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Best Proximity Point Theorems for Proximal Multi-valued Contractions

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Abstract. In this paper, we introduce new classes of proximal multi-valued contractions in a metric space and proximal multi-valued nonexpansive mappings in a Banach space and show the existence of best proximity points for both classes. Further, for proximal multi-valued nonexpansive mappings, we prove a best proximity point theorem on starshape sets. As a consequence, we also obtain some new fixed point theorems. Finally, we give some examples to illustrate our main results.

1. Introduction

Fixed point theory is one of the most powerful and prolific tools of mathematics and it is an important part of nonlinear analysis which can be applied to many important problems such as optimization, image and signal processing, machine learning, engineering and economics. One of the most well known fixed point theorems for multi-valued contractions was first proved by Nadler [1] which states that every multi-valued contractive mapping from a complete metric space X into nonempty closed bounded subsets of X always has a fixed point.

However, the best proximity point problem is to consider the question of what happen when T is a non-self mapping.

Let A, B be nonempty disjoint subsets of a metric space (X, d) and $T : A \to 2^B$ be a multi-valued mapping, where 2^B is the family of all nonempty subsets of B. It is noted that the fixed point equation $x \in Tx$ has no any solution because $d(x, Tx) > D(A, B)$ for all $x \in A$, where $D(A, B) = \inf\{d(a, b) : a \in A, b \in B\}$. So, it natural to ask the following question:

$$\text{Find a point } x \in A \text{ such that } d(x, Tx) = D(A, B),$$

where such a point x is known as a *best proximity point* of T.

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The best proximity point problem for non-self nonlinear mappings is an interesting topic in optimization theory (see [2–5]) and best proximity point theorems can be applied to study equilibrium point in economics (see [6–9]). Thus this topic attracts attentions of many mathematicians.

The existence of best proximity points of single-valued mappings have been studied by many authors (see [10–18]). In 2011, the concept of the proximal contraction was first introduced by Basha [2]. Later, Gabeleh [19] introduced a new concept of proximal nonexpansive mappings and proved the existence of best proximity points of such mappings. In 2015, Chen [20] proved an interesting existence theorem of proximity points for proximal nonexpansive mappings under starshape sets A and B.

For multi-valued mappings, the existence of best proximity points was established by many authors (see, for instance, [21–27]). Recently, Sarnmeta [28] introduced a new concept of proximal multi-valued mappings and proved the existence of best proximity points for such mappings when A_0 is a nonempty weakly compact convex set.

In this paper, we show the existence of a best proximity point theorem for our new concept of the proximal multi-valued mapping, which is called the *proximal multivalued contraction* with respect A_0 and the *proximal multi-valued nonexpansive mapping* with respect A_0 under starshape sets A_0 and B_0. Our results extend and improve some results in fixed point theory and best proximity point theory given by some authors.

2. Preliminaries

Let (X, d) be a metric space and 2^X, $CB(X)$, $P(X)$ and $K(X)$ denote the families of nonempty subsets, nonempty closed bounded subsets, nonempty proximinal bounded subsets and nonempty compact subsets of X, respectively. For each $A, B \in CB(X)$ and $x \in X$, define

$$d(x, A) = \inf\{d(x, y) : y \in A\},$$

$$D(A, B) = \inf\{d(x, y) : x \in A, y \in B\},$$

$$H(A, B) = \max\left\{\sup_{x\in A} d(x, B), \sup_{y\in B} d(y, A)\right\}.$$

The mapping H is called *the Pompeiu-Hausdorff metric* or *Hausdorf metric* on $CB(X)$.

Let A and B be nonempty subsets of a metric space (X, d). Further, we denote by A_0 and B_0 the following sets:

$$A_0 = \{x \in A : d(x, y) = D(A, B) \text{ for some } y \in B\},$$

$$B_0 = \{y \in B : d(x, y) = D(A, B) \text{ for some } x \in A\}.$$

A nonempty subset A of a linear space X is called a *p-starshape set* if there exists a point p in A such that

$$\alpha p + (1 - \alpha)x \in A \quad \text{for all } x \in A \text{ and } \alpha \in [0, 1]$$

and p is called a *center* of A.

Notice that, in a normed space $(X, \|\cdot\|)$, if both of A and B are closed and A_0 is nonempty, then A_0 is a closed set. In a starshape set, if A is a p-starshape set, B is a q-starshaped set and $\|p - q\| = D(A, B)$, then A_0 is a p-starshape set and B_0 is a q-starshaped set (see [20]).

Definition 2.1. [30] Let (A, B) be a pair of nonempty subsets of a metric space (X, d). The pair (A, B) is said to be *a semi-sharp proximinal pair* if, for each $x \in A$, there exists at most one x^* in B such that

$$d(x, x^*) = D(A, B).$$

Definition 2.2. [31] Let (A, B) be a pair of nonempty subsets of a metric space (X, d) with $A \neq \emptyset$. Then the pair (A, B) is said to have the *weak P-property* if, for all $x_1, x_2 \in A_0$ and $y_1, y_2 \in B_0$,

$$\left.\begin{aligned} d(x_1, y_1) &= D(A, B) \\ d(x_2, y_2) &= D(A, B) \end{aligned}\right\} \Longrightarrow d(x_1, x_2) \leq d(y_1, y_2).$$

In Definition 2.2, if $d(x_1, x_2) = d(y_1, y_2)$, then (A, B) have the P-property (see [32]). It is clear that the weak P-property is weaker than the P-property and (A, B) has the P-property if and only if both (A, B) and (B, A) have the weak P-property. Moreover, if a pair (A, B) has the weak P-property, then (B, A) must be a semi-sharp proximinal pair. Obviously, a semi-sharp proximinal pair (A, B) is not necessarily to have the weak P-property.

3. The Proximal Multi-valued Contraction

In this section, we first introduce a new concept of contraction multi-valued mapping, called proximal multivalued contraction with respect A_0, and give an example of this type of mapping.

Definition 3.1. Let (A, B) be a pair of nonempty subsets of a metric space (X, d). A mapping $T : A \to 2^B$ is said to be *a proximal multi-valued contraction* with respect to A_0 if there exists $\alpha \in (0, 1)$ such that, for each $x_1, x_2 \in A_0$, two sets $U_{x_1} := \{y \in A_0 : d(y, Tx_1) = D(A, B)\}$ and $U_{x_2} := \{y \in A_0 : d(y, Tx_2) = D(A, B)\}$ are nonempty closed and bounded and

$$H(U_{x_1}, U_{x_2}) \le \alpha d(x_1, x_2).$$

Remark 1. In Definition 3.1, if $B = A$ and $T : A \to CB(A)$ is a multi-valued mapping, then $U_x = Tx$ for all $x \in A$. It follows that T is a multi-valued contraction.

The following example is an example of proximal multi-valued contraction with respect to A_0 which is not a proximal multi-valued contraction with respect to A:

Example 3.2. Let $X = \mathbb{R}^2$ with the usual norm,

$$A = \{(0, y) : y \in [-2, -1] \cup [0, 2]\},$$
$$B = ((1, \infty) \times (-3, 3)) \cup \{(1, y) : y \in [0, 1]\}$$

and $T : A \to 2^B$ be a multi-valued mapping defined by

$$T(0, y) = \begin{cases} [1, \infty) \times \left\{\frac{y}{2}\right\}, & \text{if } y \in [0, 2], \\ (1, \infty) \times \left[\frac{y}{2} - 1, \frac{y}{4} - 1\right], & \text{if } y \in [-2, -1]. \end{cases}$$

It is not hard to see that, $A_0 = \{(0, y) : y \in [0, 1]\}$ and $B_0 = \{(1, y) : y \in [0, 1]\}$.

Now, we can show that T is a proximal multi-valued contraction with respect to A_0. Let $x_1, x_2 \in \{0\} \times [0, 2]$. Then $U_{x_1} = \{(0, \frac{x_1}{2})\}$ and $U_{x_2} = \{(0, \frac{x_2}{2})\}$. It follows that

$$H(U_{x_1}, U_{x_2}) \le \frac{1}{2} d(x_1, x_2).$$

We note that, if $x_1 = (0, \frac{1}{2})$ and $x_2 = (0, -1)$, then

$$V_{x_1} = \{x \in A : d(x, Tx_1) = D(A, B)\} = \left\{\left(0, \frac{1}{4}\right)\right\}$$

and

$$V_{x_2} = \{x \in A : d(x, Tx_2) = D(A, B)\} = \left\{(0, y) : y \in \left[\frac{-3}{2}, \frac{-5}{4}\right]\right\}.$$

We know that $d(x_1, x_2) = \frac{3}{2} < \frac{7}{6} = H(V_{x_1}, V_{x_2})$. Therefore, T is not a proximal multi-valued contraction with respect to A.

To prove the our main results, we need the following lemmas:

Lemma 3.3. *Let (A, B) be a pair of nonempty subsets of a metric space (X, d) such that A_0 is nonempty. Suppose that $T : A \to 2^B$ is a mapping such that, for each $x \in A_0$, $Tx \cap B_0$ is nonempty. Then we have the following:*

(1) *for all $x \in A_0$, U_x is a nonempty set;*
(2) *if A_0 is closed and $x \in A_0$, then U_x is closed;*
(3) *for each $x \in A_0$, $Tx \cap B_0$ is bounded if and only if U_x is bounded.*

Proof. (1) Let $x \in A_0$. Since $Tx \cap B_0$ is nonempty, there exist $v \in Tx \cap B_0$ and $u \in A_0$ such that

$$d(u, v) = D(A, B).$$

It follows that $d(u, Tx) = D(A, B)$ and hence U_x is a nonempty set.

(2) To show that U_x is closed, let $\{y_n\}$ be a sequence in U_x such that $y_n \to y$. From $y_n \in A_0$ and $d(y_n, Tx) = D(A, B)$ for each $n \in \mathbb{N}$, we have $d(y_n, Tx) \to D(A, B)$ as $n \to \infty$. Therefore, $d(y, Tx) = D(A, B)$. Since A_0 is closed, we have $y \in A_0$, which implies $y \in U_x$ and so U_x is closed.

(3) To show U_x is bounded, we suppose that U_x is unbounded. So, for each $n \in \mathbb{N}$, there exist x_n, y_n in U_x such that $d(x_n, y_n) \geq n$. Since $x_n, y_n \in U_x$, there exist $x'_n, y'_n \in Tx \cap B_0$ such that $d(x_n, x'_n) \leq D(A, B) + \frac{1}{n}$ and $d(y_n, y'_n) \leq D(A, B) + \frac{1}{n}$ for each $n \in \mathbb{N}$. From

$$n \leq d(x_n, y_n) \leq d(x_n, x'_n) + (x'_n, y'_n) + d(y'_n, y_n),$$

we have

$$n - 2D(A, B) - \frac{2}{n} \leq d(x'_n, y'_n) \text{ for each } n \in \mathbb{N}.$$

Thus $Tx \cap B_0$ is unbounded, which is a contradiction. Therefore, U_x is bounded.

Conversely, we want to show that $Tx \cap B_0$ is bounded. Suppose that $Tx \cap B_0$ is unbounded. So, for each $n \in \mathbb{N}$, there exist $x_n, y_n \in Tx \cap B_0$ such that $d(x_n, y_n) \geq n$. Since $x_n, y_n \in Tx \cap B_0$, there exist $x'_n, y'_n \in U_x$ such that $d(x_n, x'_n) = D(A, B)$ and $d(y_n, y'_n) = D(A, B)$ for all $n \in \mathbb{N}$. From

$$n \leq d(x_n, y_n) \leq d(x_n, x'_n) + (x'_n, y'_n) + d(y'_n, y_n),$$

we have

$$n - 2D(A, B) \leq d(x'_n, y'_n) \text{ for each } n \in \mathbb{N}.$$

Therefore, U_x is unbounded, which is a contradiction. Hence $Tx \cap B_0$ is bounded. This completes the proof. □

First, we prove the existence of a best proximity point for the proximal multi-valued contraction mapping.

Theorem 3.4. *Let (X, d) be a complete metric space and (A, B) a pair of nonempty subsets of X such that A_0 nonempty and closed. Assume that $T : A \to 2^B$ satisfies the following conditions:*

(i) *T is an α-proximal multi-valued contraction with respect to A_0;*
(ii) *for each $x \in A_0$, $Tx \cap B_0$ is nonempty and bounded.*

Then there exists $x^* \in A_0$ such that $d(x^*, Tx^*) = D(A, B)$.

Proof. Let $x_0 \in A_0$. By Lemma 3.3 (1), we have U_{x_0} is a nonempty set. Let $x_1 \in U_{x_0}$. Then $x_1 \in A_0$ an so U_{x_1} is nonempty. By Lemma 3.3 (2), (3), U_x is closed and bounded for each $x \in A_0$. Choose $x_2 \in U_{x_1}$ such that

$$d(x_2, x_1) \leq H(U_{x_1}, U_{x_0}) + \alpha.$$

Continuing this process, we get a sequence $\{x_n\}$ in A_0 such that $d(x_{n+1}, Tx_n) = D(A, B)$ and

$$d(x_{n+1}, x_n) \leq H(U_{x_n}, U_{x_{n-1}}) + \alpha^n \text{ for all } n \in \mathbb{N}.$$

Next, we show that $\{x_n\}$ is a Cauchy sequence and its limit is a best proximity point of T. By the definition of T, we have

$$\begin{aligned} d(x_{n+1}, x_n) &\leq H(U_{x_n}, U_{x_{n-1}}) + \alpha^n \\ &\leq \alpha d(x_n, x_{n-1}) + \alpha^n \\ &\leq \alpha(H(U_{x_{n-1}}, U_{x_{n-2}}) + \alpha^{n-1}) + \alpha^n \\ &= \alpha H(U_{x_{n-1}}, U_{x_{n-2}}) + 2\alpha^n \\ &\ \ \cdots \\ &\leq \alpha^n d(x_1, x_0) + n\alpha^n. \end{aligned}$$

Since $\Sigma_{n=0}^{\infty}\alpha^n < \infty$ and $\Sigma_{n=0}^{\infty} n\alpha^n < \infty$, it follows that $\{x_n\}$ is a Cauchy sequence in A_0. Since A_0 is closed, there exists $x^* \in A_0$ such that $x_n \to x^*$ as $n \to \infty$. By Lemma 3.3 (1), it follows that U_{x^*} is nonempty. Thus there exists $x_n' \in U_{x^*}$ such that

$$d(x_{n+1}, x_n') \leq H(U_{x_n}, U_{x^*}) + \frac{1}{n} \leq \alpha d(x_n, x^*) + \frac{1}{n},$$

which implies that

$$\lim_{n\to\infty} d(x_{n+1}, x_n') = 0.$$

Therefore, $x_n' \to x^*$. Since U_{x^*} is closed, it follows that $x^* \in U_{x^*}$, that is, $d(x^*, Tx^*) = D(A, B)$. This completes the proof. □

Obviously, If T is a mapping from A to $CB(A)$ then we have a fixed point theorem which is directly derived from Theorem 3.4.

Corollary 3.5. *Let A be a nonempty closed subset of a complete metric space (X, d) and $T : A \to CB(A)$ be a multi-valued contraction. Then T has a fixed point.*

4. The Proximal Multi-valued Nonexpansive Mapping

In this section, we introduce a proximal multi-valued nonexpansive mapping with respect to A_0 and prove the existence of best proximity points for such mapping on starshape sets in Banach spaces.

Definition 4.1. Let (A, B) be a pair of nonempty subsets of a normed space X. A mapping $T : A \to 2^B$ is said to be *proximal multi-valued nonexpansive* with respect to A_0. If, for each $x_1, x_2 \in A_0$, two sets $U_{x_1} := \{y \in A_0 : d(y, Tx_1) = D(A, B)\}$ and $U_{x_2} := \{y \in A_0 : d(y, Tx_2) = D(A, B)\}$ are nonempty closed and bounded and

$$H(U_{x_1}, U_{x_2}) \leq \|x_1 - x_2\|.$$

Remark 2. In Definition 4.1, If $B = A$ and $T : A \to CB(A)$ is a multi-valued mapping, then $U_x = Tx$ for all $x \in A$. So, T is a multi-valued nonexpansive mapping.

Theorem 4.2. *Let X be a Banach space, (A, B) be a pair of nonempty subsets of X such that A_0 is a $p-$starshaped set, B_0 is a q-starshaped set with $\|p - q\| = D(A, B)$. Assume that A_0 is a compact set and (B_0, A_0) is a semi-sharp proximinal pair. Suppose that a multi-valued mapping $T : A \to P(B)$ satisfies the following conditions:*

(i) *T is proximal multivalued nonexpansive with respect to A_0;*
(ii) *for each $x \in A_0$, $Tx \cap B_0$ is nonempty.*

Then there exists x^* in A_0 such that $d(x^*, Tx^*) = D(A, B)$.

Proof. For each $n \in \mathbb{N}$, define $T_n : A_0 \to P(B)$ by

$$T_n x = a_n q + (1 - a_n)Tx \text{ for each } x \in A_0,$$

where $\{a_n\}$ is a sequence in $(0, 1)$ such that $\lim_{n\to\infty} a_n = 0$. Let $x \in A_0$. From (ii), there exist $x' \in Tx$ and $x' \in B_0$. Then there exists $w \in A_0$ such that

$$\|w - x'\| = D(A, B).$$

Set $w_n^* = a_n p + (1 - a_n)w$ and $x_n^* = a_n q + (1 - a_n)x'$. Then $x_n^* \in T_n x$ and

$$\begin{aligned}\|w_n^* - x_n^*\| &= \|a_n p + (1 - a_n)w - a_n q - (1 - a_n)x'\| \\ &\le a_n\|p - q\| + (1 - a_n)\|w - x'\| \\ &= D(A, B),\end{aligned}$$

which implies $x_n^* \in B_0$. Therefore, $T_n x \cap B_0$ is nonempty. Now, define

$$U_x^n = \{w \in A_0 : d(w, T_n x) = D(A, B)\}.$$

Now, we show that, for each $x \in A_0$,

$$U_x^n = a_n p + (1 - a_n)U_x \text{ for each } n \in \mathbb{N}.$$

To show this, let $n \in \mathbb{N}$ be fixed and $w \in U_x^n$. Since $T_n x$ is a proximinal subset of B, there exist y in $T_n x$ such that

$$\|w - y\| = d(w, T_n x) = D(A, B). \tag{1}$$

Hence $y \in B_0$. Since $x' \in B_0$, there exist $w' \in A_0$ such that

$$\|w' - x'\| = d(w', Tx) = D(A, B).$$

So, $w' \in U_x$. Set $v = a_n q + (1 - a_n)w'$, we obtain

$$\begin{aligned}\|v - y\| &= \|a_n p + (1 - a_n)w' - a_n q - (1 - a_n)x'\| \\ &\le a_n\|p - q\| + (1 - a_n)\|w' - x'\| \\ &= D(A, B).\end{aligned} \tag{2}$$

Since (B, A) is a semi-sharp proximinal pair, it follows from (1) and (2) that

$$w = v = a_n p + (1 - a_n)w'.$$

So, $w_n \in a_n p + (1 - a_n)U_x$ and hence $U_x^n \subseteq a_n p + (1 - a_n)U_x$.

Now, let $y_n \in a_n p + (1 - a_n)U_x$. Then we have $y_n = a_n p + (1 - a_n)y'$ for some $y' \in U_x$. Since $y' \in U_x$ and Tx is proximinal, there exist $x' \in Tx$ such that

$$\|y' - x'\| = d(y', Tx) = D(A, B).$$

Set $x_n^* = a_n q + (1 - a_n)x' \in T_n x$, we have

$$\begin{aligned}d(y_n, T_n x) &\le \|y_n - x_n^*\| \\ &= \|a_n p + (1 - a_n)y' - (a_n q + (1 - a_n)x')\| \\ &\le a_n\|p - q\| + (1 - a_n)\|y' - x'\| \\ &= D(A, B),\end{aligned}$$

which implies $d(y_n, T_n x) = D(A, B)$. So, $y_n \in U_x^n$ and hence

$$a_n p + (1 - a_n)U_x \subseteq U_x^n.$$

Therefore, $U_x^n = a_n p + (1 - a_n)U_x$ for each $n \in \mathbb{N}$. By the compactness of A_0, it follows that U_x^n is also compact. From Lemma 3.3 (3), $T_n x \cap B_0$ is bounded. Let $x_1, x_2 \in A_0$. Since T is proximal multi-valued nonexpansive with respect A_0, we have

$$H(U_{x_1}^n, U_{x_2}^n) = (1 - a_n)H(U_{x_1}, U_{x_2}) \leq (1 - a_n)\|x_1 - x_2\|.$$

Hence T_n is proximal multi-valued contraction with respect A_0. By Theorem 3.4, there exist $x_n^* \in A_0$ such that

$$d(x_n^*, T_n x_n^*) = D(A, B).$$

Since A_0 is compact, without loss of generality, we assume that there exist $x^* \in A_0$ such that $x_n^* \to x^*$ as $n \to \infty$. Since $x_n^* \in U_{x_n^*}^n$, we have

$$x_n^* = a_n p + (1 - a_n)x_n' \text{ for some } x_n' \in U_{x_n^*},$$

which implies

$$\|x_n^* - x_n'\| = a_n\|x_n' - p\| \to 0 \text{ as } n \to \infty$$

since $a_n \to 0$. Thus it follows that $\lim_{n\to\infty} x_n' = x^*$. By Lemma 3.3 (1), U_{x^*} is nonempty. Then there exist $u_n \in U_{x^*}$ such that

$$\|x_n' - u_n\| = d(x_n', U_{x^*}) \leq H(U_{x_n^*}, U_{x^*}) \leq \|x_n^* - x^*\| \to 0 \text{ as } n \to \infty.$$

It follows that $u_n \to x^*$. Since U_{x^*} is closed, $x^* \in U_{x^*}$. Therefore, $d(x^*, Tx^*) = D(A, B)$. This completes the proof. □

It is clear that, if a pair (A, B) has the weak P-property, then (B, A) is a semi-sharp proximinal pair. So, we have the following result:

Corollary 4.3. *Let X be a Banach space, (A, B) be a pair of nonempty subsets of X such that A_0 is a p-starshaped set, B_0 is a q-starshaped set and $\|p - q\| = D(A, B)$. Assume that A_0 is a compact set and (A, B) has the weak P-property. Suppose that a multi-valued mapping $T : A \to P(B)$ satisfies the following conditions:*

(i) *T is proximal multi-valued nonexpansive with respect to A_0;*
(ii) *for each $x \in A_0$, $Tx \cap B_0$ is nonempty.*

Then there exists x^* in A_0 such that $d(x^*, Tx^*) = D(A, B)$.

The following result is a fixed point theorem which is directly obtained by Theorem 4.2:

Corollary 4.4. *Let A be a nonempty p-starshaped compact subset of a Banach space X and $T : A \to P(A)$ be a multi-valued nonexpansive mapping. Then T has a fixed point.*

The following example illustrates the preceding theorem:

Example 4.5. Let $X = \mathbb{R}^3$ with the norm $\|(x, y, z)\| = |x| + |y| + |z|$,

$$A = \{(x, 0, 0) : x \in [0, 2]\} \cup \{(0, y, 0) : y \in [-1, 1]\},$$
$$B_1 = \left\{(x, y, 2) : x \in \left[0, \frac{3}{2}\right], y \in [-2, 2]\right\},$$
$$B_2 = \{(x, y, z) : z - x = 2, x \in [-2, 0], y \in [-, 1, 1]\},$$
$$B = B_1 \cup B_2$$

and $T : A \to P(B)$ be a multi-valued mapping defined by

$$T(x,y,z) = \begin{cases} [\frac{2x}{3}, \frac{3x}{4}] \times [-\frac{x^2}{2}, \frac{x^2}{2}] \times \{0\}, & \text{if } y = 0, \\ \Big\{(x',y',z') \in B_2 : x' \in [-|y|-1, |y|], y' \in \Big[-\frac{y^2}{2}, \frac{y^2}{2}\Big]\Big\}, & \text{if } y \neq 0. \end{cases}$$

We see that the following properties are satisfied:

(1) A is a $(0,0,0)$-starshape set, and B is a $(0,y,2)$-starshaped set, where $y \in [-1,1]$;
(2) $A_0 = \Big\{(x,0,0) : x \in \Big[0, \frac{3}{2}\Big]\Big\} \cup \{(0,y,0) : y \in [-1,1]\}$ is a compact set;
(3) $B_0 = \Big\{(x,0,0) : x \in \Big[0,0,\frac{3}{2}\Big]\Big\} \cup B_2$;
(4) for each $(x,y,z) \in A_0$, $T(x,y,z) \cap B_0$ is nonempty;
(5) (B_0, A_0) is a semi-sharp proximinal pair, (A_0, B_0) is not a semi-sharp proximinal pair from

$$\|(0,0,0)-(-1,0,-1)\| = D(A,B) = \|(0,0,0)-(0,0,2)\|, \text{ but } (-1,0,-1) \neq (0,0,2);$$

(6) $(0,0,0)$ is a best proximity point of T.

It can be shown that T is a proximal multi-valued nonexpansive mapping with respect to A_0. It is easy to see that, if $x_1 = (2,0,0)$ and $x_2 = (0,0,0)$, then

$$Tx_1 = \{(0,0,0)\}, \quad Tx_2 = \Big[\frac{4}{3}, \frac{3}{2}\Big] \times [-2,2] \times \{0\}$$

and

$$\|x_1 - x_2\| = 2, \quad \|Tx_1 - Tx_2\| = \frac{7}{2}.$$

So, T is not a non-self nonexpansive mapping.

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