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Two Extensions of the Stone Duality to the Category of Zero-Dimensional Hausdorff Spaces

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Abstract. Extending the Stone Duality Theorem, we prove two duality theorems for the category **ZHaus** of zero-dimensional Hausdorff spaces and continuous maps. They extend also the Tarski Duality Theorem; the latter is even derived from one of them. We prove as well two new duality theorems for the category **EDTych** of extremally disconnected Tychonoff spaces and continuous maps. Also, we describe two categories which are dually equivalent to the category **ZComp** of zero-dimensional Hausdorff compactifications of zero-dimensional Hausdorff spaces and obtain as a corollary the Dwinger Theorem about zero-dimensional compactifications of a zero-dimensional Hausdorff space.

1. Introduction

In 1937, M. Stone [17] proved that there exists a bijective correspondence T_l between the class of all (up to homeomorphism) zero-dimensional locally compact Hausdorff spaces and the class of all (up to isomorphism) generalized Boolean algebras (or, equivalently, Boolean rings with or without unit). In the class of compact zero-dimensional Hausdorff spaces (briefly, *Stone spaces*) this bijection can be extended to a dual equivalence $\mathsf{T} : \mathbf{Stone} \longrightarrow \mathbf{Boole}$ between the category **Stone** of Stone spaces and continuous maps and the category **Boole** of Boolean algebras and Boolean homomorphisms; this is the classical Stone Duality. In 1964, H. P. Doctor [10] showed that the Stone bijection T_l can be even extended to a dual equivalence between the category of zero-dimensional locally compact Hausdorff spaces and perfect maps between them and the category of generalized Boolean algebras and suitable morphisms between them. Later on, G. Dimov [6, 7] extended the Stone Duality to the category of zero-dimensional locally compact Hausdorff spaces and continuous maps.

In this article, which was inspired by the recent paper [4] of G. Bezhanishvili, P. J. Morandi and B. Olberding, we describe two extensions of the Stone Duality to the category **ZHaus** of zero-dimensional Hausdorff spaces and continuous maps. Namely, we define two categories **dzBoole** and **mzMaps**, and prove that there exist dual equivalences $F : \mathbf{ZHaus} \longrightarrow \mathbf{dzBoole}$ and $\mathsf{F} : \mathbf{ZHaus} \longrightarrow \mathbf{mzMaps}$. Using the restrictions of F and F to the category **D** of discrete spaces and continuous maps, we show that our duality theorems extend the Tarski Duality as well. Moreover, with the help of the restriction of the dual equivalence

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F to the category **D**, we obtain a new proof of the Tarski Duality Theorem. The restrictions of F and F to the category **EDTych** of extremally disconnected Tychonoff spaces and continuous maps give us two duality theorems for the category **EDTych**. We introduce as well two other categories, namely, the categories **zBoole** and **zMaps**, and show that they are dually equivalent to the category **ZComp** of zero-dimensional Hausdorff compactifications of zero-dimensional Hausdorff spaces. As a corollary, we obtain the Dwinger Theorem [11] about zero-dimensional compactifications of a zero-dimensional Hausdorff space. Let us note that the category **ZComp** is a full subcategory of the category **Comp** of all Hausdorff compactifications of Tychonoff spaces defined in [4].

The paper is organized as follows. Section 2 contains all preliminary facts and definitions which are used in this paper.

In Section 3, we introduce the notions of *Boolean z-algebra* and *Boolean dz-algebra*, define the category **zBoole** having as objects all Boolean z-algebras, as well as its full subcategory **dzBoole** whose objects are all Boolean dz-algebras. Here we prove our first duality theorem for the category **ZHaus** by showing that there exist contravariant functors F : **ZHaus** $\longrightarrow$ **dzBoole** and G : **dzBoole** $\longrightarrow$ **ZHaus** that yield a dual equivalence between the categories **ZHaus** and **dzBoole** (see Theorem 3.15).

In the next Section 4, we introduce the notions of *Boolean z-map* and *maximal Boolean z-map*, define the category **zMaps** having as objects all Boolean z-maps and its full subcategory **mzMaps** whose objects are all maximal Boolean z-maps. In Theorem 4.8 we show that the categories **dzBoole** and **mzMaps** are equivalent. This implies immediately that the categories **ZHaus** and **mzMaps** are dually equivalent (see Theorem 4.9 which is our second duality theorem for the category **ZHaus**). The corresponding dual equivalences are denoted by F : **ZHaus** $\longrightarrow$ **mzMaps** and G : **mzMaps** $\longrightarrow$ **ZHaus**.

In Section 5 we describe the subcategories of the categories **dzBoole** and **mzMaps** which are isomorphic to the category **Boole** (see Propositions 5.1 and 5.3) and show that the dual equivalences F, G, F and G are extensions of the classical Stone dual equivalences T : **Stone** $\longrightarrow$ **Boole** and S : **Boole** $\longrightarrow$ **Stone**.

In Section 6 we describe the subcategories of the categories **dzBoole** and **mzMaps** which are isomorphic to the category **D** (see Proposition 6.1), prove that the corresponding restrictions of F, G, F and G lead to one and the same dual equivalence

$$\mathsf{A} : \mathbf{Caba} \longrightarrow \mathbf{Set}$$

which is slightly different from the classical Tarski dual equivalence At : **Caba** $\longrightarrow$ **Set**, and show that it implies the Tarski Duality Theorem (see Propositions 6.3). Hence, both of our duality theorems extend the Tarski Duality Theorem. Moreover, since in the proof of our Theorem 3.15 we do not use the Tarski Duality Theorem, we obtain in such a way a new proof of the latter one.

In Section 7 we regard the restrictions of F, G, F and G to the category **EDTych** and obtain two duality theorems for the category **EDTych** (see Theorems 7.2 and 7.4). The categories which a dually equivalent to the category **EDTych** are simpler than the categories **dzBoole** and **mzMaps**; their objects are all complete Boolean z-algebras and all complete Boolean z-maps, respectively, although one could expect that their objects should be all complete Boolean dz-algebras and all complete Boolean mz-maps, respectively.

In the last Section 8, we show that the category **zBoole** is dually equivalent to the category **ZComp** (see Theorem 8.5). Then we prove that the categories **ZComp** and **zMaps** are dually equivalent (see Theorem 8.9). In 8.11 we show that both of these results imply the Dwinger Theorem [11] which describes the ordered set of all, up to equivalence, zero-dimensional compactifications of a zero-dimensional Hausdorff space X.

We want to add that in the continuation [8] of this paper, we show how the extension of the Stone Duality Theorem to the category of zero-dimensional locally compact Hausdorff spaces and continuous maps obtained by Dimov in [6, 7] can be derived from any of our duality theorems 3.15 and 4.9, and prove two new duality theorems for this category.

We now fix the notation.

Throughout, $(B, \wedge, \vee, ^*, 0, 1)$ will denote a Boolean algebra unless indicated otherwise; we do not assume that $0 \neq 1$. With some abuse of language, we shall usually identify algebras with their universe, if no confusion can arise.

We denote by **2** the two-element Boolean algebra.

If A is a Boolean algebra, then $A^+ \stackrel{\mathrm{df}}{=} A \setminus \{0\}$, At($A$) is the set of all atoms of A and Ult(A) is the set of all ultrafilters in A.

If X is a set, we denote by $P(X)$ the power set of X; clearly, $(P(X), \cup, \cap, \setminus, \emptyset, X)$ (= $(P(X), \subseteq)$) is a complete atomic Boolean algebra.

If X is a topological space, we denote by CO(X) the set of all clopen (= closed and open) subsets of X. Obviously, $(\mathrm{CO}(X), \cup, \cap, \setminus, \emptyset, X)$ (= $(\mathrm{CO}(X), \subseteq)$) is a Boolean algebra.

If M is a subset of X, we denote by cl(M) the closure of M in X and by int(M) the interior of M in X.

If $\mathcal{C}$ is a category, we denote by $|\mathcal{C}|$ the class of the objects of $\mathcal{C}$ and by $\mathcal{C}(X, Y)$ the set of all $\mathcal{C}$-morphisms between two $\mathcal{C}$-objects X and Y.

We denote by:

- **Set** the category of sets and functions,
- **Top** the category of topological spaces and continuous maps,
- **ZHaus** the category of zero-dimensional Hausdorff spaces and continuous maps,
- **D** the category of discrete spaces and continuous maps,
- **Stone** the category of compact Hausdorff zero-dimensional spaces (= *Stone spaces*) and continuous maps,
- **EDTych** the category of extremally disconnected Tychonoff spaces and continuous maps,
- **Boole** the category of Boolean algebras and Boolean homomorphisms,
- **Caba** the category of complete atomic Boolean algebras and complete Boolean homomorphisms.

The main reference books for all notions which are not defined here are [1, 11, 12, 16].

2. Preliminaries

We start with recalling briefly the Stone Duality Theorem and the Tarski Duality Theorem; we also fix the notation.

2.1. *The Stone duality.* We will denote by $\mathsf{CO} : \mathbf{Top} \longrightarrow \mathbf{Boole}$ the contravariant functor which assigns to every $X \in |\mathbf{Top}|$ the Boolean algebra $(\mathrm{CO}(X), \subseteq)$ and to every $f \in \mathbf{Top}(X, Y)$, the Boolean homomorphism $\mathsf{CO}(f) : \mathsf{CO}(Y) \longrightarrow \mathsf{CO}(X)$ defined by $\mathsf{CO}(f)(U) \stackrel{\mathrm{df}}{=} f^{-1}(U)$, for every $U \in \mathrm{CO}(Y)$.

Now we will briefly describe the Stone duality [17] between the categories **Boole** and **Stone** using its presentation given in [14]. We will define two contravariant functors

$$\mathsf{S} : \mathbf{Boole} \longrightarrow \mathbf{Stone} \quad \text{and} \quad \mathsf{T} : \mathbf{Stone} \longrightarrow \mathbf{Boole}.$$

For any Boolean algebra A, we let the space $\mathsf{S}(A)$ to be the set

$$\mathbf{Boole}(A, \mathbf{2})$$

endowed with a topology $\mathcal{T}_A$ having as a closed base the family $\{s_A(a) \mid a \in A\}$, where

$$s_A(a) \stackrel{\mathrm{df}}{=} \{x \in \mathbf{Boole}(A, \mathbf{2}) \mid x(a) = 1\}, \tag{1}$$

for every $a \in A$; then $\mathsf{S}(A) = (\mathbf{Boole}(A, \mathbf{2}), \mathcal{T}_A)$ is a Stone space. (Many times in this paper, we will write $\mathsf{S}(A)$ instead of its underlying set $\mathbf{Boole}(A, \mathbf{2})$. It will be clear from the context what we mean in every concrete case.)

Note that the family $\{s_A(a) \mid a \in A\}$ is also an open base for the space $\mathsf{S}(A)$.

If $\varphi \in \mathbf{Boole}(A,B)$, then we define $\mathsf{S}(\varphi) : \mathsf{S}(B) \longrightarrow \mathsf{S}(A)$ by the formula $\mathsf{S}(\varphi)(y) \stackrel{\text{df}}{=} y \circ \varphi$ for every $y \in \mathsf{S}(B)$. It is easy to see that S is a contravariant functor.

The contravariant functor T is defined to be the restriction of the contravariant functor CO to the category **Stone**.

For every $X \in |\mathbf{Stone}|$, the map $t_X : X \longrightarrow \mathsf{S}(\mathsf{T}(X))$, $x \mapsto (\hat{x} : \mathsf{CO}(X) \longrightarrow \mathbf{2})$, where $\hat{x}(U) = 1 \Leftrightarrow x \in U$, is a homeomorphism and

$$t : \mathsf{Id}_{\mathbf{Stone}} \longrightarrow \mathsf{S} \circ \mathsf{T},\ \ X \mapsto t_X,$$

is a natural isomorphism. Also, *the Stone map*

$$s_A : A \longrightarrow \mathsf{T}(\mathsf{S}(A)),\ \ a \mapsto s_A(a), \tag{2}$$

is a **Boole**-isomorphism and

$$s : \mathsf{Id}_{\mathbf{Boole}} \longrightarrow \mathsf{T} \circ \mathsf{S},\ \ A \mapsto s_A,$$

is natural isomorphism. Thus $\langle \mathsf{T}, \mathsf{S}, t, s\rangle : \mathbf{Stone} \longrightarrow \mathbf{Boole}$ is an adjoint dual equivalence (in the sense of [16]).

Note that the transition between the above description of the Stone Duality and the usual one dealing with ultrafilters can be easily done using the following well-known assertion: a subset $\mathfrak{u}$ of a Boolean algebra A is an ultrafilter if and only if $\mathfrak{u} = \varphi^{-1}(1)$ for a (unique) Boolean homomorphism $\varphi : A \longrightarrow \mathbf{2}$ (see, e.g., [15, Propositions 2.2 and 2.6]).

2.2. *The Tarski duality*. The Tarski Duality between the categories **Set** and **Caba** consists of two contravariant functors

$$\mathsf{P} : \mathbf{Set} \longrightarrow \mathbf{Caba} \quad \text{and} \quad \mathsf{At} : \mathbf{Caba} \longrightarrow \mathbf{Set}$$

which are defined as follows. For every set X,

$$\mathsf{P}(X) \stackrel{\text{df}}{=} (P(X), \subseteq).$$

If $f \in \mathbf{Set}(X,Y)$, then $\mathsf{P}(f) : \mathsf{P}(Y) \longrightarrow \mathsf{P}(X)$ is defined by the formula

$$\mathsf{P}(f)(M) \stackrel{\text{df}}{=} f^{-1}(M),$$

for every $M \in P(Y)$. Further, for every $B \in |\mathbf{Caba}|$,

$$\mathsf{At}(B) \stackrel{\text{df}}{=} \mathrm{At}(B);$$

if $\sigma \in \mathbf{Caba}(B,B')$, then $\mathsf{At}(\sigma) : \mathsf{At}(B') \longrightarrow \mathsf{At}(B)$ is defined by the formula

$$\mathsf{At}(\sigma)(x') \stackrel{\text{df}}{=} \bigwedge\{b \in B \mid x' \leq \sigma(b)\},$$

for every $x' \in \mathrm{At}(B')$.

For each set X, we have a bijection $\eta_X : X \longrightarrow \mathsf{At}(\mathsf{P}(X))$, given by $\eta_X(x) \stackrel{\text{df}}{=} \{x\}$ for every $x \in X$, and

$$\eta : \mathsf{Id}_{\mathbf{Set}} \longrightarrow \mathrm{At} \circ \mathsf{P},\ \ X \mapsto \eta_X,$$

is a natural isomorphism.

For each $B \in |\mathbf{Caba}|$, we have a **Caba**-isomorphism

$$\varepsilon_B : B \longrightarrow \mathsf{P}(\mathsf{At}(B)),$$

given by $\varepsilon_B(b) \stackrel{\text{df}}{=} \{x \in \mathrm{At}(B) \mid x \leq b\}$ for each $b \in B$, and

$$\varepsilon : \mathsf{Id}_{\mathbf{Caba}} \longrightarrow \mathsf{P} \circ \mathrm{At},\ \ B \mapsto \varepsilon_B,$$

is a natural isomorphism. Note that $\varepsilon_B^{-1}(M) = \bigvee_B M$, for all $M \subseteq \mathrm{At}(B)$.

Thus $\langle \mathsf{P}, \mathsf{At}, \eta, \varepsilon\rangle : \mathbf{Set} \longrightarrow \mathbf{Caba}$ is an adjoint dual equivalence.

The following assertion is well known (because $\mathsf{At}(\sigma)$ is the restriction to $\mathrm{At}(B')$ of the lower (or, left) adjoint for σ (see [15, Theorem 4.2])), but we will present here its short proof.

Lemma 2.3. *Let $\sigma \in \mathbf{Caba}(B, B')$. Then, for every $b \in B$ and each $x' \in \mathrm{At}(B')$, $(x' \leq \sigma(b)) \Leftrightarrow (\mathsf{At}(\sigma)(x') \leq b)$.*

Proof. Since $\mathsf{At}(\sigma)(x') = \bigwedge\{b \in B \mid x' \leq \sigma(b)\}$, we obtain immediately that $(x' \leq \sigma(b)) \Rightarrow (\mathsf{At}(\sigma)(x') \leq b)$. Suppose now that $\mathsf{At}(\sigma)(x') \leq b$. Then $\sigma(\mathsf{At}(\sigma)(x')) \leq \sigma(b)$. Since $\sigma(\mathsf{At}(\sigma)(x')) = \sigma(\bigwedge\{c \in B \mid x' \leq \sigma(c)\}) = \bigwedge\{\sigma(c) \mid c \in B, x' \leq \sigma(c)\} \geq x'$, we obtain that $x' \leq \sigma(b)$. □

2.4. *Some special types of Boolean homomorphisms.* In the proofs of our duality theorems we will use very often some special Boolean homomorphisms. We give here the list of these homomorphisms and the corresponding notation.

Let $\alpha \in \mathbf{Boole}(A, B)$ and $x \in \mathrm{At}(B)$. Then it is easy to see that the map

$$\alpha_x : A \longrightarrow \mathbf{2}$$

defined by $\alpha_x(a) = 1 \Leftrightarrow x \leq \alpha(a)$, where $a \in A$, is a Boolean homomorphism. We put

$$X_\alpha \stackrel{\mathrm{df}}{=} \{\alpha_x \mid x \in \mathrm{At}(B)\}.$$

Note that if α is a complete Boolean homomorphism, then, for every $x \in \mathrm{At}(B)$, α_x is a complete Boolean homomorphism as well. We put

$$h_\alpha : \mathrm{At}(B) \longrightarrow X_\alpha, \;\; x \mapsto \alpha_x.$$

It is easy to see that *if every atom of B is a meet of some elements of $\alpha(A)$, then h_α is a bijection.*

If $A = B$ and $\alpha = id_B$, then we have that $\alpha_x(b) = 1 \Leftrightarrow x \leq b$, where $b \in B$. In this case, for simplicity, we will write $\dot{x}$ instead of α_x, $\dot{X}_B$ instead of X_α and $\dot{h}_B$ instead of h_α. Hence,

$$\dot{x} : B \longrightarrow \mathbf{2}$$

is defined by $\dot{x}(b) = 1 \Leftrightarrow x \leq b$, for all $b \in B$,

$$\dot{X}_B \stackrel{\mathrm{df}}{=} \{\dot{x} \mid x \in \mathrm{At}(B)\}$$

and

$$\dot{h}_B : \mathrm{At}(B) \longrightarrow \dot{X}_B, \;\; x \mapsto \dot{x}.$$

Note that every $\dot{x}$ is a complete Boolean homomorphism and $\dot{h}_B$ is a bijection; also, $\dot{X}_B$ is the set of all isolated points of $\mathsf{S}(B)$.

Further, if X is a set, $B = \mathsf{P}(X)$, A is a Boolean subalgebra of B and α is the inclusion map, then, obviously, the map α_x is defined by $\alpha_x(U) = 1 \Leftrightarrow x \in U$, where $U \in A$. In order to simplify the notation, for such A and B, we will write $\hat{x}$ (and, sometimes, even $\hat{x}_A$) instead of α_x. (*Note that every $\hat{x}$ is a complete Boolean homomorphism.*) Thus, in such a case, by

$$\hat{x} : A \longrightarrow \mathbf{2}$$

we will understand the map defined by $\hat{x}(U) = 1 \Leftrightarrow x \in U$, where $U \in A$; also, we will write $\hat{X}_A$ instead of X_α, and $\hat{h}_{X,A}$ instead of h_α, i.e.,

$$\hat{X}_A = \{\hat{x} : A \longrightarrow \mathbf{2} \mid x \in X\}$$

and

$$\hat{h}_{X,A} : X \longrightarrow \hat{X}_A, \;\; x \mapsto \hat{x}.$$

Note that if the family A *T_0-separates the points of* X (i.e., for every $x, y \in X$ such that $x \neq y$, there exists $U \in A$ with $|U \cap \{x, y\}| = 1$), then *the map $\hat{h}_{X,A}$ is a bijection.*

If X is a topological space and $A = (\mathrm{CO}(X), \subseteq)$, *we will simply write $\hat{X}$ instead of $\hat{X}_A$, and $\hat{h}_X$ instead of $\hat{h}_{X,A}$, i.e.,*

$$\hat{h}_X : X \longrightarrow \hat{X}, \;\; x \mapsto \hat{x}.$$

Obviously, *if X is a zero-dimensional Hausdorff space, then $\hat{h}_X$ is a bijection.*

Definition 2.5. Let X be a Tychonoff space. A *compactification* of X is any dense embedding $c : X \longrightarrow Y$, where Y is a compact Hausdorff space. Often we will write (Y, c) instead of c and will say that (Y, c) is a compactification of X.

Two compactifications (Y_i, c_i), $i = 1, 2$, of X are called *equivalent* if there exists a homeomorphism $f : Y_1 \longrightarrow Y_2$ such that $f \circ c_1 = c_2$. Clearly, this defines an equivalence relation in the class of all compactifications of X; the equivalence class of a compactification (Y, c) of X will be denoted by $[(Y, c)]$. We write $(Y_1, c_1) \leq (Y_2, c_2)$ and say that the compactification (Y_2, c_2) is *larger* than the compactification (Y_1, c_1) if there exists a continuous mapping $f : Y_2 \longrightarrow Y_1$ such that $f \circ c_2 = c_1$. This relation is a *preorder* (i.e., it is reflexive and transitive). The equivalence relation associated with this preorder (i.e., (Y_1, c_1) is larger than (Y_2, c_2) and conversely) coincides with the relation of equivalence defined above. Setting for every two compactifications (Y_i, c_i), $i = 1, 2$, of X,

$$[(Y_1, c_1)] \leq [(Y_2, c_2)] \text{ iff } (Y_1, c_1) \leq (Y_2, c_2),$$

we obtain a well-defined relation on the set of all, up to equivalence, compactifications of X; it is already an order.

Definition 2.6. ([11]) Let X be a zero-dimensional Hausdorff space. A Boolean algebra A is called *admissible for X* (or, a *Boolean base for X*) if A is a Boolean subalgebra of the Boolean algebra $\mathsf{CO}(X)$ and A is an open base for X. The set of all admissible Boolean algebras for X will be denoted by $\mathcal{BA}(X)$.

Notation 2.7. The set of all, up to equivalence, zero-dimensional compactifications of a zero-dimensional Hausdorff space X will be denoted by $\mathcal{K}_0(X)$. The order on $\mathcal{K}_0(X)$ induced by the order " $\leq$ " on the set of all, up to equivalence, compactifications of X (defined in 2.5) will be denoted again by " $\leq$".

Theorem 2.8. ([11]) *Let X be a zero-dimensional Hausdorff space. Then the ordered sets $(\mathcal{BA}(X), \subseteq)$ and $(\mathcal{K}_0(X), \leq)$ are isomorphic. The isomorphism δ between these two ordered sets is the following one: for every $A \in \mathcal{BA}(X)$, $\delta(A) \stackrel{\text{df}}{=} [(\mathsf{S}(A), e_A)]$, with $e_A : X \longrightarrow \mathsf{S}(A)$ defined by $e_A(x) \stackrel{\text{df}}{=} (\hat{x} : A \longrightarrow \mathbf{2})$, for every $x \in X$ (see 2.4 for the notation $\hat{x}$).*

For every zero-dimensional Hausdorff space X, the ordered set $(\mathcal{BA}(X), \subseteq)$ has a greatest element, namely the Boolean algebra $\mathsf{CO}(X)$. Thus, by the Dwinger Theorem 2.8, the ordered set $(\mathcal{K}_0(X), \leq)$ also has a greatest element. It is denoted by $(\beta_0 X, \beta_0)$. This fact was discovered earlier by B. Banaschewski [2] and $(\beta_0 X, \beta_0)$ is said to be *the Banaschewski compactification of X*. Clearly, $(\beta_0 X, \beta_0) = \delta(\mathsf{CO}(X))$, i.e. $\beta_0 X = \mathsf{S}(\mathsf{CO}(X))$ and $\beta_0 = e_{\mathsf{CO}(X)}$.

Theorem 2.9. ([2]) *Let X_i, $i = 1, 2$, be zero-dimensional Hausdorff spaces and (cX_2, c) be a zero-dimensional compactification of X_2. Then for every continuous function $f : X_1 \longrightarrow X_2$ there exists a unique continuous function $g : \beta_0 X_1 \longrightarrow cX_2$ such that $g \circ \beta_0 = c \circ f$. In other words, one has the commutative diagram*

$$\begin{tikzcd}
\beta_0 X_1 \arrow[r, "g"] & cX_2 \\
X_1 \arrow[u, "\beta_0"] \arrow[r, "f"] & X_2 \arrow[u, "c"']
\end{tikzcd}$$

2.10. A set F in a topological space X is *regular closed* (or a *closed domain* [12]) if it is the closure of its interior in X: $F = \mathrm{cl}(\mathrm{int}(F))$. The collection $\mathrm{RC}(X)$ of all regular closed sets in X becomes a Boolean algebra, with the Boolean operations $\vee, \wedge, {}^*, 0, 1$ given by

$$F \vee G = F \cup G, \qquad F \wedge G = \mathrm{cl}(\mathrm{int}(F \cap G)), \qquad F^* = \mathrm{cl}(X \setminus F), \qquad 0 = \emptyset, \qquad 1 = X.$$

The Boolean algebra $\mathrm{RC}(X)$ is actually complete, with the infinite joins and meets given by

$$\bigvee_{i \in I} F_i = \mathrm{cl}(\bigcup_{i \in I} F_i) = \mathrm{cl}(\bigcup_{i \in I} \mathrm{int}(F_i)) = \mathrm{cl}(\mathrm{int}(\bigcup_{i \in I} F_i)), \quad \bigwedge_{i \in I} F_i = \mathrm{cl}(\mathrm{int}(\bigcap_{i \in I} F_i)).$$

We will need as well the following well-known statement (see, e.g., [5], p.271, and, for a proof, [18]).

Lemma 2.11. *Let X be a dense subspace of a topological space Y. Then the functions*

$$r : \mathrm{RC}(Y) \longrightarrow \mathrm{RC}(X),\ F \mapsto F \cap X,$$

and

$$e : \mathrm{RC}(X) \longrightarrow \mathrm{RC}(Y),\ G \mapsto \mathrm{cl}_Y(G),$$

are inverse to each other Boolean isomorphisms. (We will sometimes write $r_{X,Y}$ (resp., $e_{X,Y}$) instead of r (resp., e).)

3. The First Duality Theorem for the Category ZHaus

The classical Stone Duality shows that the whole information about a Stone space X is contained in the Boolean algebra $\mathsf{CO}(X)$, i.e., knowing the Boolean algebra $\mathsf{CO}(X)$, we can reconstruct the space X up to homeomorphism. If X is not compact, i.e., X is only a zero-dimensional Hausdorff space, then the Boolean algebra $\mathsf{CO}(X)$ is not enough for reconstructing the space X. Indeed, by the Dwinger Theorem 2.8, the Banaschewski compactification $(\beta_0 X, \beta_0)$ is the Stone dual of $\mathsf{CO}(X)$ and thus, by the Stone duality, $\mathsf{CO}(\beta_0 X)$ and $\mathsf{CO}(X)$ are isomorphic Boolean algebras. However, if we regard, together with the Boolean algebra $\mathsf{CO}(X)$, the set $\beta_0(X)$ (i.e., the image of X under the map β_0) which is a subset of $\mathsf{S}(\mathsf{CO}(X))$, then the space X will be homeomorphic (via the map $\beta_0 : X \longrightarrow \beta_0 X$) to the set $\beta_0(X)$ endowed with the subspace topology of $\mathsf{S}(\mathsf{CO}(X))$. Moreover, the trace of $\mathsf{CO}(\beta_0 X)$ on $\beta_0(X)$ will be precisely $\mathsf{CO}(\beta_0(X))$. In this way we see that the pair $(\mathsf{CO}(X), \beta_0(X))$, where $\beta_0(X)$ is regarded only as a set, is enough for the reconstruction of the space X up to homeomorphism. The algebraic description of such pairs is given below in Definition 3.6 (see also Example 3.9 which confirms that the algebraic notion introduced by us is adequate). Since the set $\beta_0(X)$ is a dense subset of the Banaschewski compactification $\beta_0 X$ of X, we first describe algebraically the pairs $(\mathsf{CO}(X), Y)$, where Y is a dense subset of $\mathsf{S}(\mathsf{CO}(X))$. We call them *Boolean z-algebras* (see Definition 3.1 below). They will help us to describe in Section 8 the category of all zero-dimensional compactifications of zero-dimensional Hausdorff spaces which is just a subcategory of the category of all compactifications of Tychonoff spaces introduced and described in [4]. The algebraic notion which corresponds to the pairs $(\mathsf{CO}(X), \beta_0(X))$ is introduced under the name *Boolean dz-algebra* because, firstly, every Boolean dz-algebra is a Boolean z-algebra, and secondly, with the letter "d" we want to refer to the Dwinger Theorem 2.8. With the help of this notion we will obtain our first duality theorem for the category **ZHaus**.

Definition 3.1. A pair (A, X), where A is a Boolean algebra and $X \subseteq \mathbf{Boole}(A, \mathbf{2})$, is called a *Boolean z-algebra* (briefly, *z-algebra*; abbreviated as ZA) if for each $a \in A^+$ there exists $x \in X$ such that $x(a) = 1$.

Clearly, the definition of Boolean z-algebras can be expressed on the language of ultrafilters as follows: a pair (A, X), where A is a Boolean algebra and $X \subseteq \mathrm{Ult}(A)$, is called a *Boolean z-algebra* if for each $a \in A^+$ there exists $u \in X$ such that $a \in u$.

Using the definition of the space $\mathsf{S}(A)$ (see 2.1), where A is a Boolean algebra, we obtain immediately the following result:

Fact 3.2. *A pair (A, X) is a z-algebra if and only if A is a Boolean algebra and X is a dense subset of $\mathsf{S}(A)$.*

Notation 3.3. If A is a Boolean algebra and $X \subseteq \mathbf{Boole}(A, \mathbf{2})$, we set

$$s_A^X(a) \stackrel{\mathrm{df}}{=} X \cap s_A(a)$$

for each $a \in A$ (see (1) for s_A), defining in such a way a map

$$s_A^X : A \longrightarrow \mathsf{P}(X),\ \ a \mapsto s_A^X(a).$$

Fact 3.4. *A pair (A, X) is a z-algebra if and only if A is a Boolean algebra, $X \subseteq \mathbf{Boole}(A, \mathbf{2})$ and $s_A^X : A \longrightarrow \mathsf{P}(X)$ is a Boolean monomorphism.*

Proof. Suppose that (A, X) is a ZA. Then, by Fact 3.2, X is a dense subset of $K \stackrel{\text{df}}{=} \mathsf{S}(A)$ and thus $\mathrm{cl}_K(s_A^X(a)) = s_A(a)$ for each $a \in A$. Therefore, using the fact that s_A is a Boolean isomorphism, we obtain that s_A^X is a Boolean monomorphism.

Conversely, if s_A^X is a Boolean monomorphism, then $s_A^X(a) \neq \emptyset$ for each $a \in A^+$. Thus X is dense in $\mathsf{S}(A)$, which implies that (A, X) is a ZA. □

Fact 3.5. *Let (A, X) be a z-algebra. Then $s_A^X(A) \subseteq \mathrm{CO}(X)$ and the subspace topology on X induced by $\mathsf{S}(A)$ coincides with the topology on X generated by the base $s_A^X(A)$.*

Proof. Set $Y \stackrel{\text{df}}{=} \mathsf{S}(A)$. Then $\mathrm{CO}(Y) = s_A(A)$ and $\mathrm{CO}(Y)$ is a base for Y. Regarding X as a subspace of Y and using the fact that $s_A^X(A) = X \cap s_A(A)$, we obtain that $s_A^X(A)$ is a base for the subspace topology on X induced by Y and $s_A^X(A) \subseteq \mathrm{CO}(X)$. Hence, the topology on X generated by the base $s_A^X(A)$ coincides with the subspace topology on X induced by Y. □

When (A, X) is a z-algebra, having in mind Fact 3.5, we will denote by $\bar{s}_A^X$ that restriction of the function s_A^X whose domain is A and whose codomain is $\mathsf{CO}(X)$, i.e.,

$$\bar{s}_A^X : A \longrightarrow \mathsf{CO}(X).$$

Definition 3.6. A z-algebra (A, X) is called a *Boolean dz-algebra* (briefly, *dz-algebra*; abbreviated as DZA) if $s_A^X(A) = \mathrm{CO}(X)$, where X is regarded as a subspace of $\mathsf{S}(A)$.

Clearly, on the language of ultrafilters, the definition of a Boolean dz-algebra can be expressed as follows: a z-algebra (A, X) is called a *Boolean dz-algebra* if for every $U \in \mathrm{CO}(X)$, where X is regarded as a subspace of $\mathsf{S}(A)$, there exists $a \in A$ such that $U = \{\mathfrak{u} \in X \mid a \in \mathfrak{u}\}$. From now on, we will not make such translations.

Now, using Fact 3.4, we obtain immediately the following result:

Fact 3.7. *A z-algebra (A, X) is a DZA if and only if the map $\bar{s}_A^X : A \longrightarrow \mathsf{CO}(X)$ is a Boolean isomorphism (regarding X as a subspace of $\mathsf{S}(A)$).*

Example 3.8. Let A be a Boolean algebra. Then $(A, \mathbf{Boole}(A, \mathbf{2}))$ is a dz-algebra. (The dz-algebras of this type will be called *compact Boolean dz-algebras* (or, simply, *compact dz-algebras*). As we will see in Proposition 5.1 below, they correspond to the Boolean algebras and thus, by the Stone Duality, to compact zero-dimensional Hausdorff spaces.)

Indeed, setting $X \stackrel{\text{df}}{=} \mathbf{Boole}(A, \mathbf{2})$, we have that (A, X) is a z-algebra, $\bar{s}_A^X = s_A$ and thus $\bar{s}_A^X(A) = \mathrm{CO}(X)$. Hence, (A, X) is a DZA.

Example 3.9. Let X be a zero-dimensional Hausdorff space and $A \in \mathcal{BA}(X)$ (see Definition 2.6). Then the pair $(A, \hat{X}_A)$ is a z-algebra, the pair $(\mathsf{CO}(X), \hat{X})$ is a dz-algebra and the map $\hat{h}_{X,A} : X \longrightarrow \hat{X}_A$ is a homeomorphism (see 2.4 for the notation).

Indeed, the pair $(A, \hat{X}_A)$ is a z-algebra since for every $U \in A^+$ there exists $x \in U$ and thus $\hat{x}(U) = 1$. Also, we have to show that $\hat{h}_{X,A}$ is a homeomorphism. The family A T_0-separates the points of X because A is a base for the Hausdorff space X. Hence, by 2.4, $\hat{h}_{X,A}$ is a bijection. The family $\hat{X}_A \cap \mathsf{CO}(\mathsf{S}(A)) = \hat{X}_A \cap s_A(A) = s_A^{\hat{X}_A}(A)$ is a base for $\hat{X}_A$ and, for every $U \in A$, $s_A^{\hat{X}_A}(U) = \{\hat{x} \in \hat{X}_A \mid \hat{x}(U) = 1\} = \{\hat{x} \in \hat{X}_A \mid x \in U\} = \hat{h}_{X,A}(U)$; thus, $\hat{h}_{X,A}^{-1}(s_A^{\hat{X}_A}(U)) = U$. This shows that $\hat{h}_{X,A}$ is a continuous and open bijection and, therefore, it is a homeomorphism. Finally, if $A = \mathsf{CO}(X)$, then, since $\hat{h}_X = \hat{h}_{X,A}$ is a homeomorphism, $\hat{h}_X(\mathsf{CO}(X)) = \mathsf{CO}(\hat{X})$. Thus, $s_A^{\hat{X}}(\mathsf{CO}(X)) = \mathrm{CO}(\hat{X})$, i.e., $(\mathsf{CO}(X), \hat{X})$ is a DZA.

Example 3.10. The pair $(B, \dot{X}_B)$, where $B \in |\mathbf{Caba}|$, is a dz-algebra (see 2.4 for the notation $\dot{X}_B$). (The dz-algebras of this type will be called *Boolean T-algebras* (or, simply, *T-algebras*)).

Indeed, for every $b \in B^+$, there exists $x \in \mathrm{At}(B)$ such that $x \leq b$. Then $\dot{x}(b) = 1$. Thus, $(B, \dot{X}_B)$ is a z-algebra. For every $x \in \mathrm{At}(B)$, we have that $s_B^{\dot{X}_B}(x) = \{\dot{x}\}$. Hence, $\dot{X}_B$ is a discrete subspace of $\mathsf{S}(B)$. Therefore,

$CO(\dot{X}_B) = P(\dot{X}_B)$. By 2.4, the function $\dot{h}_B : \mathrm{At}(B) \longrightarrow \dot{X}_B$, $x \mapsto \dot{x}$, is a bijection. Also, if $M \subseteq \mathrm{At}(B)$ and $b_M = \bigvee M$, then $M = \{x \in \mathrm{At}(B) \mid x \leq b_M\}$. Finally, for every $b \in B$, $s_B^{\dot{X}_B}(b) = \{\dot{x} \in \dot{X}_B \mid \dot{x}(b) = 1\} = \{\dot{x} \in \dot{X}_B \mid x \leq b\}$. Thus, $s_B^{\dot{X}_B}(B) = P(\dot{X}_B)$. This shows that $(B, \dot{X}_B)$ is a dz-algebra.

In Fact 3.13 below we will present an equivalent definition of the notion of dz-algebra which will be purely algebraic. For doing this we will need a definition, namely, Definition 3.11. The idea behind it comes from the Dwinger Theorem 2.8 and the following well-known elementary topological fact: if X is a set and $\mathcal{O}_1$, $\mathcal{O}_2$ are two topologies on it with bases $\mathcal{B}_1$ and $\mathcal{B}_2$, respectively, then the topology $\mathcal{O}_1$ is coarser than the topology $\mathcal{O}_2$ (i.e., $\mathcal{O}_1 \subseteq \mathcal{O}_2$) if, and only if, for every $x \in X$ and every $U \in \mathcal{B}_1$ which contains x, there exists $V \in \mathcal{B}_2$ such that $x \in V \subseteq U$.

Definition 3.11. Let $C \in |\mathbf{Caba}|$ and A, B be Boolean subalgebras of C. If for every $a \in A$ and any $x \in \mathrm{At}(C)$ such that $x \leq a$ there exists $b \in B$ with $x \leq b \leq a$, then we will say that A *is t-coarser than* B *in* C or that B *is t-finer than* A in C; in this case we will write $A \preceq_C B$. We will say that the Boolean algebras A and B are *t-equal in* C if $A \preceq_C B$ and $B \preceq_C A$.

The following assertion is obvious:

Fact 3.12. *Let X be a set and A, B be Boolean subalgebras of the Boolean algebra $\mathsf{P}(X)$. Let $\mathcal{O}_A$ (resp., $\mathcal{O}_B$) be the topology on X generated by the base A (resp., B). Then A and B are t-equal in $\mathsf{P}(X)$ if and only if the topologies $\mathcal{O}_A$ and $\mathcal{O}_B$ coincide.*

Fact 3.13. *A z-algebra (A, X) is a DZA if and only if it satisfies the following condition:*
(Dw) If B is a Boolean subalgebra of $\mathsf{P}(X)$ and B is t-equal to $s_A^X(A)$ in $\mathsf{P}(X)$, then $B \subseteq s_A^X(A)$.

Proof. Suppose that the ZA (A, X) satisfies condition (Dw). By Fact 3.5, we have that $s_A^X(A)$ is a base for X and $s_A^X(A) \subseteq CO(X)$. Then the Fact 3.12 shows that the Boolean algebras $s_A^X(A)$ and $\mathsf{CO}(X)$ are t-equal in $\mathsf{P}(X)$. Thus, by condition (Dw), we obtain that $CO(X) \subseteq s_A^X(A)$. Therefore, $s_A^X(A) = CO(X)$, i.e., (A, X) is a DZA.

Conversely, suppose that (A, X) is a DZA. If B is a Boolean subalgebra of $\mathsf{P}(X)$ and B is t-equal to $s_A^X(A)$ in $\mathsf{P}(X)$, then $B \subseteq \mathsf{CO}(X)$. Therefore, $B \subseteq s_A^X(A)$. This shows that (A, X) satisfies condition (Dw). □

Now, we are ready to formulate and prove our first duality theorem for the category **ZHaus**. The proof of the next assertion is obvious.

Proposition 3.14. *There is a category* **zBoole** *whose objects are all z-algebras and whose morphisms between any two* **zBoole***-objects (A, X) and (A', X') are all pairs (φ, f) such that $\varphi \in$* **Boole***(A, A'), $f \in$* **Set***(X', X) and $x' \circ \varphi = f(x')$ for every $x' \in X'$. The composition $(\varphi', f') \circ (\varphi, f)$ between two* **zBoole***-morphisms $(\varphi, f) : (A, X) \longrightarrow (A', X')$ and $(\varphi', f') : (A', X') \longrightarrow (A'', X'')$ is defined to be the* **zBoole***-morphism $(\varphi' \circ \varphi, f \circ f') : (A, X) \longrightarrow (A'', X'')$; the identity morphism of a* **zBoole***-object (A, X) is defined to be (id_A, id_X).*

We denote by **dzBoole** the full subcategory of the category **zBoole** whose objects are all dz-algebras.

Theorem 3.15. *The categories* **ZHaus** *and* **dzBoole** *are dually equivalent.*

Proof. We will first define a contravariant functor

$$F : \mathbf{ZHaus} \longrightarrow \mathbf{dzBoole}.$$

For every $X \in |\mathbf{ZHaus}|$, let

$$F(X) \stackrel{\mathrm{df}}{=} (\mathsf{CO}(X), \hat{X}).$$

Then Example 3.9 shows that $F(X) \in |\mathbf{dzBoole}|$. Further, for $f \in \mathbf{ZHaus}(X, Y)$, set

$$F(f) \stackrel{\mathrm{df}}{=} (\mathsf{CO}(f), \hat{f}),$$

where

$$\hat{f} : \hat{X} \longrightarrow \hat{Y}$$

is defined by

$$\hat{f}(\hat{x}) \stackrel{\text{df}}{=} \widehat{f(x)}$$

for every $x \in X$. We will show that $F(f) \in \mathbf{dzBoole}(F(Y), F(X))$. We need only to prove that $\hat{x} \circ \mathsf{CO}(f) = \widehat{f(x)}$ for every $x \in X$. So, let $x \in X$. Then, for every $U \in \mathrm{CO}(Y)$, we have that $(\hat{x} \circ \mathsf{CO}(f))(U) = 1 \Leftrightarrow \hat{x}(f^{-1}(U)) = 1 \Leftrightarrow x \in f^{-1}(U) \Leftrightarrow f(x) \in U \Leftrightarrow \widehat{f(x)}(U) = 1$. Therefore, $\hat{x} \circ \mathsf{CO}(f) = \hat{f}(\hat{x})$, for every $x \in X$. Thus, $F(f) \in \mathbf{dzBoole}(F(Y), F(X))$.

It is easy to see that F is a contravariant functor.

Now we will define a contravariant functor

$$G : \mathbf{dzBoole} \longrightarrow \mathbf{ZHaus}$$

and will prove that the functors $F \circ G$ and $G \circ F$ are naturally isomorphic to the corresponding identity functors.

For every $(A, X) \in |\mathbf{dzBoole}|$, we set

$$G(A, X) \stackrel{\text{df}}{=} X,$$

where X is regarded as a subspace of $\mathsf{S}(A)$. Then, clearly, $G(A, X) \in |\mathbf{ZHaus}|$. If $(\varphi, f) : (A, X) \longrightarrow (A', X')$ is a $\mathbf{dzBoole}$-morphism, we put

$$G(\varphi, f) \stackrel{\text{df}}{=} f.$$

Let us show that $G(\varphi, f)$ is a continuous function. We have that $X' \subseteq \mathsf{S}(A')$ and $X \subseteq \mathsf{S}(A)$. For every $x' \in X'$, $\mathsf{S}(\varphi)(x') = x' \circ \varphi = f(x')$. Thus, f is a restriction of the continuous function $\mathsf{S}(\varphi)$. Hence, $f : X' \longrightarrow X$ is a continuous function. Therefore, G is well-defined. Now it is easy to see that G is a contravariant functor.

We will show that the functors $F \circ G$ and $\mathsf{Id}_{\mathbf{dzBoole}}$ are naturally isomorphic.

Let $(A, X) \in |\mathbf{dzBoole}|$. Then $F(G(A, X)) = F(X) = (\mathsf{CO}(X), \hat{X})$, where X is regarded as a subspace of $\mathsf{S}(A)$. By Fact 3.7, the map $\bar{s}_A^X : A \longrightarrow \mathsf{CO}(X)$ is a Boolean isomorphism. We put $\check{\imath}_X \stackrel{\text{df}}{=} \hat{h}_X^{-1}$ (recall that, by 2.4, $\hat{h}_X$ is a bijection). Hence,

$$\check{\imath}_X : \hat{X} \longrightarrow X, \ \ \hat{x} \mapsto x,$$

for every $x \in X$. Also, for every $x \in X$, $\hat{x} \circ \bar{s}_A^X = \check{\imath}_X(\hat{x})$. Indeed, for every $a \in A$, $\hat{x}(\bar{s}_A^X(a)) = 1 \Leftrightarrow x \in s_A(a) \Leftrightarrow x(a) = 1$, and thus $\hat{x} \circ \bar{s}_A^X = x = \check{\imath}_X(\hat{x})$. This shows that the map $(\bar{s}_A^X, \check{\imath}_X) : (A, X) \longrightarrow (\mathsf{CO}(X), \hat{X})$ is a $\mathbf{dzBoole}$-morphism and, moreover, it is a $\mathbf{dzBoole}$-isomorphism. We put $s'_{(A,X)} \stackrel{\text{df}}{=} (\bar{s}_A^X, \check{\imath}_X)$. Then

$$s'_{(A,X)} : (A, X) \longrightarrow (F \circ G)(A, X)$$

is a $\mathbf{dzBoole}$-isomorphism. Let now $(\varphi, f) : (A, X) \longrightarrow (A', X')$ be a $\mathbf{dzBoole}$-morphism. We will show that the diagram

$$\begin{tikzcd}
(A, X) \arrow[r, "(\varphi, f)"] \arrow[d, "s'_{(A,X)}"'] & (A', X') \arrow[d, "s'_{(A',X')}"] \\
(\mathsf{CO}(X), \hat{X}) \arrow[r, "(F \circ G)(\varphi, f)"] & (\mathsf{CO}(X'), \widehat{X'})
\end{tikzcd}$$

is commutative. Indeed, we have that

$$s'_{(A',X')} \circ (\varphi, f) = (\bar{s}_{A'}^{X'}, \check{\imath}_{X'}) \circ (\varphi, f) = (\bar{s}_{A'}^{X'} \circ \varphi, f \circ \check{\imath}_{X'})$$

and

$$(F \circ G)(\varphi, f) \circ s'_{(A,X)} = (\mathsf{CO}(f), \hat{f}) \circ (\bar{s}_A^X, \check{\imath}_X) = (\mathsf{CO}(f) \circ \bar{s}_A^X, \check{\imath}_X \circ \hat{f}).$$

Thus, we have to show that

$$\bar{s}^{X'}_{A'} \circ \varphi = \mathsf{CO}(f) \circ \bar{s}^{X}_{A} \quad \text{and} \quad f \circ \check{\imath}_{X'} = \check{\imath}_X \circ \hat{f}.$$

Let $a \in A$. Then

$$\begin{aligned} \mathsf{CO}(f)(\bar{s}^X_A(a)) &= f^{-1}(X \cap s_A(a)) \\ &= \{x' \in X' \mid f(x') \in s_A(a)\} \\ &= \{x' \in X' \mid f(x')(a) = 1\} \\ &= \{x' \in X' \mid x'(\varphi(a)) = 1\} \\ &= X' \cap s_{A'}(\varphi(a)) \\ &= \bar{s}^{X'}_{A'}(\varphi(a)). \end{aligned}$$

Also, for every $x' \in X'$, $\check{\imath}_X(\hat{f}(\widehat{x'})) = \check{\imath}_X(\widehat{f(x')}) = f(x') = f(\check{\imath}_{X'}(\widehat{x'}))$. Hence,

$$s' : \mathsf{Id}_{\mathbf{dzBoole}} \longrightarrow F \circ G, \ \ (A, X) \mapsto s'_{(A,X)},$$

is a natural isomorphism.

Finally, we will show that the functors $G \circ F$ and $\mathsf{Id}_{\mathbf{ZHaus}}$ are naturally isomorphic. Let $X \in |\mathbf{ZHaus}|$. Then $G(F(X)) = G(\mathsf{CO}(X), \hat{X}) = \hat{X}$, where $\hat{X}$ is regarded as a subspace of $\mathsf{S}(\mathsf{CO}(X))$. By Example 3.9, $\hat{h}_X$ is a homeomorphism. Let $f : X \longrightarrow Y$ be a **ZHaus**-morphism. Then $G(F(f)) = \hat{f}$, and we have to show that the diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\hat{h}_X"'] & Y \arrow[d, "\hat{h}_Y"] \\ \hat{X} \arrow[r, "\hat{f}"] & \hat{Y} \end{tikzcd}$$

is commutative. For every $x \in X$, we have $\hat{h}_Y(f(x)) = \widehat{f(x)} = \hat{f}(\hat{x}) = \hat{f}(\hat{h}_X(x))$. Therefore,

$$\hat{h} : \mathsf{Id}_{\mathbf{ZHaus}} \longrightarrow G \circ F, \ \ X \mapsto \hat{h}_X,$$

is a natural isomorphism. All this shows that the categories **ZHaus** and **dzBoole** are dually equivalent. □

4. The Second Duality Theorem for the Category ZHaus

Now we will define a new category **mzMaps** and will show, using the Tarski duality, that it is equivalent to the category **dzBoole**. This will imply immediately that the category **mzMaps** is dually equivalent to the category **ZHaus**. The category **mzMaps** is similar to the category **MDeVe**, constructed in [4] as a category dually equivalent to the category **Tych** of Tychonoff spaces and continuous maps.

The key for the transition from the category **dzBoole** to the category **mzMaps** is the replacement of the component X of a dz-algebra (A, X) with the complete atomic Boolean algebra $\mathsf{P}(X)$ from which the set X can be reconstructed in the form $\mathrm{At}(\mathsf{P}(X))$. Thus, using the Tarski Duality which shows that every complete atomic Boolean algebra B is isomorphic to $\mathsf{P}(\mathrm{At}(B))$, we can take as a second component of a dz-algebra (A, X) the complete atomic Boolean algebra $B = \mathsf{P}(X)$ instead of the set X. This idea we borrow in fact from [4] although it is implicit there because in [4] there is no duality theorem similar to our Theorem 3.15. We realize it with the help of the notions of *Boolean z-map* and *maximal Boolean z-map* (see Definition 4.1 below) which are similar to the notions of *de Vries extension* and *maximal de Vries extension*, respectively, introduced in [4]. The similarity between Boolean z-maps and de Vries extensions can be seen looking at their definitions, while that one between maximal Boolean z-maps and maximal de Vries extensions is only in the fact that they are maximal elements of the preordered classes of all Boolean z-maps and all de Vries extensions, respectively. More precisely: as it is proved in [4], the de Vries extensions correspond to the compactifications of Tychonoff spaces, and the maximal de Vries extensions correspond to the Stone-Čech compactifications of Tychonoff spaces. In Section 8 below we will show that our Boolean z-maps correspond to the zero-dimensional compactifications of zero-dimensional Hausdorff spaces, and our maximal Boolean z-maps correspond to the Banaschewski compactifications of zero-dimensional Hausdorff spaces. It is well known that, firstly, the Stone-Čech compactification βX of a zero-dimensional Hausdorff space X is equivalent to the Banaschewski compactification $\beta_0 X$ of X (i.e., βX is zero-dimensional) if, and only if, the space X is strongly zero-dimensional, and, secondly, the class of strongly zero-dimensional Hausdorff spaces is a proper subclass of the class of zero-dimensional Hausdorff spaces (see, e.g., [12, Theorems 6.2.7 and 6.2.12, and the Dowker Example 6.2.20]). So that, instead of saying that the notions of maximal Boolean z-map and maximal de Vries extension are similar, it is better to say that they are parallel to each other.

Definition 4.1. Let A be a Boolean algebra and $B \in |\mathbf{Caba}|$. A Boolean monomorphism $\alpha : A \longrightarrow B$ is said to be a *Boolean z-map* (briefly, *z-map*) if every atom of B is a meet of some elements of $\alpha(A)$. A z-map $\alpha : A \longrightarrow B$ is called a *maximal Boolean z-map* (briefly, *mz-map*) if $\mathrm{CO}(X_\alpha) = s_A^{X_\alpha}(A)$, where X_α is regarded as a subspace of $\mathsf{S}(A)$ (see 2.4 and 3.3 for the notation).

Example 4.2. Let $X \in |\mathbf{ZHaus}|$, $A \in \mathcal{BA}(X)$ (see Definition 2.6 for this notation) and $i_A : A \hookrightarrow \mathsf{P}(X)$ be the inclusion monomorphism. Then i_A is a z-map.

Indeed, since A is a base for the Hausdorff space X, we have that for every $x \in X$, $\{x\} = \bigcap\{U \in A \mid x \in U\}$. Hence, i_A is a z-map.

Example 4.3. The map $id_B : B \longrightarrow B$, $b \mapsto b$, where $B \in |\mathbf{Caba}|$, is a mz-map. (The mz-maps of this type will be called *Boolean T-maps* (or, simply, *T-maps*)).

Indeed, it is obvious that id_B is a z-map. Setting $\alpha \stackrel{\mathrm{df}}{=} id_B$, we obtain, as in 2.4, that $X_\alpha = \dot{X}_B$. In Example 3.10, we proved that $\dot{X}_B$ is a discrete subspace of $\mathsf{S}(B)$ (and, thus, $\mathrm{CO}(\dot{X}_B) = P(\dot{X}_B)$) and $s_B^{\dot{X}_B}(B) = P(\dot{X}_B)$. This shows that id_B is a mz-map.

Example 4.4. Let $A \in |\mathbf{Boole}|$. Then the map $s_A^{\mathsf{S}(A)} : A \longrightarrow \mathsf{P}(\mathsf{S}(A))$, is a mz-map. (The mz-maps of this type will be called *compact mz-maps*. As we will see in Proposition 5.3 below, they correspond to the Boolean algebras and thus, by the Stone Duality, to compact zero-dimensional Hausdorff spaces.)

Indeed, by Example 3.8, $(A, \mathsf{S}(A))$ is a dz-algebra. Thus $s_A^{\mathsf{S}(A)}(A) = \mathsf{CO}(\mathsf{S}(A))$. Since $\mathsf{S}(A)$ is a Hausdorff space, and $\mathsf{CO}(\mathsf{S}(A))$ is a base for $\mathsf{S}(A)$, we obtain that $s_A^{\mathsf{S}(A)}$ is a z-map. Set $\alpha \stackrel{\mathrm{df}}{=} s_A^{\mathsf{S}(A)}$. Then $X_\alpha = \{\alpha_x : A \longrightarrow \mathbf{2} \mid x \in \mathsf{S}(A)\}$ and, for every $x \in \mathsf{S}(A)$ and every $a \in A$, $\alpha_x(a) = 1 \Leftrightarrow x \in \alpha(a) \Leftrightarrow x(a) = 1$. Thus, $\alpha_x \equiv x$. Hence, $X_\alpha = \mathsf{S}(A)$. Then $s_A^{X_\alpha}(A) = s_A^{\mathsf{S}(A)}(A) = \mathsf{CO}(\mathsf{S}(A)) = \mathsf{CO}(X_\alpha)$. Therefore, $s_A^{\mathsf{S}(A)}$ is a mz-map.

Example 4.5. Let (A, X) be a dz-algebra. Then $s_A^X : A \longrightarrow \mathsf{P}(X)$ is a mz-map (see Notation 3.3 for s_A^X).

Indeed, notice first that $\mathsf{P}(X) \in |\mathbf{Caba}|$ and, by Fact 3.4, s_A^X is a Boolean monomorphism. Furthermore, by Fact 3.5, the topology on X generated by the base $s_A^X(A)$ is a T_2-topology. Thus, for every $x \in X$, we have that $\{x\} = \bigcap\{s_A^X(a) \mid x(a) = 1\}$. Hence, s_A^X is a z-map. Set $\alpha \stackrel{\mathrm{df}}{=} s_A^X$ and $B \stackrel{\mathrm{df}}{=} \mathsf{P}(X)$. Then $\alpha : A \longrightarrow B$ and $\mathrm{At}(B) = X$. Since (A, X) is a dz-algebra, we have that $\alpha(A) = \mathrm{CO}(X)$. Using the notation from 2.4, we obtain that for every $x \in X = \mathrm{At}(B)$ and every $a \in A$, $\alpha_x(a) = 1 \Leftrightarrow x \leq \alpha(a) \Leftrightarrow x \in s_A(a) \Leftrightarrow x(a) = 1$. Thus, $x = \alpha_x$ for every $x \in X$. Hence $X \equiv X_\alpha$ and, therefore, $s_A^{X_\alpha}(A) = s_A^X(A) = \mathrm{CO}(X) = \mathrm{CO}(X_\alpha)$. This shows that s_A^X is a mz-map.

We will present an equivalent definition of the notion of mz-map as well. It express the definition of an mz-map in purely algebraic terms and is analogous to Fact 3.13. Its straightforward proof is left to the reader.

Proposition 4.6. *Let A be a Boolean algebra and $B \in |\mathbf{Caba}|$. A z-map $\alpha : A \longrightarrow B$ is an mz-map if and only if for every Boolean subalgebra C of B which is t-equal to $\alpha(A)$ in B, we have that $C \subseteq \alpha(A)$.*

The proof of the next assertion is obvious.

Proposition 4.7. *There is a category* **zMaps** *whose objects are all z-maps and whose morphisms between any two* **zMaps***-objects $\alpha : A \longrightarrow B$ and $\alpha' : A' \longrightarrow B'$ are all pairs (φ, σ) such that $\varphi \in$* **Boole**(A, A')*, $\sigma \in$* **Caba**(B, B') *and $\alpha' \circ \varphi = \sigma \circ \alpha$. The composition $(\varphi', \sigma') \circ (\varphi, \sigma)$ between two* **zMaps***-morphisms $(\varphi, \sigma) : \alpha \longrightarrow \alpha'$ and $(\varphi', \sigma') : \alpha' \longrightarrow \alpha''$ is defined to be the* **zMaps***-morphism $(\varphi' \circ \varphi, \sigma' \circ \sigma) : \alpha \longrightarrow \alpha''$; the identity map of a* **zMaps***-object $\alpha : A \longrightarrow B$ is defined to be (id_A, id_B).*

We denote by **mzMaps** the full subcategory of the category **zMaps** whose objects are all mz-maps.

Theorem 4.8. *The categories* **mzMaps** *and* **dzBoole** *are equivalent.*

Proof. We start with defining a functor $F' : \mathbf{dzBoole} \longrightarrow \mathbf{mzMaps}$.

For every $(A, X) \in |\mathbf{dzBoole}|$, set

$$F'(A, X) \stackrel{\text{df}}{=} s_A^X$$

(see Notation 3.3 for s_A^X). Then, by Example 4.5, $F'(A, X) \in |\mathbf{mzMaps}|$.

For every $(\varphi, f) \in \mathbf{dzBoole}((A, X), (A', X'))$, set

$$F'(\varphi, f) \stackrel{\text{df}}{=} (\varphi, \mathsf{P}(f)).$$

We have that $x' \circ \varphi = f(x')$ for every $x' \in X'$. Having this in mind, we obtain that for every $a \in A$, $(\mathsf{P}(f) \circ s_A^X)(a) = f^{-1}(\{x \in X \mid x(a) = 1\}) = \{x' \in X' \mid f(x')(a) = 1\} = \{x' \in X' \mid (x' \circ \varphi)(a) = 1\} = (s_{A'}^{X'} \circ \varphi)(a)$. Hence, $\mathsf{P}(f) \circ s_A^X = s_{A'}^{X'} \circ \varphi$. Since $\mathsf{P}(f) \in \mathbf{Caba}(\mathsf{P}(X), \mathsf{P}(X'))$, we obtain that $F'(\varphi, f) \in \mathbf{mzMaps}(F'(A, X), F'(A', X'))$. Now it is easy to see that F' is a functor.

Further, we will define a functor $G' : \mathbf{mzMaps} \longrightarrow \mathbf{dzBoole}$.

For every $(\alpha : A \longrightarrow B) \in |\mathbf{mzMaps}|$, we set, in the notation from 2.4,

$$G'(\alpha) \stackrel{\text{df}}{=} (\mathsf{CO}(X_\alpha), \widehat{X_\alpha}),$$

where X_α is regarded as a subspace of $\mathsf{S}(A)$. Hence $X_\alpha = \{\alpha_x : A \longrightarrow \mathbf{2} \mid x \in \mathrm{At}(B)\}$ and $\widehat{X_\alpha} = \{\widehat{\alpha_x} : \mathsf{CO}(X_\alpha) \longrightarrow \mathbf{2} \mid \alpha_x \in X_\alpha\}$. Obviously, $X_\alpha \in |\mathbf{ZHaus}|$. It is now clear that $G'(\alpha) = F(X_\alpha)$ (where F is the contravariant functor defined in the proof of Theorem 3.15) and, therefore, by Theorem 3.15, $G'(\alpha) \in |\mathbf{dzBoole}|$.

Let $(\varphi, \sigma) \in \mathbf{mzMaps}(\alpha, \alpha')$, where $\alpha : A \longrightarrow B$ and $\alpha' : A' \longrightarrow B'$. We set

$$G'(\varphi, \sigma) \stackrel{\text{df}}{=} (\mathsf{CO}(f_\sigma), \widehat{f_\sigma}),$$

where $f_\sigma : X_{\alpha'} \longrightarrow X_\alpha$ is defined by $\alpha'_{x'} \mapsto \alpha_{\mathsf{At}(\sigma)(x')}$ and $\widehat{f_\sigma} : \widehat{X_{\alpha'}} \longrightarrow \widehat{X_\alpha}$ is defined by $\widehat{\alpha'_{x'}} \mapsto \widehat{f_\sigma(\alpha'_{x'})}$. Clearly, $G'(\varphi, \sigma) = F(f_\sigma)$, so that we need only to show that f_σ is a continuous map between the sets $X_{\alpha'}$ and X_α supplied with the subspace topology from the spaces $\mathsf{S}(A')$ and $\mathsf{S}(A)$, respectively. Let $a \in A$ and $x' \in \mathrm{At}(B')$. Then, using Lemma 2.3, we obtain that

$$\begin{aligned}
(f_\sigma(\alpha'_{x'}))(a) = 1 \quad & \Leftrightarrow \alpha_{\mathsf{At}(\sigma)(x')}(a) = 1 \\
& \Leftrightarrow \mathsf{At}(\sigma)(x') \leq \alpha(a) \\
& \Leftrightarrow x' \leq \sigma(\alpha(a)) \\
& \Leftrightarrow x' \leq \alpha'(\varphi(a)) \\
& \Leftrightarrow \alpha'_{x'}(\varphi(a)) = 1 \\
& \Leftrightarrow (\alpha'_{x'} \circ \varphi)(a) = 1 \\
& \Leftrightarrow (\mathsf{S}(\varphi)(\alpha'_{x'}))(a) = 1.
\end{aligned}$$

Thus

$$f_\sigma : X_{\alpha'} \longrightarrow X_\alpha \text{ is a restriction of the map } \mathsf{S}(\varphi) : \mathsf{S}(A') \longrightarrow \mathsf{S}(A).$$

This implies the continuity of f_σ. Now, using Theorem 3.15, we conclude that $G'(\varphi, \sigma)$ is a **dzBoole**-morphism between $G'(\alpha)$ and $G'(\alpha')$. Having all this in mind, it is easy to see that G' is a functor.

We will now prove that $F' \circ G' \cong \mathsf{Id}_{\mathbf{mzMaps}}$.

Let $(\alpha : A \longrightarrow B) \in |\mathbf{mzMaps}|$. Then $(F' \circ G')(\alpha) = F'(\mathsf{CO}(X_\alpha), \widehat{X_\alpha}) = s_{\mathsf{CO}(X_\alpha)}^{\widehat{X_\alpha}}$ and $s_{\mathsf{CO}(X_\alpha)}^{\widehat{X_\alpha}} : \mathsf{CO}(X_\alpha) \longrightarrow \mathsf{P}(\widehat{X_\alpha})$, where X_α is regarded as a subspace of $\mathsf{S}(A)$. By 2.4, the map $h_\alpha : \mathrm{At}(B) \longrightarrow X_\alpha$, $x \mapsto \alpha_x$, is a bijection. Then, clearly, the map $h_\alpha^P : \mathsf{P}(\mathrm{At}(B)) \longrightarrow \mathsf{P}(X_\alpha)$, $M \mapsto \{h_\alpha(x) \mid x \in M\}$, is a Boolean isomorphism. Again by 2.4, the map $\hat{h}_{X_\alpha} : X_\alpha \longrightarrow \widehat{X_\alpha}$, $\alpha_x \mapsto \widehat{\alpha_x}$, for all $x \in \mathrm{At}(B)$, is a bijection. Then the map $\hat{h}_{X_\alpha}^P : \mathsf{P}(X_\alpha) \longrightarrow \mathsf{P}(\widehat{X_\alpha})$, $M \mapsto \{\hat{h}_{X_\alpha}(\alpha_x) \mid \alpha_x \in M\}$, is a Boolean isomorphism. Put $\widehat{\varepsilon_B} \stackrel{\text{df}}{=} \hat{h}_{X_\alpha}^P \circ h_\alpha^P \circ \varepsilon_B$ (see 2.2 for the notation ε_B). Then

$$\widehat{\varepsilon_B} : B \longrightarrow \mathsf{P}(\widehat{X_\alpha}), \quad b \mapsto \{\widehat{\alpha_x} \mid x \in \mathrm{At}(B), x \leq b\}$$

is a Boolean isomorphism. Since α is an mz-map, we have that $s_A^{X_\alpha}(A) = \mathrm{CO}(X_\alpha)$. Thus the map

$$\bar{s}_A^{X_\alpha} : A \longrightarrow \mathsf{CO}(X_\alpha),\ \ a \mapsto X_\alpha \cap s_A(a),$$

is a Boolean isomorphism. Put

$$\varepsilon'_\alpha \stackrel{\mathrm{df}}{=} (\bar{s}_A^{X_\alpha}, \widehat{\varepsilon_B}).$$

We will show that $\varepsilon'_\alpha \in \mathbf{mzMaps}(\alpha, (F' \circ G')(\alpha))$. We need only to prove that the diagram

$$\begin{tikzcd}
A \arrow[r, "\bar{s}_A^{X_\alpha}"] \arrow[d, "\alpha"'] & \mathsf{CO}(X_\alpha) \arrow[d, "s^{\widehat{X_\alpha}}_{\mathrm{CO}(X_\alpha)}"] \\
B \arrow[r, "\widehat{\varepsilon_B}"] & \mathsf{P}(\widehat{X_\alpha})
\end{tikzcd}$$

is commutative. Let $a \in A$. Then $s^{\widehat{X_\alpha}}_{\mathrm{CO}(X_\alpha)}(\bar{s}_A^{X_\alpha}(a)) = s^{\widehat{X_\alpha}}_{\mathrm{CO}(X_\alpha)}(\{\alpha_y \in X_\alpha \mid \alpha_y(a) = 1\}) = \{\widehat{\alpha_x} \in \widehat{X_\alpha} \mid \widehat{\alpha_x}(\{\alpha_y \in X_\alpha \mid y \leq \alpha(a)\}) = 1\} = \{\widehat{\alpha_x} \in \widehat{X_\alpha} \mid \alpha_x \in \{\alpha_y \in X_\alpha \mid y \leq \alpha(a)\}\} = \{\widehat{\alpha_x} \in \widehat{X_\alpha} \mid x \leq \alpha(a)\} = \widehat{\varepsilon_B}(\alpha(a))$. Obviously, this implies that ε'_α is an **mzMaps**-isomorphism.

Let $\alpha : A \longrightarrow B$ and $\alpha' : A' \longrightarrow B'$ be **mzMaps**-objects and $(\varphi, \sigma) : \alpha \longrightarrow \alpha'$ be an **mzMaps**-morphism. We will prove that the diagram

$$\begin{tikzcd}
\alpha \arrow[r, "(\varphi,\sigma)"] \arrow[d, "\varepsilon'_\alpha"'] & \alpha' \arrow[d, "\varepsilon'_{\alpha'}"] \\
s^{\widehat{X_\alpha}}_{\mathrm{CO}(X_\alpha)} \arrow[r, "F'(G'(\varphi,\sigma))"] & s^{\widehat{X_{\alpha'}}}_{\mathrm{CO}(X_{\alpha'})}
\end{tikzcd}$$

is commutative. We have that

$$F'(G'(\varphi, \sigma)) = F'(\mathsf{CO}(f_\sigma), \widehat{f_\sigma}) = (\mathsf{CO}(f_\sigma), \mathsf{P}(\widehat{f_\sigma}))$$

and

$$\varepsilon'_{\alpha'} \circ (\varphi, \sigma) = (\bar{s}_{A'}^{X_{\alpha'}}, \widehat{\varepsilon_{B'}}) \circ (\varphi, \sigma) = (\bar{s}_{A'}^{X_{\alpha'}} \circ \varphi, \widehat{\varepsilon_{B'}} \circ \sigma).$$

Also

$$F'(G'(\varphi, \sigma)) \circ \varepsilon'_\alpha = (\mathsf{CO}(f_\sigma), \mathsf{P}(\widehat{f_\sigma})) \circ (\bar{s}_A^{X_\alpha}, \widehat{\varepsilon_B}) = (\mathsf{CO}(f_\sigma) \circ \bar{s}_A^{X_\alpha}, \mathsf{P}(\widehat{f_\sigma}) \circ \widehat{\varepsilon_B}).$$

Hence, we have to show that

$$\bar{s}_{A'}^{X_{\alpha'}} \circ \varphi = \mathsf{CO}(f_\sigma) \circ \bar{s}_A^{X_\alpha} \quad \text{and} \quad \widehat{\varepsilon_{B'}} \circ \sigma = \mathsf{P}(\widehat{f_\sigma}) \circ \widehat{\varepsilon_B}.$$

Let $a \in A$. Then

$$\begin{aligned}
\mathsf{CO}(f_\sigma)(\bar{s}_A^{X_\alpha}(a)) &= f_\sigma^{-1}(\{\alpha_x \in X_\alpha \mid \alpha_x(a) = 1\}) \\
&= \{\alpha'_{x'} \in X_{\alpha'} \mid f_\sigma(\alpha'_{x'}) \in \{\alpha_x \in X_\alpha \mid \alpha_x(a) = 1\}\} \\
&= \{\alpha'_{x'} \in X_{\alpha'} \mid \alpha'_{x'}(\varphi(a)) = 1\} \\
&= \bar{s}_{A'}^{X_{\alpha'}}(\varphi(a)).
\end{aligned}$$

So, $\bar{s}_{A'}^{X_{\alpha'}} \circ \varphi = \mathsf{CO}(f_\sigma) \circ \bar{s}_A^{X_\alpha}$. Let now $b \in B$. Then, using Lemma 2.3, we obtain that

$$\begin{aligned}
\mathsf{P}(\widehat{f_\sigma})(\widehat{\varepsilon_B}(b)) &= \widehat{f_\sigma}^{-1}(\{\widehat{\alpha_x} \in \widehat{X_\alpha} \mid x \leq b\} \\
&= \{\widehat{\alpha'_{x'}} \in \widehat{X_{\alpha'}} \mid \widehat{f_\sigma}(x') \in \{\widehat{\alpha_x} \in \widehat{X_\alpha} \mid x \leq b\}\} \\
&= \{\widehat{\alpha'_{x'}} \in \widehat{X_{\alpha'}} \mid \widehat{f_\sigma(\alpha'_{x'})} \in \{\widehat{\alpha_x} \in \widehat{X_\alpha} \mid x \leq b\}\} \\
&= \{\widehat{\alpha'_{x'}} \in \widehat{X_{\alpha'}} \mid \mathsf{At}(\sigma)(x') \leq b\} \\
&= \{\widehat{\alpha'_{x'}} \in \widehat{X_{\alpha'}} \mid x' \leq \sigma(b)\} = \widehat{\varepsilon_{B'}}(\sigma(b)).
\end{aligned}$$

Hence, $\widehat{\varepsilon_{B'}} \circ \sigma = \mathsf{P}(\widehat{f_\sigma}) \circ \widehat{\varepsilon_B}$. This shows that $\varepsilon'_{\alpha'} \circ (\varphi, \sigma) = F'(G'(\varphi, \sigma)) \circ \varepsilon'_\alpha$. Therefore,

$$\varepsilon' : \mathsf{Id}_{\mathbf{mzMaps}} \longrightarrow F' \circ G', \ \ \alpha \mapsto \varepsilon'_\alpha,$$

is a natural isomorphism.

Finally, we will prove that $G' \circ F' \cong \mathsf{Id}_{\mathbf{dzBoole}}$.

Let $(A, X) \in |\mathbf{dzBoole}|$. Then $G'(F'(A, X)) = G'(s^X_A) = (\mathsf{CO}(X), \hat{X})$, where X is regarded as a subspace of $\mathsf{S}(A)$. Indeed, putting $\alpha \stackrel{\text{df}}{=} s^X_A$, we obtain, as in Example 4.4, that $\alpha_x \equiv x$ for every $x \in X$, and, hence,

$$X_\alpha \equiv X.$$

Thus, $\hat{x} : \mathsf{CO}(X) \longrightarrow \mathbf{2}$ is defined by $\hat{x}(U) = 1 \Leftrightarrow x \in U$, for $U \in \mathsf{CO}(X)$, and $\hat{X} = \{\hat{x} \mid x \in X\}$. Obviously, we have that $G'(F'(A, X)) = F(G(A, X))$, where F and G are the contravariant functors defined in the proof of Theorem 3.15. Hence, we can use the **dzBoole**-isomorphism

$$s'_{(A,X)} : (A, X) \longrightarrow G'(F'(A, X))$$

defined there by $s'_{(A,X)} \stackrel{\text{df}}{=} (\bar{s}^X_A, \check{\imath}_X)$, where $\check{\imath}_X : \hat{X} \longrightarrow X$, $\hat{x} \mapsto x$.

Let $(\varphi, f) \in \mathbf{dzBoole}((A, X), (A', X'))$. Then $G'(F'(\varphi, f)) = G'(\varphi, \mathsf{P}(f)) = (\mathsf{CO}(f_\sigma), \widehat{f_\sigma})$, where $\sigma \stackrel{\text{df}}{=} \mathsf{P}(f)$, $f_\sigma : X_{\alpha'} \longrightarrow X_\alpha$, $\alpha = F'(A, X) = s^X_A$ and $\alpha' = F'(A', X') = s^{X'}_{A'}$. Since $X_{\alpha'} \equiv X'$ and $X_\alpha \equiv X$, we obtain that $f_\sigma(x') = \mathsf{At}(\sigma)(x') = \mathsf{At}(\mathsf{P}(f))(x') = f(x')$, i.e., $f_\sigma \equiv f$. Thus $G'(F'(\varphi, f)) = (\mathsf{CO}(f), \hat{f}) = F(G(\varphi, f))$. Thus the proof of the commutativity of the diagram

$$\begin{tikzcd}
(A, X) \arrow[r, "{(\varphi,f)}"] \arrow[d, "s'_{(A,X)}"'] & (A', X') \arrow[d, "s'_{(A',X')}"] \\
G'(F'(A, X)) \arrow[r, "{(G' \circ F')(\varphi,f)}"] & G'(F'(A', X'))
\end{tikzcd}$$

proceeds as in the proof of Theorem 3.15. Therefore,

$$s' : \mathsf{Id}_{\mathbf{dzBoole}} \longrightarrow G' \circ F', \ \ (A, X) \mapsto s'_{(A,X)},$$

is a natural isomorphism.

All this shows that the categories **mzMaps** and **dzBoole** are equivalent. □

Obviously, Theorems 3.15 and 4.8 imply the following theorem:

Theorem 4.9. *The categories* **ZHaus** *and* **mzMaps** *are dually equivalent.*

Proof. We put $\mathsf{F}_0 \stackrel{\text{df}}{=} F' \circ F$ and $\mathsf{G}_0 \stackrel{\text{df}}{=} G \circ G'$. Then

$$\mathsf{F}_0 : \mathbf{ZHaus} \longrightarrow \mathbf{mzMaps} \ \text{ and } \ \mathsf{G}_0 : \mathbf{mzMaps} \longrightarrow \mathbf{ZHaus}.$$

Clearly, they are dual equivalences. In the rest of this proof, we will find the explicit descriptions of these contravariant functors, as well as the descriptions of the natural isomorphisms $\tilde{\eta}^0 : \mathsf{Id}_{\mathbf{ZHaus}} \longrightarrow \mathsf{G}_0 \circ \mathsf{F}_0$ and $\tilde{\varepsilon}^0 : \mathsf{Id}_{\mathbf{mzMaps}} \longrightarrow \mathsf{F}_0 \circ \mathsf{G}_0$. Moreover, we will define two new contravariant functors

$$\mathsf{F} : \mathbf{ZHaus} \longrightarrow \mathbf{mzMaps} \ \text{ and } \ \mathsf{G} : \mathbf{mzMaps} \longrightarrow \mathbf{ZHaus}$$

which are simpler than F_0 and G_0 but are again dual equivalences.

For every $X \in |\mathbf{ZHaus}|$, we have that

$$\mathsf{F}_0(X) = F'(\mathsf{CO}(X), \hat{X}) = s^{\hat{X}}_{\mathsf{CO}(X)}.$$

For every $f \in \mathbf{ZHaus}(X,Y)$,

$$\mathsf{F}_0(f) = F'(\mathsf{CO}(f), \hat{f}) = (\mathsf{CO}(f), \mathsf{P}(\hat{f}))$$

(see the beginning of the proof of Theorem 3.15 for the notation $\hat{f}$).

For every $(\alpha : A \longrightarrow B) \in |\mathbf{mzMaps}|$,

$$\mathsf{G}_0(\alpha) = G(\mathsf{CO}(X_\alpha), \widehat{X_\alpha}) = \widehat{X_\alpha},$$

where X_α is regarded as a subspace of $\mathsf{S}(A)$.

For $(\varphi, \sigma) \in \mathbf{mzMaps}(\alpha, \alpha')$,

$$\mathsf{G}_0(\varphi, \sigma) = G(\mathsf{CO}(f_\sigma), \widehat{f_\sigma}) = \widehat{f_\sigma}$$

(see the definition of G' in the proof of Theorem 4.8 for the notation f_σ and $\widehat{f_\sigma}$).

Now, for every $X \in |\mathbf{ZHaus}|$, we have that

$$(\mathsf{G}_0 \circ \mathsf{F}_0)(X) = \mathsf{G}_0(s^{\hat{X}}_{\mathsf{CO}(X)}) = \widehat{X_\alpha}, \text{ where } \alpha = s^{\hat{X}}_{\mathsf{CO}(X)}.$$

Hence $X_\alpha = \{\alpha_{\hat{x}} : \mathsf{CO}(X) \longrightarrow \mathbf{2} \mid \hat{x} \in \hat{X}\}$ and, for every $U \in \mathsf{CO}(X)$ and every $x \in X$, $\alpha_{\hat{x}}(U) = 1 \Leftrightarrow \hat{x} \in \alpha(U) \Leftrightarrow \hat{x}(U) = 1$. Thus, $\alpha_{\hat{x}} = \hat{x}$ for every $x \in X$. Hence, $X_\alpha = \hat{X}$ and $(\mathsf{G}_0 \circ \mathsf{F}_0)(X) = \widehat{X_\alpha} = \hat{\hat{X}}$. According to the general theorem about compositions of adjoint functors (see, e.g., [16, Theorem IV.8.1]), we have that for every $X \in |\mathbf{ZHaus}|$, $\tilde{\eta}^0_X : X \longrightarrow (\mathsf{G}_0 \circ \mathsf{F}_0)(X)$ is defined by the formula $\tilde{\eta}^0_X = (G(s'_{F(X)}))^{-1} \circ \hat{h}_X$ (see Theorem 3.15 for $\hat{h}$ and s'). Since $F(X) = (\mathsf{CO}(X), \hat{X})$ and $s'_{F(X)} = (\bar{s}^{\hat{X}}_{\mathsf{CO}(X)}, \check{\imath}_{\hat{X}})$, where $\check{\imath}_{\hat{X}} : \hat{\hat{X}} \longrightarrow \hat{X}$, $\hat{\hat{x}} \mapsto \hat{x}$, we obtain that $G(s'_{F(X)}) = \check{\imath}_{\hat{X}}$. Thus,

$$\tilde{\eta}^0_X(x) = \{\hat{x}\},$$

for every $x \in X$. Finally, note that $\mathsf{CO}(\hat{X}) = \alpha(\mathsf{CO}(X))$ (because $\alpha = s^{\hat{X}}_{\mathsf{CO}(X)}$ is an mz-map) and thus $\hat{x} : \mathsf{CO}(\hat{X}) \longrightarrow \mathbf{2}$ is defined by $\hat{x}(\alpha(U)) = 1 \Leftrightarrow \hat{x} \in s^{\hat{X}}_{\mathsf{CO}(X)}(U) \Leftrightarrow \hat{x}(U) = 1 \Leftrightarrow x \in U$, for every $x \in X$ and every $U \in \mathsf{CO}(X)$.

We will now describe the natural isomorphism $\tilde{\varepsilon}^0 : \mathsf{Id}_{\mathbf{mzMaps}} \longrightarrow \mathsf{F}_0 \circ \mathsf{G}_0$. For $(\alpha : A \longrightarrow B) \in |\mathbf{mzMaps}|$, we have that

$$(\mathsf{F}_0 \circ \mathsf{G}_0)(\alpha) = \mathsf{F}_0(\widehat{X_\alpha}) = s^{\widehat{\widehat{X_\alpha}}}_{\mathsf{CO}(\widehat{X_\alpha})}$$

and $s^{\widehat{\widehat{X_\alpha}}}_{\mathsf{CO}(\widehat{X_\alpha})} : \mathsf{CO}(\widehat{X_\alpha}) \longrightarrow \mathsf{P}(\widehat{\widehat{X_\alpha}})$, where $\widehat{\widehat{X_\alpha}} = \{\widehat{\widehat{\alpha_x}} : \mathsf{CO}(\widehat{X_\alpha}) \longrightarrow \mathbf{2} \mid \widehat{\alpha_x} \in \widehat{X_\alpha}\}$ and, for $U \in \mathsf{CO}(\widehat{X_\alpha})$, $\widehat{\widehat{\alpha_x}}(U) = 1 \Leftrightarrow \widehat{\alpha_x} \in U$. Thus $\tilde{\varepsilon}^0_\alpha : \alpha \longrightarrow s^{\widehat{\widehat{X_\alpha}}}_{\mathsf{CO}(\widehat{X_\alpha})}$. The cited above theorem about compositions of adjoint functors gives us that $\tilde{\varepsilon}^0_\alpha = F'(s'_{G'(\alpha)}) \circ \varepsilon'_\alpha$. We have that $G'(\alpha) = (\mathsf{CO}(X_\alpha), \widehat{X_\alpha})$ and thus $s'_{G'(\alpha)} = (\bar{s}^{\widehat{X_\alpha}}_{\mathsf{CO}(X_\alpha)}, \check{\imath}_{\widehat{X_\alpha}})$, where

$$\check{\imath}_{\widehat{X_\alpha}} : \widehat{\widehat{X_\alpha}} \longrightarrow \widehat{X_\alpha},\ \ \widehat{\widehat{\alpha_x}} \mapsto \widehat{\alpha_x},\ \text{ and } \bar{s}^{\widehat{X_\alpha}}_{\mathsf{CO}(X_\alpha)} : \mathsf{CO}(X_\alpha) \longrightarrow \mathsf{P}(\widehat{X_\alpha}).$$

Then $F'(s'_{G'(\alpha)}) = (\bar{s}^{\widehat{X_\alpha}}_{\mathsf{CO}(X_\alpha)}, \mathsf{P}(\check{\imath}_{\widehat{X_\alpha}}))$. Hence,

$$\begin{aligned}
\tilde{\varepsilon}^0_\alpha &= (\bar{s}^{\widehat{X_\alpha}}_{\mathsf{CO}(X_\alpha)}, \mathsf{P}(\check{\imath}_{\widehat{X_\alpha}})) \circ \varepsilon'_\alpha \\
&= (\bar{s}^{\widehat{X_\alpha}}_{\mathsf{CO}(X_\alpha)}, \mathsf{P}(\check{\imath}_{\widehat{X_\alpha}})) \circ (\bar{s}^{X_\alpha}_A, \widehat{\varepsilon_B}) \\
&= (\bar{s}^{\widehat{X_\alpha}}_{\mathsf{CO}(X_\alpha)} \circ \bar{s}^{X_\alpha}_A, \mathsf{P}(\check{\imath}_{\widehat{X_\alpha}}) \circ \widehat{\varepsilon_B}),
\end{aligned}$$

where $\widehat{\varepsilon_B} : B \longrightarrow \mathsf{P}(\widehat{X_\alpha})$, $b \mapsto \{\widehat{\alpha_x} \mid x \in \mathrm{At}(B), x \leq b\}$.

Now we will define the contravariant functors $\mathsf{F} : \mathbf{ZHaus} \longrightarrow \mathbf{mzMaps}$ and $\mathsf{G} : \mathbf{mzMaps} \longrightarrow \mathbf{ZHaus}$.

For every $X \in |\mathbf{ZHaus}|$, we put

$$\mathsf{F}(X) \stackrel{\text{df}}{=} i_X,$$

where $i_X : \mathsf{CO}(X) \longrightarrow \mathsf{P}(X)$ is the inclusion map. Set $\alpha \stackrel{\text{df}}{=} i_X$. Obviously, α is a z-map. Further, for every $x \in X = \mathrm{At}(\mathsf{P}(X))$, $\alpha_x : \mathsf{CO}(X) \longrightarrow \mathbf{2}$ and $\alpha_x(U) = 1 \Leftrightarrow x \in \alpha(U)$, for every $U \in \mathsf{CO}(X)$. Since $\alpha(U) = U$, we obtain that $\alpha_x = \hat{x}$ and thus $X_\alpha = \hat{X}$. For every $U \in \mathsf{CO}(X)$, we have that $s^{\hat{X}}_{\mathsf{CO}(X)}(U) = \{\hat{x} \mid x \in X,\ \hat{x}(U) = 1\} = \{\hat{x} \mid x \in U\} = \hat{U} = \hat{h}_X(U)$. Thus $s^{\hat{X}}_{\mathsf{CO}(X)}(\mathsf{CO}(X)) = \hat{h}_X(\mathsf{CO}(X)) = \mathsf{CO}(\hat{X})$ because $\hat{h}_X : X \longrightarrow \hat{X}$ is a homeomorphism (as it is shown in Example 3.9). Hence, i_X is an mz-map.

For $f \in \mathbf{ZHaus}(X, Y)$, we set

$$\mathsf{F}(f) \stackrel{\text{df}}{=} (\mathsf{CO}(f), \mathsf{P}(f)).$$

Obviously, $\mathsf{F}(f)$ is a **mzMaps**-morphism.

For $(\alpha : A \longrightarrow B) \in |\mathbf{mzMaps}|$, we put

$$\mathsf{G}(\alpha) \stackrel{\text{df}}{=} X_\alpha.$$

Clearly, the set X_α endowed with the subspace topology from the space $\mathsf{S}(A)$ is a **ZHaus**-object.

For $(\varphi, \sigma) \in \mathbf{mzMaps}(\alpha, \alpha')$, we set

$$\mathsf{G}(\varphi, \sigma) \stackrel{\text{df}}{=} f_\sigma.$$

The fact that f_σ is a continuous map was proved in Theorem 4.8 after the definition of G' on the morphisms.

We define a natural isomorphism $\tau : \mathsf{F}_0 \longrightarrow \mathsf{F}$ by

$$\tau_X \stackrel{\text{df}}{=} (id_{\mathsf{CO}(X)}, \check{i}^P_X)$$

for every $X \in |\mathbf{ZHaus}|$, where $\check{i}_X : \hat{X} \longrightarrow X$, $\hat{x} \mapsto x$, and $\check{i}^P_X : \mathsf{P}(\hat{X}) \longrightarrow \mathsf{P}(X)$, $\hat{M} \mapsto \{\check{i}_X(\hat{m}) \mid \hat{m} \in \hat{M}\}$, (i.e., $\check{i}^P_X(\hat{M}) = M$, for every $M \subseteq X$). Indeed, it is obvious that for every $X \in |\mathbf{ZHaus}|$, $\tau_X : \mathsf{F}_0(X) \longrightarrow \mathsf{F}(X)$ is a **mzMaps**-isomorphism and that, for every $f \in \mathbf{ZHaus}(X, X')$, the diagram

$$\begin{tikzcd}
\mathsf{F}_0(X') \arrow[r, "\mathsf{F}_0(f)"] \arrow[d, "\tau_{X'}"'] & \mathsf{F}_0(X) \arrow[d, "\tau_X"] \\
\mathsf{F}(X') \arrow[r, "\mathsf{F}(f)"] & \mathsf{F}(X)
\end{tikzcd}$$

is commutative.

Now we define a natural isomorphism $\tau' : \mathsf{G}_0 \longrightarrow \mathsf{G}$ by

$$\tau'_\alpha \stackrel{\text{df}}{=} \check{i}_{X_\alpha}$$

for every $\alpha \in |\mathbf{mzMaps}|$. Indeed, for every $X \in |\mathbf{ZHaus}|$, the map $\check{i}_X : \hat{X} \longrightarrow X$ is a homeomorphism since $\check{i}_X = \hat{h}_X^{-1}$ and the map $\hat{h}_X : X \longrightarrow \hat{X}$ is a homeomorphism; hence, $\tau'_\alpha : \mathsf{G}_0(\alpha) \longrightarrow \mathsf{G}(\alpha)$ is a **ZHaus**-isomorphism. Also, it is clear that, for every $(\varphi, \sigma) \in \mathbf{mzMaps}(\alpha, \alpha')$, the diagram

$$\begin{tikzcd}
\mathsf{G}_0(\alpha') \arrow[r, "\mathsf{G}_0(\varphi,\sigma)"] \arrow[d, "\tau'_{\alpha'}"'] & \mathsf{G}_0(\alpha) \arrow[d, "\tau'_\alpha"] \\
\mathsf{G}(\alpha') \arrow[r, "\mathsf{G}(\varphi,\sigma)"] & \mathsf{G}(\alpha)
\end{tikzcd}$$

is commutative.

Hence, we obtain that $\tau' * \tau : \mathsf{G}_0 \circ \mathsf{F}_0 \longrightarrow \mathsf{G} \circ \mathsf{F}$, where $(\tau' * \tau)_X = \tau'_{\mathsf{F}(X)} \circ \mathsf{G}_0(\tau_X^{-1})$ for every $X \in |\mathbf{ZHaus}|$, is a natural isomorphism (see, e.g., [1, Exercise 6A]) and thus

$$\tilde{\eta} = (\tau' * \tau) \circ \tilde{\eta}^0 : \mathsf{Id}_{\mathbf{ZHaus}} \longrightarrow \mathsf{G} \circ \mathsf{F}$$

is a natural isomorphism. Analogously, $\tau * \tau' : \mathsf{F}_0 \circ \mathsf{G}_0 \longrightarrow \mathsf{F} \circ \mathsf{G}$, where $(\tau * \tau')_\alpha = \tau_{\mathsf{G}(\alpha)} \circ \mathsf{F}_0((\tau'_\alpha)^{-1})$ for every $\alpha \in |\mathbf{mzMaps}|$, is a natural isomorphism and thus

$$\tilde{\varepsilon} = (\tau * \tau') \circ \tilde{\varepsilon}^0 : \mathsf{Id}_{\mathbf{mzMaps}} \longrightarrow \mathsf{F} \circ \mathsf{G}$$

is a natural isomorphism. Therefore, F and G are dual equivalences. It is now easy to obtain that, for every $X \in |\mathbf{ZHaus}|$ and every $x \in X$,

$$\tilde{\eta}_X(x) = \{\hat{x}\}$$

and, for every $(\alpha : A \longrightarrow B) \in |\mathbf{mzMaps}|$,

$$\tilde{\varepsilon}_\alpha = (\bar{s}_A^{X_\alpha}, \varepsilon_B^\alpha),$$

where $\varepsilon_B^\alpha : B \longrightarrow \mathsf{P}(X_\alpha)$, $b \mapsto \{\alpha_x \mid x \in \mathrm{At}(B), x \leq b\}$, for every $b \in B$. □

5. The Dual Equivalences F, G, F and G are Extensions of the Stone Dual Equivalences T and S

In this section we will show that our dual equivalences F, G, F and G can be regarded as extensions of the Stone dual equivalences T and S. For doing this we will first describe the subcategories of the categories **dzBoole** and **mzMaps** which are isomorphic to the category **Boole**.

We start with realizing our plan for the dual equivalences F, G.

Let us denote by **kBoole** the full subcategory of the category **dzBoole** having as objects all compact dz-algebras (see Example 3.8 for this notion).

Proposition 5.1. *The categories* **Boole** *and* **kBoole** *are isomorphic.*

Proof. Define a functor $E : \mathbf{Boole} \longrightarrow \mathbf{kBoole}$ by setting $E(A) \stackrel{\text{df}}{=} (A, \mathbf{Boole}(A, \mathbf{2}))$, for every $A \in |\mathbf{Boole}|$ (see Example 3.8 (or 2.1) for the notation), and $E(\varphi) \stackrel{\text{df}}{=} (\varphi, \mathsf{S}(\varphi))$, for every **Boole**-morphism φ. Then, by Example 3.8, $E(A) \in |\mathbf{kBoole}|$ for every $A \in |\mathbf{Boole}|$. If $\varphi \in \mathbf{Boole}(A, A')$, then $(\mathsf{S}(\varphi))(x') = x' \circ \varphi$ for every $x' \in \mathbf{Boole}(A', \mathbf{2})$ (see 2.1). Hence $E(\varphi) \in \mathbf{kBoole}(E(A), E(A'))$.

Define also a functor $E^{-1} : \mathbf{kBoole} \longrightarrow \mathbf{Boole}$ by setting

$$E^{-1}(A, \mathbf{Boole}(A, \mathbf{2})) \stackrel{\text{df}}{=} A,$$

for every $(A, \mathbf{Boole}(A, \mathbf{2})) \in |\mathbf{kBoole}|$, and

$$E^{-1}(\varphi, f) \stackrel{\text{df}}{=} \varphi,$$

for every **kBoole**-morphism (φ, f). It is easy to see that $E \circ E^{-1} = \mathsf{Id}_{\mathbf{kBoole}}$ and $E^{-1} \circ E = \mathsf{Id}_{\mathbf{Boole}}$. (Indeed, it is enough to notice that if (φ, f) is a **kBoole**-morphism then, by the definition of $\mathsf{S}(\varphi)$ (see 2.1), we have that $f = \mathsf{S}(\varphi)$.) Thus E and E^{-1} are isomorphisms. □

Proposition 5.2. *Let* $E^s : \mathbf{Stone} \hookrightarrow \mathbf{ZHaus}$ *and* $E^a : \mathbf{kBoole} \hookrightarrow \mathbf{dzBoole}$ *be the inclusion functors. Then*

$$F(E^s(|\mathbf{Stone}|)) \subseteq |\mathbf{kBoole}| \ \ \textit{and} \ \ G(E^a(|\mathbf{kBoole}|)) \subseteq |\mathbf{Stone}|.$$

Thus the restrictions $F_s : \mathbf{Stone} \longrightarrow \mathbf{kBoole}$ *and* $G_s : \mathbf{kBoole} \longrightarrow \mathbf{Stone}$ *of F and G, respectively, are dual equivalences. Also,* $\mathsf{T} = E^{-1} \circ F_s$ *and* $\mathsf{S} = G_s \circ E$*. Thus,* T *and* S *are dual equivalences. Finally,* $F \circ E^s = E^a \circ E \circ \mathsf{T}$ *and* $E^s \circ \mathsf{S} = G \circ E^a \circ E$*. Therefore, the dual equivalences F and G are extensions of the dual equivalences* T *and* S*, respectively. (See Theorem 3.15, Proposition 5.1 and 2.1 for the notation.)*

Proof. Let $X \in |\mathbf{Stone}|$. Then $F(E^s(X)) = F(X) = (\mathsf{CO}(X), \hat{X})$. Since X is compact, we have, as it is well-known, that $\hat{X} = \mathbf{Boole}(\mathsf{CO}(X), \mathbf{2})$. (Indeed, for every $\varphi \in \mathbf{Boole}(\mathsf{CO}(X), \mathbf{2})$, $\bigcap\{U \in \mathsf{CO}(X) \mid \varphi(U) = 1\}$ is a singleton.) Thus $F(E^s(X)) \in |\mathbf{kBoole}|$. Further, for every $(A, \mathbf{Boole}(A, \mathbf{2})) \in |\mathbf{kBoole}|$, $G(E^a(A, \mathbf{Boole}(A, \mathbf{2})) = G(A, \mathbf{Boole}(A, \mathbf{2})) = \mathsf{S}(A)$ and, as it is proved by M. Stone [17], $\mathsf{S}(A) \in |\mathbf{Stone}|$. Thus, Theorem 3.15 implies that F_s and G_s are dual equivalences. The equalities $\mathsf{T} = E^{-1} \circ F_s$ and $\mathsf{S} = G_s \circ E$ are obvious and hence, $\mathsf{S} \circ \mathsf{T} = G_s \circ E \circ E^{-1} \circ F_s = G_s \circ F_s \cong \mathsf{Id}_{\mathbf{Stone}}$; analogously, $\mathsf{T} \circ \mathsf{S} \cong \mathsf{Id}_{\mathbf{Boole}}$. Therefore, T and S are dual equivalences. Finally, we have that $E^a \circ E \circ \mathsf{T} = E^a \circ E \circ E^{-1} \circ F_s = E^a \circ F_s = F \circ E^s$ and $E^s \circ \mathsf{S} = E^s \circ G_s \circ E = G \circ E^a \circ E$. □

$$\begin{tikzcd}
\mathbf{ZHaus} \arrow[r, bend left, "F"] \arrow[r, phantom, "\simeq"] & \mathbf{dzBoole} \arrow[l, bend left, "G"] \\
& \mathbf{kBoole} \arrow[u, hook, "E^a"'] \arrow[dl, bend left, "G_s"] \\
\mathbf{Stone} \arrow[uu, hook, "E^s"] \arrow[ur, bend left, "F_s"] \arrow[ur, phantom, "\simeq"] \arrow[r, bend left, "\mathsf{T}"] \arrow[r, phantom, "\simeq"] & \mathbf{Boole} \arrow[u, "E"'] \arrow[l, bend left, "\mathsf{S}"]
\end{tikzcd}$$

We are now going to work with the dual equivalences F and G.

Let **kMaps** be the full subcategory of the category **mzMaps** having as objects all compact mz-maps (see Example 4.4 for this notion).

Proposition 5.3. *The categories* **Boole** *and* **kMaps** *are isomorphic.*

Proof. Let us define a functor $K : \mathbf{Boole} \longrightarrow \mathbf{kMaps}$ by setting $K(A) \stackrel{\text{df}}{=} s_A^{\mathsf{S}(A)}$ for every $A \in |\mathbf{Boole}|$, and $K(\varphi) \stackrel{\text{df}}{=} (\varphi, \mathsf{P}(\mathsf{S}(\varphi)))$, for every $\varphi \in \mathbf{Boole}(A, A')$. Then Example 4.4 shows that K is well-defined on the objects. For proving that $K(\varphi)$ is a **kMaps**-morphism, we have to verify the equality $s_{A'}^{\mathsf{S}(A')} \circ \varphi = \mathsf{P}(\mathsf{S}(\varphi)) \circ s_A^{\mathsf{S}(A)}$. Let $a \in A$. Then $(\mathsf{P}(\mathsf{S}(\varphi)) \circ s_A^{\mathsf{S}(A)})(a) = (\mathsf{S}(\varphi))^{-1}(s_A^{\mathsf{S}(A)}(a)) = \{x' \in \mathsf{S}(A') \mid (\mathsf{S}(\varphi))(x') \in s_A^{\mathsf{S}(A)}(a)\} = \{x' \in \mathsf{S}(A') \mid x'(\varphi(a)) = 1\} = (s_{A'}^{\mathsf{S}(A')} \circ \varphi)(a)$. Hence, K is well-defined on morphisms as well. Obviously, K is a functor. (Note that the use of the contravariant functors S and T can be easily avoided; we used them just for a simplification of the notation.)

Let us now define a functor $K^{-1} : \mathbf{kMaps} \longrightarrow \mathbf{Boole}$ by setting $K^{-1}(s_A^{\mathsf{S}(A)}) \stackrel{\text{df}}{=} A$ for every $A \in |\mathbf{Boole}|$, and $K^{-1}(\varphi, \sigma) \stackrel{\text{df}}{=} \varphi$ for every **kMaps**-morphism (φ, σ). Then, obviously, K^{-1} is a well-defined functor. It is clear that $K^{-1} \circ K = \mathsf{Id}_{\mathbf{Boole}}$ and $(K \circ K^{-1})(s_A^{\mathsf{S}(A)}) = s_A^{\mathsf{S}(A)}$ for every $A \in |\mathbf{Boole}|$. For every **kMaps**-morphism (φ, σ), we have $(K \circ K^{-1})(\varphi, \sigma) = K(\varphi) = (\varphi, \mathsf{P}(\mathsf{S}(\varphi))$. Since $s_A^{\mathsf{S}(A)} \upharpoonright A = s_A : A \longrightarrow \mathsf{CO}(\mathsf{S}(A))$ is a Boolean isomorphism, the above calculation shows that $\sigma | \mathsf{CO}(\mathsf{S}(A)) \equiv \mathsf{P}(\mathsf{S}(\varphi)) | \mathsf{CO}(\mathsf{S}(A))$. Since every atom of $\mathsf{P}(\mathsf{S}(A))$ (i.e., every element of $\mathsf{S}(A)$) is a meet in $\mathsf{P}(\mathsf{S}(A))$ of some elements of $\mathsf{CO}(\mathsf{S}(A))$ and σ is a complete homomorphisms, we see that σ is uniquely determined by its restriction on $\mathsf{CO}(\mathsf{S}(A))$. Therefore, $\sigma \equiv \mathsf{P}(\mathsf{S}(\varphi))$. Thus, $K \circ K^{-1} = \mathsf{Id}_{\mathbf{kMaps}}$. Hence, the categories **Boole** and **kMaps** are isomorphic. □

Now, using arguments similar to those used in the proof of Proposition 5.2, we obtain the following assertion:

Proposition 5.4. *Let* $E^m : \mathbf{kMaps} \hookrightarrow \mathbf{mzMaps}$ *be the inclusion functor. Then*

$$\mathsf{F}(E^s(|\mathbf{Stone}|)) \subseteq |\mathbf{kMaps}| \ \text{ and } \ \mathsf{G}(E^m(|\mathbf{kMaps}|)) \subseteq |\mathbf{Stone}|.$$

Thus the restrictions $\mathsf{F}_s : \mathbf{Stone} \longrightarrow \mathbf{kMaps}$ *and* $\mathsf{G}_s : \mathbf{kMaps} \longrightarrow \mathbf{Stone}$ *of* F *and* G*, respectively, are dual equivalences. Also,* $\mathsf{T} = K^{-1} \circ \mathsf{F}_s$ *and* $\mathsf{S} = \mathsf{G}_s \circ K$*. Thus,* T *and* S *are dual equivalences. Finally,* $\mathsf{F} \circ E^s = E^m \circ K \circ \mathsf{T}$ *and* $E^s \circ \mathsf{S} = \mathsf{G} \circ E^m \circ K$*. Therefore, the dual equivalences* F *and* G *are extensions of the dual equivalences* T *and* S*, respectively. (See Theorem 4.9, Propositions 5.3 and 5.2, and 2.1 for the notation.)*

$$\begin{tikzcd}
\mathbf{ZHaus} \arrow[r, bend left, "\mathsf{F}"] \arrow[r, phantom, "\simeq"] & \mathbf{mzMaps} \arrow[l, bend left, "\mathsf{G}"] \\
 & \mathbf{kMaps} \arrow[u, hook, "E^m"] \arrow[dl, bend left, "G_s"] \\
\mathbf{Stone} \arrow[uu, hook, "E^s"] \arrow[ur, bend left, "F_s"] \arrow[ur, phantom, "\simeq"] \arrow[r, bend left, "\mathsf{T}"] \arrow[r, phantom, "\simeq"] & \mathbf{Boole} \arrow[l, bend left, "\mathsf{S}"] \arrow[u, "K"']
\end{tikzcd}$$

6. The Restriction of F to the Category D Implies the Tarski Duality

In this section we are going to show that with our duality theorems 3.15 and 4.9 we extend the Tarski Duality Theorem as well. Moreover, we will obtain as a corollary of our Theorem 3.15 a slightly different version of the Tarski Duality Theorem which, maybe, is new. Then we will show that it implies easily the classical version of the Tarski Duality Theorem.

It is clear that the category **D** of discrete spaces and continuous maps is a full subcategory of the category **ZHaus**. Using the duality theorems proved in Sections 3 and 4, we will find two categories dually equivalent to the category **D**. Since, obviously, the categories **D** and **Set** are isomorphic, we will obtain in this way two categories dually equivalent to the category **Set**. Both of them will lead to one and the same dual equivalence $\mathsf{A} : \mathbf{Caba} \longrightarrow \mathbf{Set}$ which will be slightly different from the Tarski dual equivalence $\mathsf{At} : \mathbf{Caba} \longrightarrow \mathbf{Set}$. From it we will easily obtain the Tarski Duality Theorem. Hence, both of our duality theorems extend the Tarski Duality Theorem. Since in the proof of our Theorem 3.15 we have not used the Tarski Duality Theorem, we will obtain in this way a new proof of the latter one.

Let us denote by **TBoole** the full subcategory of the category **dzBoole** having as objects all T-algebras (see Example 3.10 for this notion), and let **TMaps** be the full subcategory of the category **mzMaps** having as objects all T-maps (see Example 4.3 for this notion).

Proposition 6.1. *The categories* **TBoole** *and* **TMaps** *are dually equivalent to the category* **D** *(and, thus, to the category* **Set***).*

Proof. Using the notation from the proofs of Theorems 3.15 and 4.9, it is enough to show that $F(|\mathbf{D}|) \subseteq |\mathbf{TBoole}|$, $G(|\mathbf{TBoole}|) \subseteq |\mathbf{D}|$, $\mathsf{F}(|\mathbf{D}|) \subseteq |\mathbf{TMaps}|$ and $\mathsf{G}(|\mathbf{TMaps}|) \subseteq |\mathbf{D}|$.

We have that for every $X \in |\mathbf{D}|$, $F(X) = (\mathsf{CO}(X), \hat{X}) = (\mathsf{P}(X), \hat{X}) = (B, \dot{X}_B)$, where $B \stackrel{\text{df}}{=} \mathsf{P}(X)$, and, obviously, $(B, \dot{X}_B) \in |\mathbf{TBoole}|$. Also, $\mathsf{F}(X) = i_X = id_{\mathsf{P}(X)} \in |\mathbf{TMaps}|$. Further, for every $(B, \dot{X}_B) \in |\mathbf{TBoole}|$, $G(B, \dot{X}_B) = \dot{X}_B$, where $\dot{X}_B$ is regarded as a subspace of $\mathsf{S}(B)$. Then, as it was shown in Example 3.10, $\dot{X}_B \in |\mathbf{D}|$. Finally, for every $id_B \in |\mathbf{TMaps}|$, $\mathsf{G}(id_B) = X_{id_B} = \dot{X}_B \in |\mathbf{D}|$. Now Theorems 3.15 and 4.9 show that the restrictions $F_d : \mathbf{D} \longrightarrow \mathbf{TBoole}$, $G_d : \mathbf{TBoole} \longrightarrow \mathbf{D}$, $\mathsf{F}_d : \mathbf{D} \longrightarrow \mathbf{TMaps}$, $\mathsf{G}_d : \mathbf{TMaps} \longrightarrow \mathbf{D}$ of the contravariant functors F, G, F and G, respectively, are all dual equivalences. □

Corollary 6.2. *For every* **TBoole***-morphism (σ, f) between any two* **TBoole***-objects $(B, \dot{X}_B)$ and $(B', \dot{X}_{B'})$, we have that $\sigma \in$* **Caba**(B, B').

Proof. For every $f \in \mathbf{D}(X, Y)$, we have that $F_d(f) = F(f) = (\mathsf{CO}(f), \hat{f}) = (\mathsf{P}(f), \hat{f})$. Since $\mathsf{P}(f)$ is a **Caba**-morphism and F_d is full, faithful and isomorphism-dense, our assertion follows. □

We can prove this assertion directly, as well. Suppose that σ is not a complete homomorphism. Then there exists a set $\{b_j \mid j \in J\} \subseteq B$ such that, with $b \stackrel{\text{df}}{=} \bigvee_{j\in J} b_j$, $\sigma(b) \gneq \bigvee_{j\in J} \sigma(b_j)$. Thus, there exists $y \in \mathrm{At}(B')$ such that $y \leq \sigma(b)$ but $y \not\leq b'$, where $b' \stackrel{\text{df}}{=} \bigvee_{j\in J} \sigma(b_j)$. Then $\dot{y}(b') = 0$ and $\dot{y}(\sigma(b)) = 1$. Since $\dot{y}$ is a complete homomorphism (see 2.4), we have that $0 = \dot{y}(b') = \dot{y}(\bigvee_{j\in J}\sigma(b_j)) = \bigvee_{j\in J}\dot{y}(\sigma(b_j)) = \bigvee_{j\in J}(\dot{y}\circ\sigma)(b_j))$. Hence, $\bigvee_{j\in J}(\dot{y}\circ\sigma)(b_j)) \neq (\dot{y}\circ\sigma)(\bigvee_{j\in J}b_j)$. Since $\dot{y}\circ\sigma$ is a complete homomorphism (because $\dot{y}\circ\sigma = f(\dot{y}) \in \dot{X}_B$), we obtain a contradiction. Therefore, $\sigma \in \mathbf{Caba}(B, B')$.

6.3. Using the above Corollary, we can define a functor

$$H : \mathbf{TBoole} \longrightarrow \mathbf{Caba}$$

setting $H(B, \dot{X}_B) \stackrel{\text{df}}{=} B$ and $H(\sigma, f) \stackrel{\text{df}}{=} \sigma$. Let us also define a functor

$$H^{-1} : \mathbf{Caba} \longrightarrow \mathbf{TBoole}$$

by $H^{-1}(B) \stackrel{\text{df}}{=} (B, \dot{X}_B)$ and, for any $\sigma \in \mathbf{Caba}(B, B')$, $H^{-1}(\sigma) \stackrel{\text{df}}{=} (\sigma, f^\sigma)$, where the function

$$f^\sigma : \dot{X}_{B'} \longrightarrow \dot{X}_B$$

is defined by

$$f^\sigma(\dot{y}) \stackrel{\text{df}}{=} \dot{y}\circ\sigma,$$

for every $\dot{y} \in \dot{X}_{B'}$. We need to show that $f^\sigma(\dot{y})$ belongs to $\dot{X}_B$. Indeed, setting $x \stackrel{\text{df}}{=} \bigwedge\{a \in B \mid y \leq \sigma(a)\}$, we have that $x \in \mathrm{At}(B)$ and, using Lemma 2.3, we obtain that for every $b \in B$, $\dot{x}(b) = 1 \Leftrightarrow x \leq b \Leftrightarrow \bigwedge\{a \in B \mid y \leq \sigma(a)\} \leq b \Leftrightarrow y \leq \sigma(b) \Leftrightarrow \dot{y}(\sigma(b)) = 1$. Thus $f^\sigma(\dot{y}) = \dot{y}\circ\sigma = \dot{x} \in \dot{X}_B$. Hence, the functor H^{-1} is well defined. One sees immediately that the compositions of the functors H and H^{-1} are equal to the corresponding identity functors. Therefore, H and H^{-1} are isomorphisms. Denoting by $I : \mathbf{D} \longrightarrow \mathbf{Set}$ the obvious forgetful functor, we obtain that I is an isomorphism and $H\circ F_d\circ I^{-1} = \mathsf{P}$. Now we set

$$\mathsf{A} \stackrel{\text{df}}{=} I\circ G_d\circ H^{-1}.$$

Using Proposition 6.1, we obtain that $\mathsf{P}\circ\mathsf{A} = (H\circ F_d\circ I^{-1})\circ(I\circ G_d\circ H^{-1}) = H\circ(F_d\circ G_d)\circ H^{-1} \cong H\circ\mathsf{Id}_{\mathbf{TBoole}}\circ H^{-1} = \mathsf{Id}_{\mathbf{Caba}}$ and, similarly, $\mathsf{A}\circ\mathsf{P} \cong \mathsf{Id}_{\mathbf{Set}}$. Thus, *the contravariant functors*

$$\mathsf{P} : \mathbf{Set} \longrightarrow \mathbf{Caba} \text{ and } \mathsf{A} : \mathbf{Caba} \longrightarrow \mathbf{Set}$$

are dual equivalences. Note that for every $B \in |\mathbf{Caba}|$,

$$\mathsf{A}(B) = \dot{X}_B,$$

where $\dot{X}_B = \{\dot{x} : B \longrightarrow \mathbf{2} \mid x \in \mathrm{At}(B)\}$, $\dot{x}(b) = 1 \Leftrightarrow x \leq b$, and, for every $\sigma \in \mathbf{Caba}(B, B')$,

$$\mathsf{A}(\sigma) = f^\sigma$$

(see the definition of f^σ here above). It is easy to see that $\dot{h} : \mathsf{At} \longrightarrow \mathsf{A}$, where for every $B \in |\mathbf{Caba}|$, $\dot{h}_B$ is the bijection defined in 2.4, is a natural isomorphism. Thus, $\mathsf{P}\circ\mathsf{At} \cong \mathsf{P}\circ\mathsf{A} \cong \mathsf{Id}_{\mathbf{Caba}}$ and, similarly, $\mathsf{At}\circ\mathsf{P} \cong \mathsf{Id}_{\mathbf{Set}}$. Therefore, $\mathsf{At} : \mathbf{Caba} \longrightarrow \mathbf{Set}$ and $\mathsf{P} : \mathbf{Set} \longrightarrow \mathbf{Caba}$ are dual equivalences, *obtaining in such a way a new proof of the Tarski Duality Theorem.*

Finally, defining a functor $H_1 : \mathbf{TMaps} \longrightarrow \mathbf{Caba}$ by $H_1(id_B) \stackrel{\text{df}}{=} B$ and $H_1(\sigma, \sigma) \stackrel{\text{df}}{=} \sigma$, and a functor $H_1^{-1} : \mathbf{Caba} \longrightarrow \mathbf{TMaps}$ by $H^{-1}(B) \stackrel{\text{df}}{=} id_B$ and $H_1^{-1}(\sigma) \stackrel{\text{df}}{=} (\sigma, \sigma)$ (note that Example 3.10 shows that H_1^{-1} is well defined), we obtain that the compositions of the functors H_1 and H_1^{-1} are equal to the corresponding identity functors. Therefore, H_1 and H_1^{-1} are isomorphisms. Obviously, we get that $H_1\circ\mathsf{F}_d\circ I^{-1} = \mathsf{P}$ and $\mathsf{A} = I\circ\mathsf{G}_d\circ H_1^{-1}$. Hence, working with the contravariant functors F_d and G_d, we come to the same dual equivalences $\mathsf{P} : \mathbf{Set} \longrightarrow \mathbf{Caba}$ and $\mathsf{A} : \mathbf{Caba} \longrightarrow \mathbf{Set}$.

7. Two Duality Theorems for the Category EDTych of Extremally Disconnected Spaces

Now, using our duality theorems 3.15 and 4.9, we will obtain duality theorems for the category **EDTych** of extremally disconnected Tychonoff spaces and continuous maps.

Definition 7.1. A dz-algebra (resp., z-algebra) (A, X) is said to be *complete dz-algebra* (resp., *complete z-algebra*) if A is a complete Boolean algebra. Let us denote by **dzCBoole** the full subcategory of the category **dzBoole** having as objects all complete dz-algebras. Let **zCBoole** be the full subcategory of the category **zBoole** having as objects of all complete z-algebras, and let **EDTych** be the category of extremally disconnected Tychonoff spaces and continuous maps.

Theorem 7.2. *The categories* **EDTych** *and* **zCBoole** *are dually equivalent.*

Proof. Since **EDTych** is a subcategory of **ZHaus**, we can regard the restriction F_{ed} of the contravariant functor $F : \mathbf{ZHaus} \longrightarrow \mathbf{dzBoole}$ to **EDTych**. Analogously, we can regard the restriction G_{ed} of the contravariant functor $G : \mathbf{dzBoole} \longrightarrow \mathbf{ZHaus}$ to **dzCBoole**. Recall that F and G were defined in the proof of Theorem 3.15. We will show that $F_{ed}(|\mathbf{EDTych}|) \subseteq |\mathbf{dzCBoole}|$ and $G_{ed}(|\mathbf{dzCBoole}|) \subseteq |\mathbf{EDTych}|$. Indeed, for every $X \in |\mathbf{EDTych}|$, we have that $\mathrm{CO}(X) = \mathrm{RC}(X)$ and thus $F_{ed}(X) = (\mathsf{CO}(X), \hat{X}) = (\mathrm{RC}(X), \hat{X})$. Hence, $F(X) \in |\mathbf{dzCBoole}|$. If $(A, X) \in |\mathbf{dzCBoole}|$, then $G_{ed}(A, X) = X$. Since, by Fact 3.2, X is a dense subspace of the extremally disconnected space $\mathsf{S}(A)$, we obtain that X is an extremally disconnected space (see, e.g., [12, Exercise 6.2.G.(c)]). Thus, $G_{ed}(A, X) \in |\mathbf{EDTych}|$. Now, Theorem 3.15 implies that

$$F_{ed} : \mathbf{EDTych} \longrightarrow \mathbf{dzCBoole} \ \text{ and } \ G_{ed} : \mathbf{dzCBoole} \longrightarrow \mathbf{EDTych}$$

are dual equivalences. Finally, we will show that the categories **dzCBoole** and **zCBoole** coincide. Indeed, if $(A, X) \in |\mathbf{zCBoole}|$, then, using Lemma 2.11, we obtain that $s_A^X(A) = X \cap s_A(A) = X \cap \mathsf{CO}(\mathsf{S}(A)) = X \cap \mathrm{RC}(\mathsf{S}(A)) = \mathrm{RC}(X) = \mathsf{CO}(X)$. Therefore, (A, X) is a dz-algebra. Thus, the categories **EDTych** and **zCBoole** are dually equivalent. □

Definition 7.3. An mz-map (resp., z-map) $\alpha : A \longrightarrow B$ is said to be *complete mz-map* (resp., *complete z-map*) if A is a complete Boolean algebra. Let us denote by **cmzMaps** the full subcategory of the category **mzMaps** having as objects all complete mz-maps, and by **czMaps** the full subcategory of the category **zMaps** having as objects all complete z-maps.

Theorem 7.4. *The categories* **EDTych** *and* **czMaps** *are dually equivalent.*

Proof. Let us denote by F_{ed} the restriction of the contravariant functor $\mathsf{F} : \mathbf{ZHaus} \longrightarrow \mathbf{mzMaps}$ to **EDTych**, and by G_{ed} the restriction of the contravariant functor $\mathsf{G} : \mathbf{mzMaps} \longrightarrow \mathbf{ZHaus}$ to **cmzMaps**. Recall that F and G were defined in the proof of Theorem 4.9. We are going to show that $\mathsf{F}_{ed}(|\mathbf{EDTych}|) \subseteq |\mathbf{cmzMaps}|$ and $\mathsf{G}_{ed}(|\mathbf{cmzMaps}|) \subseteq |\mathbf{EDTych}|$. Indeed, for every $X \in |\mathbf{EDTych}|$, we have that $\mathsf{F}_{ed}(X) = i_X$, where $i_X : \mathsf{CO}(X) \hookrightarrow \mathsf{P}(X)$ is the inclusion map. Since $\mathrm{CO}(X) = \mathrm{RC}(X)$, we obtain that $\mathsf{F}_{ed}(X) \in |\mathbf{cmzMaps}|$. Let now $(\alpha : A \longrightarrow B) \in |\mathbf{cmzMaps}|$. Then $\mathsf{G}_{ed}(\alpha) = X_\alpha$. We will show that X_α is a dense subspace of $\mathsf{S}(A)$. Indeed, if $a \in A^+$ then $\alpha(a) \neq 0$ and, hence, there exists $x \in \mathrm{At}(B)$ such that $x \leq \alpha(a)$; this, however, means that $\alpha_x(a) = 1$, i.e., $\alpha_x \in s_A(a) \cap X_\alpha$. So, X_α is a dense subspace of $\mathsf{S}(A)$. Thus, $\mathsf{G}_{ed}(\alpha) \in |\mathbf{EDTych}|$. Now, Theorem 4.9 implies that

$$\mathsf{F}_{ed} : \mathbf{EDTych} \longrightarrow \mathbf{cmzMaps} \ \text{ and } \ \mathsf{G}_{ed} : \mathbf{cmzMaps} \longrightarrow \mathbf{EDTych}$$

are dual equivalences. Finally, we will show that the categories **cmzMaps** and **czMaps** coincide. Indeed, let $\alpha : A \longrightarrow B$ be a complete z-map. Then A is a complete Boolean algebra and, hence, $\mathsf{S}(A)$ is extremally disconnected. As we have already seen, X_α is a dense subspace of $\mathsf{S}(A)$, and thus X_α is also extremally disconnected. Now, using Lemma 2.11, we obtain that $s_A^{X_\alpha}(A) = X_\alpha \cap s_A(A) = X_\alpha \cap \mathsf{CO}(\mathsf{S}(A)) = X_\alpha \cap \mathrm{RC}(\mathsf{S}(A)) = \mathrm{RC}(X_\alpha) = \mathsf{CO}(X_\alpha)$. Therefore, α is an mz-map. This shows that $\mathbf{cmzMaps} \equiv \mathbf{czMaps}$. Hence, the categories **EDTych** and **czMaps** are dually equivalent. □

8. Two Duality Theorems for the Category of Zero-Dimensional Hausdorff Compactifications of Zero-Dimensional Spaces

Recall first the following assertion from [4]:

Proposition 8.1. ([4]) *There is a category* **Comp** *whose objects are Hausdorff compactifications $c : X \longrightarrow Y$ and whose morphisms between any two* **Comp** *-objects $c : X \longrightarrow Y$ and $c' : X' \longrightarrow Y'$ are all pairs (f, g), where $f : X \longrightarrow X'$ and $g : Y \longrightarrow Y'$ are continuous maps such that $g \circ c = c' \circ f$. The composition of two morphisms (f_1, g_1) and (f_2, g_2) is defined to be $(f_2 \circ f_1, g_2 \circ g_1)$. The identity map of a* **Comp***-object $c : X \longrightarrow Y$ is defined to be $id_c \stackrel{\text{df}}{=} (id_X, id_Y)$.*

Definition 8.2. We will denote by **ZComp** the full subcategory of the category **Comp** whose objects are all compactifications $c : X \longrightarrow Y$ for which Y is a zero-dimensional space. By **BZComp** we will denote the full subcategory of the category **ZComp** whose objects are all Banaschewski compactifications $\beta_0 : X \longrightarrow \beta_0 X$ and all compactifications $c : X \longrightarrow cX$ which are **ZComp**-isomorphic to them.

Remark 8.3. Note that Example 3.2 from [4] shows that there exist **ZComp**-objects $c : X \longrightarrow Y$ and $c' : X \longrightarrow Y'$ which are isomorphic in **ZComp** but not equivalent as compactifications. On the other hand, as it is shown in [4], any two equivalent compactifications of a space X are isomorphic in **Comp**.

Proposition 8.4. *Let $c : X \longrightarrow Y$ be a* **ZComp***-object. If c is isomorphic to the Banaschewski compactification $\beta_0 : X \longrightarrow \beta_0 X$ in* **ZComp***, then c is equivalent to β_0.*

Proof. The proof is analogous to that of Theorem 3.3 from [4]. The only difference is that the Banaschewski Theorem 2.9 has to be used. □

Proposition 8.4 and the last sentence in Example 8.3 show that in the definition of the category **BZComp** we can write "equivalent" instead of "**ZComp**-isomorphic".

Theorem 8.5. *The categories* **ZComp** *and* **zBoole** *are dually equivalent.*

Proof. We start with defining a contravariant functor

$$\Phi : \mathbf{ZComp} \longrightarrow \mathbf{zBoole}.$$

For every $(c : X \longrightarrow Y) \in |\mathbf{ZComp}|$, set $A_c \stackrel{\text{df}}{=} c^{-1}(\mathrm{CO}(Y))$, $\hat{X}_c \stackrel{\text{df}}{=} \hat{X}_{A_c}$ (see 2.4 for the notation), and

$$\Phi(c) \stackrel{\text{df}}{=} (A_c, \hat{X}_c).$$

Then, by Example 3.9, $\Phi(c) \in |\mathbf{zBoole}|$.

Let now $c : X \longrightarrow Y$ and $c' : X' \longrightarrow Y'$ be **ZComp**-objects and (f, g) be a **ZComp**-morphism between c and c'. Set

$$\Phi(f, g) \stackrel{\text{df}}{=} (\pi_f, \hat{f}_{cc'}),$$

where $\pi_f : A_{c'} \longrightarrow A_c$ is defined by $\pi_f(U) \stackrel{\text{df}}{=} f^{-1}(U)$ for every $U \in A_{c'}$, and

$$\hat{f}_{cc'} : \hat{X}_c \longrightarrow \widehat{X'}_{c'}$$

is defined by $\hat{f}_{cc'}(\hat{x}) \stackrel{\text{df}}{=} \widehat{f(x)}$ for every $x \in X$. Arguing as in the proof of Theorem 3.15, we obtain that $\Phi(f, g) \in \mathbf{zBoole}(\Phi(c'), \Phi(c))$. Now it is easy to see that Φ is a contravariant functor.

We define $\Psi : \mathbf{zBoole} \longrightarrow \mathbf{ZComp}$ as follows: for every $(A, X) \in |\mathbf{zBoole}|$, set

$$\Psi(A, X) \stackrel{\text{df}}{=} c_{(A,X)},$$

where, regarding X as a subspace of $\mathsf{S}(A)$, $c_{(A,X)} : X \hookrightarrow \mathsf{S}(A)$ is the embedding of X in $\mathsf{S}(A)$; for every $(\varphi, f) \in \mathbf{zBoole}((A,X),(A',X'))$, we put

$$\Psi(\varphi, f) \stackrel{\text{df}}{=} (f, \mathsf{S}(\varphi)).$$

By Fact 3.2, $c_{(A,X)}$ is a dense embedding and thus $\Psi(A,X)$ is a **ZComp**-object. Since for every $x' \in X'$, $\mathsf{S}(\varphi)(x') = x' \circ \varphi = f(x')$, we obtain that $\Psi(\varphi, f)$ is a **ZComp**-morphism. Hence, Ψ is well-defined. Obviously, it is a contravariant functor.

Let $(A,X) \in |\mathbf{zBoole}|$. Then $\Phi(\Psi(A,X)) = (A_{c_{(A,X)}}, \hat{X}_{c_{(A,X)}})$, $A_{c_{(A,X)}} = X \cap s_A(A) = s_A^X(A)$ and $\hat{X}_{c_{(A,X)}} = \{\hat{x} : s_A^X(A) \longrightarrow \mathbf{2} \mid x \in X\}$. Working like in the proof of Theorem 3.15, we define a map $\check{\imath}_X^{c_{(A,X)}} : \hat{X}_{c_{(A,X)}} \longrightarrow X$ by $\check{\imath}_X^{c_{(A,X)}}(\hat{x}) \stackrel{\text{df}}{=} x$, for every $x \in X$, and set $s''_{(A,X)} \stackrel{\text{df}}{=} (\bar{s}_A^X, \check{\imath}_X^{c_{(A,X)}})$. Then, like in Theorem 3.15, we show that $s''_{(A,X)} : (A,X) \longrightarrow (\Phi \circ \Psi)(A,X)$ is a **zBoole**-isomorphism and, moreover,

$$s'' : \mathsf{Id}_{\mathbf{zBoole}} \longrightarrow \Phi \circ \Psi, \ \ (A,X) \mapsto s''_{(A,X)},$$

is a natural isomorphism.

Let now $(c : X \longrightarrow Y) \in |\mathbf{ZComp}|$. Then $(\Psi \circ \Phi)(c) = c_{(A_c, \hat{X}_c)}$ and $c_{(A_c,\hat{X}_c)} : \hat{X}_c \hookrightarrow \mathsf{S}(A_c)$. Obviously, the map $\rho_c : A_c \longrightarrow \mathsf{CO}(Y)$, $c^{-1}(U) \mapsto U$, is a Boolean isomorphism. Hence, the map $\mathsf{S}(\rho_c) : \mathsf{S}(\mathsf{T}(Y)) \longrightarrow \mathsf{S}(A_c)$ is a homeomorphism. By Example 3.9, the map $\hat{h}_{X,A_c} : X \longrightarrow \hat{X}_c$ is a homeomorphism. Now it is easy to show that the map $\varkappa_c \stackrel{\text{df}}{=} (\hat{h}_{X,A_c}, \mathsf{S}(\rho_c) \circ t_Y) : c \longrightarrow c_{(A_c,\hat{X}_c)}$ is a **ZComp**-isomorphism (see 2.1 for the notation t_Y). Finally, it is not difficult to prove that

$$\varkappa : \mathsf{Id}_{\mathbf{ZComp}} \longrightarrow \Psi \circ \Phi, \ \ c \mapsto \varkappa_c,$$

is a natural isomorphism. Therefore, the categories **ZComp** and **zBoole** are dually equivalent. □

Corollary 8.6. *The categories* **BZComp** *and* **dzBoole** *are dually equivalent.*

Proof. We will use the notation from the proof of Theorem 8.5.

Let $(\beta_0 : X \longrightarrow \beta_0 X) \in |\mathbf{BZComp}|$. Then, the Dwinger Theorem 2.8 implies that $\Phi(\beta_0) = (\mathsf{CO}(X), \hat{X})$. Thus, by Example 3.9, $\Phi(\beta_0) \in |\mathbf{dzBoole}|$.

Conversely, if $(A,X) \in |\mathbf{dzBoole}|$, then $\Psi(A,X) = c_{(A,X)}$, where $c_{(A,X)} : X \hookrightarrow \mathsf{S}(A)$. By the definition of a dz-algebra, we have that $s_A^X(A) = \mathsf{CO}(X)$. Thus the trace of $\mathsf{CO}(\mathsf{S}(A))$ on X is $\mathsf{CO}(X)$. Now the Dwinger Theorem 2.8 implies that $c_{(A,X)}$ is equivalent to the Banaschewski compactification $\beta_0 : X \longrightarrow \beta_0 X$. Therefore, $\Psi(A,X) \in |\mathbf{BZComp}|$.

The rest follows from Theorem 8.5. □

It is easy to see that the categories **BZComp** and **ZHaus** are equivalent. Thus, using Corollary 8.6, we obtain a new proof of Theorem 3.15. Note, however, that in the proof of Theorem 8.5 we used many parts of the proof of Theorem 3.15.

We will denote by **EDComp** the full subcategory of the category **ZComp** having as objects all compactifications $c : X \longrightarrow Y$, for which $Y \in |\mathbf{EDTych}|$.

Corollary 8.7. *The categories* **EDComp** *and* **zCBoole** *are dually equivalent.*

Proof. Having in mind Theorem 8.5, it is enough to show that $\Phi(|\mathbf{EDComp}|) \subseteq |\mathbf{zCBoole}|$ and $\Psi(|\mathbf{zCBoole}|) \subseteq |\mathbf{EDComp}|$. Let $(c : X \longrightarrow Y) \in |\mathbf{EDComp}|$. Then, using Lemma 2.11, we obtain (in the notation from the proof of Theorem 8.5) that $A_c = c^{-1}(\mathsf{CO}(Y)) = c^{-1}(\mathrm{RC}(Y)) = \mathrm{RC}(X)$. Thus, $\Phi(c) \in |\mathbf{zCBoole}|$. Let now $(A,X) \in |\mathbf{zCBoole}|$. Then A is a complete Boolean algebra and, hence, $\mathsf{S}(A) \in |\mathbf{EDTych}|$. This shows that $\Psi(A,X) \in |\mathbf{EDComp}|$. So, the proof is completed. □

Corollary 8.8. *The categories* **EDComp** *and* **EDTych** *are equivalent.*

Proof. This follows immediately from Theorem 7.2 and Corollary 8.7. □

Note that Corollary 8.8 can be also proved with the help of the fact that if $(c : X \longrightarrow Y) \in |\mathbf{EDComp}|$ then X is extremally disconnected and c is equivalent (as a compactification of X) to the Stone-Čech compactification $\beta : X \longrightarrow \beta X$ of X (see [13] or [12]).

Now we will show, using the Tarski duality, that the category **zMaps** is dually equivalent to the category **ZComp**. The category **zMaps** is similar to the category **DeVe**, constructed in [4] as a category dually equivalent to the category **Comp** of Hausdorff compactifications of Tychonoff spaces.

Theorem 8.9. *The categories* **ZComp** *and* **zMaps** *are dually equivalent.*

Proof. We will utilize the notation introduced in the proof of Theorem 8.5.

We start with defining a contravariant functor

$$\Phi' : \mathbf{ZComp} \longrightarrow \mathbf{zMaps}.$$

For every $(c : X \longrightarrow Y) \in |\mathbf{ZComp}|$, we set

$$\Phi'(c) \stackrel{\text{df}}{=} s_{A_c}^{\hat{X}_c}.$$

Then it is easy to see that $\Phi'(c) \in |\mathbf{zMaps}|$.

For every $(f, g) \in \mathbf{ZComp}(c, c')$, we set

$$\Phi'(f, g) \stackrel{\text{df}}{=} (\pi_f, \mathsf{P}(\hat{f}_{cc'})).$$

It is not difficult to obtain that $\Phi'(f, g) \in \mathbf{zMaps}(\Phi'(A', X'), \Phi'(A, X))$. Now it is easy to see that Φ' is a contravariant functor.

Our next aim is to define a contravariant functor

$$\Psi' : \mathbf{zMaps} \longrightarrow \mathbf{ZComp}.$$

Let $(\alpha : A \longrightarrow B) \in |\mathbf{zMaps}|$. We put

$$\Psi'(\alpha) \stackrel{\text{df}}{=} c_\alpha, \text{ where } c_\alpha : X_\alpha \hookrightarrow \mathsf{S}(A)$$

(see 2.4 for the notation X_α). Obviously, $\Psi'(c) \in |\mathbf{ZComp}|$.

Let now $(\varphi, \sigma) \in \mathbf{zMaps}(\alpha, \alpha')$. Then it is easy to show that $\mathsf{S}(\varphi)(X_{\alpha'}) \subseteq X_\alpha$. Let $S_\varphi : X_{\alpha'} \longrightarrow X_\alpha$ be the restriction of $\mathsf{S}(\varphi)$. We put

$$\Psi'(\varphi, \sigma) \stackrel{\text{df}}{=} (S_\varphi, \mathsf{S}(\varphi)).$$

Then it is not difficult to prove that $\Psi'(\varphi, \sigma) \in \mathbf{ZComp}(\Psi'(\alpha'), \Psi'(\alpha))$ and that Ψ' is a contravariant functor.

Let $(\alpha : A \longrightarrow B) \in |\mathbf{zMaps}|$. Then $\Phi'(\Psi'(\alpha)) = s_{A_{c_\alpha}}^{\hat{X}_{c_\alpha}}$ and $s_{A_{c_\alpha}}^{\hat{X}_{c_\alpha}} : A_{c_\alpha} \longrightarrow \mathsf{P}(\hat{X}_{c_\alpha})$. We have that $A_{c_\alpha} = c_\alpha^{-1}(\mathsf{CO}(\mathsf{S}(A))) = X_\alpha \cap \mathsf{T}(\mathsf{S}(A)) = s_A^{X_\alpha}(A)$. Thus $\bar{s}_A^{X_\alpha} : A \longrightarrow A_{c_\alpha}$ is a Boolean isomorphism. Since α is a z-map, the map $h_\alpha : \mathrm{At}(B) \longrightarrow X_\alpha$, $x \mapsto \alpha_x$, is a bijection (see 2.4). Also, the map $\hat{h}_{X_\alpha, A_{c_\alpha}} : X_\alpha \longrightarrow \hat{X}_{c_\alpha}$, $\alpha_x \mapsto \widehat{\alpha_x}$, for all $x \in \mathrm{At}(B)$, where $\widehat{\alpha_x} : A_{c_\alpha} \longrightarrow \mathbf{2}$, is a bijection (see 2.4). Setting $k_\alpha \stackrel{\text{df}}{=} \hat{h}_{X_\alpha, A_{c_\alpha}} \circ h_\alpha$ and $k_\alpha^P : \mathsf{P}(\mathrm{At}(B)) \longrightarrow \mathsf{P}(\hat{X}_{c_\alpha})$, $M \mapsto \{k_\alpha(m) \mid m \in M\}$, we obtain that k_α^P is a bijection. Then the map $\varepsilon_B^{c_\alpha} \stackrel{\text{df}}{=} k_\alpha^P \circ \varepsilon_B$ is a bijection (see 2.2 for the notation ε_B) and $\varepsilon_B^{c_\alpha} : B \longrightarrow \mathsf{P}(\hat{X}_{c_\alpha})$, $b \mapsto \{\widehat{\alpha_x} \mid x \in \mathrm{At}(B), x \leq b\}$. Now we put $v_\alpha \stackrel{\text{df}}{=} (s_A^{X_\alpha}, \varepsilon_B^{c_\alpha})$. It is easy to see that $v_\alpha : \alpha \longrightarrow \Phi'(\Psi'(\alpha))$ is a **zMaps**-isomorphism. One routinely verifies that

$$v : \mathsf{Id}_{\mathbf{zMaps}} \longrightarrow \Phi' \circ \Psi', \quad \alpha \mapsto v_\alpha,$$

is a natural isomorphism.

Let $(c : X \longrightarrow Y) \in |\mathbf{ZComp}|$. Then $\Psi'(\Phi'(c)) = c_\alpha$, where $\alpha \stackrel{\text{df}}{=} s_{A_c}^{\hat{X}_c}$. Thus $c_\alpha : X_\alpha \hookrightarrow \mathsf{S}(A_c)$, where $A_c = c^{-1}(\mathsf{CO}(Y))$ and $X_\alpha = \{\alpha_{\hat{x}} \mid \hat{x} \in \hat{X}_c\}$. We have that for every $U \in A_c$, $\alpha_{\hat{x}}(U) = 1 \Leftrightarrow \hat{x} \leq \alpha(U) \Leftrightarrow \hat{x} \in$

$s_{A_c}^{\hat{X}_c}(U) \Leftrightarrow \hat{x}(U) = 1$. Thus, $\alpha_{\hat{x}} \equiv \hat{x}$ for every $x \in X$. Hence, $X_\alpha = \hat{X}_c$, i.e., $c_\alpha : \hat{X}_c \hookrightarrow \mathsf{S}(A_c)$. As we noted in the proof of Theorem 8.5, the maps $\mathsf{S}(\rho_c) : \mathsf{S}(\mathsf{T}(Y)) \longrightarrow \mathsf{S}(A_c)$ (where $\rho_c : A_c \longrightarrow \mathsf{CO}(Y)$, $c^{-1}(U) \mapsto U$) and $\hat{h}_{X,A_c} : X \longrightarrow \hat{X}_c$, $x \mapsto \hat{x}$, are homeomorphisms. Now it is easy to see that the map $\xi_c \stackrel{\mathrm{df}}{=} (\hat{h}_{X,A_c}, \mathsf{S}(\rho_c) \circ t_Y) : c \longrightarrow \Psi'(\Phi'(c))$ is a **ZComp**-isomorphism (see 2.1 for the notation t_Y). Finally, a routine verification shows that

$$\xi : \mathsf{Id}_{\mathbf{ZComp}} \longrightarrow \Psi' \circ \Phi', \quad c \mapsto \xi_c,$$

is a natural isomorphism.

All this proves that the categories **ZComp** and **zMaps** are dually equivalent. □

Corollary 8.10. *The categories* **BZComp** *and* **mzMaps** *are dually equivalent.*

Proof. Of course, this assertion follows from Corollary 8.6 and Theorem 4.8. A short direct proof, which is similar to the proof of Corollary 8.6, can be obtained as follows (we will use the notation from the proof of Theorem 8.9).

Let $(\beta_0 : X \longrightarrow \beta_0 X) \in |\mathbf{BZComp}|$. Then $\Phi'(\beta_0) = s_{\mathsf{CO}(X)}^{\hat{X}}$. Since, by Example 3.9, $(\mathsf{CO}(X), \hat{X})$ is a dz-algebra, Example 4.5 shows that $\Phi'(\beta_0) \in |\mathbf{mzMaps}|$.

Conversely, if $(\alpha : A \longrightarrow B) \in |\mathbf{mzMaps}|$, then $\Psi'(\alpha) = c_\alpha$, where $c_\alpha : X_\alpha \hookrightarrow \mathsf{S}(A)$. Since α is a mz-map, we have that $s_A^{X_\alpha}(A) = \mathsf{CO}(X_\alpha)$. Thus the trace of $\mathsf{CO}(\mathsf{S}(A))$ on X_α is $\mathsf{CO}(X_\alpha)$. Now the Dwinger Theorem 2.8 implies that c_α is equivalent to the Banaschewski compactification $\beta_0 : X \longrightarrow \beta_0 X$. Therefore, $\Psi'(\alpha) \in |\mathbf{BZComp}|$.

The rest follows from Theorem 8.9. □

A remark analogous to that one given after the proof of Corollary 8.6 can be also made here: using Corollary 8.10, we can obtain a second proof of Theorem 4.9. The direct proof of it given by us in this paper is, however, more natural because it reveals clearly the connection between our two duality theorems.

8.11. We are now going to derive the Dwinger Theorem 2.8 from our Theorems 8.5 and 8.9. In what follows, we will use the notation from their proofs.

Let us fix a space $X \in |\mathbf{ZHaus}|$. Then, obviously, the map $\lambda : \mathcal{BA}(X) \longrightarrow |\mathbf{zBoole}|$, $A \mapsto (A, \hat{X}_A)$, is an injection. (Note that, by Example 3.9, λ is a well-defined function.) Thus, the map $\lambda_0 \stackrel{\mathrm{df}}{=} \lambda \upharpoonright \mathcal{BA}(X) : \mathcal{BA}(X) \longrightarrow \lambda(\mathcal{BA}(X))$ is a bijection. We have that $\Psi(\lambda(A)) = c_{(A,\hat{X}_A)}$, where $c_{(A,\hat{X}_A)} : \hat{X}_A \hookrightarrow \mathsf{S}(A)$ is the embedding of $\hat{X}_A$ in $\mathsf{S}(A)$. We set $c_A \stackrel{\mathrm{df}}{=} c_{(A,\hat{X}_A)} \circ \hat{h}_{X,A}$ and $\Delta(A) \stackrel{\mathrm{df}}{=} [c_A]$. Then

$$c_A : X \longrightarrow \mathsf{S}(A), \ x \mapsto \hat{x}, \quad \text{and} \quad \Delta : \mathcal{BA}(X) \longrightarrow \mathcal{K}_0(X).$$

For every $(c : X \longrightarrow Y) \in \mathcal{K}_0(X)$, we set $\Delta'([c]) \stackrel{\mathrm{df}}{=} \lambda_0^{-1}(\Phi(c))$. Thus

$$\Delta'([c]) = A_c = c^{-1}(\mathsf{CO}(Y)) \in \mathcal{BA}(X) \quad \text{and} \quad \Delta' : \mathcal{K}_0(X) \longrightarrow \mathcal{BA}(X).$$

Note that the map Δ' is well-defined. Indeed, if $c_1 \in [c]$, where $c_1 : X \longrightarrow Y_1$, then there exists a homeomorphism $f : Y \longrightarrow Y_1$ such that $c_1 = f \circ c$. Hence $c_1^{-1}(\mathsf{CO}(Y_1)) = c^{-1}(f^{-1}(\mathsf{CO}(Y_1))) = c^{-1}(\mathsf{CO}(Y))$.

Now, for every $A \in \mathcal{BA}(X)$,

$$\Delta'(\Delta(A)) = A.$$

Indeed, we have that $\Delta'(\Delta(A)) = A_{c_A} = c_A^{-1}(\mathsf{T}(\mathsf{S}(A))) = \hat{h}_{X,A}^{-1}(\bar{s}_A^{\hat{X}_A}(A))$ and, for every $U \in A$, $\hat{h}_{X,A}^{-1}(\bar{s}_A^{\hat{X}_A}(U)) = U$ (see the proof of Example 3.9).

Further, for every $(c : X \longrightarrow Y) \in \mathcal{K}_0(X)$, $\Delta(\Delta'([c])) = \Delta(A_c) = [c_{A_c}]$, where $c_{A_c} : X \longrightarrow \mathsf{S}(A)$. At the end of the proof of Theorem 8.9 we have shown that the map $(\hat{h}_{X,A_c}, \mathsf{S}(\rho_c) \circ t_Y) : c \longrightarrow c_{(A_c,\hat{X}_c)}$ is a **ZComp**-isomorphism. Using the definition of the map c_{A_c}, we obtain that the map $(\hat{h}_{X,A_c}^{-1}, id_{\mathsf{S}(A_c)}) : c_{(A_c,\hat{X}_c)} \longrightarrow c_{A_c}$ is also a **ZComp**-isomorphism. Thus the diagram

$$\begin{tikzcd}
X \arrow[r, "\hat{h}_{X,A_c}"'] \arrow[rr, bend left, "\mathrm{id}_X"] \arrow[d, "c"] & \hat{X}_c \arrow[r, "\hat{h}^{-1}_{X,A_c}"'] \arrow[d, "c_{(A_c,\hat{X}_c)}"] & X \arrow[d, "c_{A_c}"] \\
Y \arrow[r, "\mathsf{S}(\rho_c)\circ t_Y"'] & \mathsf{S}(A_c) \arrow[r, "id_{\mathsf{S}(A_c)}"'] & \mathsf{S}(A_c)
\end{tikzcd}$$

is commutative. It shows that the compactifications c and c_{A_c} of X are equivalent (since $c_{A_c} = (\mathsf{S}(\rho_c)\circ t_Y)\circ c$). Thus,

$$\Delta(\Delta'([c])) = [c].$$

Therefore, Δ and Δ' are bijections.

Let now $c_1 : X \longrightarrow Y_1$ and $c_2 : X \longrightarrow Y_2$ be compactifications of X, and $c_1 \leq c_2$. Then there exists a continuous map $g : Y_2 \longrightarrow Y_1$ such that $c_1 = g\circ c_2$. Thus, $(id_X, g) \in \mathbf{ZComp}(c_2, c_1)$. Then $A_{c_1} = c_1^{-1}(\mathsf{CO}(Y_1)) = c_2^{-1}(g^{-1}(\mathsf{CO}(Y_1))) \subseteq c_2^{-1}(\mathsf{CO}(Y_2)) = A_{c_2}$. Therefore,

$$\Delta'([c_1]) \leq \Delta'([c_2]).$$

Let now $A, A' \in \mathcal{BA}(X)$ and A be a subalgebra of A'; denote by $i : A \longrightarrow A'$ the inclusion monomorphism. For every $x \in X$, set $f(\hat{x}_{A'}) = \hat{x}_A$ (see 2.4 for the notation). Then $f : \hat{X}_{A'} \longrightarrow \hat{X}_A$, $f(\hat{x}_{A'}) = \hat{x}_{A'}\circ i$ and thus $(i, f) \in \mathbf{zBoole}(\lambda(A), \lambda(A'))$. Therefore $\Psi(i, f) : \Psi(\lambda(A')) \longrightarrow \Psi(\lambda(A))$ is a $\mathbf{ZComp}$-morphism. We have that $\Psi(i, f) = (f, \mathsf{S}(i))$. Hence, the diagram

$$\begin{tikzcd}
X \arrow[r, "\hat{h}_{X,A'}"'] \arrow[rr, bend left, "\mathrm{c_{A'}}"] \arrow[d, "id_X"] & \hat{X}_{A'} \arrow[r, hook] \arrow[d, "f"] & \mathsf{S}(A') \arrow[d, "\mathsf{S}(i)"] \\
X \arrow[r, "\hat{h}_{X,A}"] \arrow[rr, bend right, "\mathrm{c_A}"'] & \hat{X}_A \arrow[r, hook] & \mathsf{S}(A)
\end{tikzcd}$$

is commutative. It shows that

$$\Delta(A) \leq \Delta(A').$$

Therefore, Δ and Δ' are isomorphisms between the ordered sets $(\mathcal{BA}(X), \subseteq)$ and $(\mathcal{K}_0(X), \leq)$. Thus, the Dwinger Theorem is proved.

For deriving the Dwinger Theorem from Theorem 8.9, we use the same maps Δ and Δ' but find another expressions for them. For every $(c : X \longrightarrow Y) \in \mathcal{K}_0(X)$, we have that $\Phi'(c) = s_{A_c}^{\hat{X}_c}$ and thus $\Delta'([c]) = \mathrm{dom}(\Phi'(c))$. Also, for every $A \in \mathcal{BA}(X)$, we have, by Example 4.2, that the map $i_A : A \hookrightarrow \mathsf{P}(X)$ is a z-map. Set $\alpha \stackrel{\mathrm{df}}{=} i_A$. Then $\Psi'(\alpha) = c_\alpha$, where $c_\alpha : X_\alpha \hookrightarrow \mathsf{S}(A)$, and $X_\alpha \equiv \hat{X}_A$. Thus $c_\alpha \equiv c_{(A,\hat{X}_A)}$ and $\Delta(A) = c_A = \Psi'(\alpha)\circ \hat{h}_{X,A}$. Then we prove exactly as above that Δ and Δ' are bijections, and that Δ' is monotone. Finally, let $A, A' \in \mathcal{BA}(X)$ and $A \subseteq A'$. Denote by $i : A \hookrightarrow A'$ the inclusion monomorphism and set $\alpha \stackrel{\mathrm{df}}{=} i_A$, $\alpha' \stackrel{\mathrm{df}}{=} i_{A'}$. Then $(i, id_{\mathsf{P}(X)}) \in \mathbf{zMaps}(\alpha, \alpha')$ and thus $\Psi'(i, id_{\mathsf{P}(X)}) \in \mathbf{ZComp}(c_{\alpha'}, c_\alpha)$. We have that $\Psi'(i, id_{\mathsf{P}(X)}) = (S_i, \mathsf{S}(i))$, where $S_i : \hat{X}_{A'} \longrightarrow \hat{X}_A$ is the restriction of $\mathsf{S}(i) : \mathsf{S}(A') \longrightarrow \mathsf{S}(A)$. Writing in the last diagram S_i instead of f, we obtain a new commutative diagram which shows again that $\Delta(A) \leq \Delta(A')$. Thus, the second proof of the Dwinger Theorem is completed.

Let us note that an interesting generalization of the Dwinger Theorem 2.8 was obtained in [3].

9. Conclusion

In this paper we defined two categories **dzBoole** and **mzMaps** which are dually equivalent to the category **ZHaus**. In our next paper [9] we will describe the subcategories of the categories **dzBoole** and **mzMaps** which are dually equivalent to the category **SZHaus** of strongly zero-dimensional Hausdorff spaces and continuous maps and the category **NZHaus** of normal zero-dimensional Hausdorff spaces and continuous maps. This, in particular, will help us to understand better the famous Dowker Example [12, Example 6.2.20].

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