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Some New Separation Axioms in Fuzzy Soft Topological Spaces

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Abstract. In this paper, a new form of separation axioms called r-fuzzy soft T_i; $(i = 0, 1, 2, 3, 4)$, r-fuzzy soft regular and r-fuzzy soft normal axioms are introduced in a fuzzy soft topological space based on the paper Aygünoğlu et al. [7]. Also, the relations of these axioms with each other are investigated with the help of examples. Furthermore, some fuzzy soft invariance properties, namely fuzzy soft topological property and hereditary property are specified.

1. Introduction and Preliminaries

The concept of soft set theory has been initiated by Molodtsov [18] as a general mathematical tool for modeling uncertainties. By a soft set we mean a pair (F, E), where E is a set interpreted as the set of parameters and the mapping $F : E \longrightarrow P(X)$ is referred to as the soft structure on X. After the introduction of the notion of soft sets several researchers improved this concept. Soft sets theory received the attention of the topologists who always seeking to generalize and apply the topological notions. One of the most important characteristics of qualitative properties of spatial data and probably the most essential aspect of space is topology and topological relationships. Topological relations between spatial objects such as meet and overlap are the relationships that are invariant with respect to particular transformations due to homeomorphism. Hassan and Ghareeb [11] gave the fundamental concepts and properties of a soft spatial region. They provided a theoretical framework for both dominant ontologies used in GIS. It should be noted that a rich potential for applications of soft set theory to topology in several directions leads to rapid progress of research (see, for example, [3,4,5]). Maji et al. [16] introduced the concept of fuzzy soft set which combines fuzzy sets [22] and soft sets [18]. Soft set and fuzzy soft set theories have a rich potential for applications in several directions. So far, lots of spectacular and creative researches about the theories of soft set and fuzzy soft set have been considered by some scholars [1,2,6-8,13,14,19]. Also, Aygünoğlu et al. [7] studied the topological structure of fuzzy soft sets based on Šostak's paper [21]. Shabir et al. [20] and Georgiou et al. [10] defined some soft separation axioms, soft θ-continuity and soft connectedness in soft topological spaces using (ordinary) points of a topological space X. Hussain and Ahmad [12] introduced and studied soft separation axioms using soft points defined by Zorlutuna et al. [24]. Later, Zakari et al. [23] introduced the notion of soft weak structures as a generalization of soft topology, generalized soft topology and soft minimal structures and discussed some of its properties with some separation axioms and compactness in it.

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In this work, the concepts of r-fuzzy soft T_i; $(i = 0, 1, 2, 3, 4)$, r-fuzzy soft regular and r-fuzzy soft normal axioms are introduced in a fuzzy soft topological space based on the paper of Aygünoğlu et al. [7]. Also, some fuzzy soft invariance properties namely fuzzy soft topological property and fuzzy soft hereditary property are specified. Moreover, the relations of these axioms with each other are investigated with the help of examples. Throughout this paper, X refers to an initial universe, E is the set of all parameters for X and $A \subseteq E$, I^X is the set of all fuzzy sets on X (where, $I = [0, 1]$, $I_0 = (0, 1]$) and for $\alpha \in I$, $\underline{\alpha}(x) = \alpha$, for all $x \in X$. A fuzzy point x_t for $t \in I_0$ is an element of I^X such that $x_t(y) = t$, if y=x and x_t(y)=0, if y≠ x. The family of all fuzzy points in X is denoted by $P_t(X)$. For $\lambda \in I^X$ a fuzzy point $x_t \in \lambda$ if and only if $t < \lambda(x)$.

Definition 1.1. ([7]) A fuzzy soft set f_A over X is a mapping from E into I^X such that $f_A(e)$ is a fuzzy set on X, for each $e \in A$ and $f_A(e) = \underline{0}$, if $e \notin A$, where $\underline{0}$ is zero function on X. The fuzzy set $f_A(e)$, for each $e \in E$, is called an element of the fuzzy soft set f_A. $\widetilde{(X, E)}$ denotes the collection of all fuzzy soft sets on X and is called a fuzzy soft universe [15].

Definition 1.2. ([17]) A fuzzy soft point e_{x_t} over X is a fuzzy soft set defined as follows: $e_{x_t}(k) = x_t$, if $k = e$ and $e_{x_t}(k) = \underline{0}$, if $k \in E - \{e\}$, where x_t is a fuzzy point. A fuzzy soft point e_{x_t} is said to belong to a fuzzy soft set f_A, denoted by $e_{x_t}\tilde{\in} f_A$ if $t < f_A(e)(x)$. Two fuzzy soft points e_{x_t} and k_{y_s} are said to be distinct, denoted by $e_{x_t} \neq k_{y_s}$ if $x \neq y$ or $e \neq k$. The family of all fuzzy soft points in X is denoted by $\widetilde{P_t(X)}$.

Definition 1.3. ([7]) A mapping $\tau : E \longrightarrow [0, 1]^{\widetilde{(X,E)}}$ is called a fuzzy soft topology on X if it satisfies the following conditions for each $e \in E$:

(O1) $\tau_e(\Phi) = \tau_e(\widetilde{E}) = 1$.

(O2) $\tau_e(f_A \sqcap g_B) \geq \tau_e(f_A) \wedge \tau_e(g_B), \ \ \forall f_A, g_B \in \widetilde{(X, E)}$.

(O3) $\tau_e(\bigsqcup_{i\in\Delta}(f_A)_i) \geq \bigwedge_{i\in\Delta} \tau_e((f_A)_i), \forall (f_A)_i \in \widetilde{(X, E)}, i \in \Delta$.

Then the pair (X, τ_E) is called a fuzzy soft topological space. The value $\tau_e(f_A)$ is interpreted as the degree of openness of f_A with respect to parameter $e \in E$.

All definitions and properties of fuzzy soft sets and fuzzy soft topology are found in [1,6,7,9,14,16].

2. Fuzzy Soft T_i; $(i = 0, 1)$ Spaces

Definition 2.1. A fuzzy soft topological space (X, τ_E) is said to be r-fuzzy soft T_0-space if for each $e_{x_t}, k_{y_s} \in \widetilde{P_t(X)}$ such that $e_{x_t} \neq k_{y_s}$, there exist at least one f_A or $g_B \in \widetilde{(X, E)}$ with $\tau_e(f_A) \geq r$, $\tau_e(g_B) \geq r$ such that $e_{x_t}\tilde{\in} f_A$, $k_{y_s}\tilde{\notin} f_A$ or $k_{y_s}\tilde{\in} g_B$, $e_{x_t}\tilde{\notin} g_B$ for each $e \in E$, $r \in I_0$.

Definition 2.2. A fuzzy soft topological space (X, τ_E) is said to be r-fuzzy soft T_1-space if for each $e_{x_t}, k_{y_s} \in \widetilde{P_t(X)}$ such that $e_{x_t} \neq k_{y_s}$, there exist f_A, $g_B \in \widetilde{(X, E)}$ with $\tau_e(f_A) \geq r$, $\tau_e(g_B) \geq r$ such that $e_{x_t}\tilde{\in} f_A$, $k_{y_s}\tilde{\notin} f_A$ and $k_{y_s}\tilde{\in} g_B$, $e_{x_t}\tilde{\notin} g_B$ for each $e \in E$, $r \in I_0$.

Example 2.3. Let $X = \{x, y\}$ be a classical set and $E = \{e_1, e_2\}$ be the parameter set of X. Define f_E, g_E, h_E, k_E, p_E and $q_E \in \widetilde{(X, E)}$ as follows: $f_E = \{(e_1, \{\frac{x}{0.1}, \frac{y}{0.9}\}), (e_2, \{\frac{x}{0.9}, \frac{y}{0.1}\})\}$, $g_E = \{(e_1, \{\frac{x}{0.9}, \frac{y}{0.1}\}), (e_2, \{\frac{x}{0.1}, \frac{y}{0.9}\})\}$, $h_E = \{(e_1, \{\frac{x}{0.9}, \frac{y}{0.1}\}), (e_2, \{\frac{x}{0.9}, \frac{y}{0.1}\})\}$, $k_E = \{(e_1, \{\frac{x}{0.1}, \frac{y}{0.9}\}), (e_2, \{\frac{x}{0.1}, \frac{y}{0.9}\})\}$, $p_E = \{(e_1, \{\frac{x}{0.1}, \frac{y}{0.1}\}), (e_2, \{\frac{x}{0.1}, \frac{y}{0.1}\})\}$, $q_E = \{(e_1, \{\frac{x}{0.9}, \frac{y}{0.9}\}), (e_2, \{\frac{x}{0.9}, \frac{y}{0.9}\})\}$. Define fuzzy soft topology $\tau_E : E \longrightarrow [0, 1]^{\widetilde{(X,E)}}$ as follows:

$$\tau_{e_1}(w_E) = \begin{cases} 1 & \text{if} \quad w_E \in \{\Phi, \widetilde{E}\}, \\ \frac{1}{4} & \text{if} \quad w_E \in \{f_E, h_E\}, \\ \frac{1}{3} & \text{if} \quad w_E \in \{g_E, k_E\}, \\ \frac{1}{2} & \text{if} \quad w_E \in \{p_E, f_E \sqcap h_E, f_E \sqcap k_E, g_E \sqcap h_E, g_E \sqcap k_E\}, \\ \frac{2}{3} & \text{if} \quad w_E \in \{q_E, f_E \sqcup h_E, f_E \sqcup k_E, g_E \sqcup h_E, g_E \sqcup k_E\}, \\ 0 & \text{otherwise}, \end{cases}$$

$$\tau_{e_2}(w_E) = \begin{cases} 1 & \text{if} \quad w_E \in \{\Phi, \widetilde{E}\}, \\ \frac{1}{5} & \text{if} \quad w_E \in \{f_E, h_E\}, \\ \frac{1}{4} & \text{if} \quad w_E \in \{g_E, k_E\}, \\ \frac{1}{3} & \text{if} \quad w_E \in \{p_E, f_E \sqcap h_E, f_E \sqcap k_E, g_E \sqcap h_E, g_E \sqcap k_E\}, \\ \frac{1}{2} & \text{if} \quad w_E \in \{q_E, f_E \sqcup h_E, f_E \sqcup k_E, g_E \sqcup h_E, g_E \sqcup k_E\}, \\ 0 & \text{otherwise.} \end{cases}$$

Then for $t, s \in (0.1, 0.9)$, (X, τ_E) is $\frac{1}{6}$-fuzzy soft T_1-space and (X, τ_E) is $\frac{1}{6}$-fuzzy soft T_0-space.

The following implication hold:

r-fuzzy soft T_1-space $\Rightarrow$ r-fuzzy soft T_0-space.

In general the converse is not true.

Example 2.4. Let $X = \{x, y\}$ be a classical set and $E = \{e_1, e_2\}$ be the parameter set of X. Define f_E, g_E, h_E, p_E and $q_E \in \widetilde{(X, E)}$ as follows: $f_E = \{(e_1, \{\frac{x}{0.1}, \frac{y}{0.9}\}), (e_2, \{\frac{x}{0.9}, \frac{y}{0.1}\})\}$, $g_E = \{(e_1, \{\frac{x}{0.1}, \frac{y}{0.1}\}), (e_2, \{\frac{x}{0.9}, \frac{y}{0.1}\})\}$, $h_E = \{(e_1, \{\frac{x}{0.1}, \frac{y}{0.1}\}), (e_2, \{\frac{x}{0.1}, \frac{y}{0.9}\})\}$, $p_E = \{(e_1, \{\frac{x}{0.1}, \frac{y}{0.1}\}), (e_2, \{\frac{x}{0.1}, \frac{y}{0.1}\})\}$, $q_E = \{(e_1, \{\frac{x}{0.1}, \frac{y}{0.9}\}), (e_2, \{\frac{x}{0.9}, \frac{y}{0.9}\})\}$. Define fuzzy soft topology $\tau_E : E \longrightarrow [0, 1]^{\widetilde{(X,E)}}$ as follows:

$$\tau_{e_1}(w_E) = \begin{cases} 1 & \text{if} \quad w_E \in \{\Phi, \widetilde{E}\}, \\ \frac{1}{3} & \text{if} \quad w_E \in \{f_E, g_E, h_E\}, \\ \frac{1}{2} & \text{if} \quad w_E \in \{p_E, f_E \sqcap g_E\}, \\ \frac{2}{3} & \text{if} \quad w_E \in \{q_E, f_E \sqcup g_E, g_E \sqcup h_E\}, \\ 0 & \text{otherwise,} \end{cases}$$

$$\tau_{e_2}(w_E) = \begin{cases} 1 & \text{if} \quad w_E \in \{\Phi, \widetilde{E}\}, \\ \frac{1}{4} & \text{if} \quad w_E \in \{f_E, g_E, h_E\}, \\ \frac{1}{3} & \text{if} \quad w_E \in \{p_E, f_E \sqcap g_E\}, \\ \frac{1}{2} & \text{if} \quad w_E \in \{q_E, f_E \sqcup g_E, g_E \sqcup h_E\}, \\ 0 & \text{otherwise.} \end{cases}$$

Then for $t, s \in (0.1, 0.9)$, (X, τ_E) is $\frac{1}{5}$-fuzzy soft T_0-space and (X, τ_E) is not $\frac{1}{5}$-fuzzy soft T_1-space.

Definition 2.5. Let (X, τ_E) be a fuzzy soft topological space, $Y \subseteq X$ and $F \subseteq E$. Define a mapping $\Im : F \longrightarrow [0, 1]^{\widetilde{(Y,F)}}$ as follows:

$$\Im_f(g_F) = \bigvee \{\tau_f(f_E) \ : f_E \in \widetilde{(X, E)}\ , f_E|_Y = g_F\}, \quad \forall\ g_F \in \widetilde{(Y, F)}, f \in F.$$

Then $\Im_F$ is said to be the fuzzy soft relative topology on Y. The pair $(Y, \Im_F)$ is called a fuzzy soft subspace of (X, τ_E).

Theorem 2.6. *Let (X, τ_E) be a fuzzy soft topological space, $Y \subseteq X$ and $F \subseteq E$. If (X, τ_E) is r-fuzzy soft T_0-space, then $(Y, \Im_F)$ is r-fuzzy soft T_0-space.*

Proof. Let $e_{x_t}, k_{y_s} \in \widetilde{P_t(Y)}$ such that $e_{x_t} \neq k_{y_s}$. Then $e_{x_t}, k_{y_s} \in \widetilde{P_t(X)}$ such that $e_{x_t} \neq k_{y_s}$. Since (X, τ_E) is r-fuzzy soft T_0-space, there exist at least one f_E or $g_E \in \widetilde{(X, E)}$ with $\tau_e(f_E) \geq r$, $\tau_e(g_E) \geq r$ such that $e_{x_t} \tilde{\in} f_E$, $k_{y_s} \tilde{\notin} f_E$ or $k_{y_s} \tilde{\in} g_E$, $e_{x_t} \tilde{\notin} g_E$ for all $e \in E$ and $r \in I_0$. Therefore $e_{x_t} \tilde{\in} f_E|_Y = h_F$ with $\Im_f(h_F) \geq r$ and $k_{y_s} \tilde{\notin} f_E|_Y = h_F$. Thus $(Y, \Im_F)$ is r-fuzzy soft T_0-space. □

We state the following result without proof in view of above theorem.

Corollary 2.7. *Let (X, τ_E) be a fuzzy soft topological space, $Y \subseteq X$ and $F \subseteq E$. If (X, τ_E) is r-fuzzy soft T_1-space, then $(Y, \Im_F)$ is r-fuzzy soft T_1-space.*

Definition 2.8. Let (X, τ_E) and (Y, τ^*_F) be fuzzy soft topological space's. Then a fuzzy soft mapping φ_ψ from $\widetilde{(X,E)}$ into $\widetilde{(Y,F)}$ is called a fuzzy soft open if $\tau_e(f_A) \leq \tau^*_{\psi(e)}(\varphi_\psi(f_A))$ for all $f_A \in \widetilde{(X,E)}$ and $e \in E$.

Theorem 2.9. *Let (X, τ_E) and (Y, τ^*_F) be fuzzy soft topological space's. If a fuzzy soft mapping φ_ψ from $\widetilde{(X,E)}$ into $\widetilde{(Y,F)}$ is fuzzy soft open, bijective and (X, τ_E) is r-fuzzy soft T_1-space, then (Y, τ^*_F) is r-fuzzy soft T_1-space.*

Proof. Let (X, τ_E) be r-fuzzy soft T_1-space and $k^1_{x_t}, k^2_{y_s} \in \widetilde{P_t(Y)}$ such that $k^1_{x_t} \neq k^2_{y_s}$. Then $\varphi^{-1}_\psi(k^1_{x_t}), \varphi^{-1}_\psi(k^2_{y_s}) \in \widetilde{P_t(X)}$ such that $\varphi^{-1}_\psi(k^1_{x_t}) \neq \varphi^{-1}_\psi(k^2_{y_s})$ (by φ_ψ is bijective mapping). Since (X, τ_E) is r-fuzzy soft T_1-space, there exist f_E and $g_E \in \widetilde{(X,E)}$ with $\tau_e(f_E) \geq r$ and $\tau_e(g_E) \geq r$ such that $\varphi^{-1}_\psi(k^1_{x_t}) \tilde{\in} f_E$, $\varphi^{-1}_\psi(k^2_{y_s}) \tilde{\notin} f_E$ and $\varphi^{-1}_\psi(k^2_{y_s}) \tilde{\in} g_E$, $\varphi^{-1}_\psi(k^1_{x_t}) \tilde{\notin} g_E$ for all $e \in E$ and $r \in I_0$. This implies $k^1_{x_t} \tilde{\in} \varphi_\psi(f_E)$, $k^2_{y_s} \tilde{\notin} \varphi_\psi(f_E)$ and $k^2_{y_s} \tilde{\in} \varphi_\psi(g_E)$, $k^1_{x_t} \tilde{\notin} \varphi_\psi(g_E)$. Since φ_ψ is fuzzy soft open mapping, $\tau^*_{\psi(e)}(\varphi_\psi(f_E)) \geq r$ and $\tau^*_{\psi(e)}(\varphi_\psi(g_E)) \geq r$. Thus (Y, τ^*_F) is r-fuzzy soft T_1-space. □

We state the following result without proof in view of above theorem.

Corollary 2.10. *Let (X, τ_E) and (Y, τ^*_F) be fuzzy soft topological space's. If a fuzzy soft mapping φ_ψ from $\widetilde{(X,E)}$ into $\widetilde{(Y,F)}$ is fuzzy soft open, bijective and (X, τ_E) is r-fuzzy soft T_0-space, then (Y, τ^*_F) is r-fuzzy soft T_0-space.*

Proposition 2.11. *Let (X, τ_E) be a fuzzy soft topological space. Then a mapping τ_e for each $e \in E$, defines a fuzzy topology on X, denoted by $\Im_e$ and defined as follows: $\Im_e(f_A(e)) = \tau_e(f_A)\ \ \forall\ \ f_A \in \widetilde{(X,E)},\ \ e \in E$. The pair $(X, \Im_e)$ is called a fuzzy topological space in the sense of Šostak.*

Proof. Let (X, τ_E) be a fuzzy soft topological space.

(1) Since $\tau_e(\Phi) = \tau_e(\widetilde{E}) = 1$, from the definition of $\Im_e$ we have the following: $\Im_e(\Phi(e)) = \tau_e(\Phi) = 1$ and $\Im_e(\widetilde{E}(e)) = \tau_e(\widetilde{E}) = 1$. This implies $\Im_e(\underline{0}) = \Im_e(\underline{1}) = 1$.

(2) Let $f_A(e),\ g_B(e) \in I^X$ for some $f_A,\ g_B \in \widetilde{(X,E)},\ \ e \in E$. Then $\tau_e(f_A \sqcap g_B) \geq \tau_e(f_A) \wedge \tau_e(g_B),\ \ \forall f_A, g_B \in \widetilde{(X,E)}$ and $(f_A \sqcap g_B)(e) = f_A(e) \wedge g_B(e)$. From the definition of $\Im_e$ we have the following, $\Im_e((f_A \sqcap g_B)(e)) \geq \Im_e(f_A(e)) \wedge \Im_e(g_B(e))$. Hence $\Im_e(f_A(e) \wedge g_B(e)) \geq \Im_e(f_A(e)) \wedge \Im_e(g_B(e))$.

(3) Let $\{(f_A)_i(e) :\ (f_A)_i(e) \in I^X,\ i \in \Delta\}$ be a collection of fuzzy sets in X for some $(f_A)_i \in \widetilde{(X,E)},\ i \in \Delta$ and $e \in E$. Then $\tau_e(\bigsqcup_{i\in\Delta}(f_A)_i) \geq \bigwedge_{i\in\Delta} \tau_e((f_A)_i)$, $\forall (f_A)_i \in \widetilde{(X,E)}, i \in \Delta$ and $(\bigsqcup_{i\in\Delta}(f_A)_i)(e) = \bigvee_{i\in\Delta}(f_A)_i(e)$. From the definition of $\Im_e$ we have, $\Im_e((\bigsqcup_{i\in\Delta}(f_A)_i)(e)) \geq \bigwedge_{i\in\Delta} \Im_e((f_A)_i(e))$. This implies

$$\Im_e(\bigvee_{i\in\Delta}(f_A)_i(e)) \geq \bigwedge_{i\in\Delta} \Im_e((f_A)_i(e)).$$

Then a mapping τ_e for each $e \in E$, defines a fuzzy topology in Šostak's sense. □

Theorem 2.12. *Let (X, τ_E) be a fuzzy soft topological space, $e \in E$ and $r \in I_0$. If (X, τ_E) is r-fuzzy soft T_0-space, then (X, τ_e) is r-fuzzy T_0-space.*

Proof. Let (X, τ_E) be a fuzzy soft topological space and $e_{x_t}, k_{y_s} \in \widetilde{P_t(X)}$ such that $e_{x_t} \neq k_{y_s}$. Then $x_t, y_s \in P_t(X)$ such that $x_t \neq y_s$ and for any $e \in E$, τ_e is a fuzzy topology on X. Since (X, τ_E) is r-fuzzy soft T_0-space, there exist at least one f_E or $g_E \in \widetilde{(X,E)}$ with $\tau_e(f_E) \geq r$, $\tau_e(g_E) \geq r$ such that $e_{x_t} \tilde{\in} f_E$, $k_{y_s} \tilde{\notin} f_E$ or $k_{y_s} \tilde{\in} g_E$, $e_{x_t} \tilde{\notin} g_E$ for all $e \in E$ and $r \in I_0$. This implies $x_t \in f_e$, $y_s \notin f_e$ or $y_s \in g_e$, $x_t \notin g_e$ with $\tau_e(f_e) \geq r$, $\tau_e(g_e) \geq r$. Thus (X, τ_e) is r-fuzzy T_0-space, for each $e \in E$, $r \in I_0$. □

We state the following result without proof in view of above theorem.

Corollary 2.13. *Let (X, τ_E) be a fuzzy soft topological space, $e \in E$ and $r \in I_0$. If (X, τ_E) is r-fuzzy soft T_1-space, then (X, τ_e) is r-fuzzy T_1-space.*

3. Fuzzy Soft Hausdorff Spaces

Definition 3.1. Two fuzzy soft sets f_A, g_B in $\widetilde{(X,E)}$ are said to be fuzzy soft disjoint, written $f_A \sqcap g_B = \Phi$, if $f_e \wedge g_e = \underline{0}$, for each $e \in E$.

Definition 3.2. A fuzzy soft topological space (X, τ_E) is said to be r-fuzzy soft Hausdorff space or r-fuzzy soft T_2-space if for each $e_{x_t}, k_{y_s} \in \widetilde{P_t(X)}$ such that $e_{x_t} \neq k_{y_s}$, there exist $f_A, g_B \in \widetilde{(X,E)}$ with $\tau_e(f_A) \geq r$, $\tau_e(g_B) \geq r$ such that $e_{x_t} \tilde{\in} f_A$, $k_{y_s} \tilde{\in} g_B$ and $f_A \sqcap g_B = \Phi$ for each $e \in E$, $r \in I_0$.

Example 3.3. Let $X = \{x, y\}$ be a classical set and $E = \{e_1, e_2\}$ be the parameter set of X. Define f_E, g_E, h_E, k_E and $q_E \in \widetilde{(X,E)}$ as follows: $f_E = \{(e_1, \{\frac{x}{0.0}, \frac{y}{0.9}\}), (e_2, \{\frac{x}{0.9}, \frac{y}{0.0}\})\}$, $g_E = \{(e_1, \{\frac{x}{0.9}, \frac{y}{0.0}\}), (e_2, \{\frac{x}{0.0}, \frac{y}{0.9}\})\}$, $h_E = \{(e_1, \{\frac{x}{0.9}, \frac{y}{0.0}\}), (e_2, \{\frac{x}{0.9}, \frac{y}{0.0}\})\}$, $k_E = \{(e_1, \{\frac{x}{0.0}, \frac{y}{0.9}\}), (e_2, \{\frac{x}{0.0}, \frac{y}{0.9}\})\}$, $q_E = \{(e_1, \{\frac{x}{0.9}, \frac{y}{0.9}\}), (e_2, \{\frac{x}{0.9}, \frac{y}{0.9}\})\}$. Define fuzzy soft topology $\tau_E : E \longrightarrow [0,1]^{\widetilde{(X,E)}}$ as follows:

$$\tau_{e_1}(w_E) = \begin{cases} 1 & \text{if} \quad w_E \in \{\Phi, \widetilde{E}\}, \\ \frac{1}{4} & \text{if} \quad w_E \in \{f_E, h_E\}, \\ \frac{1}{3} & \text{if} \quad w_E \in \{g_E, k_E\}, \\ \frac{1}{2} & \text{if} \quad w_E \in \{f_E \sqcap h_E, f_E \sqcap k_E, g_E \sqcap h_E, g_E \sqcap k_E\}, \\ \frac{2}{3} & \text{if} \quad w_E \in \{q_E, f_E \sqcup h_E, f_E \sqcup k_E, g_E \sqcup h_E, g_E \sqcup k_E\}, \\ 0 & \text{otherwise,} \end{cases}$$

$$\tau_{e_2}(w_E) = \begin{cases} 1 & \text{if} \quad w_E \in \{\Phi, \widetilde{E}\}, \\ \frac{1}{5} & \text{if} \quad w_E \in \{f_E, h_E\}, \\ \frac{1}{4} & \text{if} \quad w_E \in \{g_E, k_E\}, \\ \frac{1}{2} & \text{if} \quad w_E \in \{f_E \sqcap h_E, f_E \sqcap k_E, g_E \sqcap h_E, g_E \sqcap k_E\}, \\ \frac{2}{3} & \text{if} \quad w_E \in \{q_E, f_E \sqcup h_E, f_E \sqcup k_E, g_E \sqcup h_E, g_E \sqcup k_E\}, \\ 0 & \text{otherwise.} \end{cases}$$

Then for $t, s \in (0.0, 0.9)$, (X, τ_E) is $\frac{1}{6}$-fuzzy soft Hausdorff space.

The following implications hold:

r-fuzzy soft T_2-space $\Rightarrow$ r-fuzzy soft T_1-space $\Rightarrow$ r-fuzzy soft T_0-space.

In general the converses are not true.

Example 3.4. Let $X = \{x, y\}$ be a classical set and $E = \{e_1, e_2\}$ be the parameter set of X. Define f_E, g_E, h_E, k_E, p_E and $q_E \in \widetilde{(X,E)}$ as follows: $f_E = \{(e_1, \{\frac{x}{0.1}, \frac{y}{0.9}\}), (e_2, \{\frac{x}{0.9}, \frac{y}{0.1}\})\}$, $g_E = \{(e_1, \{\frac{x}{0.9}, \frac{y}{0.1}\}), (e_2, \{\frac{x}{0.1}, \frac{y}{0.9}\})\}$, $h_E = \{(e_1, \{\frac{x}{0.9}, \frac{y}{0.1}\}), (e_2, \{\frac{x}{0.9}, \frac{y}{0.1}\})\}$, $k_E = \{(e_1, \{\frac{x}{0.1}, \frac{y}{0.9}\}), (e_2, \{\frac{x}{0.1}, \frac{y}{0.9}\})\}$, $p_E = \{(e_1, \{\frac{x}{0.1}, \frac{y}{0.1}\}), (e_2, \{\frac{x}{0.1}, \frac{y}{0.1}\})\}$, $q_E = \{(e_1, \{\frac{x}{0.9}, \frac{y}{0.9}\}), (e_2, \{\frac{x}{0.9}, \frac{y}{0.9}\})\}$. Define fuzzy soft topology $\tau_E : E \longrightarrow [0,1]^{\widetilde{(X,E)}}$ as follows:

$$\tau_{e_1}(w_E) = \begin{cases} 1 & \text{if} \quad w_E \in \{\Phi, \widetilde{E}\}, \\ \frac{1}{4} & \text{if} \quad w_E \in \{f_E, h_E\}, \\ \frac{1}{3} & \text{if} \quad w_E \in \{g_E, k_E\}, \\ \frac{1}{2} & \text{if} \quad w_E \in \{p_E, f_E \sqcap h_E, f_E \sqcap k_E, g_E \sqcap h_E, g_E \sqcap k_E\}, \\ \frac{2}{3} & \text{if} \quad w_E \in \{q_E, f_E \sqcup h_E, f_E \sqcup k_E, g_E \sqcup h_E, g_E \sqcup k_E\}, \\ 0 & \text{otherwise,} \end{cases}$$

$$\tau_{e_2}(w_E) = \begin{cases} 1 & \text{if} \quad w_E \in \{\Phi, \widetilde{E}\}, \\ \frac{1}{5} & \text{if} \quad w_E \in \{f_E, h_E\}, \\ \frac{1}{4} & \text{if} \quad w_E \in \{g_E, k_E\}, \\ \frac{1}{3} & \text{if} \quad w_E \in \{p_E, f_E \sqcap h_E, f_E \sqcap k_E, g_E \sqcap h_E, g_E \sqcap k_E\}, \\ \frac{2}{3} & \text{if} \quad w_E \in \{q_E, f_E \sqcup h_E, f_E \sqcup k_E, g_E \sqcup h_E, g_E \sqcup k_E\}, \\ 0 & \text{otherwise.} \end{cases}$$

Then for $t, s \in (0.1, 0.9)$, (X, τ_E) is $\frac{1}{5}$-fuzzy soft T_1-space and (X, τ_E) is not $\frac{1}{5}$-fuzzy soft T_2-space.

Theorem 3.5. *Let (X, τ_E) be a fuzzy soft topological space, $e \in E$ and $r \in I_0$. If (X, τ_E) is r-fuzzy soft Hausdorff space, then (X, τ_e) is r-fuzzy Hausdorff space.*

Proof. Let (X, τ_E) be a fuzzy soft topological space and $e_{x_t}, k_{y_s} \in \widetilde{P_t(X)}$ such that $e_{x_t} \neq k_{y_s}$. Then $x_t, y_s \in P_t(X)$ such that $x_t \neq y_s$ and for any $e \in E$, τ_e is a fuzzy topology on X. Since (X, τ_E) is r-fuzzy soft Hausdorff space, there exist $f_A, g_B \in \widetilde{(X,E)}$ with $\tau_e(f_A) \geq r$, $\tau_e(g_B) \geq r$ such that $e_{x_t} \tilde{\in} f_A$, $k_{y_s} \tilde{\in} g_B$ and $f_A \sqcap g_B = \Phi$ for all $e \in E$ and $r \in I_0$. This implies $x_t \in f_A(e)$, $y_s \in g_B(k)$ with $\tau_e(f_A(e)) \geq r$, $\tau_e(g_B(k)) \geq r$ such that $f_A(e) \bigwedge g_B(k) = \underline{0}$. This proves that (X, τ_e) is r-fuzzy Hausdorff space, for each $e \in E$ and $r \in I_0$. □

Theorem 3.6. *A fuzzy soft subspace $(Y, \Im_F)$ of r-fuzzy soft Hausdorff space (X, τ_E) is r-fuzzy soft Hausdorff.*

Proof. Let $e_{x_t}, k_{y_s} \in \widetilde{P_t(Y)}$ such that $e_{x_t} \neq k_{y_s}$. Then $e_{x_t}, k_{y_s} \in \widetilde{P_t(X)}$ such that $e_{x_t} \neq k_{y_s}$. Since (X, τ_E) is r-fuzzy soft Hausdorff space, there exist $f_E, g_E \in \widetilde{(X,E)}$ with $\tau_e(f_E) \geq r$, $\tau_e(g_E) \geq r$ such that $e_{x_t} \tilde{\in} f_E$, $k_{y_s} \tilde{\in} g_E$ and $f_E \sqcap g_E = \Phi$ for all $e \in E$ and $r \in I_0$. Therefore $e_{x_t} \tilde{\in} f_E|_Y = m_F$ with $\Im_f(m_F) \geq r$ and $k_{y_s} \tilde{\in} g_E|_Y = n_F$ with $\Im_f(n_F) \geq r$ and $m_F \sqcap n_F = \Phi$. Thus $(Y, \Im_F)$ is r-fuzzy soft Hausdorff space. □

Theorem 3.7. *Let (X, τ_E) and (Y, τ^*_F) be fuzzy soft topological space's. If a fuzzy soft mapping φ_ψ from $\widetilde{(X,E)}$ into $\widetilde{(Y,F)}$ is fuzzy soft continuous, injective and (Y, τ^*_F) is r-fuzzy soft Hausdorff space, then (X, τ_E) is r-fuzzy soft Hausdorff.*

Proof. Let (Y, τ^*_F) be r-fuzzy soft Hausdorff space and $e_{x_t}, k_{y_s} \in \widetilde{P_t(X)}$ such that $e_{x_t} \neq k_{y_s}$. Then $\varphi_\psi(e_{x_t}), \varphi_\psi(k_{y_s}) \in \widetilde{P_t(Y)}$ such that $\varphi_\psi(e_{x_t}) \neq \varphi_\psi(k_{y_s})$ (by φ_ψ is injective mapping). Since (Y, τ^*_F) is r-fuzzy soft Hausdorff space, there exist f_F and $g_F \in \widetilde{(Y,F)}$ with $\tau^*_f(f_F) \geq r$ and $\tau^*_f(g_F) \geq r$ such that $\varphi_\psi(e_{x_t}) \tilde{\in} f_F$, $\varphi_\psi(k_{y_s}) \tilde{\in} g_F$ and $f_F \sqcap g_F = \Phi$. This implies $e_{x_t} \tilde{\in} \varphi^{-1}_\psi(f_F)$, $k_{y_s} \tilde{\in} \varphi^{-1}_\psi(g_F)$ and $\varphi^{-1}_\psi(f_F) \sqcap \varphi^{-1}_\psi(g_F) = \varphi^{-1}_\psi(f_F \sqcap g_F) = \Phi$. Since φ_ψ is fuzzy soft continuous, $\tau_e(\varphi^{-1}_\psi(f_F)) \geq r$ *and* $\tau_e(\varphi^{-1}_\psi(g_F)) \geq r$. Thus (X, τ_E) is r-fuzzy soft Hausdorff. □

Theorem 3.8. *Let (X, τ_E) be a fuzzy soft topological space. If (X, τ_E) is r-fuzzy soft Hausdorff space and for any $e_{x_t}, e_{y_s} \in \widetilde{P_t(X)}$ such that $e_{x_t} \neq e_{y_s}$ and $t, s \in [0.5, 1)$, there exist $f_A, g_B \in \widetilde{(X,E)}$ with $\tau_e(f^c_A) \geq r$, $\tau_e(g^c_B) \geq r$ such that $e_{x_t} \tilde{\in} f_A$, $e_{y_s} \tilde{\notin} f_A$ and $e_{y_s} \tilde{\in} g_B$, $e_{x_t} \tilde{\notin} g_B$ with $f_A \sqcup g_B = \widetilde{E}$ for all $e \in E$, $r \in I_0$.*

Proof. Since (X, τ_E) is r-fuzzy soft Hausdorff space and $e_{x_t}, e_{y_s} \in \widetilde{P_t(X)}$ such that $e_{x_t} \neq e_{y_s}$, there exist $h_E, k_E \in \widetilde{(X,E)}$ with $\tau_e(h_E) \geq r$, $\tau_e(k_E) \geq r$ such that $e_{x_t} \tilde{\in} h_E$, $e_{y_s} \tilde{\in} k_E$ and $h_E \sqcap k_E = \Phi$ for all $e \in E$ and $r \in I_0$. Since $h_E \sqsubseteq k^c_E$ and $k_E \sqsubseteq h^c_E$, hence $e_{x_t} \tilde{\in} k^c_E$ and $e_{y_s} \tilde{\in} h^c_E$. Put $k^c_E = f_A$, this gives $e_{x_t} \tilde{\in} f_A$, $e_{y_s} \tilde{\notin} f_A$. Also, put $h^c_E = g_B$, this gives $e_{y_s} \tilde{\in} g_B$, $e_{x_t} \tilde{\notin} g_B$. Moreover, $f_A \sqcup g_B = k^c_E \sqcup h^c_E = \widetilde{E}$. □

Theorem 3.9. *Let (X, τ_E) and (Y, τ^*_F) be fuzzy soft topological space's. If a fuzzy soft mapping φ_ψ from $\widetilde{(X,E)}$ into $\widetilde{(Y,F)}$ is fuzzy soft open, bijective and (X, τ_E) is r-fuzzy soft Hausdorff space, then (Y, τ^*_F) is r-fuzzy soft Hausdorff space.*

Proof. Let (X, τ_E) be r-fuzzy soft Hausdorff space and $k^1_{x_t}, k^2_{y_s} \in \widetilde{P_t(Y)}$ such that $k^1_{x_t} \neq k^2_{y_s}$. Then $\varphi^{-1}_\psi(k^1_{x_t}), \varphi^{-1}_\psi(k^2_{y_s}) \in \widetilde{P_t(X)}$ such that $\varphi^{-1}_\psi(k^1_{x_t}) \neq \varphi^{-1}_\psi(k^2_{y_s})$ (by φ_ψ is bijective mapping). Since (X, τ_E) is r-fuzzy soft Hausdorff space, there exist f_E and $g_E \in \widetilde{(X,E)}$ with $\tau_e(f_E) \geq r$ and $\tau_e(g_E) \geq r$ such that $\varphi^{-1}_\psi(k^1_{x_t}) \tilde{\in} f_E$, $\varphi^{-1}_\psi(k^2_{y_s}) \tilde{\in} g_E$ and $f_E \sqcap g_E = \Phi$ for all $e \in E$ and $r \in I_0$. This implies $k^1_{x_t} \tilde{\in} \varphi_\psi(f_E)$, $k^2_{y_s} \tilde{\in} \varphi_\psi(g_E)$ and $\varphi_\psi(f_E) \sqcap \varphi_\psi(g_E) = \varphi_\psi(f_E \sqcap g_E) = \Phi$. Since φ_ψ is fuzzy soft open, $\tau^*_{\psi(e)}(\varphi_\psi(f_E)) \geq r$ and $\tau^*_{\psi(e)}(\varphi_\psi(g_E)) \geq r$. Thus (Y, τ^*_F) is r-fuzzy soft Hausdorff. □

4. Fuzzy Soft Regular, Normal and T_i; $(i = 3, 4)$ Spaces

Definition 4.1. A fuzzy soft topological space (X, τ_E) is said to be r-fuzzy soft regular space if for each $e_{x_t} \in \widetilde{P_t(X)}$ and $h_C \in \widetilde{(X, E)}$ with $\tau_e(h_C^c) \geq r$ such that $e_{x_t} \tilde{\notin} h_C$, there exist $f_A, g_B \in \widetilde{(X, E)}$ with $\tau_e(f_A) \geq r$, $\tau_e(g_B) \geq r$ such that $e_{x_t} \tilde{\in} f_A$, $h_C \sqsubseteq g_B$ and $f_A \sqcap g_B = \Phi$ for each $e \in E$, $r \in I_0$.

Example 4.2. Let $X = \{x, y, z\}$ and $E = \{e_1, e_2\}$ be the parameter set of X. Define f_E, $g_E \in \widetilde{(X, E)}$ as follows: $f_E = \{(e_1, \{\frac{x}{1.0}, \frac{y}{0.0}, \frac{z}{1.0}\}), (e_2, \{\frac{x}{0.0}, \frac{y}{1.0}, \frac{z}{0.0}\})\}$, $g_E = \{(e_1, \{\frac{x}{0.0}, \frac{y}{1.0}, \frac{z}{0.0}\}), (e_2, \{\frac{x}{1.0}, \frac{y}{0.0}, \frac{z}{1.0}\})\}$. Define fuzzy soft topology $\tau_E : E \longrightarrow [0, 1]^{\widetilde{(X,E)}}$ as follows:

$$\tau_{e_1}(m_E) = \begin{cases} 1 & \text{if} \quad m_E \in \{\Phi, \widetilde{E}\}, \\ \frac{1}{2} & \text{if} \quad m_E = f_E, \\ \frac{2}{3} & \text{if} \quad m_E = g_E, \\ 0 & \text{otherwise,} \end{cases}$$

$$\tau_{e_2}(m_E) = \begin{cases} 1 & \text{if} \quad m_E \in \{\Phi, \widetilde{E}\}, \\ \frac{1}{3} & \text{if} \quad m_E = f_E, \\ \frac{1}{2} & \text{if} \quad m_E = g_E, \\ 0 & \text{otherwise.} \end{cases}$$

Then (X, τ_E) is $\frac{1}{4}$-fuzzy soft regular space.

Theorem 4.3. *A fuzzy soft subspace $(Y, \Im_F)$ of r-fuzzy soft regular space (X, τ_E) is r-fuzzy soft regular.*

Proof. Let $e_{x_t} \in \widetilde{P_t(Y)}$ and $m_F \in \widetilde{(Y, F)}$ with $\Im_f(m_F^c) \geq r$ such that $e_{x_t} \tilde{\notin} m_F$. Then $e_{x_t} \in \widetilde{P_t(X)}$ and $m_F \in \widetilde{(X, E)}$ with $\tau_e(m_F^c) \geq r$ such that $e_{x_t} \tilde{\notin} m_F$. Since (X, τ_E) is r-fuzzy soft regular space, there exist f_E, $g_E \in \widetilde{(X, E)}$ with $\tau_e(f_E) \geq r$, $\tau_e(g_E) \geq r$ such that $e_{x_t} \tilde{\in} f_E$, $m_F \sqsubseteq g_E$ and $f_E \sqcap g_E = \Phi$ for all $e \in E$ and $r \in I_0$. Therefore $e_{x_t} \tilde{\in} f_E|_Y = h_F$ with $\Im_f(h_F) \geq r$ and $m_F \sqsubseteq g_E|_Y = k_F$ with $\Im_f(k_F) \geq r$ and $h_F \sqcap k_F = \Phi$. Thus $(Y, \Im_F)$ is r-fuzzy soft regular space. □

Theorem 4.4. *Let (X, τ_E) and (Y, τ_F^*) be fuzzy soft topological space's. If a fuzzy soft mapping φ_ψ from $\widetilde{(X, E)}$ into $\widetilde{(Y, F)}$ is fuzzy soft open, bijective and (X, τ_E) is r-fuzzy soft regular space, then (Y, τ_F^*) is r-fuzzy soft regular space.*

Proof. Let (X, τ_E) be r-fuzzy soft regular space, $k_{x_t} \in \widetilde{P_t(Y)}$ and $h_F \in \widetilde{(Y, F)}$ with $\tau_f^*(h_F^c) \geq r$ such that $k_{x_t} \tilde{\notin} h_F$. Then $\varphi_\psi^{-1}(k_{x_t}) \in \widetilde{P_t(X)}$ and $\varphi_\psi^{-1}(h_F) \in \widetilde{(X, E)}$ such that $\varphi_\psi^{-1}(k_{x_t}) \tilde{\notin} \varphi_\psi^{-1}(h_F)$ (by φ_ψ is bijective mapping). Since (X, τ_E) is r-fuzzy soft regular space, there exist f_E and $g_E \in \widetilde{(X, E)}$ with $\tau_e(f_E) \geq r$ and $\tau_e(g_E) \geq r$ such that $\varphi_\psi^{-1}(k_{x_t}) \tilde{\in} f_E$, $\varphi_\psi^{-1}(h_F) \sqsubseteq g_E$ and $f_E \sqcap g_E = \Phi$ for all $e \in E$ and $r \in I_0$. This implies $k_{x_t} \tilde{\in} \varphi_\psi(f_E)$, $h_F \sqsubseteq \varphi_\psi(g_E)$ and $\varphi_\psi(f_E) \sqcap \varphi_\psi(g_E) = \varphi_\psi(f_E \sqcap g_E) = \Phi$. Since φ_ψ is fuzzy soft open, then $\tau^*_{\psi(e)}(\varphi_\psi(f_E)) \geq r$ and $\tau^*_{\psi(e)}(\varphi_\psi(g_E)) \geq r$. Thus, (Y, τ_F^*) is r-fuzzy soft regular. □

Definition 4.5. A fuzzy soft topological space (X, τ_E) is said to be r-fuzzy soft T_3-space if it is r-fuzzy soft regular space and r-fuzzy soft T_1-space.

The following corollary follows form Corollary 2.7 and Theorem 4.3.

Corollary 4.6. *A fuzzy soft subspace $(Y, \Im_F)$ of r-fuzzy soft T_3-space (X, τ_E) is r-fuzzy soft T_3.*

Definition 4.7. A fuzzy soft topological space (X, τ_E) is said to be r-fuzzy soft normal space if for each $h_C, k_D \in \widetilde{(X, E)}$ with $\tau_e(h_C^c) \geq r$, $\tau_e(k_D^c) \geq r$ such that $h_C \sqcap k_D = \Phi$, there exist f_A, $g_B \in \widetilde{(X, E)}$ with $\tau_e(f_A) \geq r$, $\tau_e(g_B) \geq r$ such that $h_C \sqsubseteq f_A$, $k_D \sqsubseteq g_B$ and $f_A \sqcap g_B = \Phi$ for each $e \in E$, $r \in I_0$.

Example 4.8. Let $X = \{x, y, z, w\}$ be a classical set and $E = \{e_1, e_2\}$ be the parameter set of X. Define f_E, $g_E \in \widetilde{(X,E)}$ as follows: $f_E = \{(e_1, \{\frac{x}{1.0}, \frac{y}{0.0}, \frac{z}{0.0}, \frac{w}{1.0}\}), (e_2, \{\frac{x}{0.0}, \frac{y}{1.0}, \frac{z}{1.0}, \frac{w}{0.0}\})\}$, $g_E = \{(e_1, \{\frac{x}{0.0}, \frac{y}{1.0}, \frac{z}{1.0}, \frac{w}{0.0}\}), (e_2, \{\frac{x}{1.0}, \frac{y}{0.0}, \frac{z}{0.0}, \frac{w}{1.0}\})\}$. Define fuzzy soft topology $\tau_E : E \longrightarrow [0,1]^{\widetilde{(X,E)}}$ as follows:

$$\tau_{e_1}(m_E) = \begin{cases} 1 & \text{if} \quad m_E \in \{\Phi, \widetilde{E}\}, \\ \frac{2}{3} & \text{if} \quad m_E = f_E, \\ \frac{1}{3} & \text{if} \quad m_E = g_E, \\ 0 & \text{otherwise,} \end{cases}$$

$$\tau_{e_2}(m_E) = \begin{cases} 1 & \text{if} \quad m_E \in \{\Phi, \widetilde{E}\}, \\ \frac{1}{3} & \text{if} \quad m_E = f_E, \\ \frac{2}{3} & \text{if} \quad m_E = g_E, \\ 0 & \text{otherwise.} \end{cases}$$

Then (X, τ_E) is $\frac{1}{4}$-fuzzy soft normal space.

Theorem 4.9. *A fuzzy soft subspace $(Y, \Im_F)$ of r-fuzzy soft normal space (X, τ_E) is r-fuzzy soft normal.*

Proof. Let $m_F, n_F \in \widetilde{(Y,F)}$ with $\Im_f(m_F^c) \geq r$, $\Im_f(n_F^c) \geq r$ such that $m_F \sqcap n_F = \Phi$. Then $m_F, n_F \in \widetilde{(X,E)}$ with $\tau_e(m_F^c) \geq r$, $\tau_e(n_F^c) \geq r$ such that $m_F \sqcap n_F = \Phi$. Since (X, τ_E) is r-fuzzy soft normal space, there exist f_E, $g_E \in \widetilde{(X,E)}$ with $\tau_e(f_E) \geq r$, $\tau_e(g_E) \geq r$ such that $m_F \sqsubseteq f_E$, $n_F \sqsubseteq g_E$ and $f_E \sqcap g_E = \Phi$ for all $e \in E$ and $r \in I_0$. Therefore $m_F \sqsubseteq f_E|_Y = h_F$ with $\Im_f(h_F) \geq r$ and $n_F \sqsubseteq g_E|_Y = k_F$ with $\Im_f(k_F) \geq r$ and $h_F \sqcap k_F = \Phi$. Thus $(Y, \Im_F)$ is r-fuzzy soft normal space. □

Theorem 4.10. *Let (X, τ_E) and (Y, τ_F^*) be fuzzy soft topological space's. If a fuzzy soft mapping φ_ψ from $\widetilde{(X,E)}$ into $\widetilde{(Y,F)}$ is fuzzy soft continuous, injective and (Y, τ_F^*) is r-fuzzy soft normal space, then (X, τ_E) is r-fuzzy soft normal space.*

Proof. Let (Y, τ_F^*) be r-fuzzy soft normal space and $h_E, k_E \in \widetilde{(X,E)}$ with $\tau_e(h_E^c) \geq r$, $\tau_e(k_E^c) \geq r$ such that $h_E \sqcap k_E = \Phi$. Then $\varphi_\psi(h_E)$, $\varphi_\psi(k_E) \in \widetilde{(Y,F)}$ such that $\varphi_\psi(h_E) \sqcap \varphi_\psi(k_E) = \Phi$ (by φ_ψ is injective mapping). Since (Y, τ_F^*) is r-fuzzy soft normal space, there exist f_F and $g_F \in \widetilde{(Y,F)}$ with $\tau_f^*(f_F) \geq r$ and $\tau_f^*(g_F) \geq r$ such that $\varphi_\psi(h_E) \sqsubseteq f_F$, $\varphi_\psi(k_E) \sqsubseteq g_F$ and $f_F \sqcap g_F = \Phi$. This implies $h_E \sqsubseteq \varphi_\psi^{-1}(f_F)$, $k_E \sqsubseteq \varphi_\psi^{-1}(g_F)$ and $\varphi_\psi^{-1}(f_F) \sqcap \varphi_\psi^{-1}(g_F) = \varphi_\psi^{-1}(f_F \sqcap g_F) = \Phi$. Since φ_ψ is fuzzy soft continuous, then $\tau_e(\varphi_\psi^{-1}(f_F)) \geq r$ and $\tau_e(\varphi_\psi^{-1}(g_F)) \geq r$. Thus, (X, τ_E) is r-fuzzy soft normal space. □

Definition 4.11. A fuzzy soft topological space (X, τ_E) is said to be r-fuzzy soft T_4-space if it is r-fuzzy soft normal space and r-fuzzy soft T_1-space.

The following corollary follows form Corollary 2.7 and Theorem 4.9.

Corollary 4.12. *A fuzzy soft subspace $(Y, \Im_F)$ of r-fuzzy soft T_4-space (X, τ_E) is r-fuzzy soft T_4.*

Theorem 4.13. *Every r-fuzzy soft T_4-space (X, τ_E) is r-fuzzy soft T_2-space if $\tau_e(e_{x_t}^c) \geq r$ for each $e_{x_t} \in \widetilde{P_t(X)}$, $e \in E$, $r \in I_0$.*

Proof. Let (X, τ_E) be r-fuzzy soft T_4-space and $e_{x_t}, k_{y_s} \in \widetilde{P_t(X)}$ such that $e_{x_t} \neq k_{y_s}$. Then for each $e_{x_t}, k_{y_s} \in \widetilde{P_t(X)}$ we have $\tau_e(e_{x_t}^c) \geq r$, $\tau_e(k_{y_s}^c) \geq r$ and $k_{y_s} \sqcap e_{x_t} = \Phi$. Since (X, τ_E) is r-fuzzy soft normal space, there exist f_E, $g_E \in \widetilde{(X,E)}$ with $\tau_e(f_E) \geq r$, $\tau_e(g_E) \geq r$ such that $e_{x_t} \sqsubseteq f_E$, $k_{y_s} \sqsubseteq g_E$ and $f_E \sqcap g_E = \Phi$ for all $e \in E$ and $r \in I_0$. Thus, $e_{x_t} \tilde{\in} f_E$, $k_{y_s} \tilde{\in} g_E$. Therefore, (X, τ_E) is r-fuzzy soft T_2-space. □

5. Conclusion

Hussain and Ahmad [12] defined and studied some soft separation axioms using soft points defined by Zorlutuna et al. [24]. In the present work, the concepts of r-fuzzy soft T_i; $(i = 0, 1, 2, 3, 4)$, r-fuzzy soft regular and r-fuzzy soft normal axioms are introduced, which are generalization of the concepts introduced in [12] and the relations of these axioms with each other are investigated with the help of examples. Moreover, some fuzzy soft invariance properties are specified. These separation axioms would be useful for the development of the theory of fuzzy soft topology to solve the complicated problems containing uncertainties in engineering, medical, environment and in general man-machine systems of various types.

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