Filomat 35:6 (2021), 1823–1839
https://doi.org/10.2298/FIL2106823S

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

$D_{p,q}$-Classical Orthogonal Polynomials: An Algebraic Approach

Mabrouk Sghaier[a], Mohamed Zaatra[b], Mehdi Mechri[c]

[a]*Institut Supeur d'Informatique de Medenine. Medenine 4119, Tunisia.*
[b]*Institut Supeur des Sciences et Techniques des Eaux de Gab Gab072, Tunisia.*
[c]*Facults Sciences de Gab 6029 route de Medenine Gab Tunisia.*

Abstract. In this paper, we introduce a new technical method for the study of the $D_{p,q}$-classical orthogonal polynomials where $D_{p,q}$ is the (p,q)-difference operator, using basically an algebraic approach. Some new characterizations are given. The approach has been illustrated with three examples.

1. Introduction

In recent years, Corcino [4] studied the (p,q)-extension of the binomial coefficients and also derived some properties. Duran et al [8] considered (p,q)-analogues of Bernoulli polynomials, Euler polynomials and Genocchi polynomials and acquired the (p,q)-analogues of known earlier formulae. Duran and Acikgoz [7] gave (p,q)-analogues of the Apostol-Bernoulli, Euler and Genocchi polynomials and derived their some properties. See also [2, 6]. In [18], the authors introduceeneral (p,q)-Sturm-Liouville difference equation whose solutions are (p,q)-analogues of classical orthogonal polynomials.

In this paper, we establish some new characterizations concerning the $D_{p,q}$-classical orthogonal polynomials with the method of the dual sequence.

The structure of this paper is as follows: Section 2 is devoted to preliminary results and notations to be used in the sequel. In section 3, we present four properties concerning the $D_{p,q}$-classical orthogonal polynomials. The first one is a $D_{p,q}$-distributional equation of Pearson type fulfilled by its associated form. The second is a second order (p,q)-difference equation satisfied by this sequence. The third is the so-called Rodrigues formula involving the form itself which allows us to determine the coefficients of the second-order recurrence relation fulfilled by the $D_{p,q}$-classical orthogonal sequences. The fourth, we obtain the coefficients in the three-term recurrence relation for the orthogonal polynomials solutions of the $D_{p,q}$-distributional equation. Finally, in the last section, we illustrate our results applying them to some known families of $D_{p,q}$-classical orthogonal polynomials.

2010 *Mathematics Subject Classification*. Primary 33C45; Secondary 42C05
Keywords. Keywords: Orthogonal polynomials, (p,q)-difference operator, (p,q)-classical polynomial, Rodrigues formula, Recurrence relation.
Received: 05 May 2020; Accepted: 22 October 2020
Communicated by Miodrag Spalević
Email addresses: sghaier.mabrookbelhedi580@gmail.com (Mabrouk Sghaier), medzaatra@yahoo.fr (Mohamed Zaatra), mehdi_mechri@yahoo.fr (Mehdi Mechri)

2. Preliminaries and notations

Let $\mathcal{P}$ be the vector space of polynomials with complex coefficients and let $\mathcal{P}'$ be its dual. We denote by $\langle u, f\rangle$ the action of $u \in \mathcal{P}'$ on $f \in \mathcal{P}$. In particular, we denote by $(u)_n := \langle u, x^n\rangle$, $n \geq 0$, the moments of u. For instance, for any form u, any polynomial g and any $a \in \mathbb{C}\backslash\{0\}$, we let $Du = u'$, gu, $h_a u$ and $x^{-1}u$ be the forms defined by duality

$$\langle u', f\rangle = -\langle u, f'\rangle\,,\ \langle gu, f\rangle = \langle u, gf\rangle\,,\ \langle h_a u, f\rangle = \langle u, h_a f\rangle\,,$$

$$\left\langle x^{-1}u, f\right\rangle = \langle u, \theta_0 f\rangle\,,\ f \in \mathcal{P},$$

where $(h_a f)(x) = f(ax)$ and $\left(\theta_0 f\right)(x) = \dfrac{f(x) - f(0)}{x}$.

Let $\{P_n\}_{n\geq 0}$ be a sequence of monic polynomials with $\deg(P_n) = n$, $n \geq 0$ and let $\{u_n\}_{n\geq 0}$ be its dual sequence, $u_n \in \mathcal{P}'$ defined by

$$< u_n, P_m >:= \delta_{n,m}\ , n\ ,\ m \geq 0\ .$$

Let us recall some results [17].

Lemma 1.1. *For any $u \in \mathcal{P}'$ and any integer $m \geq 1$, the following statements are equivalent*

(i) $< u, P_{m-1} >\neq 0\ ,\ < u, P_n >= 0\ ,\ n \geq m\ .$

(ii) $\exists \lambda_\nu \in \mathbb{C}\ , 0 \leq \nu \leq m-1\ , \lambda_{m-1} \neq 0$ such that $u = \displaystyle\sum_{\nu=0}^{m-1} \lambda_\nu u_\nu\ .$

As a consequence, the dual sequence $\{u_n^{[1]}\}_{n\geq 0}$ of $\{P_n^{[1]}\}_{n\geq 0}$ where $P_n^{[1]}(x) = (n+1)^{-1}P'_{n+1}(x)$, $n \geq 0$ is given by

$$(u_n^{[1]})' = -(n+1)u_{n+1}\ ,\quad n \geq 0\ . \tag{1}$$

Similarly, the dual sequence $\{\widetilde{u}_n\}_{n\geq 0}$ of $\{\widetilde{P}_n\}_{n\geq 0}$ where $\widetilde{P}_n(x) = a^{-n}P_n(ax), n \geq 0, a \neq 0$ is given by

$$\widetilde{u}_n = a^n(h_{a^{-1}}u_n)\ ,\quad n \geq 0\ .$$

The form u is called regular if we can associate with it a sequence $\{P_n\}_{n\geq 0}$ such that

$$< u, P_m P_n >= r_n \delta_{n,m}\ ,\quad n\ ,\ m \geq 0\ ;\ r_n \neq 0\ ,\ n \geq 0\ . \tag{2}$$

The sequence $\{P_n\}_{n\geq 0}$ is then said orthogonal with respect to u. In this case, we have

$$u_n = r_n^{-1} P_n u_0\ ,\ n \geq 0\ . \tag{3}$$

According to Favard's Theorem, a MOPS is characterized by the following three-term recurrence relation [3]

$$\begin{aligned} &P_0(x) = 1\ ,\qquad P_1(x) = x - \beta_0\ ,\\ &P_{n+2}(x) = (x - \beta_{n+1})P_{n+1}(x) - \gamma_{n+1}P_n(x)\ ,\quad n \geq 0\ . \end{aligned} \tag{4}$$

Let us introduce the (p,q)-difference operator [15]

$$(D_{p,q}f)(x) = \frac{(h_p f)(x) - (h_q f)(x)}{(p-q)x}\ ,\qquad f \in \mathcal{P}\ ,\quad 0 < |q| < |p| \leq 1\ .$$

Remark 1. When $p \longrightarrow 1$, we again meet the Hahn's operator [10].

From the definition, we obtain

$$D_{p,q} = \frac{1}{p-q}\theta_0 \circ (h_p - h_q) \ .$$

On account of the last equation we have,

$${}^tD_{p,q} = \frac{1}{p-q}(h_p - h_q)x^{-1} \ ,$$

where ${}^tD_{p,q}$ denotes the transposed of $D_{p,q}$. We can define $D_{p,q}$ from $\mathcal{P}'$ to $\mathcal{P}'$ by $D_{p,q} := -{}^tD_{p,q}$ so that

$$\langle D_{p,q}u, f\rangle = -\langle u, D_{p,q}f\rangle, \quad f \in \mathcal{P} \ , \quad u \in \mathcal{P}' \ .$$

In particular, this yields

$$(D_{p,q}u)_n = -[n](u)_{n-1} \ , \quad n \geq 0 \ ,$$

where

$$(u)_{-1} = 0 \quad and \quad [n] := \frac{p^n - q^n}{p-q} \ .$$

Lemma 1.2. *The following formulas hold*

$$(D_{p,q}f_1f_2)(x) = h_pf_1(D_{p,q}f_2) + h_qf_2(D_{p,q}f_1) \ , \quad f_1, f_2 \in \mathcal{P} \ , \tag{5}$$

$$D_{p,q} \circ h_{p^{-1}q^{-1}} = p^{-1}q^{-1}D_{p^{-1},q^{-1}} \quad in \ \mathcal{P} \ , \tag{6}$$

$$D_{p,q} \circ h_a = ah_a \circ D_{p,q} \quad in \ \mathcal{P} \ , \ a \in \mathbb{C}\backslash\{0\} \ , \tag{7}$$

$$h_{p^{-1}q^{-1}} \circ D_{p,q} = D_{p^{-1},q^{-1}} \quad in \ \mathcal{P} \ , \tag{8}$$

$$h_{p^{-1}q^{-1}} \circ D_{p,q} = p^{-1}q^{-1}D_{p^{-1},q^{-1}} \quad in \ \mathcal{P}' \ , \tag{9}$$

$$D_{p,q} \circ h_a = a^{-1}h_a \circ D_{p,q} \quad in \ \mathcal{P}' \ , \ a \in \mathbb{C}\backslash\{0\} \ , \tag{10}$$

$$D_{p,q} \circ h_{p^{-1}q^{-1}} = D_{p^{-1},q^{-1}} \quad in \ \mathcal{P}' \ , \tag{11}$$

$$D_{p,q} \circ D_{p^{-1},q^{-1}} = p^{-1}q^{-1}D_{p^{-1},q^{-1}} \circ D_{p,q} \quad in \ \mathcal{P} \ , \tag{12}$$

$$D_{p,q} \circ D_{p^{-1},q^{-1}} = pqD_{p^{-1},q^{-1}} \circ D_{p,q} \quad in \ \mathcal{P}' \ , \tag{13}$$

$$\text{The operator } D_{p,q} \text{ is injective in } \mathcal{P}' \ , \tag{14}$$

$$D_{p,q}(gu) = (h_{q^{-1}}g)(D_{p,q}u) + q^{-1}(D_{p^{-1},q^{-1}}g)(h_pu), \ g \in \mathcal{P} \ , \ u \in \mathcal{P}'. \tag{15}$$

Proof. The relation (5) is well known [14]. It is easy to prove (6) – (7), then (8) is a consequence of them. From the definition of $D_{p,q}$, $h_{p^{-1}q^{-1}}$ and (6), we obtain directly (9). Likewise, the relation (10) is obtained from (7) and (11) from (8). It is easy to prove (12), then (13) is deduced. The property (14) is evident from the definition. Finally, we have

$$\begin{aligned}
\langle D_{p,q}(gu), f > &= -\langle u, g(D_{p,q}f)\rangle \\
&= -\langle u, D_{p,q}(h_{q^{-1}}g)f - (h_pf)D_{p,q}(h_{q^{-1}}g)\rangle \ from\,(5) \\
&= \langle D_{p,q}u, (h_{q^{-1}}g)f\rangle + q^{-1}\langle h_pu, fh_{p^{-1}q^{-1}}(D_{p,q}g)\rangle \ from\,(7) \\
&= \langle D_{p,q}u, (h_{q^{-1}}g)f\rangle + q^{-1}\langle h_pu, f(D_{p^{-1},q^{-1}}g)\rangle \ from\,(8)
\end{aligned}$$

Therefore, we obtain (15). □

Now, consider a MOPS $\{P_n\}_{n\geq 0}$ as above in section 1 and let

$$P_n^{[1]}(x, p, q) := \frac{1}{[n+1]}(D_{p,q}P_{n+1})(x), \quad n \geq 0. \tag{16}$$

Denoting by $\{u_n^{[1]}(p,q)\}_{n\geq 0}$ the dual sequence of $\{P_n^{[1]}(.,p,q)\}_{n\geq 0}$.
LEMMA 1.3.

$$D_{p,q}(u_n^{[1]}(p,q)) = -[n+1]u_{n+1}\ , \quad n \geq 0\ . \tag{17}$$

Proof. From the definition $< u_n^{[1]}, P_m^{[1]} >= \delta_{n,m},\ n\ ,\ m \geq 0$, we have

$$< D_{p,q}(u_n^{[1]}(p,q)), P_{m+1} >= -[m+1]\delta_{n,m}\ .$$

Therefore,

$$\begin{aligned} < D_{p,q}(u_n^{[1]}(p,q)), P_{n+1} > &= -[n+1]\ , \\ < D_{p,q}(u_n^{[1]}(p,q)), P_m > &= 0\ , \quad m \geq n+2\ ,\ n \geq 0\ . \end{aligned}$$

By virtue of Lemma 1.1, we have

$$D_{p,q}(u_n^{[1]}(p,q)) = \sum_{\nu=0}^{n+1} \lambda_{n,\nu} u_\nu\ , \quad n \geq 0\ .$$

But,

$$< D_{p,q}(u_n^{[1]}(p,q)), P_\mu >= \lambda_{n,\mu}, \quad 0 \leq \mu \leq n+1\ ,$$

and

$$\begin{aligned} \lambda_{n,\mu} &= 0\ , \quad 0 \leq \mu \leq n\ , \\ \lambda_{n,n+1} &= -[n+1]\ , \quad n \geq 0\ . \end{aligned}$$

Hence, the desired result follows. □

DEFINITION 1.4. *An MOPS $\{P_n\}_{n\geq 0}$ is called $D_{p,q}$-classical if $\{P_n^{[1]}(.,p,q)\}_{n\geq 0}$ is also a MOPS. In this case, the form u_0 is called $D_{p,q}$-classical form.*

3. The $D_{p,q}$-classical orthogonal polynomials

TEOREM 2.1. *For any MOPS $\{P_n\}_{n\geq 0}$ the following statements are equivalent*
(a) *The sequence $\{P_n\}_{n\geq 0}$ is $D_{p,q}$-classical.*
(b) *There exist two polynomials Φ (monic) and Ψ with* $\deg(\Phi) \leq 2$ *and* $\deg(\Psi) = 1$ *fulfilling*

$$\Psi'(0) - \frac{p^{1-n}}{2}[n]\Phi''(0) \neq 0\ ,\ n \geq 0\ , \tag{18}$$

and such that the associated regular form u_0 satisfies

$$D_{p,q}\Big(h_{p^{-1}}(\Phi u_0)\Big) + \Psi u_0 = 0\ . \tag{19}$$

For the proof we need the following result.
LEMMA 2.2. [16] *Let be u a regular form and ϕ a polynomial such that $\phi u = 0$. Then necessarily $\phi = 0$.*

Proof. (of Theorem 2.1) $(a) \Rightarrow (b)$ From the assumption, we have

$$u_n = r_n^{-1} P_n u_0 \ , \ n \geq 0 \ , \tag{20}$$

and

$$u_n^{[1]}(p,q) = (r_n^{[1]})^{-1} P_n^{[1]}(.,p,q) u_0^{[1]}(p,q) \ , \ n \geq 0 \ . \tag{21}$$

Substitution of (20) and (21) into (17) gives

$$D_{p,q}\Big(P_n^{[1]}(.,p,q) u_0^{[1]}(p,q)\Big) = -X_n P_{n+1} u_0 \ , \ n \geq 0 \ , \tag{22}$$

where

$$X_n = \frac{r_n^{[1]}}{r_{n+1}} [n+1] \ , \ n \geq 0 \ . \tag{23}$$

Using formula (15), equation (22) can reads as for $n \geq 0$

$$\begin{aligned} &\Big(h_{q^{-1}} P_n^{[1]}(.,p,q)\Big) D_{p,q} u_0^{[1]}(p,q) \\ &\qquad + q^{-1}\Big(D_{p^{-1},q^{-1}} P_n^{[1]}(.,p,q)\Big) h_p\Big(u_0^{[1]}(p,q)\Big) = -X_n P_{n+1} u_0 \ . \end{aligned} \tag{24}$$

For $n = 0$ (respectively, for $n = 1$), equation (24) becomes

$$D_{p,q} u_0^{[1]}(p,q) = -\gamma_1^{-1} P_1 u_0 \ , \tag{25}$$

$$\Big(h_{q^{-1}} P_1^{[1]}(.,p,q)\Big) D_{p,q} u_0^{[1]}(p,q) + q^{-1} h_p u_0^{[1]}(p,q) = -(p+q) \frac{r_1^{[1]}}{r_2} P_2 u_0 \ . \tag{26}$$

Substitution of (25) into (26) gives

$$h_p u_0^{[1]}(p,q) = K\Phi u_0 \ , \tag{27}$$

where

$$K\Phi(x) = q\gamma_1^{-1}\Big(h_{q^{-1}} P_1^{[1]}(x,p,q)\Big) P_1 - q(p+q) \frac{r_1^{[1]}}{r_2} P_2(x) \ .$$

(K is constant to make Φ monic)

Applying $h_{p^{-1}}$ to (27), we get

$$u_0^{[1]}(p,q) = h_{p^{-1}}(K\Phi u_0) \ . \tag{28}$$

Substitution of (28) into (25) gives (19), where

$$\Psi(x) = \frac{1}{\gamma_1 K} P_1(x) \ . \tag{29}$$

Now, taking into account (25), (27) and (29), the equation (24) can be written as (for $n \geq 0$)

$$\Big\{q^{-1}\Big(D_{p^{-1},p^{-1}} P_n^{[1]}(.,p,q)\Big)\Phi + K^{-1} X_n P_{n+1} - \Psi\Big(h_{q^{-1}} P_n^{[1]}(.,p,q)\Big)\Big\} u_0 = 0 \ .$$

But, by the regulariry of u_0, we have from the Lemma 2.2

$$q^{-1}\Big(D_{p^{-1},q^{-1}} P_n^{[1]}(.,p,q)\Big)\Phi + K^{-1} X_n P_{n+1} - \Psi\Big(h_{q^{-1}} P_n^{[1]}(.,p,q)\Big) = 0 \ , \ n \geq 0 \ .$$

Taking into account that $\deg(\Phi) \leq 2$ and $\deg(\Psi) = 1$, we obtain (18) by identifying the highest degree coefficients.
$(b) \Rightarrow (a)$ Let us prove that the sequence $\{P_n^{[1]}(.,p,q)\}_{n\geq 0}$ is orthogonal with respect to

$$\vartheta = h_{p^{-1}}(\Phi u_0)\ . \tag{30}$$

Let $m \leq n-1$. From (16) and (15), we have

$$\begin{aligned}\langle \vartheta, P_m(x)P_n^{[1]}(x,p,q)\rangle &= -\frac{1}{[n+1]}\langle D_{p,q}(P_m\vartheta), P_{n+1}(x)\rangle \\ &= -\frac{1}{[n+1]}\langle \Big(h_{q^{-1}}P_m\Big)D_{p,q}\vartheta + q^{-1}\Big(D_{p^{-1},q^{-1}}P_m\Big)h_p\vartheta, P_{n+1}\rangle\ .\end{aligned}$$

Taking into account that $\{P_n\}_{n\geq 0}$ is orthogonal with respect to u_0 and that

$$D_{p,q}\vartheta = -\Psi u_0\ , \tag{31}$$

where Ψ is a polynomial of first degree, we get

$$\langle \vartheta, P_mP_n^{[1]}(.,p,q)\rangle = -\frac{q^{-1}}{[n+1]}\langle h_p\vartheta, (D_{p^{-1},q^{-1}}P_m\Big)(x)P_{n+1}(x)\rangle\ .$$

Using (30), the orthogonality of $\{P_n\}_{n\geq 0}$ with respect to u_0 and the fact that $\deg(\Phi) \leq 2$, we obtain

$$\langle \vartheta, P_m(x)P_n^{[1]}(x,p,q)\rangle = -\frac{q^{-1}}{[n+1]}\langle u_0, \Phi(x)(D_{p^{-1},q^{-1}}P_m\Big)(x)P_{n+1}(x)\rangle = 0\ .$$

For $m = n$, a second use of (15) gives

$$\langle \vartheta, P_n(x)P_n^{[1]}(x,p,q)\rangle = -\frac{1}{[n+1]}\langle \Big(h_{q^{-1}}P_n\Big)D_{p,q}\vartheta + q^{-1}\Big(D_{p^{-1},q^{-1}}P_n\Big)h_p\vartheta, P_{n+1}(x)\rangle\ . \tag{32}$$

Using (31) and the fact that $\{P_n\}_{n\geq 0}$ is orthogonal with respect to u_0, we get

$$\langle \Big(h_{q^{-1}}P_n\Big)D_{p,q}\vartheta, P_{n+1}\rangle = -q^{-n}r_{n+1}\Psi'(0)\ , \tag{33}$$

where r_{n+1} is given in (3).
Owing to (30), we have

$$\langle q^{-1}\Big(D_{p^{-1},q^{-1}}P_n\Big)h_p\vartheta, P_{n+1}\rangle = \frac{1}{2}q^{-n}p^{1-n}r_{n+1}[n]\Phi''(0)\ . \tag{34}$$

Substitution of (33) and (34) into (32) gives

$$\langle \vartheta, P_nP_n^{[1]}(.,p,q)\rangle = -\frac{q^{-n}}{[n+1]}\Big\{\frac{p^{1-n}}{2}[n]\Phi''(0) - \Psi'(0)\Big\}r_{n+1}\ .$$

On account of condition (18), the last equation implies that

$$\langle \vartheta, P_nP_n^{[1]}(.,p,q)\rangle \neq 0\ ,\ n \geq 0\ .$$

So, the sequence $\{P_n^{[1]}(.,p,q)\}_{n\geq 0}$ is orthogonal with respect to the form ϑ. $\square$

Corollary 2.3. *If $\{P_n\}_{n\geq 0}$ is $D_{p,q}-$classical, the sequence $\{P_n^{[m]}(.,p,q)\}_{n\geq 0}$ is $D_{p,q}-$classical for any $m \geq 1$ and we have*

$$D_{p,q}\Big(h_{p^{-1}}(\Phi_m u_0^{[m]}(p,q))\Big) + \Psi_m u_0^{[m]}(p,q) = 0 \,, \tag{35}$$

with

$$q^{mt}\Phi_m(x) = \Big(h_{q^m}\Phi\Big)(x) \,, \tag{36}$$

$$q^{mt}\Psi_m(x) = p^m\Big(h_{p^m}\Psi\Big)(x) - p\sum_{\nu=0}^{m-1}\Big(D_{p,q}\circ h_{q^\nu p^{m-\nu}}\Phi\Big)(x) \,, \tag{37}$$

$$u_0^{[m]}(p,q) = q^{-\frac{1}{2}m(m-1)t}\zeta_m\Big(\prod_{\nu=0}^{m-1} h_{q^\nu p^{m-\nu}}\Phi\Big)h_{p^{-m}}u_0 \,,\; t = \deg(\Phi) \,, \tag{38}$$

where ζ_m is defined by the condition $(u_0^{[m]}(p,q))_0 = 1$.

Proof. Suppose $m = 1$. The form u_0 satisfies (19). Multiplying both sides by Φ and on account of (15), we get

$$D_{p,q}\Big(\Big(h_q\Phi\Big)\Big(h_{p^{-1}}(\Phi u_0)\Big)\Big) + \Big(\Psi - q^{-1}D_{p^{-1},q^{-1}}(h_q\Phi)\Big)\Phi u_0 = 0 \,.$$

Then, from (27) we obtain

$$D_{p,q}\Big(\Big(h_q\Phi\Big)u_0^{[1]}(p,q)\Big) + \Big(\Psi - q^{-1}D_{p^{-1},q^{-1}}(h_q\Phi)\Big)\Big(h_p(u_0^{[1]}(p,q))\Big) = 0 \,.$$

Applying $h_{p^{-1}}$ to the previous equation and taking into account (7), (10) and the formula

$$h_a(gu) = (h_{a^{-1}}g)(h_a u) \,,\quad g\in\mathcal{P} \,,\; u\in\mathcal{P}' \,,\; a\in\mathbb{C}\backslash\{0\} \,, \tag{39}$$

we get

$$D_{p,q}\Big(h_{p^{-1}}\Big(h_q\Phi\Big)u_0^{[1]}(p,q)\Big)\Big) + p\Big(h_p\Psi - D_{p,q}\Phi\Big)u_0^{[1]}(p,q) = 0 \,.$$

Therefore (35) – (38) are valid for $m = 1$. By induction, we can easily obtain the general case. □

Proposition 2.4. *Let $\{P_n\}_{n\geq 0}$ be orthogonal with respect to u_0. The form u_0 is a $D_{p,q}$-classical if and only if there exist two polynomials Φ and Ψ with $\deg(\Phi) \leq 2$, $\deg(\Psi) = 1$ and a sequence $\{\lambda_n\}_{n\geq 0}$, $\lambda_n \neq 0$, $n \geq 0$ such that*

$$\begin{aligned}\Phi(x)\Big(D_{p,q}\circ D_{p^{-1},q^{-1}}P_{n+1}\Big)(x) - p^{-1}\Psi(x)\Big(h_p\circ D_{p^{-1},q^{-1}}P_{n+1}\Big)(x)\\ = \lambda_n P_{n+1}(x) \,,\; n\geq 0 \,.\end{aligned} \tag{40}$$

Proof. The condition is necessary. Then u_0 fulfils (19). By Euclidean division, we get for $n \geq 0$

$$\begin{aligned}\Phi(x)\Big(D_{p,q}\circ D_{p^{-1},q^{-1}}P_{n+1}\Big)(x) - p^{-1}\Psi(x)\Big(h_p\circ D_{p^{-1},q^{-1}}P_{n+1}\Big)(x)\\ = \lambda_n P_{n+1}(x) + \sum_{\nu=0}^{n}\theta_{n,\nu}P_(x) \,.\end{aligned} \tag{41}$$

From the assumption, one has $\lambda_n \neq 0$ and from (41), we have for $0 \leq m \leq n$

$$\langle u_0, \Big(\Phi\Big(D_{p,q}\circ D_{p^{-1},q^{-1}}\Big)P_{n+1} - p^{-1}\Psi\Big(h_p\circ D_{p^{-1},q^{-1}}\Big)P_{n+1}\Big)P_m\rangle = \theta_{n,m}\langle u_0, P_m^2\rangle \,.$$

But,

$$\begin{aligned}&\langle u_0, \Big(\Phi\Big(D_{p,q}\circ D_{p^{-1},q^{-1}}P_{n+1}\Big)\Big)P_m\rangle\\ &\qquad = \langle h_{p^{-1}}\Big(\Phi u_0\Big), h_p\Big(\Big(D_{p,q}\circ D_{p^{-1},q^{-1}}P_{n+1}\Big)\Big)h_p P_m\rangle \,.\end{aligned}$$

Then, from (7) we get

$$\begin{aligned}&\langle u_0, \Big(\Phi\Big(D_{p,q} \circ D_{p^{-1},q^{-1}} P_{n+1}\Big)\Big) P_m\rangle \\ &\qquad = p^{-1}\langle h_{p^{-1}}\Big(\Phi u_0\Big), D_{p,q}\Big(\Big(h_p o D_{p^{-1},q^{-1}} P_{n+1}\Big)\Big) h_p P_m\rangle \ ,\end{aligned}$$

what implies from (5) and (7) – (8)

$$\begin{aligned}&\langle u_0, \Big(\Phi\Big(D_{p,q} \circ D_{p^{-1},q^{-1}} P_{n+1}\Big)\Big) P_m\rangle \\ &\qquad = -p^{-1}\{\langle D_{p,q}\Big(h_{p^{-1}}\Big(\Phi u_0\Big)\Big), \Big(\Big(h_p o D_{p^{-1},q^{-1}} P_{n+1}\Big)\Big) P_m\rangle \\ &\qquad\quad + \langle h_{p^{-1}}\Big(\Phi u_0\Big), \Big(D_{p,q} P_{n+1}\Big)\Big(D_{p,q} P_m\Big)\rangle\} \ .\end{aligned}$$

Therefore, from (19) we have

$$-p^{-1}\langle h_{p^{-1}}\Big(\Phi u_0\Big), \Big(D_{p,q} P_{n+1}\Big)\Big(D_{p,q} P_m\Big)\rangle = \theta_{n,m}\langle u_0, P_m^2\rangle \ , \ 0 \le m \le n \ .$$

Hence $\theta_{n,m} = 0$, $0 \le m \le n$, since the sequence $\{P_n^{[1]}(.,p,q)\}_{n\ge 0}$ is orthogonal with respect to $h_{p^{-1}}(\Phi u_0)$. Here each P_{n+1} fulfils (40).

Conversely, let $\{P_n\}_{n\ge 0}$ be orthogonal with respect to u_0 and such that P_{n+1} fulfils (40). Then, we have

$$\langle u_0, \Phi\Big(D_{p,q} \circ D_{p^{-1},q^{-1}} P_{n+1}\Big) - p^{-1}\Psi\Big(h_p \circ D_{p^{-1},q^{-1}} P_{n+1}\Big)\rangle = 0 \ , \ n \ge 0 \ ,$$

or,

$$\langle D_{p,q}\Big(h_{p^{-1}}\Phi u_0\Big) + \Psi u_0, \Big(h_p \circ D_{p^{-1},q^{-1}} P_{n+1}\Big)\rangle = 0 \ , \ n \ge 0 \ ,$$

what implies

$$\langle D_{p^{-1},q^{-1}}\Big(D_{p,q}\Big(h_{p^{-1}}\Phi u_0\Big) + \Psi u_0\Big), h_p P_{n+1}\rangle = 0 \ , \ n \ge 0 \ .$$

Hence,

$$D_{p^{-1},q^{-1}}\Big(D_{p,q}\Big(h_{p^{-1}}\Phi u_0\Big) + \Psi u_0\Big) = 0 \ .$$

Thus, u_0 verifies (19) and $\{P_n\}_{n\ge 0}$ is $D_{p,q}$-classical sequence. □

Corollary 2.5. [12] *Let $\{P_n\}_{n\ge 0}$ be orthogonal with respect to u_0. The form u_0 is a $D_{p,q}$-classical if and only if there exist two polynomials σ and τ with* $\deg(\sigma) \le 2$, $\deg(\tau) = 1$ *and a sequence $\{\varrho_n\}_{n\ge 0}$, $\varrho_n \ne 0$, $n \ge 0$ such that*

$$\sigma(x)\Big(D_{p,q}^2 P_{n+1}\Big)(x) + \tau(x)\Big(h_p \circ D_{p,q} P_{n+1}\Big)(x) = \varrho_n\Big(h_{pq} P_{n+1}\Big)(x) \ , \ n \ge 0 \ ,$$

with

$$\sigma(x) = (pq)^{-t}\Phi(pqx) \ , \ \tau(x) = -q(pq)^{-t}\Psi(pqx) \ , \ \varrho_n = (pq)^{-t}\lambda_n \ .$$

Proof. Taking into account the relation (8), the equation (40) is reduced to

$$\begin{aligned}&\Phi(x)\Big(D_{p,q} \circ h_{p^{-1}q^{-1}} \circ D_{p,q} P_{n+1}\Big)(x) - p^{-1}\Psi(x)\Big(h_{q^{-1}} \circ D_{p,q} P_{n+1}\Big)(x) \\ &\qquad = \lambda_n P_{n+1}(x) \ , \ n \ge 0 \ .\end{aligned}$$

Then, from (7) the last equation becomes (for $n \ge 0$)

$$p^{-1}q^{-1}\Phi(x)\Big(h_{p^{-1}q^{-1}} \circ D_{p,q}^2 P_{n+1}\Big)(x) - p^{-1}\Psi(x)\Big(h_{q^{-1}} \circ D_{p,q} P_{n+1}\Big)(x) = \lambda_n P_{n+1}(x) \ .$$

What implies

$$\begin{aligned}&h_{p^{-1}q^{-1}}\{\Big(h_{pq}\Phi\Big)(x)\Big(D_{p,q}^2 P_{n+1}\Big)(x) - q\Big(h_{pq}\Psi\Big)(x)\Big(h_p \circ D_{p,q} P_{n+1}\Big)(x)\} \\ &\qquad = \lambda_n P_{n+1}(x) \ , \ n \ge 0 \ .\end{aligned}$$

Hence, the desired result. □

Lemma 2.6. *Consider the sequence* $\{\tilde{P}_n\}_{n\geq 0}$ *obtained by shifting* P_n *i.e.* $\tilde{P}_n(x) = a^{-n}P_n(ax) = a^{-n}(h_aP_n)(x)$, $n \geq 0$, $a \neq 0$. *If* u_0 *satisfies* (19), *then* $\tilde{u}_0 = h_{a^{-1}}u_0$ *fulfils the equation*

$$D_{p,q}\Big(h_{p^{-1}}(\tilde{\Phi}\tilde{u}_0)\Big) + \tilde{\Psi}\tilde{u}_0 = 0 \ , \tag{42}$$

where $\tilde{\Phi}(x) = a^{-t}\Phi(ax)$, $\quad \tilde{\Psi}(x) = a^{1-t}\Psi(ax)$, $\quad t = \deg(\Phi)$.

Proof. From (10) and (39), we have

$$\begin{aligned} D_{p,q}\Big(h_{p^{-1}}(\Phi u_0)\Big) &= D_{p,q}\Big(h_{p^{-1}}(\Phi(h_a\tilde{u}_0)\Big) \\ &= a^{-1}h_a\Big(D_{p,q}\Big(h_{p^{-1}}(\Phi(ax)\tilde{u}_0)\Big)\Big) \ . \end{aligned}$$

Further

$$\begin{aligned} \Psi u_0 &= \Psi(h_a\tilde{u}_0) \\ &= h_a\Big(\Psi(ax)\tilde{u}_0\Big) \ . \end{aligned}$$

Then, the equation (19) becomes

$$h_a\Big(D_{p,q}\Big(h_{p^{-1}}(\Phi(ax)\tilde{u}_0)\Big) + \Psi(ax)\tilde{u}_0\Big) = 0 \ .$$

Hence, the desired result. □

The following result allows us to characterize the $D_{p,q}$-classical sequences through the so-called Rodrigues formula. See [1, 9, 11, 13, 16, 19].

Proposition 2.7. *The orthogonal sequence* $\{P_n\}_{n\geq 0}$ *is* $D_{p,q}$*classical if and only if there exist a monic polynomial* Φ, $\deg(\Phi) \leq 2$ *and a sequence* $\{\mathcal{V}_n\}_{n\geq 0}$, $\mathcal{V}_n \neq 0$, $n \geq 0$ *such that*

$$P_n u_0 = \mathcal{V}_n D^n_{p,q}\Big(\Big(\prod_{\nu=0}^{n-1} h_{q^{\nu}p^{n-\nu}}\Phi\Big)h_{p^{-n}}u_0\Big) \ , \ n \geq 0 \tag{43}$$

with $\prod_{\nu=0}^{-1} = 1$.

Proof. Necessity. Consider $< D^n_{p,q}u_0^{[n]}(p,q), P_m >= (-1)^n < u_0^{[n]}(p,q), D^n_{p,q}P_m >, \quad n, m \geq 0$,
For $0 \leq m \leq n-1$, $n \geq 1$, we have $D^n_{p,q}P_m = 0$. For $m \geq n$, put $m = n + \mu$, $\mu \geq 0$. Then

$$< u_0^{[n]}(p,q), D^n_{p,q}P_{n+\mu} >= \prod_{\mu=1}^{n}[\mu+\nu] < u_0^{[n]}(p,q), P_\mu^{[n]}(.,p,q) >= [n]!\delta_{0,\mu} \ ,$$

where $[0]! := 1$, $[n]! := \prod_{\nu=1}^{n}[\nu]$, $n \geq 1$. Consequently, we have

$$D^n_{p,q}u_0^{[n]}(p,q) = (-1)^n[n]!u_n \ , \ n \geq 0 \ .$$

But, from the assumption $u_n = (< u_0, P_n^2 >)^{-1}P_n u_0$, $n \geq 0$ so that, in accordance with (38), we obtain (43) where

$$\mathcal{V}_n = (-1)^n q^{-\frac{1}{2}(n-1)nt}\zeta_n\frac{< u_0, P_n^2 >}{[n]!} \ , \quad n \geq 0 \ . \tag{44}$$

Sufficiency. Making $n = 1$ in (43), we get

$$P_1 u_0 = \mathcal{V}_1 D_{p,q}\Big(h_{p^{-1}}\big(\Phi u_0\big)\Big) = -\mathcal{V}_1 \Psi u_0 \ . \tag{45}$$

Therefore, the form u_0 is $D_{p,q}$-classical, since it is regular. □

The Rodrigues formula can serve for describing the $D_{p,q}$-classical sequences which are completely determined by the knowledge of the sequences $\{\beta_n\}_{n\geq 0}$ and $\{\gamma_{n+1}\}_{n\geq 0}$. It doubles the shortest way for obtaining them. Indeed, on account of (43), the recurrence relation (4) is equivalent to

$$\begin{aligned}\mathcal{V}_{n+2} D^{n+2}_{p,q}&\Big(\Big(\prod_{\nu=0}^{n+1} h_{q^\nu p^{n+2-\nu}}\Phi\Big) h_{p^{-n-2}} u_0\Big)\\ &= \mathcal{V}_{n+1}(x-\beta_{n+1}) D^{n+1}_{p,q}\Big(\Big(\prod_{\nu=0}^{n} h_{q^\nu p^{n+1-\nu}}\Phi\Big) h_{p^{-n-1}} u_0\Big)\\ &\quad -\mathcal{V}_n \gamma_{n+1} D^{n}_{p,q}\Big(\Big(\prod_{\nu=0}^{n-1} h_{q^\nu p^{n-\nu}}\Phi\Big) h_{p^{-n}} u_0\Big)\ , \quad n\geq 0.\end{aligned} \tag{46}$$

Proposition 2.8. *The sequences $\{\mathcal{V}_n\}_{n\geq 0}$, $\{\beta_n\}_{n\geq 0}$ and $\{\gamma_{n+1}\}_{n\geq 0}$ respectively fulfil the equations*

$$\begin{aligned}&p\mathcal{V}_{n+2}\{p^{2n+1}\mathcal{V}_1^{-1} + \tfrac{1}{2}\Phi''(0)[2n+2]\}\{p^{2n}\mathcal{V}_1^{-1} + \tfrac{1}{2}\Phi''(0)[2n+1]\}\\ &-q^{n+1}p^{n+1}\mathcal{V}_{n+1}\{p^{n-1}\mathcal{V}_1^{-1} + \tfrac{1}{2}\Phi''(0)[n]\} = 0\ , \quad n\geq 0\ ,\end{aligned} \tag{47}$$

$$\begin{aligned}\beta_{n+1} = &\frac{1}{p^{2n}\mathcal{V}_1^{-1} + \frac{p}{2}\Phi''(0)[2n]}\Big\{p\frac{\mathcal{V}_{n+2}}{\mathcal{V}_{n+1}}\{p^n\mathcal{V}_1^{-1}\beta_0 - \Phi'(0)[n+1]\}\{p^{2n}\mathcal{V}_1^{-1}(1+pq^{-1})\\ &+\tfrac{1}{2}\Phi''(0)\big(p[2n] + q^{-1}[2n+2]\big)\} - p^n q^n \mathcal{V}_1^{-1}\beta_0\Big\}\ , \quad n\geq 0\ ,\end{aligned} \tag{48}$$

$$\begin{aligned}\mathcal{V}_n\gamma_{n+1} = &\mathcal{V}_{n+1}\Big\{\beta_{n+1}\{p^n\mathcal{V}_1^{-1}\beta_0 - p\Phi'(0)[n]\} - p\Phi(0)[n]\Big\}\\ &-p\mathcal{V}_{n+2}\Big\{\{\Phi'(0)[n+1] - p^n\mathcal{V}_1^{-1}\beta_0\}\{p\Phi'(0)[n] - p^n\mathcal{V}_1^{-1}\beta_0\}\\ &+q^{-1}\Phi(0)\{p^{2n+1}\mathcal{V}_1^{-1} + \tfrac{1}{2}\Phi''(0)[2n+2]\}\Big\}\ , \quad n\geq 0\ .\end{aligned} \tag{49}$$

For the proof we need the following lemmas.

Lemma 2.9. *For any a, $b\in\mathbb{C}$ and $u\in\mathcal{P}'$, we have*

$$(ax+b)D^n_{p,q}u = D^n_{p,q}\Big((aq^n x + b)u\Big) - a[n]D^{n-1}_{p,q}\circ h_p u\ ,\ n\geq 1\ . \tag{50}$$

Proof. It is easy to prove this Lemma by induction on account of (15). □

Lemma 2.10. *We have for $n\geq 0$*

$$D_{p^{-1},q^{-1}}\Big(\prod_{\nu=0}^{n}\Big(h_{q^\nu p^{n+1-\nu}}\Phi\Big)\Big) = \Big(\prod_{\nu=0}^{n-1}\Big(h_{q^\nu p^{n-\nu}}\Phi\Big)\Big)\sum_{\nu=0}^{n}\Big(D_{p^{-1},q^{-1}}\circ h_{q^\nu p^{n+1-\nu}}\Big)\Phi\ . \tag{51}$$

Proof. We proceed by induction. For $n = 1$, we get from (5)

$$\Big(D_{p^{-1},q^{-1}}\Big(h_{p^2}\Phi h_{pq}\Phi\Big)\Big)(x) = \Big(h_p\Phi\Big)(x)\Big(D_{p^{-1},q^{-1}}\Big(h_{p^2}\Phi\Big) + D_{p^{-1},q^{-1}}\Big(h_{pq}\Phi\Big)\Big)(x)\ .$$

We assume (51) for $0 \leq m \leq n$. Therefore, according to (5), (7) and (51), we have

$$\begin{aligned}
&D_{p^{-1},q^{-1}}\Big(\prod_{\nu=0}^{n+1}\big(h_{q^{\nu}p^{n+2-\nu}}\Phi\big)\Big)\\
&= D_{p^{-1},q^{-1}}\circ h_p\Big\{\Big(\prod_{\nu=0}^{n}\big(h_{q^{\nu}p^{n+1-\nu}}\Phi\big)\Big)\big(h_{q^{n+1}}\Phi\big)\Big\}\\
&= p\Big\{h_p\circ D_{p^{-1},q^{-1}}\Big\{\Big(\prod_{\nu=0}^{n}\big(h_{q^{\nu}p^{n+1-\nu}}\Phi\big)\Big)\big(h_{q^{n+1}}\Phi\big)\Big\}\Big\}\\
&= p\Big\{h_p\Big\{\big(h_{q^n}\Phi\big)D_{p^{-1},q^{-1}}\prod_{\nu=0}^{n}\big(h_{q^{\nu}p^{n+1-\nu}}\Phi\big)\Big)\\
&+h_{p^{-1}}\Big(\prod_{\nu=0}^{n}\big(h_{q^{\nu}p^{n+1-\nu}}\Phi\big)\Big)\big(D_{p^{-1},q^{-1}}\circ h_{q^{n+1}}\big)\Phi\Big\}\Big\}\\
&= p\Big\{h_p\Big\{\big(h_{q^n}\Phi\big)\Big(\prod_{\nu=0}^{n-1}\big(h_{q^{\nu}p^{n-\nu}}\Phi\big)\Big)\sum_{\nu=0}^{n}\big(D_{p^{-1},q^{-1}}\circ h_{q^{\nu}p^{n+1-\nu}}\big)\Phi\\
&+h_{p^{-1}}\Big(\prod_{\nu=0}^{n}\big(h_{q^{\nu}p^{n+1-\nu}}\Phi\big)\Big)\big(D_{p^{-1},q^{-1}}\circ h_{q^{n+1}}\big)\Phi\Big\}\Big\}\\
&= \Big(\prod_{\nu=0}^{n}\big(h_{q^{\nu}p^{n+1-\nu}}\Phi\big)\Big)\sum_{\nu=0}^{n+1}\big(D_{p^{-1},q^{-1}}\circ h_{q^{\nu}p^{n+2-\nu}}\big)\Phi\ .
\end{aligned}$$

Hence, the desired result (51). □

Proof. (of Proposition 2.8) The proof will be carried out in three steps.
First step. From (50) we may write

$$\begin{aligned}
&D_{p,q}^{n+1}\Big\{(q^{n+1}x-\beta_{n+1})\Big(\prod_{\nu=0}^{n}h_{q^{\nu}p^{n+1-\nu}}\Phi\Big)h_{p^{-n-1}}u_0\Big\}\\
&= (x-\beta_{n+1})D_{p,q}^{n+1}\Big\{\Big(\prod_{\nu=0}^{n}h_{q^{\nu}p^{n+1-\nu}}\Phi\Big)h_{p^{-n-1}}u_0\Big\}\\
&\quad+[n+1]D_{p,q}^{n}\Big\{\Big(\prod_{\nu=0}^{n}h_{q^{\nu}p^{n-\nu}}\Phi\Big)h_{p^{-n}}u_0\Big\}\ ,\ n\geq 0\ .
\end{aligned}\tag{52}$$

Then, (46) becomes

$$\begin{aligned}
&D_{p,q}^{n}\Big\{\mathcal{V}_{n+2}D_{p,q}^{2}\Big(\Big(\prod_{\nu=0}^{n+1}h_{q^{\nu}p^{n+2-\nu}}\Phi\Big)h_{p^{-n-2}}u_0\Big)\\
&-\mathcal{V}_{n+1}D_{p,q}\Big((q^{n+1}x-\beta_{n+1})\Big(\prod_{\nu=0}^{n}h_{q^{\nu}p^{n+1-\nu}}\Phi\Big)h_{p^{-n-1}}u_0\Big)\\
&+\mathcal{V}_{n+1}[n+1]\Big(\prod_{\nu=0}^{n}h_{q^{\nu}p^{n-\nu}}\Phi\Big)h_{p^{-n}}u_0\\
&+\mathcal{V}_{n}\gamma_{n+1}\Big(\prod_{\nu=0}^{n-1}h_{q^{\nu}p^{n-\nu}}\Phi\Big)h_{p^{-n}}u_0\Big\}=0\ ,\ n\geq 0\ .
\end{aligned}$$

Hence, the next result

$$\begin{aligned} &D_{p,q}\Big\{\mathcal{V}_{n+2}D_{p,q}\Big(\Big(\prod_{\nu=0}^{n+1}h_{q^{\nu}p^{n+2-\nu}}\Phi\Big)h_{p^{-n-2}}u_0\Big)\\ &\quad-\mathcal{V}_{n+1}(q^{n+1}x-\beta_{n+1})\Big(\prod_{\nu=0}^{n}h_{q^{\nu}p^{n+1-\nu}}\Phi\Big)h_{p^{-n-1}}u_0\Big\}\\ &\qquad+\mathcal{V}_{n+1}[n+1]\Big(\prod_{\nu=0}^{n}h_{q^{\nu}p^{n-\nu}}\Phi\Big)h_{p^{-n}}u_0\\ &\qquad\quad+\mathcal{V}_{n}\gamma_{n+1}\Big(\prod_{\nu=0}^{n-1}h_{q^{\nu}p^{n-\nu}}\Phi\Big)h_{p^{-n}}u_0=0\ ,\ n\geq 0\ . \end{aligned} \tag{53}$$

Second step. We may write

$$\begin{aligned} &D_{p,q}\Big\{\prod_{\nu=0}^{n+1}\Big(h_{q^{\nu}p^{n+2-\nu}}\Phi\Big)h_{p^{-n-2}}u_0\Big\}\\ &\quad=D_{p,q}\Big\{h_{pq}\Big(\prod_{\nu=0}^{n}h_{q^{\nu}p^{n-\nu}}\Phi\Big)h_{p^{-n-2}}(\Phi u_0)\Big\}\\ &\quad=\Big(\prod_{\nu=0}^{n}h_{q^{\nu}p^{n+1-\nu}}\Phi D_{p,q}\Big(h_{p^{-n-2}}(\Phi u_0)\Big)\\ &\quad+q^{-1}D_{p^{-1},q^{-1}}\Big\{h_{pq}\Big(\prod_{\nu=0}^{n}h_{q^{\nu}p^{n-\nu}}\Phi\Big)\Big\}h_{p^{-n-1}}(\Phi u_0)\ from\ (15)\\ &\quad=\Big(\prod_{\nu=0}^{n}h_{q^{\nu}p^{n+1-\nu}}\Phi\Big)D_{p,q}\Big(h_{p^{-n-2}}(\Phi u_0)\Big)\\ &\quad+pD_{p,q}\Big\{\Big(\prod_{\nu=0}^{n}h_{q^{\nu}p^{n-\nu}}\Phi\Big)\Big\}h_{p^{-n-1}}(\Phi u_0)\ from\ (6)\\ &\quad=p^{n+1}\Big\{\Big(\prod_{\nu=0}^{n}h_{q^{\nu}p^{n+1-\nu}}\Phi\Big)\Big\}\Big\{h_{p^{-n-1}}\Big(D_{p,q}\Big(h_{p^{-1}}\Phi u_0\Big)\Big)\Big\}\\ &\quad+pD_{p,q}\Big\{\Big(\prod_{\nu=0}^{n}h_{q^{\nu}p^{n-\nu}}\Phi\Big)\Big\}h_{p^{-n-1}}(\Phi u_0)\ from\ (10)\\ &\quad=p^{n+1}\mathcal{V}_1^{-1}\Big\{\Big(\prod_{\nu=0}^{n}h_{q^{\nu}p^{n+1-\nu}}\Phi\Big)\Big\}\Big(h_{p^{-n-1}}(P_1u_0)\Big)\\ &\quad+pD_{p,q}\Big\{\Big(\prod_{\nu=0}^{n}h_{q^{\nu}p^{n-\nu}}\Phi\Big)\Big\}h_{p^{-n-1}}(\Phi u_0)\ from\ (43)\ . \end{aligned}$$

Then, from (53) we obtain

$$\begin{aligned} &D_{p,q}\Big(\Omega_n h_{p^{-n-1}}\Big(\Phi u_0\Big)\Big)+\Big(\prod_{\nu=0}^{n-1}h_{q^{\nu}p^{n-\nu}}\Phi\Big)\Big\{\mathcal{V}_{n+1}[n+1]\Big(h_{q^n}\Phi\Big)\\ &\qquad+\mathcal{V}_n\gamma_{n+1}\Big\}h_{p^{-n}}u_0=0\ ,\ n\geq 0\ , \end{aligned} \tag{54}$$

with

$$\Omega_n = \mathcal{V}_{n+2}\Big\{p^{n+1}\mathcal{V}_1^{-1}\Big(h_{p^{n+1}}P_1\Big)\Big(\prod_{\nu=0}^{n-1} h_{q^{\nu+1}p^{n-\nu}}\Phi\Big) + pD_{p,q}\Big(\prod_{\nu=0}^{n} h_{q^{\nu}p^{n-\nu}}\Phi\Big)\Big\} - \mathcal{V}_{n+1}\Big(q^{n+1}x-\beta_{n+1}\Big)\Big(\prod_{\nu=0}^{n-1} h_{q^{\nu+1}p^{n-\nu}}\Phi\Big)\,. \tag{55}$$

Further, in accordance with (10), (15) and (43), we get

$$\begin{aligned} D_{p,q}\Big(\Omega_n h_{p^{-n-1}}\big(\Phi u_0\big)\Big) &= \Big(h_{q^{-1}}\Omega_n\Big)D_{p,q}\Big(h_{p^{-n-1}}\big(\Phi u_0\big)\Big) \\ &+ q^{-1}\Big(D_{p^{-1},q^{-1}}\Omega_n\Big)h_{p^{-n}}\Big(\Phi u_0\Big) \\ &= p^n\mathcal{V}_1^{-1}\Big(h_{p^n}P_1\Big)\Big(h_{q^{-1}}\Omega_n\Big)\Big(h_{p^{-n}}u_0\Big) \\ &+ q^{-1}\Big(D_{p^{-1},q^{-1}}\Omega_n\Big)\Big(h_{p^n}\Phi\Big)\Big(h_{p^{-n}}u_0\Big)\,. \end{aligned}$$

Since $h_{p^{-n}}u_0$ is regular and taking into account the last equation, (54) and the Lemma 2.2, we can deduce

$$p^n\mathcal{V}_1^{-1}\Big(h_{p^n}P_1\Big)\Big(h_{q^{-1}}\Omega_n\Big) + q^{-1}\Big(D_{p^{-1},q^{-1}}\Omega_n\Big)\Big(h_{p^n}\Phi\Big) + \Big(\prod_{\nu=0}^{n-1} h_{q^{\nu}p^{n-\nu}}\Phi\Big)\Big\{\mathcal{V}_{n+1}[n+1]\Big(h_{q^n}\Phi\Big) + \mathcal{V}_n\gamma_{n+1}\Big\} = 0\,,\ n\geq 0\,. \tag{56}$$

Third step. From (7) – (8) and (55), we have

$$h_{q^{-1}}(\Omega_n) = \mathcal{V}_{n+2}\Big\{p^{n+1}\mathcal{V}_1^{-1}\Big(h_{q^{-1}p^{n+1}}P_1\Big)(x)\Big(\prod_{\nu=0}^{n-1}\Big(h_{q^{\nu}p^{n-\nu}}\Phi\Big)(x)\Big) + D_{p^{-1},q^{-1}}\Big(\prod_{\nu=0}^{n}\Big(h_{q^{\nu}p^{n+1-\nu}}\Phi\Big)(x)\Big)\Big\} - \mathcal{V}_{n+1}(q^n x-\beta_{n+1})\Big(\prod_{\nu=0}^{n-1}\Big(h_{q^{\nu}p^{n-\nu}}\Phi\Big)(x)\Big)\,,\ n\geq 0\,. \tag{57}$$

On account of (51), the relation (57) becomes

$$(h_{q^{-1}}\Omega_n)(x) = \Big(\prod_{\nu=0}^{n-1}\Big(h_{q^{\nu}p^{n-\nu}}\Phi\Big)(x)\Big)\Big\{\mathcal{V}_{n+2}\{p^{n+1}\mathcal{V}_1^{-1}\Big(h_{q^{-1}p^{n+1}}P_1\Big)(x) + \sum_{\nu=0}^{n}\Big(D_{p^{-1},q^{-1}}\circ h_{q^{\nu}p^{n+1-\nu}}\Phi\Big)(x)\} - \mathcal{V}_{n+1}(q^n x-\beta_{n+1})\Big\}\,,\ n\geq 0\,. \tag{58}$$

Hence,

$$\Big(h_{p^{n+1}}\Phi\Big)(x)\Big(\Omega_n\Big)(x) = \Big(\prod_{\nu=0}^{n}\Big(h_{q^{\nu}p^{n+1-\nu}}\Phi\Big)(x)\Big)\Lambda_n(x)\,,\ n\geq 0\,,$$

with

$$\Lambda_n(x) = \mathcal{V}_{n+2}\Big\{p^{n+1}\mathcal{V}_1^{-1}\Big(h_{p^{n+1}}P_1\Big)(x) + p\sum_{\nu=0}^{n}\Big(D_{p,q}\circ h_{q^{\nu}p^{n-\nu}}\Phi\Big)(x)\Big\} - \mathcal{V}_{n+1}(q^{n+1}x-\beta_{n+1})\,,\ n\geq 0\,.$$

According to (5), this yields for $n \geq 0$

$$D_{p^{-1},q^{-1}}\Big(\big(h_{p^{n+1}}\Phi\big)(x)\Omega_n(x)\Big) = h_{q^{-1}}(\Omega_n)(x)D_{p^{-1},q^{-1}}\big(h_{p^{n+1}}\Phi\big)(x) + \big(h_{p^n}\Phi\big)(x)D_{p^{-1},q^{-1}}(\Omega_n)(x) \,,$$

$$\begin{aligned} D_{p^{-1},q^{-1}}\Big\{\Big(\prod_{\nu=0}^{n}\big(h_{q^\nu p^{n+1-\nu}}\Phi\big)(x)\Big)\Lambda_n(x)\Big\} &= \Big(\prod_{\nu=0}^{n}\big(h_{q^\nu p^{n-\nu}}\Phi\big)(x)\Big)D_{p^{-1},q^{-1}}(\Lambda_n)(x) \\ &+\big(h_{q^{-1}}\Lambda_n\big)(x)D_{p^{-1},q^{-1}}\Big(\prod_{\nu=0}^{n}\big(h_{q^\nu p^{n+1-\nu}}\Phi\big)(x)\Big) \,. \end{aligned}$$

Comparing and in accordance with (51) and (58), we can deduce

$$\begin{aligned} \big(h_{p^n}\Phi\big)(x)D_{p^{-1},q^{-1}}\Omega_n(x) &= \Big(\prod_{\nu=0}^{n-1}\big(h_{q^\nu p^{n-\nu}}\Phi\big)(x)\Big)\Big\{\big(h_{q^n}\Phi\big)(x)D_{p^{-1},q^{-1}}\Lambda_n(x) \\ &+\big(h_{q^{-1}}\Lambda_n\big)(x)\sum_{\nu=1}^{n}\big(D_{p^{-1},q^{-1}}\circ h_{q^\nu p^{n+1-\nu}}\Phi\big)(x)\Big\} \,,\ n \geq 0 \,. \end{aligned}$$

Taking into account of (for $n \geq 0$)

$$\begin{aligned} \big(h_{q^{-1}}\Lambda_n\big)(x) &= \mathcal{V}_{n+2}\Big\{p^{n+1}\mathcal{V}_1^{-1}\big(h_{q^{-1}p^{n+1}}P_1\big)(x) \\ &+\sum_{\nu=0}^{n}\big(D_{p^{-1},q^{-1}}\circ h_{q^\nu p^{n+1-\nu}}\Phi\big)(x)\Big\} - \mathcal{V}_{n+1}(q^n x - \beta_{n+1}) \,, \end{aligned}$$

$$D_{p^{-1},q^{-1}}\Lambda_n(x) = \mathcal{V}_{n+2}\Big\{p^{2n+2}\mathcal{V}_1^{-1} + p\sum_{\nu=0}^{n}\big(D_{p^{-1},q^{-1}}\circ D_{p,q}\circ h_{q^\nu p^{n-\nu}}\Phi\big)(x)\Big\} - q^{n+1}\mathcal{V}_{n+1} \,,$$

the relation (56) becomes

$$\begin{aligned} &\mathcal{V}_n\gamma_{n+1} + \big(h_{q^n}\Phi\big)(x)\Big\{pq^{-1}\mathcal{V}_{n+2}\{\sum_{\nu=0}^{n}\big(D_{p^{-1},q^{-1}}\circ D_{p,q}\circ h_{q^\nu p^{n-\nu}}\Phi\big)(x) \\ &+p^{2n+1}\mathcal{V}_1^{-1}\} + p\mathcal{V}_{n+1}[n]\Big\} + \{p^n\mathcal{V}_1^{-1}\big(h_{p^n}P_1\big)(x) \\ &+q^{-1}\sum_{\nu=1}^{n}\big(D_{p^{-1},q^{-1}}\circ h_{q^\nu p^{n+1-\nu}}\Phi\big)(x)\}\Big\{\mathcal{V}_{n+2}\{p^{n+1}\mathcal{V}_1^{-1}\big(h_{q^{-1}p^{n+1}}P_1\big)(x) \\ &+\sum_{\nu=0}^{n}\big(D_{p^{-1},q^{-1}}\circ h_{q^\nu p^{n+1-\nu}}\Phi\big)(x)\} - \mathcal{V}_{n+1}(q^n x - \beta_{n+1})\Big\} = 0 \,,\ n \geq 0 \,. \end{aligned} \tag{59}$$

Lastly, writing $\Phi(x) = \frac{1}{2}\Phi''(0)x^2 + \Phi'(0)x + \Phi(0)$ and with

$$q^{-1}\sum_{\nu=1}^{n}\big(D_{p^{-1},q^{-1}}\circ h_{q^\nu p^{n+1-\nu}}\Phi\big)(x) = p\Big(\frac{1}{2}\Phi''(0)[2n]x + \Phi'(0)[n]\Big) \,,\ n \geq 0 \,,$$

$$\sum_{\nu=0}^{n}\big(D_{p^{-1},q^{-1}}\circ h_{q^\nu p^{n+1-\nu}}\Phi\big)(x) = p\Big(\frac{q^{-1}}{2}\Phi''(0)[2n+2]x + \Phi'(0)[n+1]\Big) \,,\ n \geq 0 \,,$$

$$\sum_{\nu=0}^{n}\big(D_{p^{-1},q^{-1}}\circ D_{p,q}\circ h_{q^\nu p^{n-\nu}}\Phi\big)(x) = \frac{1}{2}\Phi''(0)[2n+2] \,,\ n \geq 0 \,,$$

an easy computation leads to (47) – (49). □

Corollary 2.11. [18] *The sequences $\{\beta_n\}_{n\geq 0}$ and $\{\gamma_{n+1}\}_{n\geq 0}$ of the three-term recurrence relation* (4) *are explicity given by*

$$\begin{aligned} &\beta_n = \varpi_{1,n} - \varpi_{1,n+1} \;,\; n \geq 0 \;, \\ &\gamma_{n+1} = \varpi_{2,n} - \varpi_{2,n+1} - \beta_{n+1}\varpi_{1,n+1} \;,\; n \geq 0 \;, \end{aligned} \tag{60}$$

where

$$\begin{aligned} &\varpi_{1,n} = -[n]\tfrac{B_{n-1}}{A_{2n-2}} \;,\; n \geq 0 \;, \\ &\varpi_{2,n} = -[n]\tfrac{pq\varpi_{1,n+1}B_{n-1}-\Phi(0)[n+1]}{(p+q)A_{2n-1}} \;,\; n \geq 0 \;, \\ &A_n = p^{n-1}\mathcal{V}_1^{-1} + \tfrac{1}{2}\Phi''(0)[n] \;,\; n \geq 0 \;, \\ &B_n = p^{n-1}\mathcal{V}_1^{-1}\beta_0 - \Phi'(0)[n] \;,\; n \geq 0 \;. \end{aligned} \tag{61}$$

Proof. From (47), we have

$$\frac{\mathcal{V}_{n+1}}{\mathcal{V}_n} = q^n p^{n-1}\frac{A_{n-1}}{A_{2n}A_{2n-1}} \;,\; n \geq 0 \;.$$

Then, from the previous equation, the relations (48) and (49) become

$$\begin{aligned} \beta_n &= q^{n-1}p^{n-1}\tfrac{(p+q)A_{n-1}B_n}{A_{2n}A_{2n-2}} - q^{n-1}p^{n-2}\tfrac{\mathcal{V}_1^{-1}\beta_0}{A_{2n-2}} \;,\; n \geq 0 \;, \\ \gamma_{n+1} &= q^n p^n \tfrac{A_{n-1}B_n\beta_{n+1}}{A_{2n}A_{2n-1}} - q^n p^n \tfrac{A_{n-1}[n]}{A_{2n-1}A_{2n}}\Phi(0) \\ &\quad - q^{2n+1}p^{2n+1}\tfrac{A_nA_{n-1}B_nB_{n+1}}{A_{2n-1}A_{2n}A_{2n+1}A_{2n+2}} - q^{2n}p^{2n}\tfrac{A_{n-1}A_n}{A_{2n-1}A_{2n}A_{2n+1}}\Phi(0) \;,\; n \geq 0 \;. \end{aligned}$$

After some straightforward calculation, we obtain

$$\begin{aligned} &q^{n-1}p^{n-1}\tfrac{(p+q)A_{n-1}B_n}{A_{2n}A_{2n-2}} = B_n\{\tfrac{[n+1]}{A_{2n}} - \tfrac{[n-1]}{A_{2n-2}}\} \;, \\ &q^n p^n \tfrac{A_{n-1}B_n\beta_{n+1}}{A_{2n}A_{2n-1}} = [n]\tfrac{B_n w_{1,n+2}}{A_{2n-1}} - [n]\tfrac{B_n w_{1,n+1}}{A_{2n-1}} - \beta_{n+1}w_{1,n+1} \;, \\ &q^{2n+1}p^{2n+1}\tfrac{A_nA_{n-1}B_nB_{n+1}}{A_{2n-1}A_{2n}A_{2n+1}A_{2n+2}} = [n]\tfrac{B_n w_{1,n+2}}{A_{2n-1}} - pq[n+1]\tfrac{B_n w_{1,n+2}}{(p+q)A_{2n+1}} - [n]\tfrac{B_{n+1}w_{1,n+1}}{(p+q)A_{2n-1}} \;, \\ &q^{2n}p^{2n}\tfrac{A_{n-1}A_n}{A_{2n-1}A_{2n}A_{2n+1}} = \tfrac{[n+1][n+2]}{(p+q)A_{2n+1}} + \tfrac{pq[n][n-1]}{(p+q)A_{2n-1}} - \tfrac{[n][n+1]}{A_{2n}} \;, \\ &q^n p^n \tfrac{A_{n-1}}{A_{2n-1}A_{2n}} = \tfrac{[n+1]}{A_{2n}} - \tfrac{[n]}{A_{2n-1}} \;. \end{aligned}$$

Hence, the desired result (60). □

Remark 2. (i) If $\Phi''(0) = 0$, then from (47) − (49), we get

$$\begin{aligned} &\mathcal{V}_n = \mathcal{V}_1^n \frac{q^{\frac{n(n-1)}{2}}}{p^{n(n-1)}} \;,\; n \geq 0 \;, \\ &\beta_n = \tfrac{q^n}{p^n}\beta_0 - \tfrac{q^n}{p^{2n-1}}(1 + pq^{-1})\mathcal{V}_1\Phi'(0)[n] \;,\; n \geq 0 \;, \\ &\gamma_{n+1} = \tfrac{q^n}{p^{2n}}\mathcal{V}_1[n+1]\{\tfrac{q^n}{p^{2n-1}}\mathcal{V}_1\Big(\Phi'(0)\Big)^2 - \tfrac{q^n}{p^n}\beta_0\Phi'(0) - \Phi(0)\} \;,\; n \geq 0 \;. \end{aligned} \tag{62}$$

(ii) If $\Phi(x) = (x - c)(x - d)$, then from (47) − (49), we obtain

$$\begin{aligned} &\mathcal{V}_n = \frac{q^{\frac{n(n-1)}{2}}}{p^{\frac{n(n+1)}{2}}}\frac{\Gamma(p^{-1}\mathcal{V}_1^{-1}+p^{2-n}[n-1])}{\Gamma(p^{-1}\mathcal{V}_1^{-1}+p^{2-2n}[2n-1])} \;,\; n \geq 0 \;, \\ &\beta_n = \frac{q^{n-1}p^{n-2}R(n,c,d)}{\{p^{2n-3}\mathcal{V}_1^{-1}+[2n-2]\}\{p^{2n-1}\mathcal{V}_1^{-1}+[2n]\}} \;,\; n \geq 0 \;, \\ &\gamma_{n+1} = \frac{q^n p^n[n+1]\{p^{n-2}\mathcal{V}_1^{-1}+[n-1]\}C(n,c,d)D(n,c,d)}{\{p^{2n-2}\mathcal{V}_1^{-1}+[2n-1]\}\{p^{2n-1}\mathcal{V}_1^{-1}+[2n]\}^2\{p^{2n}\mathcal{V}_1^{-1}+[2n+1]\}} \;,\; n \geq 0 \;, \end{aligned} \tag{63}$$

where (for $n \geq 0$)

$$\begin{aligned} C(n,c,d) &= c(q-p)[n]^2 + \{p^n(c-d) + p^{n-1}(q-p)\beta_0 \mathcal{V}_1^{-1}\}[n] \\ &\quad + p^{2n-1}\mathcal{V}_1^{-1}(\beta_0 - d)\ , \\ D(n,c,d) &= -d(q-p)[n]^2 + \{p^n(c-d) - p^{n-1}(q-p)\beta_0 \mathcal{V}_1^{-1}\}[n] \\ &\quad - p^{2n-1}\mathcal{V}_1^{-1}(\beta_0 - c)\ , \\ R(n,c,d) &= (p+q)\{p^{n-1}\mathcal{V}_1^{-1} + p[n-1]\}\{p^{n-1}\mathcal{V}_1^{-1}\beta_0 + (c+d)[n]\} \\ &\quad - \mathcal{V}_1^{-1}\beta_0\{p^{2n-1}\mathcal{V}_1^{-1} + [2n]\}\ . \end{aligned} \tag{64}$$

4. Examples

Example 1. $\Phi(x) = 1$

With the choice $\mathcal{V}_1 = -(p+q)^{-1}$ and $\beta_0 = 0$, we get from (62) the following canonical case

$$\beta_n = 0\ , \quad \gamma_{n+1} = \frac{q^n}{(p+q)p^{2n}}[n+1]\ ,\ n \geq 0\ ,$$
$$D_{p,q}\Big(h_{p^{-1}}u_0\Big) + (p+q)xu_0 = 0\ .$$

We have obtained the (p,q)-Hermite polynomials [5].

On the other hand, from (43) and (62) we have

$$P_n u_0 = (-1)^n (p+q)^{-n} \frac{q^{\frac{n(n-1)}{2}}}{p^{n(n-1)}} D_{p,q}^n\Big(h_{p^{-n}}u_0\Big)\ ,\ n \geq 0\ .$$

Example 2. $\Phi(x) = x$

With the choice $\mathcal{V}_1 = qd^{-1}$ and $\beta_0 = pqd^{-1}(p-q)^{-1}(1-p^{\alpha+1}q^{-\alpha-1})$, we obtain from (62) the following canonical case

$$\begin{aligned} \beta_n &= d^{-1}(p-q)^{-1}q^n p^{1-2n}\{q^n(p+q) - p^{n+1}(p^\alpha q^{-\alpha}+1)\}\ , \\ \gamma_{n+1} &= d^{-2}(p-q)^{-2}p^{1-4n}q^{2n+1-\alpha}(p^{n+1}-q^{n+1})(p^{n+\alpha+1}-q^{n+\alpha+1})\ , \\ & D_{p,q}\Big(h_{p^{-1}}(xu_0)\Big) - \{(q^{-1}dx - p(p-q)^{-1}(1-p^{\alpha+1}q^{-\alpha-1})\}u_0 = 0\ . \end{aligned}$$

We have obtained the (p,q)-Laguerre polynomials [18].

On the other hand, from (43) and (62) we get

$$P_n u_0 = \frac{q^{n^2}}{d^n p^{\frac{n(n-3)}{2}}} D_{p,q}^n\Big(x^n h_{p^{-n}}u_0\Big)\ ,\ n \geq 0\ .$$

Example 3. $\Phi(x) = x(x-c)$

With the choice $c = p^2$, $\mathcal{V}_1^{-1} = -\frac{p}{p-q}(1-p^{\alpha+\beta+2}q^{-\alpha-\beta-2})$ and $\beta_0 = \frac{p^2(1-p^{\beta+1}q^{-\beta-1})}{1-p^{\alpha+\beta+2}q^{-\alpha-\beta-2}}$, we obtain from (63) − (64) the following canonical case

$$\begin{aligned} \beta_n &= \frac{p^{n+2}q^{n+\alpha+1}}{(p^{2n+\alpha+\beta}-q^{2n+\alpha+\beta})(p^{2n+\alpha+\beta+2}-q^{2n+\alpha+\beta+2})} \\ &\quad \times\{(p^\beta+q^\beta)(p^{2n+\alpha+\beta+1}+q^{2n+\alpha+\beta+1}) - (p+q)(p^\alpha+q^\alpha)p^{n+\beta}q^{n+\beta}\}\ , \\ \gamma_{n+1} &= p^{2n+\beta+5}q^{2n+2\alpha+\beta+3}\frac{(p^{n+1}-q^{n+1})(p^{n+\alpha+1}-q^{n+\alpha+1})(p^{n+\beta+1}-q^{n+\beta+1})}{(p^{2n+\alpha+\beta+1}-q^{2n+\alpha+\beta+1})(p^{2n+\alpha+\beta+2}-q^{2n+\alpha+\beta+2})^2} \\ &\quad \times \frac{(p^{n+\alpha+\beta+1}-q^{n+\alpha+\beta+1})}{(p^{2n+\alpha+\beta+3}-q^{2n+\alpha+\beta+3})}\ , \\ & D_{p,q}\Big(h_{p^{-1}}(x(x-p^2)u_0)\Big) + \frac{p}{p-q}(1-p^{\alpha+\beta+2}q^{-\alpha-\beta-2})\Big(x - \frac{p^2(1-p^{\beta+1}q^{-\beta-1})}{1-p^{\alpha+\beta+2}q^{-\alpha-\beta-2}}\Big)u_0 = 0\ . \end{aligned}$$

We have obtained the (p,q)-shifted Jacobi polynomials [18].
Moreover, from (43) and (63), we have

$$P_n u_0 = (-1)^n q^{n(n-1)} p^{2n} \frac{\Gamma\Big((p-q)^{-1}(p^{\alpha+\beta+2}q^{-\alpha-\beta-2}-1)+p^{2-n}[n-1]\Big)}{\Gamma\Big((p-q)^{-1}(p^{\alpha+\beta+2}q^{-\alpha-\beta-2}-1)+p^{2-2n}[2n-1]\Big)} \times D^n_{p,q}\Big(x^n(p^{n-2}x, \tfrac{q}{p})_n\Big) \ , \ n \geq 0 \ ,$$

with $(a,q)_0 = 0 \ , \ (a,q)_n = \prod_{\nu=0}^{n-1}(1-aq^{\nu}) \ , \ n \geq 1.$

Acknowledgment. Thanks are due to the referees for their valuable comments and useful suggestions and for their careful reading of the manuscript.

References

[1] F. Abdelkarim and P. Maroni, The D_w-classical polynomials, Resut. Math. 32 (1997), 1-28.
[2] M. Acikoz, S. Araci and U. Duran, Some (p,q)-analogues of Apostol type numbers and polynomials, Acta Commentationes Universitatis tartuensis de Mathematica 23 (1) (2019), 37-50.
[3] T. S. Chihara, An introduction to orthogonal polynomials, Gordon and Breach, New York, 1978.
[4] R. B. Corcino, On P,Q-Binomial coefficients, Electron. J. Combin. Number Theory, 8 (2008), ♯ A29, 1-16.
[5] U. Duran, M. Acikgoz, A. Esi and S. Araci, A Note of the $(p,q)-$Hermite Polynomials, App. Math. Inf. Sci., 12 (1) (2018), 227-231.
[6] U. Duran, M. Acikoz and S. Araci, Unified (p,q)-analog of Apostol Type Polynomials of Order α, Filomat 32 (2) (2018), 387-394.
[7] U. Duran and M. Acikoz, Apostol type (p,q)-Bernoulli, (p,q)-Euler and (p,q)-Genocchi polynomials and numbers, Communications in Mathematics and Applications 8 (1) (2017), 6-30.
[8] U. Duran, M. Acikoz and S. Araci, On (p,q)-Bernoulli, (p,q)-Euler and (p,q)-Genocchi polynomials, J. Comput. Theor. Nanosci. 13 (2016), 7833-7846.
[9] G. Garcia, F. marcellan and L. Salto, A distributional study of discrete classical orthogonal polynomials, J. Comput. Appl. Math., 57 (1995), 147-162.
[10] W. Hahn, Uber orthogonalpolynomial, die $q-$differenezenleichungen, Math. Nachr., 2 (1949), 4-34.
[11] L. Khji and P. Maroni, The H_q-classical orthogonal polynomials, Acta. Appl. Math. **71** (2002), 49-115.
[12] R. Koekeok, P.A. Lesky and R.F. Swarttouw, Hypergeometric orthogonal polynomails and thier $q-$analogues, Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2010.
[13] P. Lesky, Die Ubersetzung der Klassisken orthogonalen polynome in die Differenzenrechnung, Monotsh. Math., 65 (1961), 1-26.
[14] P. N. Sadjang, On two $(p,q)-$analogues of the Laplace transform, J. Diff. Equ. Appl., 23 (2017), 1562-1583.
[15] P. N. Sadjang, On the fundamental theorem of $(p,q)-$calculus and some $(p,q)-$Taylor theorems, Results Math, (2013), 1-15.
[16] P. Maroni, Fonctions eulennes. Polyn orthogonaux classiques. In Technique de l'Iingeur, A 154 (1994), 1-30.
[17] P. Maroni, Variations around classical orthogonal polynomials. Connected problems, put. Appl. Math., 48 (1993), 133-155.
[18] M. Masjed-Jamei, F. Soleyman, I. Area and J.J. Nieto, On $(p,q)-$classical orthogonal polynomails and thier characteriation theorems, Advances in Difference Equations, Berlin, (2017), 1-17.
[19] M. Weber and A. Erdelyi, On the finite difference analogue of Rodrigues formula, Amer. Math. Monthly, 59 (1952), 163-168.