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Generalized Derivations Vanishing on Co-Commutator Identities in Prime Rings

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Abstract. Let R be a noncommutative prime ring of char $(R) \neq 2$ with Utumi quotient ring U and extended centroid C and I a nonzero two sided ideal of R. Suppose that $F(\neq 0)$, G and H are three generalized derivations of R and $f(x_1, \ldots, x_n)$ is a multilinear polynomial over C, which is not central valued on R. If

$$F(G(f(r))f(r) - f(r)H(f(r))) = 0$$

for all $r = (r_1, \ldots, r_n) \in I^n$, then we obtain information about the structure of R and describe the all possible forms of the maps F, G and H. This result generalizes many known results recently proved by several authors ([1], [4], [5], [8], [9], [13], [15]).

1. Introduction

Throughout this paper R always denotes an associative prime ring and U be its Utumi quotient ring. The center $C = Z(U)$ is called the extended centroid of R. By d, we mean a nonzero derivation of R. For $x, y \in R$, $[x, y] = xy - yx$ is the commutator of x and y. The s_4 denotes the standard polynomial in four variables, which is $s_4(x_1, x_2, x_3, x_4) = \sum_{\sigma \in S_4} (-1)^{\sigma} x_{\sigma(1)} x_{\sigma(2)} x_{\sigma(3)} x_{\sigma(4)}$, where $(-1)^{\sigma}$ is $+1$ or -1 according to σ being an even or odd permutation in symmetric group S_4. Let S be a nonempty subset of R. An additive mapping $f : R \to R$ is said to be commuting (respectively, centralizing) on S, if $[f(x), x] = 0$ for all $x \in S$ (respectively, $[f(x), x] \in Z(R)$ for all $x \in S$). Two additive mappings $f, g : R \to R$ are said to be co-commuting (respectively, co-centralizing) on S, if $f(x)x - xg(x) = 0$ for all $x \in S$ (respectively, $f(x)x - xg(x) \in Z(R)$ for all $x \in S$).

A good number of results on co-centralizing and co-commuting maps have been obtained by a number of authors. A well known result of Posner [24] states that if $[d(x), x] \in Z(R)$ for all $x \in R$, then R must be commutative. This result further is generalized in many directions by a number of authors. For instance, Brešar proved in [3] that if d and δ are two derivations of R such that $d(x)x - x\delta(x) \in Z(R)$ for all $x \in R$, then either $d = \delta = 0$ or R is commutative. Later Lee and Wong [20] consider the situation $d(x)x - x\delta(x) \in Z(R)$ for all $x \in L$, where L is a noncentral Lie ideal of R, and obtained that either $d = \delta = 0$ or R satisfies s_4.

We know the fact that a noncentral Lie ideal L of a prime ring contains all the commutators $[x_1, x_2]$ for x_1, x_2 in some nonzero ideal of R, except when char $(R) \neq 2$ and R satisfies s_4. So, it is natural to consider

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the above situation for all commutators $x = [x_1, x_2]$ or more general case $x = f(x_1, \ldots, x_n)$ where $f(x_1, \ldots, x_n)$ is a multilinear polynomial.

In [25], Wong proved that if d and δ are two derivations of R such that $d(x)x - x\delta(x) \in Z(R)$ for all $x \in \{f(x_1, \ldots, x_n) | x_1, \ldots, x_n \in I\}$, where I is a nonzero ideal of R and $f(x_1, \ldots, x_n)$ is a multilinear polynomial which is not central valued on R, then either $d = \delta = 0$, or $\delta = -d$ and $f(x_1, \ldots, x_n)^2$ is central valued on R, except when char $(R) = 2$ and R satisfies s_4.

In [12], De Filippis and Di Vincenzo considered the situation $\delta([d(x), x]) = 0$ for all $x \in \{f(x_1, \ldots, x_n) | x_1, \ldots, x_n \in R\}$, where d and δ are two derivations of R.

Further in [11], De Filippis generalized the above result and proved the following:

Let R be a prime ring of char $(R) \neq 2$, C be the extended centroid of R and let $f(x_1, \ldots, x_n)$ be a multilinear polynomial over C not central-valued on R. Suppose that d, g and $\delta(\neq 0)$ are derivations of R such that

$$\delta(d(f(r_1, \ldots, r_n))f(r_1, \ldots, r_n) - f(r_1, \ldots, r_n)g(f(r_1, \ldots, r_n))) = 0$$

for all $r_1, \ldots, r_n \in R$. Then d and g are both inner derivations on R and one of the following holds: (1) $d = g = 0$; (2) $d = -g$ and $f(x_1, \ldots, x_n)^2$ is central-valued on R.

The main purpose of the present paper is to generalize the above result replacing derivations d, g and δ with three generalized derivations G, H and F respectively, that is,

$$F\Big(G(f(r_1, \ldots, r_n))f(r_1, \ldots, r_n) - f(r_1, \ldots, r_n)H(f(r_1, \ldots, r_n))\Big) = 0$$

for all $r_1, \ldots, r_n \in R$ and then to determined the all possible forms of the maps. The generalized derivation means an additive mapping $F : R \to R$ such that $F(xy) = F(x)y + xd(y)$ for all $x, y \in R$, where d is a derivation of R. The d is called as an associated derivation to F. For some fixed $a, b \in R$, the map $F : R \to R$ defined by $F(x) = ax + xb$ for all $x \in R$ is an example of generalized derivations. This kind of generalized derivations are called as inner generalized derivations of R. Obviously, any derivation is a generalized derivation, but the converse need not be true in general.

Our result generalize many known recent results in literature. In particular, when $F(x) = ax$ for all $x \in R$, then the situation becomes $a(G(x)x - xH(x)) = 0$ for all $x \in \{f(x_1, \ldots, x_n) | x_1, \ldots, x_n \in R\}$, which was studied by De Filippis et al. in [8].

In [9], De Filippis and Vincenzo studied the situation $d([G(x), x]) = 0$ for all $x \in \{f(x_1, \ldots, x_n) | x_1, \ldots, x_n \in R\}$, which is a special case of our result, when $F = d$ is a derivation and $G = H$ is a generalized derivation of R.

Recently in [13], the author of this paper studied the situation $F([G(x), x]) = 0$ for all $x \in \{f(x_1, \ldots, x_n) | x_1, \ldots, x_n \in R\}$, which is a special case of our main theorem.

Moreover, Carini and De Filippis [5] studied the case $\delta(G(x)x) = 0$ for all $x \in \{f(x_1, \ldots, x_n) | x_1, \ldots, x_n \in R\}$. This result is a particular case of our result, when $F = \delta$ a derivation of R and $H = 0$. This result farther extended by Dhara and Argac [14] by replacing derivation δ with a generalized derivation F of R, which is the particular result of our main result when $H = 0$.

In [1], Argac and De Fillipis proved the following theorem when F is identity map.

Theorem A. *Let R be a noncommutative prime ring with Utumi quotient ring U and extended centroid C, I a nonzero two sided ideal of R and $f(x_1, \ldots, x_n)$ be a multilinear polynomial over C, which is not central valued on R. Suppose that G and H are two nonzero generalized derivations of R such that $G(f(r))f(r) - f(r)H(f(r)) = 0$ for all $r = (r_1, \ldots, r_n) \in I^n$, then one of the following holds:*

1. *there exists $a \in U$ such that $G(x) = xa$, $H(x) = ax$ for all $x \in R$;*
2. *$f(x_1, \dots, x_n)^2$ is central valued on R, there exists $a, b \in U$ such that $G(x) = ax + xb$, $H(x) = bx + xa$ for all $x \in R$;*
3. *char $(R) = 2$ and R satisfies s_4 (standard identity of degree 4).*

In [15], Dhara et al. considered the situation when F is a derivation. More precisely, authors proved the following:

Theorem B. *Let R be a noncommutative prime ring of char $(R) \neq 2$ with Utumi quotient ring U and extended centroid C, and $f(x_1, \dots, x_n)$ be a multilinear polynomial over C, which is not central valued on R. Suppose that G and H are two generalized derivations of R and d is a nonzero derivation of R such that $d(G(f(r))f(r) - f(r)H(f(r))) = 0$ for all $r = (r_1, \dots, r_n) \in R^n$, then one of the following holds:*

1. *there exist $a, p, q, c \in U$ and $\lambda \in C$ such that $G(x) = ax + xp + \lambda x$, $H(x) = px + xq$ and $d(x) = [c, x]$ for all $x \in R$, with $[c, a - q] = 0$ and $f(x_1, \dots, x_n)^2$ is central valued on R;*
2. *there exists $a \in U$ such that $G(x) = xa$ and $H(x) = ax$ for all $x \in R$;*
3. *there exist $a, b \in U$ and $\lambda \in C$ such that $G(x) = \lambda x + xa - bx$ and $H(x) = ax + xb$ for all $x \in R$, with $b + \alpha c \in C$ for some $\alpha \in C$;*
4. *R satisfies s_4 and there exist $a, b \in U$ and $\lambda \in C$ such that $G(x) = \lambda x + xa - bx$ and $H(x) = ax + xb$ for all $x \in R$;*
5. *there exist $a', b, c \in U$ and δ a derivation of R such that $G(x) = a'x + xb - \delta(x)$, $H(x) = bx + \delta(x)$ and $d(x) = [c, x]$ for all $x \in R$, with $[c, a'] = 0$ and $f(x_1, \dots, x_n)^2$ is central valued on R.*

In the present paper, we prove the following theorem:

Main Theorem: *Let R be a noncommutative prime ring of char $(R) \neq 2$ with Utumi quotient ring U and extended centroid C and I a nonzero two sided ideal of R. Suppose that $F(\neq 0)$, G and H are three generalized derivations of R and $f(x_1, \dots, x_n)$ is a multilinear polynomial over C, which is not central valued on R. If*

$$F(G(f(r))f(r) - f(r)H(f(r))) = 0$$

for all $r = (r_1, \dots, r_n) \in I^n$, then one of the following holds:

1. *$f(x_1, \dots, x_n)^2$ is central valued on R and there exist $a, b, c, c', u \in U$ such that $F(x) = cx + xc'$, $G(x) = ax - [u, x]$ and $H(x) = [u, x] + xb$ for all $x \in R$, with $c(a - b) + (a - b)c' = 0$;*
2. *$f(x_1, \dots, x_n)^2$ is central valued on R and there exist $p, q \in U$ such that $G(x) = px + xq$ and $H(x) = qx + xp$ for all $x \in R$ and F is any generalized derivation of R;*
3. *there exists $a \in U$ such that $G(x) = xa$ and $H(x) = ax$ for all $x \in R$ and F is any generalized derivation of R;*
4. *there exist $b, c, p, q \in U$ such that $F(x) = cx$, $G(x) = px + xq$ and $H(x) = bx$ for all $x \in R$, with $c(p + q - b) = 0$ and $q - b \in C$;*
5. *there exist $a, c, u, v \in U$ such that $F(x) = xc$, $G(x) = xa$ and $H(x) = ux + xv$ with $a - u \in C$ and $(a - u - v)c = 0$;*
6. *there exist $0 \neq \lambda \in C$ and $a, c, c', u, v \in U$ such that $a + \lambda c, v + \lambda c' \in C$ and $F(x) = cx + xc'$, $G(x) = ax + xu$ and $H(x) = ux + xv$ for all $x \in R$. Moreover, in this case for some $\alpha \in C$ either $\alpha c - \lambda c^2 = -\alpha c' - \lambda c'^2 \in C$ or $\alpha c - \lambda c^2 = -\alpha c' - \lambda c'^2$, $f(x_1, \dots, x_n)^2$ is central valued;*
7. *R satisfies s_4 and there exist $c, c', p, u, v \in U$ such that $F(x) = cx + xc'$, $G(x) = px + xu$ and $H(x) = ux + xv$ for all $x \in R$;*
8. *there exist $a', b, c \in U$ and δ a derivation of R such that $F(x) = [c, x]$, $G(x) = a'x + xb - \delta(x)$, $H(x) = bx + \delta(x)$ for all $x \in R$, with $[c, a'] = 0$ and $f(x_1, \dots, x_n)^2$ is central valued on R.*

Note that recently Carini et al. [4] studied the case $F(G(u)u - uH(u)) = 0$ for all u in a noncentral Lie ideal in prime ring R.

Corollary: *Let R be a noncommutative prime ring of char $(R) \neq 2$ with Utumi quotient ring U and extended centroid C and I a nonzero two sided ideal of R. Suppose that $F(\neq 0)$, G and H are three generalized derivations of R.*

If

$$F(G(x)x - xH(x)) = 0$$

for all $x \in I$, then one of the following holds:

1. *there exists $a \in U$ such that $G(x) = xa$ and $H(x) = ax$ for all $x \in R$ and F is any generalized derivation of R;*
2. *there exist $b, c, p, q \in U$ such that $F(x) = cx$, $G(x) = (p+\lambda)x + xb$ and $H(x) = bx$ for all $x \in R$, with $F(p+\lambda) = 0$;*
3. *there exist $a, c, u, v \in U$ such that $F(x) = xc$, $G(x) = x(u+\lambda)$ and $H(x) = ux + xv$ with $F(\lambda - v) = 0$;*
4. *there exist $0 \neq \lambda \in C$ and $a, c, c', u, v \in U$ such that $a + \lambda c, v + \lambda c' \in C$ and $F(x) = cx + xc'$, $G(x) = ax + xu$ and $H(x) = ux + xv$ for all $x \in R$. Moreover, in this case for some $\alpha \in C$ either $\alpha c - \lambda c^2 = -\alpha c' - \lambda c'^2 \in C$.*
5. *R satisfies s_4 and there exist $c, c', p, u, v \in U$ such that $F(x) = cx + xc'$, $G(x) = px + xu$ and $H(x) = ux + xv$ for all $x \in R$.*

2. Some Results

The following facts are frequently used to prove our results.

Fact 2.1. *Let R be a prime ring and I a two-sided ideal of R. Then R, I and U satisfy the same generalized polynomial identities (GPIs) with coefficients in U ([6]).*

Fact 2.2. *Let R be a prime ring and I a two-sided ideal of R. Then R, I and U satisfy the same differential identities ([21]).*

Fact 2.3. *[1, Lemma 3] Let R be a noncommutative prime ring with Utumi quotient ring U and extended centroid C, and $f(x_1, \ldots, x_n)$ be a multilinear polynomial over C, which is not central valued on R. Suppose that there exist $a, b, c, q \in U$ such that $(af(r) + f(r)b)f(r) - f(r)(cf(r) + f(r)q) = 0$ for all $r = (r_1, \ldots, r_n) \in R^n$. Then one of the following holds:*

(1) $a, q \in C$ and $q - a = b - c = \alpha \in C$;

(2) $f(x_1, \ldots, x_n)^2$ is central valued on R and there exists $\alpha \in C$ such that $q - a = b - c = \alpha$;

(3) char $(R) = 2$ and R satisfies s_4.

Fact 2.4. *(See [18], [21]) Denote by $Der(U)$ the set of all derivations on U. By a derivation word Δ of R we mean an additive map in $End(U, +, .)$ of the form $\Delta = d_1 d_2 d_3 \ldots d_m$ for some derivations $d_i \in Der(U)$ of R. For $x \in R$, we denote by x^Δ the image of x under Δ, that is $x^\Delta = (\cdots (x^{d_1})^{d_2} \cdots)^{d_m}$. By a differential polynomial, we mean a generalized polynomial, with coefficients in U, of the form $\Phi(x_i^{\Delta_j})$ involving noncommutative indeterminates x_i on which the derivations words Δ_j act as unary operations. $\Phi(x_i^{\Delta_j}) = 0$ is said to be a differential identity on a subset T of U if it vanishes for any assignment of values from T to its indeterminates x_i.*

Now let D_{int} be the C-subspace of $Der(U)$ consisting of all inner derivations on U.

Let R be a prime ring and d, d' and δ be derivations of R. If d, d' and δ are linearly C-independent modulo D_{int} and $\Phi(x_i^{\Delta_j})$ is a differential identity on R, where $\Delta_j \in \{d, d', \delta\}$, then $\Phi(y_{ji})$ is a generalized polynomial identity of R, where y_{ji} are distinct indeterminates.

As a particular case, we have:

If d is a nonzero derivation on R and $\Phi(x_1, \ldots, x_n, x_1^d, \ldots, x_n^d)$ is a differential identity on a prime ring R, then one of the following holds:

(i) either $d \in D_{int}$

or (ii) R satisfies the generalized polynomial identity $\Phi(x_1, \ldots, x_n, y_1, \ldots, y_n)$

Fact 2.5. *([9, Lemma 1]) Let C be an infinite field and $m \geq 2$. If $A_1, \ldots, A_k$ are not scalar matrices in $M_m(C)$ then there exists some invertible matrix $P \in M_m(C)$ such that any matrices $PA_1P^{-1}, \ldots, PA_kP^{-1}$ have all non-zero entries.*

Fact 2.6. *Let $f(r_1,\dots,r_n)$ be the multilinear polynomial over the field C and d, δ are derivations on R.*
We shall use the notation

$$f(r_1,\dots,r_n)=r_1r_2\cdots r_n+\sum_{\sigma\in S_n,\sigma\neq id}\alpha_\sigma r_{\sigma(1)}r_{\sigma(2)}\cdots r_{\sigma(n)}$$

for some $\alpha_\sigma\in C$, and S_n denotes the symmetric group of degree n.
Then we have

$$d(f(r_1,\dots,r_n))=f^d(r_1,\dots,r_n)+\sum_i f(r_1,\dots,d(r_i),\dots,r_n),$$

where $f^d(r_1,\dots,r_n)$ be the polynomials obtained from $f(r_1,\dots,r_n)$ replacing each coefficients α_σ with $\delta(\alpha_\sigma)$. Similarly, by calculation, we have

$$\begin{aligned} d^2(f(r_1,\dots,r_n)) &= f^{d^2}(r_1,\dots,r_n)+2\sum_i f^d(r_1,\dots,d(r_i),\dots,r_n)\\ &\quad+\sum_i f(r_1,\dots,d^2(r_i),\dots,r_n)\\ &\quad+\sum_i f(r_1,\dots,d(r_i),\dots,d(r_j),\dots,r_n),\end{aligned}$$

and

$$\begin{aligned} d\delta(f(r_1,\dots,r_n)) &= f^{d\delta}(r_1,\dots,r_n)+\sum_i f^\delta(r_1,\dots,d(r_i),\dots,r_n)\\ &\quad+\sum_i f^d(r_1,\dots,\delta(r_i),\dots,r_n)+\sum_i f(r_1,\dots,d\delta(r_i),\dots,r_n)\\ &\quad+\sum_i f(r_1,\dots,d(r_i),\dots,\delta(r_j),\dots,r_n).\end{aligned}$$

Fact 2.7. *Let R satisfies a nontrivial GPI $\Phi(x_1,\dots,x_n)=0$. Then by [6], U also satisfies the same GPI i.e., $\Phi(x_1,\dots,x_n)=0$. In case C is infinite, we know that $U\otimes_C\overline{C}$ satisfies $\Phi(x_1,\dots,x_n)=0$, where $\overline{C}$ is the algebraic closure of C. Since both U and $U\otimes_C\overline{C}$ are prime and centrally closed [16, Theorems 2.5 and 3.5], we may replace R by U or $U\otimes_C\overline{C}$ according to C finite or infinite and assume that R is centrally closed over C. Then R (by [23]) is a primitive ring with nonzero socle soc(R) with C as its associated division ring. Hence R is isomorphic to a dense ring of linear transformations of a vector space V over C (By Jacobson's theorem [17, p.75]).*

Fact 2.8. *Let R be a prime ring and U be its Utumi ring of quotients and $C=Z(U)$ is the center of U. In [19], Lee proved that any generalized derivation of R can be uniquely extended to a generalized derivation of U and its form will be $g(x)=ax+d(x)$ for some $a\in U$, where d is the associated derivation.*

3. The Case: Inner Generalized Derivations

We dedicate this section, when all the generalized derivations are inner and then obtain the conclusions of the Main Theorem.

We need the following known results.

Lemma 3.1. *([10, Proposition 2.5]) Let R be a prime ring with char $(R)\neq 2$. Assume that R does not embed in $M_2(K)$, the algebra of 2×2 matrices over a field K. If there exist $a,b,c,q,v,w\in R$ such that $a(cs+sq)+(cs+sq)b=vs+sw$ for all $s\in[R,R]$, then one of the following holds:*

1. *c and q are central;*

2. *a and b are central;*
3. *b, q and w are central;*
4. *a, c and v are central;*
5. *there exists $0 \neq \alpha \in K$ such that $a + \alpha c$ and $b - \alpha q$ are central.*

In particular, from above lemma, when $v = w = 0$, we have the following:

Lemma 3.2. *Let R be a prime ring with char $(R) \neq 2$. Assume that R does not embed in $M_2(K)$, the algebra of 2×2 matrices over a field K. If there exist $a, b, c, q \in R$ such that $a(cs + sq) + (cs + sq)b = 0$ for all $s \in [R, R]$, then one of the following holds:*

1. *$c = -q \in C$;*
2. *$b, q \in C$ with $(a + b)(c + q) = 0$;*
3. *$a, b \in C$ with $a + b = 0$;*
4. *$a, c \in C$ with $(c + q)(a + b) = 0$;*
5. *there exists $0 \neq \lambda \in C$ such that $c + \lambda a$ and $q - \lambda b$ are central.*

Proof. Then by Lemma 3.1, one of the following holds:

(i) c and q are central. In this case, identity reduces to $a(c + q)s + s(c + q)b = 0$ for all $s \in [R, R]$. Then $(c + q)b \in C$ with $a(c + q) + (c + q)b = 0$. If $c + q = 0$, then conclusion (1) is obtained. If $c + q \neq 0$, then $b \in C$ and so $(a + b)(c + q) = 0$. Thus the conclusion (2) is obtained.

(ii) a and b are central. In this case $(a + b)(cs + sq) = 0$ for all $s \in [R, R]$. Then $q \in C$ and $(a + b)(c + q) = 0$. If $a + b \neq 0$, then $c + q = 0$, this is conclusion (1). If $a + b = 0$, then conclusion (3) follows.

(iii) b and q are central. In this case $(a + b)(c + q)s = 0$ for all $s \in [R, R]$. This implies $(a + b)(c + q) = 0$, which is conclusion (2).

(iv) a and c are central. In this case $s(c + q)(a + b) = 0$ for all $s \in [R, R]$, which implies $(c + q)(a + b) = 0$. Thus conclusion (4) is obtained.

(v) there exists $0 \neq \alpha \in C$ such that $a + \alpha c$ and $b - \alpha q$ are central. Therefore, $c + \lambda a$ and $q - \lambda b$ are central, where $\lambda = \alpha^{-1}$. □

Lemma 3.3. *Let R be a noncommutative prime ring with char $(R) \neq 2$, $a, b, c, q \in U$, $p(x_1, \ldots, x_n)$ be any polynomial over C, which is not an identity for R. If $c(ap(r) - p(r)q) + (ap(r) - p(r)q)b = 0$ for all $r = (r_1, \ldots, r_n) \in R^n$, then one of the following holds:*

1. *$c(a - q) + (a - q)b = 0$ and $p(x_1, \ldots, x_n)$ is central valued on R;*
2. *$a = q \in C$;*
3. *$a, b \in C$ with $a + b = 0$;*
4. *$b, q \in C$ with $(b + c)(a - q) = 0$;*
5. *$a, c \in C$ with $(a - q)(b + c) = 0$;*
6. *there exists $0 \neq \lambda \in C$ such that $a + \lambda c$ and $q + \lambda b$ are central;*
7. *R satisfies s_4.*

Proof. If $p(x_1, \ldots, x_n)$ is central valued on R, then our assumption $c(ap(r) - p(r)q) + (ap(r) - p(r)q)b = 0$ yields $\{c(a - q) + (a - q)b\}p(r) = 0$ for all $r = (r_1, \ldots, r_n) \in R^n$. Since $p(r_1, \ldots, r_n)$ is nonzero valued on R, $c(a - q) + (a - q)b = 0$ and hence we obtain our conclusion (1).

Hence, assume next that $p(x_1, \ldots, x_n)$ is not central valued on R. Let G be the additive subgroup of R generated by the set $S = \{p(x_1, \ldots, x_n) | x_1, \ldots, x_n \in R\}$. Then $S \neq \{0\}$, since $p(x_1, \ldots, x_n)$ is nonzero valued on R. By our assumption we get $c(ax - xq) + (ax - xq)b = 0$ for any $x \in G$. By [7], either $G \subseteq Z(R)$ or char $(R) = 2$ and R satisfies s_4, except when G contains a noncentral Lie ideal L of R. Since $p(x_1, \ldots, x_n)$ is not central valued on R, the first case can not occur. Since char $(R) \neq 2$, second case also can not occur. Then G contains a noncentral Lie ideal L of R. By [2, Lemma 1], there exists a noncentral two sided ideal I of R such that $[I, R] \subseteq L$. In particular, $c(a[x_1, x_2] - [x_1, x_2]q) + (a[x_1, x_2] - [x_1, x_2]q)b = 0$ for all $x_1, x_2 \in I$. By [6],

$c(a[x_1,x_2]-[x_1,x_2]q)+(a[x_1,x_2]-[x_1,x_2]q)b=0$ is a generalized polynomial identity for R and for U. Thus by Lemma 3.2, if R does not satisfy s_4, then one of the following holds:

(i) $a=q\in C$;
(ii) $b,q\in C$ with $(b+c)(a-q)=0$;
(iii) $a,b\in C$ with $a+b=0$;
(iv) $a,c\in C$ with $(a-q)(b+c)=0$;
(v) there exists $0\neq\lambda\in C$ such that $a+\lambda c$ and $q+\lambda b$ are central.

Therefore, all the conclusions are obtained. □

Lemma 3.4. *[1] Let R be a non-commutative prime ring, $a,b\in U$, $p(x_1,\ldots,x_n)$ be any polynomial over C, which is not an identity for R. If $ap(r)-p(r)b=0$ for all $r=(r_1,\ldots,r_n)\in R^n$, then one of the following holds:*

1. *$a=b\in C$;*
2. *$a=b$ and $p(x_1,\ldots,x_n)$ is central valued on R;*
3. *char $(R)=2$ and R satisfies s_4.*

Proposition 3.5. *Let R be a noncommutative prime ring of char $(R)\neq 2$ with Utumi quotient ring U and extended centroid C, I a nonzero two sided ideal of R and $f(x_1,\ldots,x_n)$ be a multilinear polynomial over C, which is not central valued on R. Suppose that for some $a,b,p,q,u,v\in U$, $F(x)=ax+xb$, $G(x)=px+xq$ and $H(x)=ux+xv$ for all $x\in R$ are three inner generalized derivations of R such that $F(G(f(r))f(r)-f(r)H(f(r)))=0$ for all $r=(r_1,\ldots,r_n)\in R^n$ and F is nonzero, then one of the following holds:*

1. *$f(x_1,\ldots,x_n)^2$ is central valued and $F(x)=ax+xb$, $G(x)=(p+q)x-[u,x]$ and $H(x)=[u,x]+x(u+v)$ for all $x\in R$, with $a(p+q-u-v)+(p+q-u-v)b=0$;*
2. *$f(x_1,\ldots,x_n)^2$ is central valued and $F(x)=ax+xb$, $G(x)=px+xq$ and $H(x)=qx+xp$ for all $x\in R$;*
3. *$F(x)=ax+xb$, $G(x)=x(p+q)$ and $H(x)=(p+q)x$ for all $x\in R$;*
4. *$F(x)=(a+b)x$, $G(x)=px+xq$ and $H(x)=(u+v)x$ for all $x\in R$, with $(a+b)(p+q-u-v)=0$ and $q-u-v\in C$;*
5. *$F(x)=x(a+b)$, $G(x)=x(p+q)$ and $H(x)=ux+xv$ with $p+q-u\in C$ and $(p+q-u-v)(a+b)=0$;*
6. *there exists $0\neq\lambda\in C$ such that $p+q-u+\lambda a$ and $v+\lambda b$ are in C. $F(x)=ax+xb$, $G(x)=(p+q-u)x+xu$ and $H(x)=ux+xv$ for all $x\in R$; Moreover, in this case either $\alpha a-\lambda a^2=-\alpha b-\lambda b^2\in C$ or $\alpha a-\lambda a^2=-\alpha b-\lambda b^2$, $f(x_1,\ldots,x_n)^2$ is central valued, for some $\alpha\in C$.*
7. *R satisfies s_4 and $F(x)=ax+xb$, $G(x)=(p+\lambda)x+xu$ and $H(x)=ux+xv$ for all $x\in R$, and $q-u=\lambda\in C$.*

By Fact 2.1, our hypothesis $F(G(x)x-xH(x))=0$ for all $x\in f(I)$ gives

$$a(pf(r)^2+f(r)(q-u)f(r)-f(r)^2v)+(pf(r)^2+f(r)(q-u)f(r)-f(r)^2v)b=0 \tag{1}$$

that is,

$$apf(r)^2+af(r)(q-u)f(r)-af(r)^2v+pf(r)^2b+f(r)(q-u)f(r)b-f(r)^2vb=0 \tag{2}$$

for all $r=(r_1,\ldots,r_n)\in R^n$.

Now we show that if $a,b\in C$ or $q-u\in C$, then conclusions of Proposition 3.5 hold true and otherwise contradiction arises. Thus we consider the following Lemmas.

Lemma 3.6. *If $a,b\in C$, then conclusions (2) and (3) of Proposition 3.5 hold true.*

Proof. Since $a,b\in C$, $F(x)=(a+b)x$ for all $x\in R$. Since $F\neq 0$, $0\neq a+b\in C$ and hence by (1)

$$pf(r)^2+f(r)(q-u)f(r)-f(r)^2v=0. \tag{3}$$

for all $r=(r_1,\ldots,r_n)\in R^n$. Then by [1, Lemma 3], one of the following holds:

(1) $p, v \in C$ and $v - p = q - u \in C$. In this case $G(x) = px + xq = x(p+q)$ and $H(x) = ux + xv = (u+v)x$ for all $x \in R$, with $p + q = u + v$. This gives conclusion (3).

(2) $f(x_1, \ldots, x_n)^2$ is central valued on R and there exists $\alpha \in C$ such that $v - p = q - u = \alpha \in C$. In this case $H(x) = ux + xv = (q - \alpha)x + x(p + \alpha) = qx + xp$ for all $x \in R$. This is conclusion (2). □

Lemma 3.7. *If $q - u \in C$, then conclusions (1), (3), (4), (5), (6) and (7) of Proposition 3.5 hold true.*

Proof. Since $q - u \in C$, (2) reduces to

$$a((p+q-u)f(r)^2 - f(r)^2 v) + ((p+q-u)f(r)^2 - f(r)^2 v)b = 0 \tag{4}$$

for all $r = (r_1, \ldots, r_n) \in R^n$. Then by Lemma 3.3, one of the following holds:

(1) $f(x_1, \ldots, x_n)^2$ is central valued and $a(p+q-u-v)+(p+q-u-v)b = 0$. In this case, we have $F(x) = ax+xb$, $G(x) = px + xq = px + x(q-u) + xu = (p+q-u)x + xu = (p+q)x - [u, x]$ and $H(x) = ux + xv = [u, x] + x(u+v)$ for all $x \in R$, which is conclusion (1).

(2) $p + q - u = v \in C$. Since $q - u \in C$, $p \in C$. Thus we have, $F(x) = ax + xb$, $G(x) = px + xq = x(p+q)$ and $H(x) = ux + xv = (u+v)x$ for all $x \in R$ with $p + q = u + v$ which is our conclusion (3).

(3) $a, b \in C$ with $a + b = 0$. In this case, $F(x) = ax + xb = (a+b)x = 0$ for all $x \in R$, a contradiction.

(4) $b, v \in C$ with $(a+b)(p+q-u-v) = 0$. In this case, $F(x) = ax + xb = (a+b)x$ for all $x \in R$, $G(x) = px + xq$ and $H(x) = ux + xv = (u+v)x$ for all $x \in R$, with $(a+b)(p+q-u-v) = 0$ and $q - u - v \in C$. This is our conclusion (4).

(5) $a, p+q-u \in C$ with $(p+q-u-v)(a+b) = 0$. Since $q-u \in C$, we have $p \in C$. Thus $F(x) = ax+xb = x(a+b)$, $G(x) = px + xq = x(p+q)$ and $H(x) = ux + xv$ with $p + q - u \in C$ and $(p+q-u-v)(a+b) = 0$. Thus conclusion (5) is obtained.

(6) there exists $0 \neq \lambda \in C$ such that $p + q - u + \lambda a$ and $v + \lambda b$ are in C. Then $F(x) = ax + xb$, $G(x) = px + xq = px + x(q-u) + xu = (p+q-u)x + xu$ and $H(x) = ux + xv$ for all $x \in R$, with $p + q - u + \lambda a, v + \lambda b \in C$, which is conclusion (6). Moreover, in this case assuming $p + q - u + \lambda a = \mu \in C$ and $v + \lambda b = \gamma \in C$, we have by hypothesis, $a\{(\mu - \lambda a)f(r)^2 - f(r)^2(\gamma - \lambda b)\} + \{(\mu - \lambda a)f(r)^2 - f(r)^2(\gamma - \lambda b)\}b = 0$, which gives $\{(\mu - \gamma)a - \lambda a^2\}f(r)^2 + f(r)^2\{(\mu - \gamma)b + \lambda b^2\} = 0$ for all $r = (r_1, \ldots, r_n) \in R^n$. Then by Lemma 3.4, one of the following holds:

- $(\mu - \gamma)a - \lambda a^2 = -(\mu - \gamma)b - \lambda b^2 \in C$;
- $(\mu - \gamma)a - \lambda a^2 = -(\mu - \gamma)b - \lambda b^2$ and $f(x_1, \ldots, x_n)^2$ is central valued.

(7) R satisfies s_4. In this case, $G(x) = px + xq = px + x(q-u) + xu = (p+q-u)x + xu = (p+\lambda)x + xu$, where $q - u = \lambda \in C$. This is our conclusion (7). □

Now to complete the proof of Proposition 3.5, we assume $a, b \notin C$ and $q - u \notin C$ and then we show a number of contradictions.

Lemma 3.8. *If $a, b \notin C$ and $q - u \notin C$, then (2) is a non-trivial generalized polynomial identity for R.*

Proof. Let $w = q - u$. By hypothesis, we have

$$\begin{aligned} \zeta(x_1, \ldots, x_n) \;=\; & apf(x_1, \ldots, x_n)^2 + af(x_1, \ldots, x_n)wf(x_1, \ldots, x_n) - af(x_1, \ldots, x_n)^2 v + pf(x_1, \ldots, x_n)^2 b \\ & + f(x_1, \ldots, x_n)wf(x_1, \ldots, x_n)b - f(x_1, \ldots, x_n)^2 vb = 0 \end{aligned} \tag{5}$$

for all $x_1, \ldots, x_n \in R$. By Fact 2.1, U satisfies $\zeta(x_1, \ldots, x_n) = 0$. Suppose that $\zeta(x_1, \ldots, x_n)$ is a trivial GPI for U. Let $T = U *_C C\{x_1, x_2, \ldots, x_n\}$, the free product of U and $C\{x_1, \ldots, x_n\}$, the free C-algebra in noncommuting indeterminates $x_1, x_2, \ldots, x_n$. Then, $\zeta(x_1, \ldots, x_n)$ is zero element in $T = U *_C C\{x_1, \ldots, x_n\}$. This implies that $\{ap, a, p, 1\}$ is linearly C-dependent. Then there exists $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in C$ such that $\alpha_1 ap + \alpha_2 a + \alpha_3 p + \alpha_4.1 = 0$. If $\alpha_1 = \alpha_3 = 0$, then $\alpha_2 \neq 0$ and so $a = -\alpha_2^{-1}\alpha_4 \in C$, a contradiction. Therefore, either $\alpha_1 \neq 0$ or $\alpha_3 \neq 0$. Without loss of generality, we assume that $\alpha_1 \neq 0$. Then $ap = \alpha a + \beta p + \gamma$, where $\alpha = -\alpha_1^{-1}\alpha_2, \beta = -\alpha_1^{-1}\alpha_3, \gamma = -\alpha_1^{-1}\alpha_4$. Then U satisfies

$$\begin{aligned} & (\alpha a + \beta p + \gamma)f(x_1, \ldots, x_n)^2 + af(x_1, \ldots, x_n)wf(x_1, \ldots, x_n) - af(x_1, \ldots, x_n)^2 v + pf(x_1, \ldots, x_n)^2 b \\ & + f(x_1, \ldots, x_n)wf(x_1, \ldots, x_n)b - f(x_1, \ldots, x_n)^2 vb = 0. \end{aligned} \tag{6}$$

This implies that $\{a, p, 1\}$ is linearly C-dependent. Then there exist $\beta_1, \beta_2, \beta_3 \in C$ such that $\beta_1 a + \beta_2 p + \beta_3 = 0$. By same argument as before, since $a \notin C$, we have $\beta_2 \neq 0$ and hence $p = \alpha' a + \beta'$ for some $\alpha', \beta' \in C$. Thus our identity becomes

$$(\alpha a + \beta\alpha' a + \beta\beta' + \gamma)f(x_1, \ldots, x_n)^2 + af(x_1, \ldots, x_n)wf(x_1, \ldots, x_n) - af(x_1, \ldots, x_n)^2 v$$
$$+(\alpha' a + \beta')f(x_1, \ldots, x_n)^2 b + f(x_1, \ldots, x_n)wf(x_1, \ldots, x_n)b - f(x_1, \ldots, x_n)^2 vb = 0. \tag{7}$$

Since $\{a, 1\}$ is linearly C-independent, we have that U satisfies

$$(\alpha + \beta\alpha')af(x_1, \ldots, x_n)^2 + af(x_1, \ldots, x_n)wf(x_1, \ldots, x_n) - af(x_1, \ldots, x_n)^2 v + \alpha' af(x_1, \ldots, x_n)^2 b = 0, \tag{8}$$

that is

$$af(x_1, \ldots, x_n)\Big((\alpha + \beta\alpha')f(x_1, \ldots, x_n) + wf(x_1, \ldots, x_n) - f(x_1, \ldots, x_n)v + \alpha' f(x_1, \ldots, x_n)b\Big) = 0 \tag{9}$$

for all $x_1, \ldots, x_n \in U$. Moreover, since $w \notin C$, the term $af(x_1, \ldots, x_n)wf(x_1, \ldots, x_n)$ can not be canceled and hence $af(x_1, \ldots, x_n)wf(x_1, \ldots, x_n) = 0$ which implies $a = 0$ or $q - u = w = 0$, a contradiction.

Similarly, we can prove that either $b \in C$ or $w = q - u \in C$, a contradiction. $\square$

Lemma 3.9. *For $R = M_m(C)$, the ring of all $m \times m$ matrices over the field C, if $a, b \notin C$ and $q - u \notin C$ such that R satisfies (2), then no conclusion of Proposition 3.5.*

Proof. By our hypothesis, R satisfies the generalized polynomial identity

$$apf(x_1, \ldots, x_n)^2 + af(x_1, \ldots, x_n)(q - u)f(x_1, \ldots, x_n) - af(x_1, \ldots, x_n)^2 v + pf(x_1, \ldots, x_n)^2 b$$
$$+f(x_1, \ldots, x_n)(q - u)f(x_1, \ldots, x_n)b - f(x_1, \ldots, x_n)^2 vb = 0. \tag{10}$$

Case-1: *When C is infinite field.*

Since $a \notin Z(R)$ and $q - u \notin Z(R)$, by Fact 2.5 there exists a C-automorphism ϕ of $M_m(C)$ such that the matrices $\phi(a)$ and $\phi(q - u)$ have all non-zero entries. Clearly, R satisfies the generalized polynomial identity

$$\phi(ap)f(x_1, \ldots, x_n)^2 + \phi(a)f(x_1, \ldots, x_n)\phi(q - u)f(x_1, \ldots, x_n) - \phi(a)f(x_1, \ldots, x_n)^2\phi(v)$$
$$+\phi(p)f(x_1, \ldots, x_n)^2\phi(b) + f(x_1, \ldots, x_n)\phi(q - u)f(x_1, \ldots, x_n)\phi(b) - f(x_1, \ldots, x_n)^2\phi(vb) = 0. \tag{11}$$

By e_{ij}, we mean the usual matrix unit with 1 in (i, j)-entry and zero elsewhere. Since $f(x_1, \ldots, x_n)$ is not central valued, by [21] (see also [22]), there exist matrices $x_1, \ldots, x_n \in M_m(C)$ and $\gamma \in C - \{0\}$ such that $f(x_1, \ldots, x_n) = \gamma e_{ij}$, with $i \neq j$. Substituting this value in (11), we have

$$\phi(a)e_{ij}\phi(q - u)e_{ij} + e_{ij}\phi(q - u)e_{ij}\phi(b) = 0 \tag{12}$$

and then left multiplying by e_{ij}, it follows $e_{ij}\phi(a)e_{ij}\phi(q - u)e_{ij} = 0$, which is a contradiction, since $\phi(a)$ and $\phi(q - u)$ have all non-zero entries.

Moreover, as $b \notin Z(R)$ and $q - u \notin Z(R)$, then by same argument as above we have a contradiction with the fact $e_{ij}\phi(q - u)e_{ij}\phi(b)e_{ij} = 0$ obtained from (12).

Case-2: *When C is finite field.*

Let K be an infinite field which is an extension of the field C. Let $\overline{R} = M_m(K) \cong R \otimes_C K$. Notice that the multilinear polynomial $f(x_1, \ldots, x_n)$ is central-valued on R if and only if it is central-valued on $\overline{R}$. Consider the generalized polynomial

$$\Psi(x_1, \ldots, x_n) =$$
$$apf(x_1, \ldots, x_n)^2 + af(x_1, \ldots, x_n)(q - u)f(x_1, \ldots, x_n) - af(x_1, \ldots, x_n)^2 v + pf(x_1, \ldots, x_n)^2 b$$
$$+f(x_1, \ldots, x_n)(q - u)f(x_1, \ldots, x_n)b - f(x_1, \ldots, x_n)^2 vb \tag{13}$$

which is a generalized polynomial identity for R.

Moreover, it is a multi-homogeneous of multi-degree $(2, \ldots, 2)$ in the indeterminates $x_1, \ldots, x_n$.

Hence the complete linearization of $\Psi(x_1,\ldots,x_n)$ yields a multilinear generalized polynomial $\Theta(x_1,\ldots,x_n,y_1,\ldots,y_n)$ in $2n$ indeterminates, moreover

$$\Theta(x_1,\ldots,x_n,x_1,\ldots,x_n) = 2^n\Psi(x_1,\ldots,x_n).$$

Clearly the multilinear polynomial $\Theta(x_1,\ldots,x_n,y_1,\ldots,y_n)$ is a generalized polynomial identity for R and $\overline{R}$ too. Since $char(C) \neq 2$ we obtain $\Psi(x_1,\ldots,x_n) = 0$ for all $x_1,\ldots,x_n \in \overline{R}$ and then conclusion follows from case-1 as above. □

Lemma 3.10. *Let R be a prime ring of char $(R) \neq 2$, C the extended centroid of R and $f(x_1,\ldots,x_n)$ a non-central multilinear polynomial over C. If R satisfies (2) and $a,b \notin C$ and $q-u \notin C$, then no conclusion of Proposition 3.5.*

Proof. Let $w = q-u$. By hypothesis, we have

$$\begin{aligned} &apf(x_1,\ldots,x_n)^2 + af(x_1,\ldots,x_n)wf(x_1,\ldots,x_n) - af(x_1,\ldots,x_n)^2v + pf(x_1,\ldots,x_n)^2b \\ &+f(x_1,\ldots,x_n)wf(x_1,\ldots,x_n)b - f(x_1,\ldots,x_n)^2vb = 0 \end{aligned} \tag{14}$$

for all $x_1,\ldots,x_n \in R$. By Fact 2.1, U satisfies (14). By Lemma 3.8, (14) is a non-trivial GPI for U. In this case by Fact 2.7, R is isomorphic to a dense ring of linear transformations of a vector space V over C.

Let $\dim_C V = m$. By density of R, then $R \cong M_m(C)$. Since $f(r_1,\ldots,r_n)$ is not central valued on R, R must be noncommutative and so $m \geq 2$. In this case, by Lemma 3.9, we get no conclusion of Proposition 3.5.

Let $\dim_C V = \infty$. Then by Lemma 2 in [26], the set $f(R)$ is dense on R. Thus by hypothesis, R satisfies

$$apx^2 + ax(q-u)x - ax^2v + px^2b + x(q-u)xb - x^2vb = 0. \tag{15}$$

By Proposition 16 in [4], we conclude that either $a,b \in C$ or $q-u \in C$, a contradiction. Hence there is no conclusion of Proposition 3.5. □

Proof of Proposition 3.5. In case $a,b \in C$ or $q-u \in C$, then by Lemma 3.6 and Lemma 3.7, we have our conclusions (1)- (7), otherwise we have no conclusion by Lemma 3.10.

In particular, we have the following Corollary which we need to prove our Main Theorem in next section.

Corollary 3.11. *Let R be a prime ring of char $(R) \neq 2$, U be its Utumi ring of quotients, C be the extended centroid of R and $f(x_1,\ldots,x_n)$ a non-central multilinear polynomial over C which is not central valued on R. If for some $a,b,p,q,u,v \in U$, R satisfies*

$$a\Big\{(pf(r)+f(r)q)f(r) - f(r)(uf(r)+f(r)v)\Big\}$$

$$+\Big\{(pf(r)+f(r)q)f(r) - f(r)(uf(r)+f(r)v)\Big\}b = 0$$

for all $r = (r_1,\ldots,r_n) \in R^n$, then either $a,b \in C$ or $q-u \in C$.

4. Proof of Main Theorem

In this section, R always be a prime ring of char $(R) \neq 2$, U be its Utumi ring of quotients and $C = Z(U)$ be the extended centroid of R. Let $f(x_1,\ldots,x_n)$ be a noncentral multilinear polynomial over C. By [19, Theorem 3], $F(x) = cx + d(x)$, $G(x) = ax + d'(x)$ and $H(x) = bx + \delta(x)$ for some $a,b,c \in U$ and d, d' and δ are three derivations of U.

By hypothesis, we have

$$c\Big(af(r)^2 + d'(f(r))f(r) - f(r)bf(r) - f(r)\delta(f(r))\Big) + d\Big(af(r)^2 + d'(f(r))f(r) - f(r)bf(r) - f(r)\delta(f(r))\Big)$$

for all $r = (r_1, \dots, r_n) \in I^n$. By Fact 2.1 and Fact 2.2, we have

$$c\Big(af(r)^2 + d'(f(r))f(r) - f(r)bf(r) - f(r)\delta(f(r))\Big) + d\Big(af(r)^2 + d'(f(r))f(r) - f(r)bf(r) - f(r)\delta(f(r))\Big) = 0 \quad (16)$$

for all $r = (r_1, \dots, r_n) \in U^n$.

Moreover, if d, d' and δ are all inner derivations, then by Proposition 3.5, we have our conclusions of Main Theorem. Thus, to prove our Main Theorem, we need to consider the following cases.

- d', δ are inner, d is outer.
- d, δ are inner, d' is outer.
- d, d' are inner, δ is outer.
- d is inner, d', δ are outer.
- d' is inner, d, δ are outer.
- δ is inner, d, d' are outer.
- d, d' and δ all are outer.

Case-1: *d', δ are inner, d is outer.*

Let $d'(x) = [p, x]$ and $\delta(x) = [q, x]$ for all $x \in R$ and for some $p, q \in U$. By (16), U satisfies

$$c((a+p)f(r)^2 - f(r)(p+q+b)f(r) + f(r)^2q) + d((a+p)f(r)^2 - f(r)(p+q+b)f(r) + f(r)^2q) = 0 \quad (17)$$

that is

$$\begin{aligned} &c((a+p)f(r)^2 - f(r)(p+q+b)f(r) + f(r)^2q) + d(a+p)f(r)^2 + (a+p)d(f(r))f(r) + (a+p)f(r)d(f(r)) \\ &-d(f(r))(p+q+b)f(r) - f(r)d(p+q+b)f(r) - f(r)(p+q+b)d(f(r)) + d(f(r))f(r)q + f(r)d(f(r))q \\ &+f(r)^2d(q) = 0. \end{aligned} \quad (18)$$

By Fact 2.4 and Fact 2.6, we can replace $d(f(r_1, \dots, r_n))$ by $f^d(r_1, \dots, r_n) + \sum_i f(r_1, \dots, y_i, \dots, r_n)$ in (18) and then U satisfies blended component

$$\begin{aligned} &(a+p)\sum_i f(r_1, \dots, y_i, \dots, r_n)f(r_1, \dots, r_n) + (a+p)f(r_1, \dots, r_n)\sum_i f(r_1, \dots, y_i, \dots, r_n) \\ &- \sum_i f(r_1, \dots, y_i, \dots, r_n)(p+q+b)f(r_1, \dots, r_n) - f(r_1, \dots, r_n)(p+q+b)\sum_i f(r_1, \dots, y_i, \dots, r_n) \\ &+ \sum_i f(r_1, \dots, y_i, \dots, r_n)f(r_1, \dots, r_n)q + f(r_1, \dots, r_n)\sum_i f(r_1, \dots, y_i, \dots, r_n)q = 0. \end{aligned}$$

In particular, for $y_1 = r_1$ and $y_2 = \dots = y_n = 0$ we have that

$$(a+p)f(r)^2 - f(r)(p+q+b)f(r) + f(r)^2q = 0$$

which is nothing but the identity

$$G(f(r))f(r) - f(r)H(f(r)) = 0$$

for all $r = (r_1, \dots, r_n) \in U^n$. Then by Theorem A, we have conclusions (2) and (3).

Case-2: *d, δ are inner, d' is outer.*

Let $d(x) = [p, x]$ and $\delta(x) = [q, x]$ for all $x \in R$ and for some $p, q \in U$. By (16), U satisfies

$$c(af(r)^2 + d'(f(r))f(r) - f(r)bf(r) - f(r)[q, f(r)]) + [p, af(r)^2 + d'(f(r))f(r) - f(r)bf(r) - f(r)[q, f(r)]] = 0. \quad (19)$$

Since d' is outer, by Fact 2.4 and Fact 2.6 we can replace $d'(f(r_1, \dots, r_n))$ by $f^{d'}(r_1, \dots, r_n) + \sum_i f(r_1, \dots, y_i, \dots, r_n)$ in (19) and then U satisfies blended component

$$c \sum_i f(r_1, \dots, y_i, \dots, r_n) f(r_1, \dots, r_n) + [p, \sum_i f(r_1, \dots, y_i, \dots, r_n) f(r_1, \dots, r_n)] = 0. \quad (20)$$

Replacing y_i with $[a', r_i]$ for some $a' \notin C$, U satisfies

$$c[a', f(r_1, \dots, r_n)] f(r_1, \dots, r_n) + [p, [a', f(r_1, \dots, r_n)] f(r_1, \dots, r_n)] = 0. \quad (21)$$

Then by [14], we conclude that $a' \in C$, a contradiction.

Case-3: *d, d' are inner, δ is outer.*

Let $d(x) = [p, x]$ and $d'(x) = [q, x]$ for all $x \in R$ and for some $p, q \in U$. By (16), U satisfies

$$c(af(r)^2 + [q, f(r)]f(r) - f(r)bf(r) - f(r)\delta(f(r))) + [p, af(r)^2 + [q, f(r)]f(r) - f(r)bf(r) - f(r)\delta(f(r))] = 0. \quad (22)$$

Since δ is outer, by Fact 2.4 and Fact 2.6, we can replace $\delta(f(r_1, \dots, r_n))$ by $f^{\delta}(r_1, \dots, r_n) + \sum_i f(r_1, \dots, y_i, \dots, r_n)$ in (22) and then U satisfies blended component

$$-cf(r_1, \dots, r_n) \sum_i f(r_1, \dots, y_i, \dots, r_n) + [p, -f(r_1, \dots, r_n)\lambda \sum_i f(r_1, \dots, y_i, \dots, r_n)] = 0. \quad (23)$$

Replacing y_i with $[q', y_i]$ for some $q' \notin C$, we obtain from above relation

$$cf(r)[q', f(r)] + [p, f(r)[q', f(r)]] = 0,$$

that is

$$F\Big(f(r)[q', f(r)]\Big) = 0$$

for all $r = (r_1, \dots, r_n) \in U^n$. By Corollary 3.11, either $c, p \in C$ or $q' \in C$. Since $q' \notin C$, we have $c, p \in C$, which implies $F(x) = cx$ for all $x \in R$. Since $F \neq 0$, $0 \neq c \in C$.

Hence our hypothesis reduces to the identity $G(f(r))f(r) - f(r)H(f(r)) = 0$ for all $r = (r_1, \dots, r_n) \in U^n$. Then by Theorem A, we have conclusions (2) and (3).

Case-4: *d is inner, d', δ are outer.*

Let $d(x) = [p, x]$ for all $x \in R$ and for some $p \in U$. By (16),

$$c(af(r)^2 + d'(f(r))f(r) - f(r)bf(r) - f(r)\delta(f(r))) + [p, af(r)^2 + d'(f(r))f(r) - f(r)bf(r) - f(r)\delta(f(r))] = 0 \quad (24)$$

for all $r = (r_1, \ldots, r_n) \in U^n$.

Sub-case-i: Assume next that d' and δ are C-independent modulo inner derivations of U. Then by Fact 2.4 and Fact 2.6, we can replace $d'(f(r_1, \ldots, r_n))$ by $f^{d'}(r_1, \ldots, r_n) + \sum_i f(r_1, \ldots, x_i, \ldots, r_n)$ and $\delta(f(r_1, \ldots, r_n))$ by $f^{\delta}(r_1, \ldots, r_n) + \sum_i f(r_1, \ldots, y_i, \ldots, r_n)$ in (24) and then U satisfies blended components

$$c \sum_i f(r_1, \ldots, x_i, \ldots, r_n) f(r_1, \ldots, r_n) + [p, \sum_i f(r_1, \ldots, x_i, \ldots, r_n) f(r_1, \ldots, r_n)] = 0. \tag{25}$$

This is same as (20) and then by same argument it leads to a contradiction.

Sub-case-ii: Assume that d' and δ are C-dependent modulo inner derivations of U, say $\alpha d' + \beta\delta = ad'_q$, where $\alpha, \beta \in C$, $q' \in U$ and $ad'_q(x) = [q', x]$ for all $x \in R$.

Since δ is outer, $\alpha \neq 0$ and hence $d'(x) = \lambda\delta(x) + [q, x]$ for all $x \in U$, where $\lambda = -\beta\alpha^{-1}$ and $q = \alpha^{-1}q'$.

From (24), we obtain

$$\begin{aligned} &c(af(r)^2 + \lambda\delta(f(r))f(r) + [q, f(r)]f(r) - f(r)bf(r) - f(r)\delta(f(r))) \\ &+[p, af(r)^2 + \lambda\delta(f(r))f(r) + [q, f(r)]f(r) - f(r)bf(r) - f(r)\delta(f(r))] = 0 \end{aligned} \tag{26}$$

By Fact 2.4 and Fact 2.6, we can replace $\delta(f(r_1, \ldots, r_n))$ by $f^{\delta}(r_1, \ldots, r_n) + \sum_i f(r_1, \ldots, y_i, \ldots, r_n)$ in (26) and then U satisfies blended components

$$\begin{aligned} &c\Big\{\lambda \sum_i f(r_1, \ldots, y_i, \ldots, r_n) f(r_1, \ldots, r_n) - f(r_1, \ldots, r_n) \sum_i f(r_1, \ldots, y_i, \ldots, r_n)\Big\} \\ &+\Big[p, \lambda \sum_i f(r_1, \ldots, y_i, \ldots, r_n) f(r_1, \ldots, r_n) - f(r_1, \ldots, r_n) \sum_i f(r_1, \ldots, y_i, \ldots, r_n)\Big] = 0. \end{aligned} \tag{27}$$

Replacing y_i with $[c', y_i]$ for some $c' \notin C$, we obtain from above relation that

$$c\Big\{\lambda[c', f(r)]f(r) - f(r)[c', f(r)]\Big\} + \Big[p, \lambda[c', f(r)]f(r) - f(r)[c', f(r)]\Big] = 0,$$

By Corollary 3.11, either $c, p \in C$ or $\lambda c' + c' = (\lambda + 1)c' \in C$.

If $c, p \in C$, then $F(x) = cx$ for all $x \in R$. Since $F \neq 0$, $c \neq 0$ and so our hypothesis reduces to $(G(f(r))f(r) - f(r)H(f(r))) = 0$ for all $r = (r_1, \ldots, r_n) \in U^n$. Then again by Theorem A, we have conclusions (2) and (3).

On the other hand, if $(\lambda + 1)c' \in C$, then since $c' \notin C$, it yields that $\lambda = -1$.

Then (27) yields

$$\begin{aligned} &c\Big\{ - \sum_i f(r_1, \ldots, y_i, \ldots, r_n) f(r_1, \ldots, r_n) - f(r_1, \ldots, r_n) \sum_i f(r_1, \ldots, y_i, \ldots, r_n)\Big\} \\ &+\Big[p, - \sum_i f(r_1, \ldots, y_i, \ldots, r_n) f(r_1, \ldots, r_n) - f(r_1, \ldots, r_n) \sum_i f(r_1, \ldots, y_i, \ldots, r_n)\Big] = 0. \end{aligned} \tag{28}$$

In particular, for $y_1 = r_1$ and $y_2 = \cdots = y_n = 0$, U satisfies

$$cf(r_1, \ldots, r_n)^2 + [p, f(r_1, \ldots, r_n)^2] = 0$$

that is

$$(c + p)f(r_1, \ldots, r_n)^2 - f(r_1, \ldots, r_n)^2 p = 0.$$

Then by Lemma 3.4, one of the following holds: (i) $c + p = p \in C$. In this case, $p \in C$ and $c = 0$ implying $F = 0$, a contradiction. (ii) $c + p = p$ and $f(r_1, \ldots, r_n)^2 \in C$. In this case $c = 0$ and hence $F(x) = [p, x]$ for all

$x \in R$ is an inner derivation of R. In this case, as d', δ are outer derivations of R, by Theorem B, we have conclusion (8) of Main Theorem.

Case-5: *d' is inner, d, δ are outer.*

Let $d'(x) = [p, x]$ for all $x \in R$ and for some $p \in U$. By (16), U satisfies

$$c(af(r)^2 + [p, f(r)]f(r) - f(r)bf(r) - f(r)\delta(f(r))) + d(af(r)^2 + [p, f(r)]f(r) - f(r)bf(r) \\ -f(r)\delta(f(r))) = 0. \tag{29}$$

Sub-case-i: Let δ and d be C-independent modulo inner derivations of U.
By Fact 2.4 and Fact 2.6, in (29), we can replace $d(f(r_1, \dots, r_n))$ with $f^d(r_1, \dots, r_n) + \sum_i f(r_1, \dots, y_i, \dots, r_n)$ and $d\delta(f(r_1, \dots, r_n))$ with

$$f^{d\delta}(r_1, \dots, r_n) + \sum_i f^{\delta}(r_1, \dots, s_i, \dots, r_n) + \sum_i f^d(r_1, \dots, y_i, \dots, r_n) \\ + \sum_i f(r_1, \dots, t_i, \dots, r_n) + \sum_i f(r_1, \dots, s_i, \dots, y_j, \dots, r_n)$$

in (29) and then U satisfies blended component

$$f(r_1, \dots, r_n) \sum_i f(r_1, \dots, t_i, \dots, r_n) = 0. \tag{30}$$

In particular, for $t_1 = r_1$ and $t_2 = \cdots = t_n = 0$, we have $f(r_1, \dots, r_n)^2 = 0$ which implies $f(r_1, \dots, r_n) = 0$, a contradiction.

Sub-case-ii: Let δ and d be C-dependent modulo inner derivations of U. Then $\delta(x) = \alpha d(x) + [q, x]$ for all $x \in U$, for some $0 \neq \alpha \in C$. From (29), U satisfies

$$c(af(r)^2 + [p, f(r)]f(r) - f(r)bf(r) - \alpha f(r)d(f(r)) - f(r)[q, f(r)]) \\ +d(af(r)^2 + [p, f(r)]f(r) - f(r)bf(r) - \alpha f(r)d(f(r)) - f(r)[q, f(r)]) = 0. \tag{31}$$

Applying Fact 2.4 and Fact 2.6 to (31), we can replace $d(f(r_1, \dots, r_n))$ with $f^d(r_1, \dots, r_n) + \sum_i f(r_1, \dots, y_i, \dots, r_n)$ and $d^2(f(r_1, \dots, r_n))$ with

$$f^{d^2}(r_1, \dots, r_n) \\ +2\sum_i f^d(r_1, \dots, y_i, \dots, r_n) + \sum_i f(r_1, \dots, t_i, \dots, r_n) + \sum_i f(r_1, \dots, y_i, \dots, y_j, \dots, r_n)$$

in (31) and then U satisfies blended component

$$\alpha f(r_1, \dots, r_n) \sum_i f(r_1, \dots, t_i, \dots, r_n) = 0. \tag{32}$$

In particular, for $t_1 = r_1$ and $t_2 = \cdots = t_n = 0$, we have $\alpha f(r_1, \dots, r_n)^2 = 0$ which implies $f(r_1, \dots, r_n) = 0$, a contradiction.

Case-6: *δ is inner, d, d' are outer.*

Let $\delta(x) = [p, x]$ for all $x \in R$, for some $p \in U$. By (16), U satisfies

$$c(af(r)^2 + d'(f(r))f(r) - f(r)bf(r) - f(r)[p, f(r)]) \\ +d(af(r)^2 + d'(f(r))f(r) - f(r)bf(r) - f(r)[p, f(r)]) = 0. \tag{33}$$

Sub-case-i: Let d and d' be C-independent modulo inner derivations of U. By applying Fact 2.4 and Fact 2.6 to the above relation, we can replace $d(f(r_1,\ldots,r_n))$ with $f^d(r_1,\ldots,r_n)+\sum_i f(r_1,\ldots,y_i,\ldots,r_n)$ and $dd'(f(r_1,\ldots,r_n))$ with

$$f^{d\delta}(r_1,\ldots,r_n)+\sum_i f^{\delta}(r_1,\ldots,s_i,\ldots,r_n)+\sum_i f^{d}(r_1,\ldots,y_i,\ldots,r_n)$$
$$+\sum_i f(r_1,\ldots,t_i,\ldots,r_n)+\sum_i f(r_1,\ldots,s_i,\ldots,y_j,\ldots,r_n)$$

in (33) and then U satisfies blended component

$$\sum_i f(r_1,\ldots,t_i,\ldots,r_n)f(r_1,\ldots,r_n)=0. \tag{34}$$

In particular, for $t_1=r_1$ and $t_2=\cdots=t_n=0$, we have $f(r_1,\ldots,r_n)^2=0$ which implies $f(r_1,\ldots,r_n)=0$, a contradiction.

Sub-case-ii: Let δ and d be C-dependent modulo inner derivations of U. Then $d'(x)=\alpha d(x)+[q,x]$ for all $x\in U$, for some $0\neq\alpha\in C$. From (33), U satisfies

$$c(af(r)^2+\alpha d(f(r))f(r)+[q,f(r)]f(r)-f(r)bf(r)-f(r)[p,f(r)])$$
$$+d(af(r)^2+\alpha d(f(r))f(r)+[q,f(r)]f(r)-f(r)bf(r)-f(r)[p,f(r)])=0. \tag{35}$$

By applying Fact 2.4 and Fact 2.6 to (35), we can replace $d(f(r_1,\ldots,r_n))$ with $f^d(r_1,\ldots,r_n)+\sum_i f(r_1,\ldots,y_i,\ldots,r_n)$ and $d^2(f(r_1,\ldots,r_n))$ with

$$f^{d^2}(r_1,\ldots,r_n)$$
$$+2\sum_i f^d(r_1,\ldots,y_i,\ldots,r_n)+\sum_i f(r_1,\ldots,t_i,\ldots,r_n)+\sum_i f(r_1,\ldots,y_i,\ldots,y_j,\ldots,r_n)$$

in (35) and then U satisfies blended component

$$\alpha\sum_i f(r_1,\ldots,t_i,\ldots,r_n)f(r_1,\ldots,r_n)=0. \tag{36}$$

In particular, for $t_1=r_1$ and $t_2=\cdots=t_n=0$, we have $\alpha f(r_1,\ldots,r_n)^2=0$ which implies $f(r_1,\ldots,r_n)=0$, a contradiction.

Case-7: *d, d' and δ all are outer.*

Sub-case-i: Let d, d' and δ be C-independent modulo inner derivations of U. In this case we rewrite (16) as

$$c(af(r)^2+d'(f(r))f(r)-f(r)bf(r)-f(r)\delta(f(r)))+d(a)f(r)^2+ad(f(r))f(r)+af(r)d(f(r))+dd'(f(r))f(r)$$
$$+d'(f(r))d(f(r))-d(f(r))bf(r)-f(r)d(b)f(r)-f(r)bd(f(r))-d(f(r))\delta(f(r))-f(r)d\delta(f(r))=0 \tag{37}$$

for all $r=(r_1,\ldots,r_n)\in U^n$.

By Fact 2.4 and Fact 2.6, we can replace $dd'(f(x_1,\ldots,x_n))$ by

$$f^{dd'}(r_1,\ldots,r_n)+\sum_i f^{d'}(r_1,\ldots,x_i,\ldots,r_n)+\sum_i f^{d}(r_1,\ldots,t_i,\ldots,r_n)$$
$$+\sum_{i\neq j} f(r_1,\ldots,t_i,\ldots,x_j\ldots,r_n)+\sum_i f(r_1,\ldots,w_i,\ldots,r_n)$$

in above equality and then U satisfies the blended component

$$\sum_i f(r_1,\ldots,w_i,\ldots,r_n)f(r_1,\ldots,r_n)=0.$$

This is same as (34) and hence by same argument as above, it leads to a contradiction.

Sub-case-ii: Let d, d' and δ be C-dependent modulo inner derivations of U i.e., $\alpha_1 d + \alpha_2 d' + \alpha_3\delta = ad_{a'}$ for some $\alpha_1, \alpha_2, \alpha_3 \in C$ and $a' \in U$. Then at least one of $\alpha_1, \alpha_2, \alpha_3$ must be nonzero. Let $\alpha_1 \neq 0$. Then we can write $d = \beta_1 d' + \beta_2\delta + ad_{a''}$ for some $\beta_1, \beta_2 \in C$ and $a'' \in U$. Then by (16), we have

$$\begin{aligned} c(af(r)^2 + d'(f(r))f(r) - f(r)bf(r) - f(r)\delta(f(r))) + \beta_1 d'(af(r)^2 + d'(f(r))f(r) - f(r)bf(r) - f(r)\delta(f(r)))\\ +\beta_2\delta(af(r)^2 + d'(f(r))f(r) - f(r)bf(r) - f(r)\delta(f(r)))\\ +[a'', af(r)^2 + d'(f(r))f(r) - f(r)bf(r) - f(r)\delta(f(r))] = 0 \end{aligned} \tag{38}$$

for all $r = (r_1, \ldots, r_n) \in U^n$.

Using Fact 2.4 and Fact 2.6, we substitute the following values in (38) $d'(f(r_1, \ldots, r_n))$ by

$$f^{d'}(r_1, \ldots, r_n) + \sum_i f(r_1, \ldots, y_i, \ldots, r_n),$$

$\delta(f(r_1, \ldots, r_n))$ by

$$f^{\delta}(r_1, \ldots, r_n) + \sum_i f(r_1, \ldots, t_i, \ldots, r_n),$$

$d'\delta(f(r_1, \ldots, r_n))$ by

$$\begin{gathered} f^{d'\delta}(r_1, \ldots, r_n) + \sum_i f^{d'}(r_1, \ldots, t_i, \ldots, r_n)\\ +\sum_i f^{\delta}(r_1, \ldots, y_i, \ldots, r_n) + \sum_{i\neq j} f(r_1, \ldots, y_i, \ldots, t_j \ldots, r_n)\\ +\sum_i f(r_1, \ldots, w_i, \ldots, r_n), \end{gathered}$$

$\delta d'(f(r_1, \ldots, r_n))$ by

$$\begin{gathered} f^{\delta d'}(r_1, \ldots, r_n) + \sum_i f^{\delta}(r_1, \ldots, y_i, \ldots, r_n)\\ +\sum_i f^{d'}(r_1, \ldots, t_i, \ldots, r_n) + \sum_{i\neq j} f(r_1, \ldots, y_i, \ldots, t_j \ldots, r_n)\\ +\sum_i f(r_1, \ldots, w'_i, \ldots, r_n), \end{gathered}$$

$\delta^2(f(r_1, \ldots, r_n))$ by

$$\begin{gathered} f^{\delta^2}(r_1, \ldots, r_n) + 2\sum_i f^{\delta}(r_1, \ldots, t_i, \ldots, r_n)\\ +\sum_i f(r_1, \ldots, z_i, \ldots, r_n) + \sum_{i\neq j} f(r_1, \ldots, t_i, \ldots, t_j, \ldots, r_n), \end{gathered}$$

and $d'^2(f(r_1, \ldots, r_n))$ by

$$\begin{gathered} f^{d'^2}(r_1, \ldots, r_n) + 2\sum_i f^{d'}(r_1, \ldots, y_i, \ldots, r_n)\\ +\sum_i f(r_1, \ldots, z'_i, \ldots, r_n) + \sum_{i\neq j} f(r_1, \ldots, y_i, \ldots, y_j, \ldots, r_n). \end{gathered}$$

Therefore, U satisfies the blended component

$$\beta_1 \sum_i f(r_1, \ldots, z'_i, \ldots, r_n) f(r_1, \ldots, r_n) = 0.$$

and

$$\beta_2 f(r_1, \ldots, r_n) \sum_i f(r_1, \ldots, w_i, \ldots, r_n) = 0.$$

If $\beta_1 \neq 0$, then from above, U satisfies

$$\sum_i f(r_1, \ldots, z'_i, \ldots, r_n) f(r_1, \ldots, r_n) = 0.$$

This is same as (34) and hence by same argument as above, it leads to a contradiction. Thus we conclude that $\beta_1 = 0$. Similarly, from above relation, we conclude that $\beta_2 = 0$. Then d is inner, a contradiction. This completes the proof of the theorem. □

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