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On the Kesten-Type Inequality for Randomly Weighted Sums With Applications to an Operational Risk Model

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Abstract. This paper considers the randomly weighted sums generated by some dependent subexponential primary random variables and some arbitrarily dependent random weights. To study the randomly weighted sums with infinitely many terms, we establish a Kesten-type upper bound for their tail probabilities in presence of subexponential primary random variables and under a certain dependence among them. Our result extends the study of Chen [5] to the dependent case. As applications, we derive some asymptotic formulas for the tail probability and the Value-at-Risk of total aggregate loss in a multivariate operational risk cell model.

1. Introduction

Throughout this paper, let $\{X_k, k \in \mathbb{N}\}$ be a sequence of identically distributed real-valued random variables (r.v.s) with common distribution F, called primary r.v.s, and let $\{\theta_k, k \in \mathbb{N}\}$ be another sequence of nonnegative, arbitrarily dependent and uniformly bounded above r.v.s, called random weights. As usual, $\{X_k, k \in \mathbb{N}\}$ and $\{\theta_k, k \in \mathbb{N}\}$ are assumed to be mutually independent. In this study we are interested in the randomly weighted sums

$$S_n^\theta = \sum_{k=1}^{n} \theta_k X_k, \qquad n \in \mathbb{N}. \tag{1}$$

Randomly weighted sums in (1) have been an attractive research topic in various areas particularly in insurance and finance, since they have been widely used in many financial products, such as bond price, insurance premium, stochastic present value of investment portfolio, ruin probability, and among many others. After the pioneering work of Tang and Tsitsiashvili [19], more and more research attention has focused on the tail behavior of randomly weighted sums of heavy-tailed primary r.v.s, most of which is related to the case of finitely many terms. Under various assumptions, the asymptotic formula

$$P(S_n^\theta > x) \sim \sum_{k=1}^{n} P(\theta_k X_k > x) \tag{2}$$

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is established for every fixed $n \in \mathbb{N}$, where the symbol $\sim$ means that the quotient of both sides tends to 1 as $x \to \infty$. Tang and Tsitsiashvili [20] firstly considered the case of independent and identically distributed (i.i.d.) subexponential primary r.v.s $\{X_k, k \in \mathbb{N}\}$ but required the random weights $\{\theta_k, k \in \mathbb{N}\}$ to be bounded away from both 0 and ∞, i.e. $\theta_k \in [a, b]$ for some $0 < a \le b < \infty$ and all $k \in \mathbb{N}$. Under these conditions, they established the asymptotic formula (2). Tang and Yuan [21] further relaxed the restriction on $\{\theta_k, k \in \mathbb{N}\}$ by only requiring that they are nonnegative, non-degenerate at zero, and bounded above, i.e. $0 \le \theta_k \le b$ for some $0 < b < \infty$ and all $k \in \mathbb{N}$. Under the dependence structure, two similar results with subexponential primary r.v.s were derived by Wang [23] and Yang et al. [25]. The former considered the case of dependent primary r.v.s but the random weights being bounded from both sides as those in [20]; whereas the latter dealt with the case allowing some certain dependence between each pair of the primary r.v. and the corresponding bounded above random weight. More results on the asymptotic tail behavior of S_n^θ can be found in [22], [31], [13], [30], [16], [8], [9], [14], [7], [29], [27], [24], and among many others. In the aspect of randomly weighted sums with infinitely many terms, almost all literature is restricted to some extremely heavy-tailed primary r.v.s, to our knowledge, such as the regularly varying tailed ones in [6], the extended regularly varying tailed ones in [31], the consistently varying tailed ones in [13].

To solve some problems concerning infinitely many terms and subexponential primary r.v.s, Chen [5] recently obtained a Kesten-type upper bound for the tail probability of randomly weighted sums in (1) under the independence structure. However, due to the increasing complexity of insurance and financial products, the independence assumption is not practical and modelling the dependence has become imperative. Motivated by Chen [5], in this paper we aim to establish a Kesten-type inequality for the tail probability of S_n^θ by allowing a certain dependence among the primary r.v.s. As applications, we can utilize such an inequality to investigate some asymptotics for the tail probability and the Value-at-Risk (VaR) of total aggregate loss in a multivariate operational risk cell model.

The rest of this paper is organized as follows. Section 2 states the main result of this paper, Section 3 presents its proof after preparing a series of lemmas, and Section 4 gives some applications in a multivariate operational risk cell model.

2. Main result

Throughout the paper, all limit relationships are according to $x \to \infty$ unless otherwise stated. For two positive functions $g_1(\cdot)$ and $g_2(\cdot)$, we write $g_1(x) \lesssim g_2(x)$ or $g_2(x) \gtrsim g_1(x)$ if $\limsup g_1(x)/g_2(x) \le 1$, write $g_1(x) \sim g_2(x)$ if $\lim g_1(x)/g_2(x) = 1$, and write $g_1(x) = o(g_2(x))$ if $\lim g_1(x)/g_2(x) = 0$. Moreover, for two positive bivariate functions $g_1(\cdot,\cdot)$ and $g_2(\cdot,\cdot)$, we write $g_1(x,t) \sim g_2(x,t)$ uniformly for all t in a nonempty set A, if

$$\lim_{x\to\infty} \sup_{t\in A} \left| \frac{g_1(x,t)}{g_2(x,t)} - 1 \right| = 0.$$

For a non-decreasing function $g : \mathbb{R} \mapsto \mathbb{R}$, denote by $g^{\leftarrow}$ the general inverse, that is, for $y \in \mathbb{R}$, $g^{\leftarrow}(y) = \inf\{x \in \mathbb{R} : g(x) \ge y\}$, where $\inf \emptyset = \infty$ by convention. For any $x \in \mathbb{R}$ and any set A, denote by $x^+ = \max\{x, 0\}$ and by 1_A the indicator function of A.

A distribution V on $\mathbb{R}_+ = [0, \infty)$ is said to be subexponential, written as $V \in \mathcal{S}$, if $\overline{V}(x) = 1 - V(x) > 0$ for all $x \ge 0$ and $\overline{V^{*n}}(x) \sim n\overline{V}(x)$ holds for all (or, equivalently, for some) $n \ge 2$, where V^{*n} is the n-fold convolution of V. More generally, a distribution V on $\mathbb{R}$ is still said to be subexponential if the distribution $V(x)1_{\{x\ge 0\}}$ is subexponential. By Lemma 1.3.5(a) of [11], if a distribution V on $\mathbb{R}$ is subexponential, then it holds that

$$\overline{V}(x+y) \sim \overline{V}(x), \tag{3}$$

for any fixed $y \in \mathbb{R}$, which defines the class of long-tailed distributions, denoted by $\mathcal{L}$. Automatically, relation (3) holds uniformly on every compact set of y. Hence, it is easy to see that there exists some positive function $h(\cdot)$, with $h(x) = o(x)$ and $h(x) \uparrow \infty$, such that relation (3) holds uniformly for all $|y| \le h(x)$. One of the most useful subclass of subexponential distributions is that of regularly varying tailed distributions.

Recall that a positive measurable function h on $\mathbb{R}_+$ is said to be regularly varying at ∞ with index $\alpha \in \mathbb{R}$, written as $h \in \mathrm{RV}_\alpha$, if $h(xy) \sim y^\alpha h(x)$ for any $y > 0$. Closely related is the rapid variation. A positive measurable function h on $\mathbb{R}_+$ is said to be rapidly varying, denoted by $h \in \mathrm{RV}_{-\infty}$, if $h(xy) = o(h(x))$ for any $y > 1$, In particular, a distribution V on $\mathbb{R}$ is said to be regularly (or rapidly) varying tailed if its tail distribution $\overline{V} \in \mathrm{RV}_{-\alpha}$ for some $\alpha > 0$ (or $\overline{V} \in \mathrm{RV}_{-\infty}$), denoted also by $V \in \mathcal{R}_{-\alpha}$ (or $V \in \mathcal{R}_{-\infty}$). The reader is referred to [11] and [12] for reviews of subexponential and regularly (rapidly) varying tailed distributions with applications to insurance and finance.

Under the independence structure, Chen [5] established a Kesten-type upper bound for the tail probability of randomly weighted infinite sums with subexponential primary r.v.s.

Theorem A *Let $\{X_k, k \in \mathbb{N}\}$ be a sequence of i.i.d. real-valued r.v.s with common distribution $F \in \mathcal{S}$, and $\{\theta_k, k \in \mathbb{N}\}$ be another sequence of nonnegative and uniformly bounded above r.v.s independent of $\{X_k, k \in \mathbb{N}\}$. Then, for any $\varepsilon > 0$, there exists a positive constant C_ε such that*

$$P\left(S_n^\theta > x\right) \leq C_\varepsilon (1+\varepsilon)^n \sum_{k=1}^n P(\theta_k X_k > x) \tag{4}$$

holds for all $n \in \mathbb{N}$ and all $x \geq 0$.

Our main result extends Theorem A to the following dependence assumption, which is proposed by Ko and Tang [17], see also Yang et al. [26].

Assumption 2.1. *There exist two positive constants M and large x_0 such that*

$$\sup_{x \geq x_0} \sup_{n \geq 1} \sup_{y \in [x_0, x]} \frac{P\left(\sum_{k=1}^n X_k > x - y | X_{n+1} = y\right)}{P\left(\sum_{k=1}^n X_k > x - y\right)} \leq M.$$

This assumption can be satisfied by most of negative dependence structures but the extremely positive dependence structures are excluded. Thus, the dependence structure above allows both positive and negative dependence among an infinite number of r.v.s to a certain extent.

Now we are ready to state our main result.

Theorem 2.1. *Let $\{X_k, k \in \mathbb{N}\}$ be a sequence of identically distributed real-valued r.v.s with common distribution $F \in \mathcal{S}$ and satisfying Assumption 2.1, and $\{\theta_k, k \in \mathbb{N}\}$ be another sequence of nonnegative and uniformly bounded above r.v.s independent of $\{X_k, k \in \mathbb{N}\}$. Then, for any $\varepsilon > 0$, there exists a positive constant C_ε such that (4) holds for all $n \in \mathbb{N}$ and all $x \geq 0$.*

3. Proof of main result

Before proving our main result, we firstly cite a series of lemmas. The first one gives a Kesten-type upper bound for the (non-weighted) sums of subexponential primary r.v.s, which is derived from [26].

Lemma 3.1. *Let $\{X_k, k \in \mathbb{N}\}$ be a sequence of identically distributed real-valued r.v.s with common distribution $F \in \mathcal{S}$ and satisfying Assumption 2.1. Then, for any $\varepsilon > 0$, there exists a positive constant C_ε such that*

$$P\left(\sum_{k=1}^n X_k > x\right) \leq C_\varepsilon (1+\varepsilon)^n \overline{F}(x)$$

holds for all $n \in \mathbb{N}$ and all $x \geq 0$.

The second lemma can also be found in [28]. For the sake of self-containedness, we present its proof below.

Lemma 3.2. *Let $X_1^{\perp}, \ldots, X_n^{\perp}$ be n i.i.d. nonnegative r.v.s with common distribution $F \in \mathcal{S}$. Then, for any function $h(x) \uparrow \infty$, any $k = 1, \ldots, n$, and $0 < a \le b < \infty$, it holds that uniformly for all $c_i \in [a, b], i = 1, \ldots, n$,*

$$P\left(\sum_{i=1}^{n} c_i X_i^{\perp} > x, h(x) < c_k X_k^{\perp} \le x\right) = o(1) \sum_{i=1}^{n} P(c_i X_i^{\perp} > x). \tag{5}$$

Proof. Clearly, the probability on the left-hand side of (5) is no more than

$$\begin{aligned}
& \int_0^x P\left(x - y < \sum_{i=1, i\neq k}^{n} c_i X_i^{\perp} \le x\right) P\left(c_k X_k^{\perp} \in dy\right) \\
& + P\left(\sum_{i=1, i\neq k}^{n} c_i X_i^{\perp} > x\right) P\left(c_k X_k^{\perp} > h(x)\right) \\
= \; & P\left(\sum_{i=1}^{n} c_i X_i^{\perp} > x\right) - \left(P\left(\sum_{i=1, i\neq k}^{n} c_i X_i^{\perp} > x\right) P\left(c_k X_k^{\perp} \le x\right) + P\left(c_k X_k^{\perp} > x\right)\right) \\
& + P\left(\sum_{i=1, i\neq k}^{n} c_i X_i^{\perp} > x\right) P\left(c_k X_k^{\perp} > h(x)\right) \\
=: \; & I_1 - I_2 + I_3.
\end{aligned}$$

By Lemma 1 of [21], we have that uniformly for all $c_i \in [a, b],\ i = 1, \ldots, n$,

$$I_1 \sim \sum_{i=1}^{n} P(c_i X_i^{\perp} > x).$$

Again by Lemma 1 of [21], uniformly for all $c_i \in [a, b],\ i = 1, \ldots, n$,

$$\begin{aligned}
I_3 & \lesssim \sum_{i=1}^{n} P(c_i X_i^{\perp} > x) \cdot \overline{F}\left(\frac{h(x)}{b}\right) \\
& = o(1) \sum_{i=1}^{n} P(c_i X_i^{\perp} > x).
\end{aligned}$$

Similarly, uniformly for all $c_i \in [a, b],\ i = 1, \ldots, n$,

$$I_2 \sim \sum_{i=1}^{n} P(c_i X_i^{\perp} > x).$$

Therefore, relation (5) follows from the above estimates. □

We now establish a uniform Kesten-type inequality for the tail probability of deterministically weighted sums, which plays an important role in the proof of Theorem 2.1.

Lemma 3.3. *Let $\{X_k, k \in \mathbb{N}\}$ be a sequence of identically distributed real-valued r.v.s with common distribution $F \in \mathcal{S}$ and satisfy Assumption 2.1. Then, for $0 < a \le b < \infty$ and for any $\varepsilon > 0$, there exists a positive constant C_ε such that*

$$P\left(\sum_{k=1}^{n} c_k X_k > x\right) \le C_\varepsilon (1 + \varepsilon)^n \sum_{k=1}^{n} P(c_k X_k > x) \tag{6}$$

holds for all $c_k \in [a, b], k = 1, \ldots, n$, all $n \in \mathbb{N}$ and all $x \ge 0$.

Proof. For each $n \in \mathbb{N}$, denote by $\mathbf{c_n} = (c_1, \ldots, c_n) \in [a,b]^n$ and $c_{(n)} = \max\{c_1, \ldots, c_n\}$. Due to the fact that $P\left(\sum_{k=1}^n c_k X_k > x\right) \leq P\left(\sum_{k=1}^n c_k X_k^+ > x\right)$ for all $x \geq 0$, without loss of generality we can assume that $\{X_k, k \in \mathbb{N}\}$ are nonnegative. Write

$$\alpha_n = \sup_{\mathbf{c_n} \in [a,b]^n} \sup_{x \geq 0} \frac{P\left(\sum_{k=1}^n c_k X_k > x\right)}{\sum_{k=1}^n P(c_k X_k > x)}. \tag{7}$$

We can assume $\alpha_n \geq 1$. Let $h(\cdot)$ be a function such that $h(x) = o(x)$, $h(x) \uparrow \infty$ and relation (3) holds uniformly for all $|y| \leq \frac{h(x)}{a}$ because of $F \in \mathcal{S} \subset \mathcal{L}$. Consider the tail probability of the weighted sum $\sum_{k=1}^{n+1} c_k X_k$ and split it as

$$\begin{aligned} P\left(\sum_{k=1}^{n+1} c_k X_k > x\right) &= P\left(\sum_{k=1}^{n+1} c_k X_k > x, c_{n+1} X_{n+1} \leq h(x)\right) \\ &\quad + P\left(\sum_{k=1}^{n+1} c_k X_k > x, h(x) < c_{n+1} X_{n+1} \leq x - h(x)\right) \\ &\quad + P\left(\sum_{k=1}^{n+1} c_k X_k > x, c_{n+1} X_{n+1} > x - h(x)\right) \\ &=: I_1 + I_2 + I_3. \end{aligned} \tag{8}$$

We start with I_1 and I_3. By $F \in \mathcal{S} \subset \mathcal{L}$ and (7), for any $\varepsilon > 0$, there exists some sufficiently large x_1 such that for all $x \geq x_1$ and uniformly for $\mathbf{c_{n+1}} \in [a,b]^{n+1}$,

$$\begin{aligned} I_1 &\leq P\left(\sum_{k=1}^{n} c_k X_k > x - h(x)\right) \\ &\leq \alpha_n \sum_{k=1}^{n} P(c_k X_k > x - h(x)) \\ &\leq \left(1 + \frac{\varepsilon}{2}\right) \alpha_n \sum_{k=1}^{n} P(c_k X_k > x), \end{aligned} \tag{9}$$

and

$$\begin{aligned} I_3 &\leq P(c_{n+1} X_{n+1} > x - h(x)) \\ &\leq \left(1 + \frac{\varepsilon}{2}\right) P(c_{n+1} X_{n+1} > x) \\ &\leq \left(1 + \frac{\varepsilon}{2}\right) \alpha_n P(c_{n+1} X_{n+1} > x). \end{aligned} \tag{10}$$

We next deal with I_2. Conditioning on X_{n+1}, there exists some sufficiently large $x_2 \geq x_1$ with $\frac{h(x_2)}{b} \geq x_0$, such that for all $x \geq x_2$ and all $\mathbf{c_{n+1}} \in [a,b]^{n+1}$,

$$\begin{aligned} I_2 &= \int_{\frac{h(x)}{c_{n+1}}}^{\frac{x-h(x)}{c_{n+1}}} P\left(\sum_{k=1}^{n} c_k X_k > x - c_{n+1} y \,\middle|\, X_{n+1} = y\right) P(X_{n+1} \in dy) \\ &\leq \int_{\frac{h(x)}{c_{n+1}}}^{\frac{x-h(x)}{c_{n+1}}} P\left(\sum_{k=1}^{n} X_k > \frac{x - c_{n+1} y}{c_{(n)}} \,\middle|\, X_{n+1} = y\right) P(X_{n+1} \in dy) \\ &\leq M \int_{\frac{h(x)}{c_{n+1}}}^{\frac{x-h(x)}{c_{n+1}}} P\left(\sum_{k=1}^{n} X_k > \frac{x - c_{n+1} y}{c_{(n)}}\right) P(X_{n+1} \in dy), \end{aligned}$$

where in the last step we used Assumption 2.1. By Lemma 3.1, there exists some positive constant $\tilde{C}_\varepsilon$ such that for all $x \geq 0$ and all $\mathbf{c_{n+1}} \in [a,b]^{n+1}$,

$$\begin{aligned} I_2 &\leq \tilde{C}_\varepsilon \left(1+\frac{\varepsilon}{2}\right)^n \int_{\frac{h(x)}{c_{n+1}}}^{\frac{x-h(x)}{c_{n+1}}} P\left(X_1 > \frac{x - c_{n+1}y}{c_{(n)}}\right) P(X_{n+1} \in dy) \\ &= \tilde{C}_\varepsilon \left(1+\frac{\varepsilon}{2}\right)^n P\left(c_{(n)}X_1^\perp + c_{n+1}X_{n+1}^\perp > x, h(x) < c_{n+1}X_{n+1}^\perp \leq x - h(x)\right), \end{aligned}$$

where $X_1^\perp$ and $X_{n+1}^\perp$ are two i.i.d. r.v.s with common distribution F. Then, by Lemma 3.2, there exists some large $x_3 \geq x_2$, irrespective of n, such that for all $x \geq x_3$ and all $\mathbf{c_{n+1}} \in [a,b]^{n+1}$,

$$\begin{aligned} I_2 &\leq \varepsilon\tilde{C}_\varepsilon \left(1+\frac{\varepsilon}{2}\right)^n \left(P(c_{(n)}X_1 > x) + P(c_{n+1}X_{n+1} > x)\right) \\ &\leq \varepsilon\tilde{C}_\varepsilon \left(1+\frac{\varepsilon}{2}\right)^n \sum_{k=1}^{n+1} P(c_k X_k > x). \end{aligned} \tag{11}$$

Plugging (9), (10) and (11) into (8) yields that for all $x \geq x_3$ and all $\mathbf{c_{n+1}} \in [a,b]^{n+1}$,

$$P\left(\sum_{k=1}^{n+1} c_k X_k > x\right) \leq \left(\left(1+\frac{\varepsilon}{2}\right)\alpha_n + \varepsilon\tilde{C}_\varepsilon \left(1+\frac{\varepsilon}{2}\right)^n\right) \sum_{k=1}^{n+1} P(c_k X_k > x).$$

This proves that

$$\sup_{\mathbf{c_{n+1}} \in [a,b]^{n+1}} \sup_{x \geq x_3} \frac{P\left(\sum_{k=1}^{n+1} c_k X_k > x\right)}{\sum_{k=1}^{n+1} P(c_k X_k > x)} \leq \left(1+\frac{\varepsilon}{2}\right)\alpha_n + \varepsilon\tilde{C}_\varepsilon \left(1+\frac{\varepsilon}{2}\right)^n.$$

When $x < x_3$, it holds uniformly for $\mathbf{c_{n+1}} \in [a,b]^{n+1}$ that

$$\frac{P\left(\sum_{k=1}^{n+1} c_k X_k > x\right)}{\sum_{k=1}^{n+1} P(c_k X_k > x)} \leq \frac{1}{\overline{F}\left(\frac{x_3}{a}\right)}.$$

Then, by the recursive inequality,

$$\begin{aligned} \alpha_{n+1} &= \left(\sup_{\mathbf{c_{n+1}} \in [a,b]^{n+1}} \sup_{x \geq x_3} + \sup_{\mathbf{c_{n+1}} \in [a,b]^{n+1}} \sup_{x < x_3}\right) \frac{P\left(\sum_{k=1}^{n+1} c_k X_k > x\right)}{\sum_{k=1}^{n+1} P(c_k X_k > x)} \\ &\leq \left(1+\frac{\varepsilon}{2}\right)\alpha_n + \varepsilon\tilde{C}_\varepsilon \left(1+\frac{\varepsilon}{2}\right)^n + \frac{1}{\overline{F}\left(\frac{x_3}{a}\right)} \\ &\leq \left(1 + n\varepsilon\tilde{C}_\varepsilon + \frac{2}{\varepsilon\overline{F}\left(\frac{x_3}{a}\right)}\right)\left(1+\frac{\varepsilon}{2}\right)^n. \end{aligned}$$

Therefore, there exists some large $n_0 \in \mathbb{N}$ such that $\alpha_{n+1} \leq (1+\varepsilon)^n$ holds for all $n \geq n_0$; and when $n < n_0$,

$$\alpha_{n+1} \leq \left(1 + n_0\varepsilon\tilde{C}_\varepsilon + \frac{2}{\varepsilon\overline{F}\left(\frac{x_3}{a}\right)}\right)(1+\varepsilon)^n.$$

We can choose $C_\varepsilon = 1 + n_0\varepsilon\tilde{C}_\varepsilon + \frac{2}{\varepsilon\overline{F}(\frac{x_3}{a})}$ to derive the Kesten-type upper bound (6). □

Proof of Theorem 2.1. By using Lemma 3.3 we can prove Theorem 2.1 along the lines of the proof of Theorem 1.2 in [5]. □

4. Applications to operational risk

The Basel II accord imposes new methods of calculating regulatory capital, among which operational risk is defined as the risk of losses resulting from inadequate or failed internal processes, people and systems, or from external events. According to Basel II accord, banks should allocate losses from operational risk to more than one business line or loss event type, which leads to the core problem of the multivariate modelling encompassing all different risk type/business line cells. In this section, we consider a multivariate operational risk model with d cells, in which all the loss severities $\{X_k^{(i)}, k \in \mathbb{N}\}_{i=1,\dots,d}$ are dependent and identically distributed with common distribution F; the random weights $\{\theta_k^{(i)}, k \in \mathbb{N}\}_{i=1,\dots,d}$, considered as the discount factors when calculating the present values of loss severities, are also identically distributed but arbitrarily dependent; and the number of loss events in the time interval $[0,t]$ for $t \geq 0$ are described by some counting processes $\{N_i(t), t \geq 0\}_{i=1,\dots,d}$, called frequency processes. Assume that the severity processes $\{X_k^{(i)}, k \in \mathbb{N}\}_{i=1,\dots,d}$, the random weights $\{\theta_k^{(i)}, k \in \mathbb{N}\}_{i=1,\dots,d}$, and the frequency processes $\{N_i(t), t \geq 0\}_{i=1,\dots,d}$ are mutually independent. However, arbitrary dependence may exist among $N_1(t), \dots, N_d(t)$. In this model, for each business line $i = 1, \dots, d$, the aggregate loss process $S_i(t)$ in $[0,t]$ constitutes a process

$$S_i(t) = \sum_{k=1}^{N_i(t)} \theta_k^{(i)} X_k^{(i)}, \qquad t \geq 0,$$

and the bank's total aggregate loss process is defined as

$$S(t) = \sum_{i=1}^{d} S_i(t), \qquad t \geq 0, \tag{12}$$

with distribution $G_t(x) = P(S(t) \leq x)$.

Such a multivariate operational risk model is to meet the requirements for the Advanced Measurement Approach, which is proposed by Basel Committee on Banking Supervision [2]. One of the most popular frequency/severity approaches satisfying the Advanced Measurement Approach standards, called Loss Distribution Approach, is widely used in banks and insurance companies, see, e.g. [1] and [32].

In this section, we will use the Loss Distribution Approach to investigate some asymptotics for the tail probability and the risk measure VaR of total aggregate loss $S(t)$ in (12). Some similar results can be found in [15]. The following result presents an asymptotic formula for the former. Denote by H the distribution of $\theta_1^{(1)} X_1^{(1)}$.

Theorem 4.1. *Consider the total aggregate loss (12) under the multivariate operational risk cell model. Assume that the nonnegative loss severities $\{X_k^{(i)}, k \in \mathbb{N}\}_{i=1,\dots,d}$ are dependent according to Assumption 2.1 and identically distributed with common distribution $F \in \mathcal{S}$; the random weights $\{\theta_k^{(i)}, k \in \mathbb{N}\}_{i=1,\dots,d}$ are identically distributed, bounded above and non-degenerate at 0; the frequency processes $\{N_i(t), t \geq 0\}_{i=1,\dots,d}$ satisfy $E\left[(1+\varepsilon_0)^{N_i(t)}\right] < \infty$ for some $\varepsilon_0 > 0$, $i = 1, \dots, d$; and the loss severities, the random weights and the frequency processes are mutually independent. Then, regardless of arbitrary dependence among $\{N_i(t), t \geq 0\}_{i=1,\dots,d}$ and among $\{\theta_k^{(i)}, k \in \mathbb{N}\}_{i=1,\dots,d}$, it holds that*

$$P(S(t) > x) \sim \overline{H}(x) \sum_{i=1}^{d} E\left[N_i(t)\right]. \tag{13}$$

Remark 4.2. *Since the random weights $\{\theta_k^{(i)}, k \in \mathbb{N}\}_{i=1,\dots,d}$ are bounded above, denoting the upper bound by $b > 0$,* *by $F \in \mathcal{S}$, Theorem 2.1 of [10] implies $H \in \mathcal{S}$. In addition, if $F \in \mathcal{R}_{-\infty}$ then for any $y > 1$,*

$$\begin{aligned}\overline{H}(xy) &= \int_0^b \frac{\overline{F}\left(\frac{xy}{u}\right)}{\overline{F}\left(\frac{x}{u}\right)} \overline{F}\left(\frac{x}{u}\right) P(\theta_1^{(1)} \in du) \\ &= o(1) \int_0^b \overline{F}\left(\frac{x}{u}\right) P(\theta_1^{(1)} \in du) \\ &= o(\overline{H}(x)),\end{aligned}$$

implying $H \in \mathcal{R}_{-\infty}$.

The risk measure VaR has been widely used as the minimum capital requirement or the reserve in banks and financial institutions. It is more practical and tractable than the tail probability in reality. By definition, the VaR of total aggregate loss can be defined as the q-quantile of G_t,

$$\mathrm{VaR}_q(S(t)) = G_t^{\leftarrow}(q) = \inf\{x \in \mathbb{R} : P(S(t) \le x) \ge q\},$$

for confidence level $q \in (0, 1)$. Actually, a closed-form expression for $G_t(x)$ is not available except for few ideal distributional assumptions. Hence, in general, $G_t^{\leftarrow}(q)$ can not be analytically calculated. The next result gives an asymptotic formula for $\mathrm{VaR}_q(S(t))$ as $q \uparrow 1$.

Theorem 4.3. *Under the conditions of Theorem 4.1, if further $F \in \mathcal{S} \cap \mathcal{R}_{-\infty}$, then it holds that as $q \uparrow 1$,*

$$\mathrm{VaR}_q(S(t)) \sim H^{\leftarrow}(q).$$

Furthermore, if $\theta_k^{(i)} = e^{-rT}$, $k \in \mathbb{N}, i = 1, \ldots, d$, where $r > 0$ is the constant force of interest, and $T > 0$ is the length of the period, then the following corollary is straightforward.

Corollary 4.4. *Under the conditions of Theorem 4.3, if $\theta_k^{(i)} = e^{-rT}$ for some $r > 0$ and $T > 0$, $k \in \mathbb{N}, i = 1, \ldots, d$, then it holds that as $q \uparrow 1$,*

$$\mathrm{VaR}_q(S(t)) \sim e^{-rT} F^{\leftarrow}(q).$$

To prove Theorems 4.1 and 4.3, we need the following two lemmas. The first one investigates the asymptotic behavior of the randomly weighted sums, which is due to [14].

Lemma 4.5. *Let $X_1, \ldots, X_n$ be n identically distributed nonnegative r.v.s with common distribution $F \in \mathcal{S}$ and satisfying Assumption 2.1, and let $\theta_1, \ldots, \theta_n$ be another n nonnegative, non-degenerate at 0, bounded above but arbitrarily dependent r.v.s, which are independent of $X_1, \ldots, X_n$. Then, it holds that*

$$P\left(\sum_{k=1}^n \theta_k X_k > x\right) \sim \sum_{k=1}^n P(\theta_k X_k > x).$$

The second lemma can be found in [4].

Lemma 4.6. *Let F_1 and F_2 be two distributions satisfying $\overline{F_1}(x) \sim \overline{F_2}(x)$. If $F_1 \in \mathcal{R}_{-\alpha}$ for some $0 < \alpha \le \infty$, then*

$$\left(\frac{1}{\overline{F_1}}\right)^{\leftarrow}(x) \sim \left(\frac{1}{\overline{F_2}}\right)^{\leftarrow}(x).$$

Now we are ready to prove Theorems 4.1 and 4.3.

Proof of Theorem 4.1. Clearly, the tail probability $P(S(t) > x)$ can be written as

$$P(S(t) > x) = \sum_{n_1=0}^{\infty} \cdots \sum_{n_d=0}^{\infty} P\left(\sum_{i=1}^d \sum_{k=1}^{n_i} \theta_k^{(i)} X_k^{(i)} > x\right) P(N_1(t) = n_1, \ldots, N_d(t) = n_d).$$

By using Lemma 4.5, we have that for any fixed $n_i \in \mathbb{N}$, $i = 1, \ldots, d$,

$$P\left(\sum_{i=1}^{d}\sum_{k=1}^{n_i}\theta_k^{(i)}X_k^{(i)} > x\right) \sim \overline{H}(x)\sum_{i=1}^{d} n_i. \tag{14}$$

For the $\varepsilon = \frac{1}{2}\left((1+\varepsilon_0)^{\frac{1}{d}} - 1\right) > 0$ and all $x > 0$, it holds that

$$\begin{aligned}
\frac{P(S(t) > x)}{\overline{H}(x)} &\le \sum_{n_1=0}^{\infty}\cdots\sum_{n_d=0}^{\infty} C_\varepsilon(1+\varepsilon)^{\sum_{i=1}^{d} n_i}\sum_{i=1}^{d} n_i P(N_1(t) = n_1, \ldots, N_d(t) = n_d)\\
&= C_\varepsilon E\left[(1+\varepsilon)^{\sum_{i=1}^{d} N_i(t)}\sum_{i=1}^{d} N_i(t)\right]\\
&\le \tilde{C}_\varepsilon E\left[(1+2\varepsilon)^{\sum_{i=1}^{d} N_i(t)}\right]\\
&\le \tilde{C}_\varepsilon \left(\prod_{i=1}^{d} E\left[(1+2\varepsilon)^{dN_i(t)}\right]\right)^{\frac{1}{d}}\\
&= \tilde{C}_\varepsilon \left(\prod_{i=1}^{d} E\left[(1+\varepsilon_0)^{N_i(t)}\right]\right)^{\frac{1}{d}}\\
&< \infty,
\end{aligned}$$

for some $C_\varepsilon > 0$ and $\tilde{C}_\varepsilon > 0$, where we used Theorem 2.1 in the first step, the generalized Hölder's inequality (see, e.g. [18]) in the fourth step, and the conditions $E\left[(1+\varepsilon_0)^{N_i(t)}\right] < \infty, i = 1, \ldots, d$ in the last step. Then, by (14), the dominated convergence theorem gives the desired relation (13). □

Proof of Theorem 4.3. By $F \in \mathcal{R}_{-\infty}$ and Remark 4.2, we know $H \in \mathcal{R}_{-\infty}$ implying $\left(\frac{1}{\overline{H}}\right)^{\leftarrow} \in \mathrm{RV}_0$ by Theorem 2.4.7(ii) of [3]. Then, it follows from Theorem 4.1 and Lemma 4.6 that as $q \uparrow 1$,

$$\begin{aligned}
\mathrm{VaR}_q(S(t)) &= \left(\frac{1}{\overline{G}_t}\right)^{\leftarrow}\left(\frac{1}{1-q}\right)\\
&\sim \left(\frac{1}{\overline{H}}\right)^{\leftarrow}\left(\frac{1}{1-q}\sum_{i=1}^{d} E[N_i(t)]\right)\\
&\sim \left(\frac{1}{\overline{H}}\right)^{\leftarrow}\left(\frac{1}{1-q}\right)\\
&= H^{\leftarrow}(q),
\end{aligned}$$

as claimed. □

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