mathbb{R}} |t|^{\overline{\lambda}+1} d\mu_N(t)\right)^{\frac{\overline{\lambda}}{\overline{\lambda}+1}}, \quad \forall N \in \mathbb{N}. \tag{67}$$

If $(SL)\int_{\mathbb{R}} |t|^{\overline{\lambda}+1} d\mu_N(t) \to 0$, then $P_N(f)(x) \to f(x)$, as $N \to +\infty$.

Proof. By Theorem 5.27. $\square$

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