Filomat 35:6 (2021), 1841–1850
https://doi.org/10.2298/FIL2106841B

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

On the Solvability of Some Nonlinear Functional Integral Equations on $L^p(\mathbb{R}_+)$

Mahmoud Bousselsal[a]

[a]*Laboratory of Nonlinear Partial Differential Equations and History of Mathematics, Department of Mathematics, ENS Mohamed El Bachir El Ibrahimi,16050, Vieux-Kouba, Algiers, (Algeria)*

Abstract. In this paper, we prove theorems on the existence of solutions in $L^p(\mathbb{R}_+)$, $1 \leq p < \infty$, for some functional integral equations. The basic tool used in the proof is the fixed point theorem due to Darbo with respect to so called measure of noncompactness. The obtained results generalize and extend several ones obtained earlier in many papers and monographs. An example which shows the applicability of our results is also included.

1. Introduction

Integral equation have a lot of applications in many branches of mathematical physics, engineering, mechanics, biology and economics see [24] and references therein. Several different techniques were proposed to study the existence of solutions of the functional integral equations in appropriate function spaces. Although all of these techniques have the same goal, they differ in the function spaces and the fixed point theorems to be applied.

Many papers in the field of functional integral equations give different sets of conditions for the existence of solutions of such equations, see for instance [2, 7, 10, 13, 16, 18] . Apart from that, integral equations are often investigated in research papers and monographs (cf. [6, 8, 11, 12, 15, 17]) and the references cited therein.

Agarwal and O'Regan [4] in 2004, proved the existence of the solutions for the nonlinear integral equation

$$x(t) = \int_0^{+\infty} k(t,s) f(t, x(s)) ds, \ \ t \in \mathbb{R}_+$$

in $C_l[0,+\infty)$, where $C_l[0,+\infty)$, denotes the space of bounded and continuous functions on $\mathbb{R}_+$ which have limit at infinity.

2010 *Mathematics Subject Classification*. 45G10, 45M99, 47H09
Keywords. Existence, the space of Lebesgue integrable functions, measure of noncompactness.
Received: 07 May 2020; Revised: 20 January 2021; Accepted: 13 March 2021
Communicated by Snežana Živković-Zlatanović
Email address: bousselsal55@gmail.com, mahmoud.bousselsal@g.ens-kouba.dz (Mahmoud Bousselsal)

In [23], the author gave the existence of an integrable solutions of the following functional integral equation

$$x(t) = f(t, x(t)) + g\left(t, \int_0^{+\infty} k(t,s) f(t, x(s)) ds\right), \ t \in \mathbb{R}_+.$$

In [20], the authors discussed the solvability the functional integral equation of convolution type

$$x(t) = f(t, x(t)) + \int_0^{+\infty} k(t-s)(Qx)(s) ds$$

using a new construction of a measure of noncompactness in $L^p(\mathbb{R}_+)$.

Next, the authors in [3] study the existence of solutions to the following general functional integral equation

$$x(t) = f(t, x(t)) + g\left(t, \int_0^{+\infty} k(t-s)(Qx)(s) ds\right)$$

using the same new construction of a measure of noncompactness in $L^p(\mathbb{R}_+)$.

In this paper, we consider the following more general integral equation

$$x(t) = f_1(t, x(t)) + f_2\left(t, (Q_1 x)(t), \int_0^{+\infty} u(t, s, (Q_2 x)(s)) ds\right). \tag{1}$$

This equation includes many important integral and functional equations that arise in nonlinear analysis and its applications. We look for solutions to (1) in $L^p(\mathbb{R}_+)$, $1 \leq p < \infty$. The main tool used in our considerations is the conjunction of the techniques of measure of noncompactness with Darbo fixed point theorem. An example is presented to show the importance and the applicability of our results.

2. Notation, Definitions and Auxiliary Facts

Definition 2.1. *The function $f(t, x, y) = f : \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is said to have the Carathéodory property if f is measurable in t for any $(x, y) \in \mathbb{R} \times \mathbb{R}$ and continuous in x, y for almost all $t \in \mathbb{R}_+$.*

Now, we are going to recall some notion about the continuity of the linear integral operator on the space $L^p = L^p(\mathbb{R}_+)$. Let $\Delta = \{(t, s) : 0 \leq s \leq t\}$ and $k : \Delta \to \mathbb{R}$ be the linear Fredholm operator $K : L^p(\mathbb{R}_+) \to L^p(\mathbb{R}_+)$ defined by $(Kx)(t) = \int_0^{+\infty} k(t, s)x(s) ds$. It is a continuous operator, and $\|Kx\|_p \leq \|K\| \, \|x\|_p$. The norm of the operator is majorized by

$$\|K\| = \sup\left(\|Kx\|_{L^p(\mathbb{R}_+)} ; \|x\|_{L^p(\mathbb{R}_+)} \leq 1\right) \text{ and hence } \|K\| < \infty.$$

Remark 2.2. *Observe that if Ω is a nonempty and mesurable subset of $\mathbb{R}_+$, then we can also consider the linear Volterra integral operator $(Kx)(t) = \int_0^t k(t, s)x(s) ds$ associated with the Lebesque space $L^p(\Omega)$, $1 \leq p \leq \infty$. Namely, if $x \in L^p(\Omega)$, $1 \leq p \leq \infty$, then we can extend x to be the whole half axis $\mathbb{R}_+$ by putting $x(t) = 0$ for $t \in \mathbb{R}_+ \backslash \Omega$. Then we can treat the operator K in the usual way (see [21]).*

Now, we will collect some definitions and basic results which will be used further on throughout the paper.

First, we denote by $L^p(\mathbb{R}_+)$ the space of Lebesgue $p-$ integrable functions on $\mathbb{R}_+$ equipped with the standard norm, $x \in L^p(\mathbb{R}_+)$, $\|x\|_p^p = \int_0^{+\infty} |x(t)|^p \, dt$

Next, we recall some basic facts concerning measure of noncompactness. Assume that $(E, \|.\|)$ is a real Banach space with zero element θ. Let $B(x, r)$ denote the closed ball centered at x and with radius r. The symbol B_r stands for the ball $B(\theta, r)$. If X is a subset of E, then $\overline{X}$ and $ConvX$ denote the closure and convex closure of X, respectively. **By** the symbols λX and $X + Y$, we denote the standard algebraic operations on sets. Moreover, we denote by M_E the familiy of all nonempty and bounded subsets of E and N_E its subfamily consisting of all relatively compact subsets.The definition of the concept of a measure of noncompactness presented bellow comes from [9].

Definition 2.3. *[9] A mapping $\mu : M_E \to \mathbb{R}_+ = [0, +\infty[$ is said to be a measure of noncompactness in E if it satisfies following conditions*

1. *The family* $\ker \mu = \{X \in M_E : \ \mu(X) = 0\}$ *is nonempty and* $\ker \mu \subset N_E$.
2. $X \subset Y \Longrightarrow \mu(X) \leq \mu(Y)$
3. $\mu(\overline{X}) = \mu(ConvX) = \mu(X)$
4. $\mu(\lambda X + (1 - \lambda)Y) \leq \lambda \mu(X) + (1 - \lambda)Y$, *for* $\lambda \in [0, 1]$
5. *If $\{X_n\}$ is a sequence of nonempty, bounded, closed subsets of E such that $X_{n+1} \subset X_n$, $(n = 1, 2, \ldots)$ and $\lim\limits_{n\to\infty} \mu(X_n) = 0$, then the set $X_\infty = \cap_{n=1}^{\infty} X_n$ is nonempty.*
   Observe that the intersection set X_∞ belongs to $\ker \mu$. *Indeed, since $\mu(X_\infty) \leq \mu(X_n)$ for any n, then we infer $\mu(X_\infty) = 0$, so $X_\infty \in$* $\ker \mu$. *For other facts concerning measures of noncompactness we refer to [9], [19].*
   In the following, we give a nonempty $X \subset L^p(\mathbb{R}_+)$ bounded, $\varepsilon > 0$, and $T > 0$. For arbitrary function $x \in X$, we let

$$\omega(x, \varepsilon) = \sup \left\{ \left(\int_0^\infty |x(t + h) - x(t)|^p \, dt \right)^{\frac{1}{p}}, |h| < \varepsilon \right\}$$

$$\omega(X, \varepsilon) = \sup \{\omega(x, \varepsilon) : \ x \in X\}$$

and

$$\omega_0(X) = \lim_{\varepsilon \to 0} \omega(X, \varepsilon).$$

Also, let

$$d_T(X) = \sup \left\{ \left(\int_T^\infty |x(t)|^p \, dt \right)^{\frac{1}{p}}, x \in X \right\}$$

and

$$d(X) = \lim_{T \to \infty} d_T(X).$$

Then, the function $\mu : M_{L^p(\mathbb{R}_+)} \to \mathbb{R}_+$ given by $\mu(X) = \omega_0(X) + d(X)$ is a measure of noncompactness on $L^p(\mathbb{R}_+)$, see ([20]) .

Darbo's fixed point theorem is a very important generalization of Schauder's fixed point theorem and includes the existence part of Banach's theorem.

Theorem 2.4. *Schauder (see [5]) Let Ω be a nonempty, bounded, closed, and convex subset of a Banach space E, Then every compact continuous map $T : \Omega \to \Omega$ has at least one fixed point.*

In the following, we state a fixed point theorem of Darbo type proved by Banas and Goebel [9]

Theorem 2.5. *(See [15], [9]) Let Ω be a nonempty, bounded, closed, and convex subset of a Banach space E, and let $T : \Omega \to \Omega$ be a continuous mapping such that a constant $k \in [0, 1)$ exists with the property*

$$\mu(TX) \leq k\mu(X)$$

for any nonempty X of Ω. Then T has a fixed point in the set Ω.

Now, we need to characterize the compact subsets of $L^p(\mathbb{R}_+)$.

Theorem 2.6. *[20] Let $\mathcal{F}$ be a bounded set in $L^p(\mathbb{R}^N)$ with $1 \le p < +\infty$. Then, $\mathcal{F}$ has a compact closure in $L^p(\mathbb{R}^N)$ if and only if $\lim_{h\to 0} \left\|\tau_h f - f\right\|_p = 0$ uniformly in $f \in \mathcal{F}$, where $\tau_h f(x) = f(x+h)$ for all $x \in \mathbb{R}^N$. In addition, for $\varepsilon > 0$, there is a bounded and measurable subset Ω of $\mathbb{R}^N$ such that $\left\|f\right\|_{L^p(\mathbb{R}^N\setminus\Omega)} < \varepsilon$ for all $f \in \mathcal{F}$.*

Corollary 2.7. *Let $\mathcal{F}$ be a bounded set in $L^p(\mathbb{R}^N)$ with $1 \le p < +\infty$. The closure of $\mathcal{F}$ in $L^p(\mathbb{R}^N)$ is compact if and only if $\lim_{h\to 0}\left(\int_0^\infty \left|f(x) - f(x+h)\right|^p dx\right)^{\frac{1}{p}} = 0$ uniformly in $f \in \mathcal{F}$. Also, for $\varepsilon > 0$, there is a constant $T > 0$ such that $\left(\int_T^\infty \left|f(x)\right|^p dx\right)^{\frac{1}{p}} < \varepsilon$ for all $f \in \mathcal{F}$.*

Lemma 2.8. *[14]. Let Ω be a Lebesque measurable subset of $\mathbb{R}^n$ and $1 \le p \le \infty$. If $\{f_n\}$ is a sequence in $L^p(\Omega)$ convergent to $f \in L^p(\Omega)$ in norm, then there exists a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ which converges to f a.e. in Ω and a function $g \in L^p(\Omega)$, such that*

$$\left|f_{n_k}(x)\right| \le g(x), \quad \text{for all } k \ge 1, \text{ a.e. } x \in \Omega.$$

Also, we need the following result which is a classical result in Topology.

Lemma 2.9. *Let E be a metric space and (x_n) a sequence in E. If there exists $x \in E$ such that any subsequence (x_{n_k}) of (x_n) converges to x, then $x_n \to x$ in E, as $n \to \infty$.*

We shall study the existence of the solutions of eq.(1) assuming some conditions are satisfied.

3. Main Results

Theorem 3.1. *Assume that the following conditions are satisfied.*

1. The function $f_1 : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ satisfies Carathéodory conditions, and there exist constant $\lambda_1 \in [0,1)$ and $a_1 \in L^p(\mathbb{R}_+)$ such that
$$\left|f_1(t,x) - f_1(s,y)\right| \le |a_1(t) - a_1(s)| + \lambda_1 \left|x - y\right|$$
for any $x, y \in \mathbb{R}$ and almost all $s, t \in \mathbb{R}_+$ with $f_1(.,0) \in L^p(\mathbb{R}_+)$.
2. The functions $u : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ and $k : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ satisfy Carathéodory conditions, and there exist $g_1, g_2 \in L^p(\mathbb{R}_+)$ and $g \in L^q(\mathbb{R}_+)$ ($\frac{1}{p} + \frac{1}{q} = 1$) such that
$$\begin{aligned} |u(t,s,x)| &\le k(t,s)|x|, \\ |u(t_1,s,x) - u(t_2,s,x)| &\le g(s)\left|g_2(t_1) - g_2(t_2)\right| \\ k(t,s) &\le g_1(t)g(s) \ \ \forall\, t,s \in \mathbb{R}_+, \forall\, x \in \mathbb{R}. \end{aligned}$$
3. The function $f_2 : \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ satisfies Carathéodory conditions, and there exist constants $\lambda_2, \lambda_3 \ge 0$ and $a_2 \in L^p(\mathbb{R}_+)$ such that
$$\left|f_2(t,x,y) - f_2(s,z,w)\right| \le |a_2(t) - a_2(s)| + \lambda_2 |x - z| + \lambda_3 \left|y - w\right|$$
for any $x, y, z, w \in \mathbb{R}$ and almost all $s, t \in \mathbb{R}_+$. $f_2(.,0,0) \in L^p(\mathbb{R}_+)$.
4. The operators Q_i, $i = 1,2$ act continuously from $L^p(\mathbb{R}_+)$ into itself and constants $b_i \in \mathbb{R}_+$, $i = 1,2$ exist such that
$\|Q_i x\|_{L^p} \le b_i \|x\|_{L^p[T,+\infty)}$ for any $x \in L^p(\mathbb{R}_+)$ and $T \in \mathbb{R}_+$.
5. There exists the nonnegative constant q_{r_0} such that the inequality $\omega_0(Q_1 X) \le q_{r_0}\omega_0(X)$ holds for all nonempty and bounded subset X of of the ball B_{r_0} where
$$r_0 = \frac{\left\|f_1(.,0)\right\|_p + \left\|f_2(.,0,0)\right\|_p}{1 - (\lambda_1 + \lambda_2 b_1 + \lambda_3 b_2 \|K\|)}.$$

6. $M = \max\{\lambda_1 + \lambda_2 b_1 + \lambda_3 b_2 \|K\|, \lambda_1 + \lambda_2 q_{r_0}\} < 1.$

Then the nonlinear integral equation (1) have at least one solution in the space $L^p(\mathbb{R}_+)$.

Proof. First, we define the operator $F : L^p(\mathbb{R}_+) \to L^p(\mathbb{R}_+)$ by

$$(Fx)(t) = f_1(t, x(t)) + f_2\left(t, (Q_1x)(t), \int_0^{+\infty} u(t, s, (Q_2x(s)))\, ds\right).$$

Setting $(F_1x)(t) = f_1(t, x(t))$ and $(F_2x)(t) = f_2\left(t, (Q_1x)(t), \int_0^{+\infty} u(t, s, (Q_2x)(s))ds\right)$.
Further considering the Carathéodory conditions, we infer that Fx is measurable for any $x \in L^p(\mathbb{R}_+)$. Now, we prove that $Fx \in L^p(\mathbb{R}_+)$ for any $x \in L^p(\mathbb{R}_+)$. We have

$$\begin{aligned}|(Fx)(t)| &\le \left|f_1(t, x) - f_1(t, 0)\right| + \left|f_1(t, 0)\right| \\ &+ \left|f_2\left(t, (Q_1x)(t), \int_0^{+\infty} u(t, s, (Q_2x)(s))ds\right) - f_2(t, 0, 0)\right| \\ &+ \left|f_2(t, 0, 0)\right|.\end{aligned}$$

By using the **Minkowski inequality**, we get

$$\begin{aligned}\left(\int_0^{+\infty} |(Fx)(t)|^p dt\right)^{\frac{1}{p}} &\le \left(\int_0^{+\infty} \left|f_1(t, x) - f_1(t, 0)\right|^p dt\right)^{\frac{1}{p}} + \left(\int_0^{+\infty} \left|f_1(t, 0)\right|^p dt\right)^{\frac{1}{p}} \\ &+ \lambda_2\left(\int_0^{+\infty} |(Q_1x)(t)|^p dt\right)^{\frac{1}{p}} + \lambda_3\left(\int_0^{+\infty} \left|\int_0^{+\infty} u(t, s, (Q_2x)(s))ds\right|^p dt\right)^{\frac{1}{p}} \\ &+ \left(\int_0^{+\infty} \left|f_2(t, 0, 0)\right|^p dt\right)^{\frac{1}{p}}.\end{aligned}$$

Then,

$$\begin{aligned}\left(\int_0^{+\infty} |(Fx)(t)|^p dt\right)^{\frac{1}{p}} &\le \left(\int_0^{+\infty} \left|f_1(t, x) - f_1(t, 0)\right|^p dt\right)^{\frac{1}{p}} + \left(\int_0^{+\infty} \left|f_1(t, 0)\right|^p dt\right)^{\frac{1}{p}} \\ &+ \lambda_2\left(\int_0^{+\infty} |(Q_1x)(t)|^p dt\right)^{\frac{1}{p}} + \lambda_3\left(\int_0^{+\infty} \left|\int_0^{+\infty} k(t, s)\, Q_2x)(s)ds\right|^p dt\right)^{\frac{1}{p}} \\ &+ \left(\int_0^{+\infty} \left|f_2(t, 0, 0)\right|^p dt\right)^{\frac{1}{p}}.\end{aligned}$$

So, By using assumptions $(1), -, (6)$ we obtain

$$\begin{aligned}\|Fx\|_p &\le \lambda_1 \|x\|_p + \left\|f_1(., 0)\right\|_p + \left\|f_2(., 0, 0)\right\|_p \\ &+ \lambda_2 b_1 \|x\|_p + \lambda_3 b_2 \|K\| \|x\|_p.\end{aligned}$$

Therefore,

$$\begin{aligned}\|Fx\|_p &\le \left\|f_1(., 0)\right\|_p + \left\|f_2(., 0, 0)\right\|_p \\ &+ (\lambda_1 + \lambda_2 b_1 + \lambda_3 b_2 \|K\|) \|x\|_p.\end{aligned} \tag{2}$$

Hence, $F(x) \in L^p(\mathbb{R}_+)$ and F is well defined and also from (2), we have $F(\overline{B}_{r_0}) \subset \overline{B}_{r_0}$, where r_0 is

$$r_0 = \frac{\left\|f_1(., 0)\right\|_p + \left\|f_2(., 0, 0)\right\|_p}{1 - (\lambda_1 + \lambda_2 b_1 + \lambda_3 b_2 \|K\|)}.$$

Now, we prove that F is continuous in $L^p(\mathbb{R}_+)$. It is enough to prove that F_2 is continuous, Indeed, Let (x_n) be a sequence in $L^p(\mathbb{R}_+)$ which converges to $x \in L^p(\mathbb{R}_+)$, since Q_i, $i = 1, 2$ are continuous for a.e. $t \in \mathbb{R}_+$ and from lemma 2.8, it follows that up a subsequence that

$$\begin{aligned}&x_{n_k} \to x, \qquad Q_i x_{n_k} \to Q_i x, \text{ for } i = 1, 2 \\ &\exists \varphi \ge 0, \quad \varphi \in L^p(\mathbb{R}_+) : \max\left\{\left|x_{n_k}(s)\right|, \left|Q_i x_{n_k}(s)\right|\right\} \le \varphi(s) \text{ a.e. on } \mathbb{R}_+.\end{aligned}$$

Since u satisfies the Carathéodory conditions, $Q_i x_{n_k} \to Q_i x$, almost everywhere on $\mathbb{R}_+$. It follows from assumption (2) that

$$u\left(t,s,Q_2 x_{n_k}\right) \longrightarrow u\left(t,x,Q_2 x\right) \text{ for almost } t,s \in \mathbb{R}_+ \tag{3}$$

and

$$\left|u\left(t,s,Q_2 x_{n_k}\right)\right| \le k\left(t,s\right)\varphi(s). \tag{4}$$

Then we have by using the Lebesque's Dominated Convergence Theorem

$$\int_0^{+\infty} u(t,s,\left(Q_2 x_{n_k}\right)(s))ds \to \int_0^{+\infty} u(t,s,\left(Q_2 x\right)(s))ds.$$

Hence for almost all $t \in \mathbb{R}_+$

$$\begin{aligned} & f_2\left(t,\left(Q_1 x_{n_k}\right)(s), \int_0^{+\infty} u(t,s,\left(Q_2 x_{n_k}\right)(s))ds\right) \\ \to \quad & f_2\left(t,\left(Q_1 x\right)(s), \int_0^{+\infty} u(t,s,\left(Q_2 x\right)(s)\right)ds. \end{aligned} \tag{5}$$

We have for almost everywhere in $\mathbb{R}_+$ the following estimate

$$\begin{aligned} & \left|f_2(t,Q_1 x_{n_k}(t), \int_0^{+\infty} u(t,s,\left(Q_2 x_{n_k}\right)(s))ds\right| \\ \le \quad & \lambda_1 g(t) + \lambda_2 g_1(t)\int_0^{+\infty} g(s)\varphi(s)ds + \left|f(t,0,0)\right| \end{aligned} \tag{6}$$

Regarding the assumptions on g, g_1 and $\left|f(t,0,0)\right|$ we get

$$\lambda_1 g(t) + \lambda_2 g_1(t)\int_0^{+\infty} g(s)\varphi(s)ds + \left|f(t,0,0)\right| \in L^p\left(\mathbb{R}_+\right). \tag{7}$$

Then from $(5),(6),(7)$ and by using the Lebesgue's Dominated Convergence Theorem, we get

$$\left\|F_2 x_{n_k} - F_2 x\right\|_{L^p} \to 0.$$

Since any sequence $\{x_n\}$ converging to x in L^p has a subsequence $\{x_{n_k}\}$ such that $\left\|F_2 x_{n_k} - F_2 x\right\|_{L^p} \to 0$, we can conclude that F_2 is a continuous operator, Further, we will show that

$$\omega_0(FX) \le \left(\lambda_1 + \lambda_2 b_1 + \lambda_3 b_2 \|K\|\right)\omega_0(X)$$

for any nonempty set $X \subset \overline{B}_{r_0}$. **To this end**, we fix an arbitrary $\varepsilon > 0$. Let us choose $x \in X$ and $t,h \in \mathbb{R}_+$ with $|h| \le \varepsilon$. we have

$$\begin{aligned} |(Fx)(t) - (Fx)(t+h)| \le & \left|f_1\left(t,x(t)\right) - f_1(t+h,x(t))\right| \\ & + \left|f_1\left(t+h,x(t)\right) - f_1\left(t+h,x(t+h)\right)\right| \\ & + \Big|f_2\left(t,(Q_1 x)(t), \textstyle\int_0^{+\infty} u(t,s,(Q_2 x)(s))ds\right) \\ & - f_2\left(t+h,(Q_1 x)(t), \textstyle\int_0^{+\infty} u(t,s,(Q_2 x)(s))ds\right)\Big| \\ & + \Big|f_2\left(t+h,(Q_1 x)(t), \textstyle\int_0^{+\infty} u(t,s,(Q_2 x)(s))ds\right) \\ & - f_2\left(t+h,\left(Q_1 x\right)(t+h), \textstyle\int_0^{+\infty} u(t,s,(Q_2 x)(s))ds\right)\Big| \\ & + \Big|f_2\left(t+h,\left(Q_1 x\right)(t+h), \textstyle\int_0^{+\infty} u(t,s,(Q_2 x)(s))ds\right) \\ & - f_2\left(t+h,\left(Q_1 x\right)(t+h), \textstyle\int_0^{+\infty} u(t+h,s,(Q_2 x)(s))ds\right)\Big|. \end{aligned}$$

Therefore

$$\begin{aligned}|(Fx)(t)-(Fx)(t+h)| &\leq |a_1(t)-a_1(t+h)| + \lambda_1 |x(t)-x(t+h)| \\ &+\lambda_2 |(Q_1x)(t+h)-(Q_1x)(t)| + |a_2(t)-a_2(t+h)| \\ &+\lambda_3 \left|\int_0^{+\infty} [u(t+h,s,(Q_2x)(s)-u(t,s,(Q_2x)(s)]\, ds\right|.\end{aligned}$$

By Minkowki's inequality, we get

$$\begin{aligned} & \left(\int_0^\infty |(Fx)(t)-(Fx)(t+h)|^p\, dt\right)^{\frac{1}{p}} \\ \leq\ & \left(\int_0^\infty |a_1(t)-a_1(t+h)|^p\, dt\right)^{\frac{1}{p}} + \lambda_1 \left(\int_0^\infty |x(t)-x(t+h)|^p\, dt\right)^{\frac{1}{p}} \\ +\ & \left(\int_0^\infty |a_2(t)-a_2(t+h)|^p\, dt\right)^{\frac{1}{p}} \\ +\ & \lambda_2 \left(\int_0^{+\infty} |(Q_1x)(t)-(Q_1x)(t+h)|^p\, dt\right)^{\frac{1}{p}} \\ +\ & \lambda_3 \left(\int_0^\infty \left|\int_0^{+\infty} (u(t,s,(Q_2x)(s))-u(t,s,(Q_2x)(s)))\, ds\right|^p dt\right)^{\frac{1}{p}}.\end{aligned}$$

Consequentle we get

$$\begin{aligned} & \left(\int_0^\infty |(Fx)(t)-(Fx)(t+h)|^p\, dt\right)^{\frac{1}{p}} \leq \\ & \left(\int_0^\infty |a_1(t)-a_1(t+h)|^p\, dt\right)^{\frac{1}{p}} + \lambda_1 \left(\int_0^\infty |x(t)-x(t+h)|^p\, dt\right)^{\frac{1}{p}} \\ & + \left(\int_0^\infty |a_2(t)-a_2(t+h)|^p\, dt\right)^{\frac{1}{p}} \\ & +\lambda_2 \left(\int_0^{+\infty} |(Q_1x)(t)-(Q_1x)(t+h)|^p\, dt\right)^{\frac{1}{p}} \\ & \lambda_3 \left(\int_0^{+\infty} \left(\int_0^{+\infty} \left|g_2(t)-g_2(t+h)\right|^q \left|g(s)\right|^q ds\right)^{\frac{p}{q}} dt\right)^{\frac{1}{p}} \|Q_2x\|_{L^p(\mathbb{R}_+)}\end{aligned}$$

Hence, we obtain

$$\begin{aligned} & \left(\int_0^\infty |(Fx)(t)-(Fx)(t+h)|^p\, dt\right)^{\frac{1}{p}} \leq \\ & +\lambda_1 \|x-\tau_h x\|_{L^1(\mathbb{R})} \\ & \|a_1-\tau_h a_1\|_{L^p(\mathbb{R}_+)} + \|a_2-\tau_h a_2\|_{L^p(\mathbb{R}_+)} + \\ & \lambda_2 \|(Q_1x)-\tau_h (Q_1x)\|_{L^p(\mathbb{R}_+)} + \\ & \lambda_3 \left\|g\right\|_{L^q(\mathbb{R}_+)} \|Q_2x\|_{L^p(\mathbb{R}_+)} \left(\int_0^{+\infty} \left|g_2(t)-g_2(t+h)\right|^p\right)^{\frac{1}{p}} dt.\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}&\omega(FX,\varepsilon) \leq \omega(a_1,\varepsilon) + \lambda_1\omega(X,\varepsilon) + \omega(a_2,\varepsilon) \\ &+\lambda_2\omega(Q_1X,\varepsilon) + \lambda_3 b_2 r_0 \left\|g\right\|_{L^q(\mathbb{R}_+)} \omega(g_2,\varepsilon).\end{aligned} \tag{8}$$

Since $\{a_1\}, \{a_2\}, \{g_2\}$ are compacts set in $L^p(\mathbb{R}_+)$, we have $\omega(a_1,\varepsilon) \longrightarrow 0, \omega(a_2,\varepsilon) \longrightarrow 0$ and $\omega(g_2,\varepsilon) \longrightarrow 0$ as $\varepsilon \longrightarrow 0$. Then, by going to the limit in (8) as $\varepsilon \longrightarrow 0$ and from assumption (5), we obtain

$$\omega_0(FX) \leq (\lambda_1 + \lambda_2 q_{r_0})\, \omega_0(X). \tag{9}$$

In the following, we fix an arbitrary number $T > 0$. Then, for an arbitrary function $x \in X$, we have

$$\begin{aligned} & \left(\int_T^\infty |F(x)(t)|^p\, dt\right)^{\frac{1}{p}} \\ & \leq \left(\int_T^\infty \left|f_1(t,x)-f_1(t,0)\right|^p dt\right)^{\frac{1}{p}} + \left(\int_T^\infty \left|f_1(t,0)\right|^p dt\right)^{\frac{1}{p}} \\ & \left(\int_T^\infty \left|f_2\left(t,(Q_1x)(t),\int_0^\infty u(t,s,(Q_2x)(s))ds\right) - f_2(t,0,0)\right|^p dt\right)^{\frac{1}{p}} \\ & + \left(\int_T^\infty \left|f_2(t,0,0)\right|^p dt\right)^{\frac{1}{p}}.\end{aligned}$$

Therefore

$$\begin{aligned}
&\left(\int_T^\infty |F(x)(t)|^p\, dt\right)^{\frac{1}{p}} \\
&\leq \lambda_1 \left(\int_T^\infty |x(t)|^p\, dt\right)^{\frac{1}{p}} + \left(\int_T^\infty \left|f_1(t,0)\right|^p dt\right)^{\frac{1}{p}} + \lambda_2 \left(\int_T^\infty |(Q_1x)(t)|^p\, dt\right)^{\frac{1}{p}} \\
&\lambda_3 \left(\int_T^\infty \left|\int_0^\infty k(t,s)(Q_2x)(s)ds\right|^p dt\right)^{\frac{1}{p}} + \left(\int_T^\infty \left|f_2(t,0,0)\right|^p dt\right)^{\frac{1}{p}}.
\end{aligned}$$

Then we have

$$\begin{aligned}
\left(\int_T^\infty |F(x)(t)|^p\, dt\right)^{\frac{1}{p}} &\leq \lambda_1 \left(\int_T^\infty |x(t)|^p\, dt\right)^{\frac{1}{p}} + \left(\int_T^\infty \left|f_1(t,0)\right|^p dt\right)^{\frac{1}{p}} \\
&+\lambda_2 b_1 \left(\int_T^\infty |x(t)|^p\, dt\right)^{\frac{1}{p}} + \lambda_3 b_2 \|K\| \left(\int_T^\infty |x(t)|^p\, dt\right)^{\frac{1}{p}} \\
&+ \left(\int_T^\infty \left|f_2(t,0,0)\right|^p dt\right)^{\frac{1}{p}}.
\end{aligned}$$

Since $\{f_1(t,0)\}$ and $\{f_2(t,0,0)\}$ are compacts in $L^p(\mathbb{R}_+)$, then, as T goes to $+\infty$, we obtain $\left(\int_T^\infty \left|f_1(t,0)\right|^p dt\right)^{\frac{1}{p}}$ and $\left(\int_T^\infty \left|f_2(t,0,0)\right|^p dt\right)^{\frac{1}{p}}$ go to 0.

Hence,

$$d(FX) \leq (\lambda_1 + \lambda_2 b_1 + \lambda_3 b_2 \|K\|)\, d(X). \tag{10}$$

So, from (9) and (10) it follows

$$\mu(FX) \leq \max\left\{\lambda_1 + \lambda_2 b_1 + \lambda_3 b_2 \|K\|, \lambda_1 + \lambda_2 q_{r_0}\right\} \mu(X). \tag{11}$$

By (11), assumption (6) and Theorem 2.5, we deduce that the operator F has a fixed point x in B_{r_0} and consequently, eq.(1) has at least one solution in $L^p(\mathbb{R}_+)$. □

4. Example

Consider the functional integral equation

$$x(t) = \frac{\cos x(t)}{t+2} + \frac{|x(t)|}{21\,(1+|x(t)|)} e^{-t} + \frac{1}{10}\int_0^{+\infty} \frac{\sin\left(|x(s)|\, e^{-|x(s)|}\right)}{e(t+3)^2(s+2)^2} ds \tag{12}$$

Eq. (12) is a special case of Eq. (1) with

$$\begin{aligned}
&f_1(t,x) = \tfrac{\cos x(t)}{t+2},\ f_2(t,x,y) = \tfrac{1}{3}x + \tfrac{1}{10}y,\ (Q_1x)(s) = \tfrac{|x(s)|}{7(1+|x(s)|)}e^{-s}, \\
&k(t,s) = \tfrac{1}{e(t+3)^2(s+2)^2},\ (Q_2x)(s) = e^{-|x(s)|}\,|x(s)|\,, |(Q_1x)(s)| \leq \tfrac{1}{7}\,|x(s)|\,, \\
&|(Q_2x)(s)| \leq |x(s)|\,.
\end{aligned}$$

In this example, hypothesis (1) holds with $a_1(t) = \frac{1}{t+2}$ and $\lambda_1 = \frac{1}{2}$, indeed, we have

$$\begin{aligned}
\left|f_1(t,x) - f_1(s,y)\right| &= \left|\tfrac{\cos x}{t+2} - \tfrac{\cos y}{s+2}\right| \\
&\leq \left|\tfrac{1}{t+2} - \tfrac{1}{s+2}\right| + \tfrac{1}{2}\left|x - y\right|.
\end{aligned}$$

In addition, $\left|f_1(t,0)\right| = \frac{1}{t+2} \in L^p(\mathbb{R}_+)$, indeed, $\left\|f_1(t,0)\right\|^p_{L^p(\mathbb{R}_+)} = \int_0^{+\infty} \frac{dx}{(1+x)^p} = \frac{1}{p-1}$ for all $p > 1$. Thus, we have $\left\|f_1(t,0)\right\|_{L^p(\mathbb{R}_+)} = \left(\frac{1}{p-1}\right)^{\frac{1}{p}}$. Further we have

$$\left|f_2(t,x,y) - f_2(s,z,w)\right| \leq \frac{1}{3}\,|x - z| + \frac{1}{10}\left|y - w\right|.$$

$a_2(t) = 0$, $\lambda_2 = \frac{1}{3}$ and $\lambda_3 = \frac{1}{10}$, $f_2(t, 0, 0) = 0 \in L^p(\mathbb{R}_+)$.

We have

$$\begin{array}{rcl} u(t,s,x) & = & k(t,s)\sin x \\ |u(t,s,x)| & \leq & k(t,s)\,|x| \\ |u(t_1,s,x) - u(t_2,s,x)| & \leq & |k(t_1,s) - k(t_2,s)|\, g(s) \end{array}$$

and

$$\begin{array}{rcl} k(t,s) & \leq & \frac{1}{e(t+3)^2} \times \frac{1}{(s+2)^2} \\ g_1(t) & = & \frac{1}{e(t+3)^2} \\ g_2(s) = g(s) & = & \frac{1}{(s+2)^2} \\ |k(t_1,s) - k(t_2,s)| & \leq & \left| \frac{1}{e(t_1+3)^2} - \frac{1}{e(t_2+3)^2} \right| \frac{1}{(s+2)^2} \end{array}$$

Q_1 and Q_2 satisfied assumption (4) of theorem 3.1 with $b_2 = 1$, $b_1 = \frac{1}{7}$. By using theorem 3.4 in [1], we have $\|K\| \leq \frac{1}{e}$.

Further, for $\varepsilon \geq 0$, $\|x\| \leq r_0$, $|h| < \varepsilon$ and by the Mean theorem, we get

$$\begin{aligned} & \left(\int_0^{+\infty} |(Q_1x)(t+h) - (Q_1x)(t+h)|^p \, dt \right)^{\frac{1}{p}} \\ \leq \; & \frac{1}{7} \left(\int_0^{+\infty} |x(t+h) - x(t+h)|^p \, dt \right)^{\frac{1}{p}} + \frac{|h|}{7} e^{-\theta h} \left(\int_0^{+\infty} e^{-pt} dt \right)^{\frac{1}{p}}, 0 < \theta < 1 \\ \leq \; & \frac{1}{7} \left(\int_0^{+\infty} |x(t+h) - x(t+h)|^p \, dt \right)^{\frac{1}{p}} + \frac{\varepsilon}{7} e^{-\theta h} \left(\int_0^{+\infty} e^{-pt} dt \right)^{\frac{1}{p}} \\ & \frac{1}{7} \left(\int_0^{+\infty} |x(t+h) - x(t+h)|^p \, dt \right)^{\frac{1}{p}} + \frac{\varepsilon}{7} M, \end{aligned}$$

where $M = \left(\int_0^{+\infty} e^{-pt} dt \right)^{\frac{1}{p}}$. Hence, from the last estimate, we get as ε goes to 0

$$\omega_0(Q_1X) \leq \frac{1}{7}\omega_0(X). \tag{13}$$

Thus, according to assumption (5) we may put $q_{r_0} = \frac{1}{7}$. Further we get $\lambda_1 + \lambda_2 b_1 + \lambda_3 b_2 \|K\| \leq \frac{1}{2} + \frac{1}{21} + \frac{1}{10e} < 1$ and $\lambda_1 + \lambda_2 q_{r_0} = \frac{1}{2} + \frac{1}{49} = 0,52 < 1$. The inequality of assumption (6) is satisfied with the constant $M < 1$. Since all of the assumptions of Theorem 3.1 are fullfilled, we deduce that the functional integral equation (12) has at least one solution belonging to the ball B_{r_0} of the space $L^p(\mathbb{R}_+)$.

Acknowledgment

The author would like to thank the referee for carefully reading the paper.

References

[1] A. Aghajani, D. O'Regan and A. Shole Haghighi, Measure of noncompactness on $L^p(\mathbb{R}^N)$ and Applications, CUBO A Math. J. vol. 17, No 01, (85-97), 2015.

[2] A. Aghajani, J. Banas, Y. Jalilian, Existence of solution for a class nonlinear Volterra singular integral, Comput. Math. Appl. 62 (2011), 1215-1227.

[3] W. Al Sayed and M.A. Darwish, On the existence of solutions of a perturbed functional integral equation in the space of Lebesgue Integrable functions on $\mathbb{R}_+$. Journal of Math. and Appl. JMA No 41, pp. 19-27 (2018).

[4] R. P. Argawal and D. O'Regan, Fredholm and Volterra integral equations with inghajani, tegrable singularieties, Hokkaido Math. J. 33 (2004), no. 2, 443-456.
[5] R. P. Argawal and D. O'Regan, Fixed point theory and applications, Cambridge University Press (2004).
[6] J. Banas, Z. Knap, On measures of weak noncompactness and nonlinear integral equations of convolution type, J. Math. Anal. 146 (2) (1990) 353-362.
[7] J. Banas, Z. Knap, Integrable solutions of a functional-integral equation, Revista Mat. Univ. Complutence de Madrid 2 (1989) 31-38.
[8] J. Banas, J. Rivero, On measures of weak noncompactness, Ann. Mat. Pure Appl. 151, (1988) 213-224.
[9] J. Banas and K. Goebel, Measures of Noncompactness In Banch spces, Lecture Notes in Pure and Applied Mathematics 60, Marcel Dekker, New York, 1980.
[10] J. Banas, R.Rzepka, An application of a measure of noncompactness in the study of asymptotic stability, Appl. Math. Lett. 16 (2003), 1-6
[11] M. Bousselsal, A. bellour and M.A. Taoudi, On the solvability of a nonlinear integro-differential equation on the half axis, Meddit. J. Math. 2016, 13, 5, pp. 2887-2896
[12] M. Bousselsal, A. Bellour and M.A Taoudi, Integrable solutions of a nonlinear integral equation related to some epidemic models, Glasnick Mat. 49, (69), 3 95406, (2014)
[13] M. Bousselsal and Jah Sidi, Integrable Solutions of a nonlinear integral equation via noncompactness measure and Krasnoselskii's fixed point theorem, International J. of Analysis, vol. 2014, article ID 280, 709, 10 pages.
[14] H. Brezis, Analyse fonctionnelle, théorie et applications, edition, Masson, (1983).
[15] G. Darbo, Puntu uniti in trasformazioni a codominio non compatto, Rend. Sem. Mat. Univ. Padova, 24 (1955), 84-92.
[16] M. Darwish, On a perturbed functional integral equation of Urysohn type, Appl. Math. Comput. 218 (2012), 8800-8805.
[17] B. Folland, Real Analysis, A Wiley-Interscience Publication, 1999.
[18] W. Gomaa El-Sayed, Nonlinear functional integral equations of convolution type, Port. Math. 54 (1997), 449-456.
[19] K. Kuratowski, Sur les espaces complets. Fund. Math. 15 (1930), 301-309
[20] H; Khosravi, R. Allahyari, A. S. Haghighi, Existence of solutions of functional integral equations of convolution type using a new construction of a measure of noncompactness on $L^p(\mathbb{R}_+)$. Appl. Math. and Comp. 260 (2015) 140-147.
[21] M.M.A. Metwali, The solvability of functional quadratic Volterra-Urysohn integral equations on the half line, Sc. Fas. Math. 61, 2018.
[22] W. Rudin, Real and complex analysis, Mc Graw-Hill, New York, 1987.
[23] N. Salhi and M.A. Taoudi, Existence of Integrable solutions of an integral equation of Harmmerstein type on an unbounded interval, Meditter. J. math. 9, (2012), 729-739.
[24] P.P. Zabrejko, A. I. Koshelev, M.A. Krasnosel'skii, S.G. Mikhlin, L.S. Rakovshchik, V.J. Stetsenko, Integral Equations, Nauka Moscow, 1968.