



## General Decay and Well-Posedness of the Cauchy Problem for the Jordan-Moore-Gibson-Thompson Equation With Memory

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**Abstract.** In this paper, we consider the Cauchy problem of a third order in time nonlinear equation known as the Jordan-Moore-Gibson-Thompson (JMGT) equation with the presence of both memory. Using the well known energy method combined with Lyapunov functionals approach, we prove a general decay result, and we show a local existence result in appropriate function spaces. Finally, we prove a global existence result for small data, and we prove the uniqueness of the generalized solution.

### 1. Introduction

In this paper, we are interested to study the following problem for the nonlinear Jordan-Moore-Gibson-Thompson equation with memory:

$$\tau u_{ttt} + u_{tt} - c^2 \Delta u - \beta \Delta u_t - \int_0^t h(t-s) \Delta u(s) ds = \frac{\partial}{\partial t} \left( \frac{1}{c^2} \frac{B}{2A} (u_t)^2 + |\nabla u|^2 \right), \quad (1)$$

where

$$(x, t) \in \mathbb{R}^n \times (0, \infty).$$

We consider the initial conditions

$$u(x, 0) = u_0, \quad u_t(x, 0) = u_1, \quad u_{tt}(x, 0) = u_2.$$

Here  $\tau, \beta$  are physical parameters,  $c$  is the speed of sound, and  $B/A$  the parameter of nonlinearity. The convolution term  $\int_0^t h(t-s) \Delta u(s) ds$  reflects the memory effect of materials due to viscoelasticity, and  $h$  is the

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relaxation function satisfying

(H1)  $h \in C^1(\mathbb{R}_+, \mathbb{R}_+)$  is a non-increasing function satisfying

$$h(0) > 0, \quad 1 - h_0 = l > 0. \quad (2)$$

where  $h_0 = G(\infty) = \int_0^\infty h(s) ds > 0$ ,  $G(t) = \int_0^t h(s) ds$  and  $h'' > 0$ .

(H2) There exists a positive non-increasing differentiable function  $\vartheta \in (\mathbb{R}_+, \mathbb{R}_+)$  satisfying

$$h'(t) \leq -\vartheta(t)h(t), \quad t \geq 0. \quad (3)$$

(H3)

$$\beta - \tau > 0. \quad (4)$$

In the field of irreversible thermodynamics (EIT) the presence of the derivative for the third time is very important and ideal. It is the theory that suggested abolishing the annoying properties in it that we do not want Diffusion of heat and speed attached to infinite speed when using the Fourier-Navier-Stokes equations. As the physical quantities such as thermodynamics were the main reason for this. Also, in the theory of (EIT), the equations of evolution with the appropriate time for relaxation it controls.

The study of models under these classic conditions is a good start for a deeper understanding of good behavior and converging behavior of linear models.

Later, Lasićka [14] showed the general decay result of equation (MGT) with memory. Then after a while, and in the absence of memory ( $h = 0$ ). In [18], the authors studied the Jordan-Moore-Gibson-Thompson equation in its full generality. Under the assumption  $0 < \tau < \beta$ , by using the contraction mapping theorem in appropriately chosen spaces, they showed a local existence result in some appropriate functional spaces, and by using some energy type estimates they proved a global existence result for small initial data by constructing an appropriate energy norm and showed that this norm remains uniformly bounded with respect to time. For further clarification and depth on this topic see ([6], [7], [8],[11],[12],[13],[15],[22]).

A complement to these results, our purpose in this paper is to give the Jordan-Moore-Gibson-Thompson equation with memory (1) for the Cauchy problem  $x \in \mathbb{R}^n$ . Under suitable assumptions, we show a local existence result in some appropriate functional spaces, second we prove a global existence result for small initial data, and we prove the uniqueness of the generalized solution. Our result is new and improves previous results in the literature. In each of the following we introduce some notations. Let  $\|\cdot\|_q$  and  $\|\cdot\|_{H^l}$  stand for the  $L^q(\mathbb{R}^n)$  - norm ( $2 \leq q \leq \infty$ ) and the  $H^l(\mathbb{R}^n)$  - norm. We define the weighted function space  $L^{1,1}(\mathbb{R}^n)$ ,  $n \geq 1$  as follows:  $u \in L^{1,1}(\mathbb{R}^n)$  if  $u \in L^1(\mathbb{R}^n)$  and

$$\|u\|_{1,1} = \int_{\mathbb{R}^n} (1 + |x|)|u(x)|dx < \infty.$$

The symbol  $[A, B] = AB - BA$  denotes the commutator, and  $c$  is a positive constant.

## 2. Preliminaries

The following lemma has been proved for instance in ([10], Lemma 4.1).

**Lemma 2.1.** Let  $1 \leq p, q, r \leq \infty$  and  $\frac{1}{p} = \frac{1}{q} + \frac{1}{r}$ . Then, we have

$$\|\nabla^k(uv)\|_{L^p} \leq c\left(\|u\|_{L^q}\|\nabla^k v\|_{L^r} + \|v\|_{L^q}\|\nabla^k u\|_{L^r}\right), t \geq 0, \quad (5)$$

and the commutator estimate

$$\|[\nabla^k, f]g\|_{L^2} \leq c\left(\|f\|_{L^\infty}\|\nabla^k g\|_{L^2} + \|g\|_{L^\infty}\|\nabla^k f\|_{L^2}\right), t \geq 0, \quad (6)$$

for some constant  $c > 0$ .

The next lemma has been proved in ([23], Lemma 3.7).

**Lemma 2.2.** Let  $M = M(t)$ , be a non-negative continuous function satisfying the inequality

$$M(t) \leq c_1 + c_2 M(t)^\kappa, \quad (7)$$

in some interval containing 0, where  $c_1, c_2 > 0$  are positive constants and  $\kappa > 1$ . If  $M(0) \leq c_1$  and

$$c_1 c_2^{1/(\kappa-1)} \leq \left(1 - \frac{1}{\kappa}\right) \kappa^{-1/(\kappa-1)}, \quad (8)$$

then in the same interval

$$M(t) \leq \frac{c_1}{1 - 1/\kappa}. \quad (9)$$

We will use the Gagliardo-Nirenberg interpolation inequality as follows.

**Lemma 2.3.** [16] Let  $N > 1$ . Let  $1 \leq p, q, r \leq \infty$  and let  $m$  be a positive integer. Then for any integer  $j$  with  $0 \leq j \leq m$ , we have

$$\|\nabla^j u\|_{L^p} \leq c \|u\|_{L^q}^{1-\alpha} \|\nabla^m u\|_{L^r}^\alpha, \quad (10)$$

where

$$\frac{1}{p} = \frac{j}{N} + \alpha \left( \frac{1}{r} + \frac{m}{N} \right) + \frac{1-\alpha}{q}, \quad (11)$$

for  $\alpha > 0$  satisfying  $j/m \leq \alpha \leq 1$  and  $c$  is a positive constant.

We also recall the decay estimates of the linearized problem associated to (1)

**Proposition 2.4.** [17] Let  $u$  be the solution of the linear problem (with  $h = 0$ )

$$\tau u_{ttt} + u_{tt} - c^2 \Delta u - \beta \Delta u_t = 0. \quad (12)$$

Assume that  $0 < \tau < \beta$ . Let  $V = (u_{tt} + \tau u_t, \nabla(u + \tau u_t), \nabla u_t)$  and assume in addition that  $V_0 \in L^1(\mathbb{R}^N) \cap H^s(\mathbb{R}^N)$ . Then, for all  $0 \leq j \leq s$ , we have

$$\|\nabla^j V(t)\|_{L^2} \leq c(1+t)^{(-N/4-j/2)} \|V_0\|_{L^1} + c e^{-ct} \|\nabla^j V_0\|_{L^2}. \quad (13)$$

Also, differentiating (12) with respect to  $t$  and following the same steps as in the proof of ([17], Theorem 5.5), we have the following result.

**Proposition 2.5.** Let  $0 \leq \tau \leq \beta$  and  $v_0, v_1, v_2 \in L^1(\mathbb{R}^N) \cap H^s(\mathbb{R}^N)$ . Also, let  $(v_1, v_2) \in L^{1,1}(\mathbb{R}^N)$  with  $\int_{\mathbb{R}^N} v_i(x) dx = 0, i = 1, 2$ . Then, for  $0 \leq j \leq s$ , the following estimate

$$\begin{aligned} \|\nabla^j V(t)\|_{L^2} &\leq c \left( \|v_0\|_{L^1} + \|v_1\|_{L^{1,1}} + \|v_2\|_{L^{1,1}} \right) (1+t)^{(-N/4-j/2)} \\ &\leq c \left( \|\nabla^j v_0\|_{L^2} + \|\nabla^j v_1\|_{L^2} + \|\nabla^j v_2\|_{L^2} \right) e^{-ct}. \end{aligned} \quad (14)$$

Here  $v_1 = w_0$  and  $v_2 = u_{tt}(t=0)$ .

### 3. General Decay

In this section, we state and prove our decay result for the energy of the system (1) using the multiplier technique.

First, we rewrite the right-hand side of equation (1) in the form

$$\frac{\partial}{\partial t} \left( \frac{1}{c^2} \frac{B}{2A} (u_t)^2 + |\nabla u|^2 \right) = \frac{1}{c^2} \frac{B}{A} u_{tt} u_t + 2 \nabla u_t \nabla u,$$

and introduce the new variables

$$\varphi = u_t, \quad \psi = u_{tt}.$$

Without loss of generality, we assume from now on

$$c = 1.$$

Then equation (1) can be rewritten as the following first order system

$$\begin{cases} u_t = \varphi \\ \varphi_t = \psi \\ \tau \psi_t = -\psi + \Delta u + \beta \Delta \varphi + \int_0^t h(t-s) \Delta u(s) ds + \frac{B}{A} \psi \varphi + 2 \nabla \varphi \nabla u. \end{cases} \quad (15)$$

We are now ready to prove the main result.

**Theorem 3.1.** *Let  $n > 1$  and  $s > n/2 + 1$ . Assume (2)-(4) and  $u_0, u_1, u_2 \in H^s(\mathbb{R}^N)$  hold. Then, there exists a small positive constant  $\alpha$ , such that if  $\Theta_s(0) \leq \alpha$ , then the local solution  $u$  to (1) given in Theorem 4.1 exists globally in time.*

The proof of Theorem 3.1 will be given through several lemmas.

#### 3.1. First order energy estimates

**Lemma 3.2.** *The energy functional  $E_1$ , defined by*

$$\begin{aligned} E_1(t) &= \frac{1}{2} \int_{\mathbb{R}^n} \left( |\varphi + \tau \psi|^2 + \tau(\beta - \tau) |\nabla \varphi|^2 + |\nabla(u + \tau \varphi)|^2 \right) dx \\ &\quad + \frac{1}{2} \left\{ (h - \tau h') \circ \nabla u + (\tau h(t) - G(t)) \int_{\mathbb{R}^n} |\nabla u|^2 dx \right\} \\ &\quad + \tau \int_{\mathbb{R}^n} \nabla \varphi \int_0^t h(t-s) \nabla u(s) ds dx, \end{aligned} \quad (16)$$

satisfies

$$\begin{aligned} E_1'(t) &= \frac{1}{2} (h' - \tau h'') \circ \nabla u - \frac{1}{2} (h(t) - \tau h'(t)) \int_{\mathbb{R}^n} |\nabla u|^2 dx \\ &\quad - (\beta - \tau) \int_{\mathbb{R}^n} |\nabla \varphi|^2 dx + R_1, \end{aligned} \quad (17)$$

where

$$R_1 = \int_{\mathbb{R}^n} \left( \frac{B}{A} \psi \varphi + 2 \nabla \varphi \nabla u \right) (\varphi + \tau \psi) dx, \quad (18)$$

and

$$(h \circ \nabla u)(t) = \int_0^t h(t-s) \|\nabla u(t) - \nabla u(s)\|_{L^2(\mathbb{R}^n)}^2 ds.$$

*Proof.* Summing up the equation (15)<sub>1</sub> and (15)<sub>2</sub>, we get

$$\begin{aligned}
 (\varphi + \tau\psi)_t &= \Delta u + \beta\Delta\varphi + \int_0^t h(t-s)\Delta u(s)ds \\
 &\quad + \frac{B}{A}\psi\varphi + 2\nabla\varphi\nabla u.
 \end{aligned} \tag{19}$$

Multiplying (19) by  $(\varphi + \tau\psi)$  and integrating by parts over  $\mathbb{R}^n$ , we obtain

$$\begin{aligned}
 \int_{\mathbb{R}^n} (\varphi + \tau\psi)_t(\varphi + \tau\psi)dx &= \int_{\mathbb{R}^n} \Delta u(\varphi + \tau\psi)dx + \beta \int_{\mathbb{R}^n} \Delta\varphi(\varphi + \tau\psi)dx \\
 &\quad + \int_{\mathbb{R}^n} (\varphi + \tau\psi) \int_0^t h(t-s)\Delta u(s)dsdx \\
 &\quad + \int_{\mathbb{R}^n} \left(\frac{B}{A}\psi\varphi + 2\nabla\varphi\nabla u\right)(\varphi + \tau\psi)dx,
 \end{aligned} \tag{20}$$

we have

$$\begin{aligned}
 \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^n} |\varphi + \tau\psi|^2 dx &= - \int_{\mathbb{R}^n} \nabla u(\nabla\varphi + \tau\nabla\psi)dx - \beta \int_{\mathbb{R}^n} \nabla\varphi(\nabla\varphi + \tau\nabla\psi)dx \\
 &\quad + \int_{\mathbb{R}^n} (\varphi + \tau\psi) \int_0^t h(t-s)\Delta u(s)dsdx \\
 &\quad + \int_{\mathbb{R}^n} \left(\frac{B}{A}\psi\varphi + 2\nabla\varphi\nabla u\right)(\varphi + \tau\psi)dx,
 \end{aligned} \tag{21}$$

and

$$\frac{1}{2} \tau(\beta - \tau) \frac{d}{dt} \int_{\mathbb{R}^n} |\nabla\varphi|^2 dx = \tau(\beta - \tau) \int_{\mathbb{R}^n} \nabla\psi\nabla\varphi dx, \tag{22}$$

and we have

$$\begin{aligned}
 \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^n} |\nabla(u + \tau\varphi)|^2 dx &= \tau \int_{\mathbb{R}^n} \nabla\psi\nabla u dx + \tau^2 \int_{\mathbb{R}^n} \nabla\varphi\nabla\varphi dx \\
 &\quad + \int_{\mathbb{R}^n} \nabla\varphi\nabla u dx + \tau \int_{\mathbb{R}^n} |\nabla\varphi|^2 dx,
 \end{aligned} \tag{23}$$

and

$$\begin{aligned}
 \int_{\mathbb{R}^n} (\varphi + \tau\psi) \int_0^t h(t-s)\Delta u(s)dsdx &= \underbrace{\int_{\mathbb{R}^n} \varphi \int_0^t h(t-s)\Delta u(s)dsdx}_{I_1} \\
 &\quad + \tau \underbrace{\int_{\mathbb{R}^n} \psi \int_0^t h(t-s)\Delta u(s)dsdx}_{I_2}.
 \end{aligned} \tag{24}$$

Now estimating  $I_1, I_2$ , we have

$$\begin{aligned}
 I_1 &= \int_{\mathbb{R}^n} \varphi \int_0^t h(t-s) \Delta u(s) ds dx \\
 &= \int_{\mathbb{R}^n} u_t \int_0^t h(t-s) \Delta u(s) ds dx \\
 &= -\frac{1}{2} \frac{d}{dt} \left\{ h \circ \nabla u - G(t) \int_{\mathbb{R}^n} |\nabla u|^2 dx \right\} \\
 &\quad + \frac{1}{2} h' \circ \nabla u - \frac{1}{2} h(t) \int_{\mathbb{R}^n} |\nabla u|^2 dx,
 \end{aligned} \tag{25}$$

and

$$\begin{aligned}
 I_2 &= \tau \int_{\mathbb{R}^n} \psi \int_0^t h(t-s) \Delta u(s) ds dx \\
 &= \tau \int_{\mathbb{R}^n} u_{tt} \int_0^t h(t-s) \Delta u(s) ds dx \\
 &= -\frac{1}{2} \frac{d}{dt} \left\{ \tau h' \circ \nabla u + \tau h(t) \int_{\mathbb{R}^n} |\nabla u|^2 dx - 2\tau \int_{\mathbb{R}^n} \varphi \int_0^t h(t-s) \Delta u(s) ds dx \right\} \\
 &\quad - \frac{1}{2} \tau h'' \circ \nabla u + \frac{1}{2} \tau h'(t) \int_{\mathbb{R}^n} |\nabla u|^2 dx.
 \end{aligned} \tag{26}$$

By replacement of (22)-(26) into (21), we get (17).  $\square$

Now we denote the functional

$$\begin{aligned}
 F_1(t) &= \int_{\mathbb{R}^n} \left( |\varphi + \tau\psi|^2 + |\nabla\varphi|^2 + |\nabla(u + \tau\varphi)|^2 + |\nabla u|^2 \right) dx - h' \circ \nabla u \\
 &= \|\varphi + \tau\psi\|_{L^2}^2 + \|\nabla(u + \tau\varphi)\|_{L^2}^2 + \|\nabla\varphi\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 + \|\nabla u\|_h^2,
 \end{aligned} \tag{27}$$

where

$$\|\nabla u\|_h^2 = - \int_0^t h'(t-s) \|\nabla u(t) - \nabla u(s)\|_{L^2}^2 ds, \quad t \geq 0$$

**Lemma 3.3.**

$$E_1(t) \sim F_1(t). \tag{28}$$

*Proof.* From (16), we have

$$\begin{aligned}
2E_1(t) &= \int_{\mathbb{R}^n} \left( |\varphi + \tau\psi|^2 + \tau(\beta - \tau)|\nabla\varphi|^2 + |\nabla(u + \tau\varphi)|^2 \right) dx \\
&\quad + \left\{ (h - \tau h') \circ \nabla u + (\tau h(t) - G(t)) \int_{\mathbb{R}^n} |\nabla u|^2 dx \right\} \\
&\quad + \tau \int_{\mathbb{R}^n} \nabla\varphi \int_0^t h(t-s)\nabla u(s) ds dx. \\
&= \int_{\mathbb{R}^n} |\varphi + \tau\psi|^2 dx + \tau(\beta - \tau) \int_{\mathbb{R}^n} |\nabla\varphi|^2 dx \\
&\quad + \tau^2 \int_{\mathbb{R}^n} |\nabla\varphi|^2 dx + 2\tau(1 - G(t)) \int_{\mathbb{R}^n} \nabla\varphi \nabla u dx \\
&\quad + (1 - G(t)) \int_{\mathbb{R}^n} |\nabla u|^2 dx + \left\{ (h - \tau h') \circ \nabla u + \tau h(t) \int_{\mathbb{R}^n} |\nabla u|^2 dx \right\} \\
&\quad + \tau \int_{\mathbb{R}^n} \nabla\varphi \int_0^t h(t-s)(\nabla u(t) - \nabla u(s)) ds dx. \\
&= \int_{\mathbb{R}^n} |\varphi + \tau\psi|^2 dx + \tau(\beta - \tau) \int_{\mathbb{R}^n} |\nabla\varphi|^2 dx \\
&\quad + (1 - G(t)) \left( \tau^2 \int_{\mathbb{R}^n} |\nabla\varphi|^2 dx + 2\tau \int_{\mathbb{R}^n} \nabla\varphi \nabla u dx + \int_{\mathbb{R}^n} |\nabla u|^2 dx \right) \\
&\quad + \left\{ (h - \tau h') \circ \nabla u + \tau h(t) \int_{\mathbb{R}^n} |\nabla u|^2 dx \right\} + \tau^2 G(t) \int_{\mathbb{R}^n} |\nabla\varphi|^2 dx \\
&\quad + \tau \int_{\mathbb{R}^n} \nabla\varphi \int_0^t h(t-s)(\nabla u(t) - \nabla u(s)) ds dx. \tag{29}
\end{aligned}$$

Since

$$\begin{aligned}
\tau \left| \int_{\mathbb{R}^n} \nabla\varphi \int_0^t h(t-s)(\nabla u(t) - \nabla u(s)) ds dx \right| &\leq \tau \delta h \circ \nabla u \\
&\quad + \frac{\tau G(t)}{2\delta} \int_{\mathbb{R}^n} |\nabla\varphi|^2 dx. \tag{30}
\end{aligned}$$

By setting  $\delta = \frac{1}{\tau}$ , we get

$$\tau \left| \int_{\mathbb{R}^n} \nabla\varphi \int_0^t h(t-s)(\nabla u(t) - \nabla u(s)) ds dx \right| \leq h \circ \nabla u + \tau^2 G(t) \int_{\mathbb{R}^n} |\nabla\varphi|^2 dx. \tag{31}$$

By (29) and (31), we obtain

$$\begin{aligned}
2E_1(t) &\geq \int_{\mathbb{R}^n} \left( (1 - G(t))|\nabla(u + \tau\varphi)|^2 + |\varphi + \tau\psi|^2 + \tau(\beta - \tau)|\nabla\varphi|^2 \right) dx \\
&\quad - \tau h' \circ \nabla u + \tau h(t) \int_{\mathbb{R}^n} |\nabla u|^2 dx. \tag{32}
\end{aligned}$$

Recall (2) and (4), we have

$$E_1(t) \geq cF_1(t). \tag{33}$$

On the other hand, by (3) it is easy to see from (29)

$$\begin{aligned} E_1(t) &\leq c \left\{ \int_{\mathbb{R}^n} \left( |\nabla(u + \tau\varphi)|^2 dx + |\varphi + \tau\psi|^2 dx + \tau(\beta - \tau)|\nabla\varphi|^2 + |\nabla u|^2 \right) dx \right. \\ &\quad \left. - h' \circ \nabla u \right\} \\ &\leq cF_1(t). \end{aligned} \quad (34)$$

Because of the boundedness of  $h(t)$ ,  $G(t)$ , and assumptions (2)-(4). Hence, by (33) and (34), we obtain

$$E_1(t) \sim F_1(t). \quad (35)$$

□

Next, we define the energy of second order.

**Lemma 3.4.** *The energy functional  $E_2$ , defined by*

$$\begin{aligned} E_2(t) &= \frac{1}{2} \int_{\mathbb{R}^n} \left( |\nabla(\varphi + \tau\psi)|^2 + \tau(\beta - \tau)|\Delta\varphi|^2 + |\Delta(u + \tau\varphi)|^2 \right) dx \\ &\quad + \frac{1}{2} \left\{ (h - \tau h') \circ \Delta u + (\tau h(t) - G(t)) \int_{\mathbb{R}^n} |\Delta u|^2 dx \right\} \\ &\quad + \tau \int_{\mathbb{R}^n} \Delta\varphi \int_0^t h(t-s)\Delta u(s) ds dx, \end{aligned} \quad (36)$$

satisfies

$$\begin{aligned} E_2'(t) &= \frac{1}{2} (h' - \tau h'') \circ \Delta u - \frac{1}{2} (h(t) - \tau h'(t)) \int_{\mathbb{R}^n} |\Delta u|^2 dx \\ &\quad - (\beta - \tau) \int_{\mathbb{R}^n} |\Delta\varphi|^2 dx + R_2, \end{aligned} \quad (37)$$

where

$$R_2 = - \int_{\mathbb{R}^n} \left( \frac{B}{A} \psi\varphi + 2\nabla\varphi\nabla u \right) \Delta(\varphi + \tau\psi). \quad (38)$$

Similarly to  $E_1$ , we have the following lemma.

**Lemma 3.5.**

$$E_2(t) \sim F_2(t), \quad (39)$$

where

$$\begin{aligned} F_2(t) &= \int_{\mathbb{R}^n} \left( |\nabla(\varphi + \tau\psi)|^2 + |\Delta\varphi|^2 + |\Delta(u + \tau\varphi)|^2 + |\Delta u|^2 \right) dx - h' \circ \Delta u \\ &= \|\nabla(\varphi + \tau\psi)\|_{L^2}^2 + \|\Delta(u + \tau\varphi)\|_{L^2}^2 + \|\Delta\varphi\|_{L^2}^2 + \|\Delta u\|_{L^2}^2 + \|\Delta u\|_h^2. \end{aligned} \quad (40)$$

We need the following lemmas.



**Lemma 3.6.** *The functional*

$$D_1(t) := \int_{\mathbb{R}^n} \nabla(u + \tau\varphi)\nabla(\varphi + \tau\psi)dx,$$

satisfies, for any  $\varepsilon_1 > 0$

$$\begin{aligned} D'_1(t) \leq & -(1 - 3\varepsilon_1) \int_{\mathbb{R}^n} |\Delta(u + \tau\varphi)|^2 dx + \int_{\mathbb{R}^n} |\nabla(\varphi + \tau\psi)|^2 dx \\ & + c(1 + \frac{1}{\varepsilon_1}) \int_{\mathbb{R}^n} |\Delta\varphi|^2 dx + \frac{c}{\varepsilon_1} \int_{\mathbb{R}^n} |\Delta u|^2 dx + \frac{c}{\varepsilon_1} h \circ \Delta u + |R_3|, \end{aligned} \tag{41}$$

with

$$R_3 = \int_{\mathbb{R}^n} \Delta(u + \tau\varphi) \left( \frac{B}{A} \psi\varphi + 2\nabla\varphi\nabla u \right) dx. \tag{42}$$

*Proof.* By differentiating  $D_1$ , then using (15), integration by parts, we obtain

$$\begin{aligned} D'_1(t) &= \int_{\mathbb{R}^n} |\nabla(\varphi + \tau\psi)|^2 dx + \int_{\mathbb{R}^n} \nabla(u + \tau\varphi)\nabla(\varphi + \tau\psi)_t dx \\ &= \int_{\mathbb{R}^n} |\nabla(\varphi + \tau\psi)|^2 dx - \int_{\mathbb{R}^n} \Delta(u + \tau\varphi)(\varphi + \tau\psi)_t dx \\ &= \int_{\mathbb{R}^n} |\nabla(\varphi + \tau\psi)|^2 dx - \int_{\mathbb{R}^n} \Delta(u + \tau\varphi)\Delta u dx - \beta \int_{\mathbb{R}^n} \Delta(u + \tau\varphi)\Delta\varphi dx \\ &\quad - \int_{\mathbb{R}^n} \Delta(u + \tau\varphi) \int_0^t h(t-s)\Delta u(s) ds dx \\ &\quad + \int_{\mathbb{R}^n} \Delta(u + \tau\varphi) \left( \frac{B}{A} \psi\varphi + 2\nabla\varphi\nabla u \right) dx, \end{aligned} \tag{43}$$

we have

$$\begin{aligned} - \int_{\mathbb{R}^n} \Delta(u + \tau\varphi)(\Delta u + \Delta\varphi) dx &= - \int_{\mathbb{R}^n} |\Delta(u + \tau\varphi)|^2 dx \\ &\quad - (\beta - \tau) \int_{\mathbb{R}^n} \Delta(u + \tau\varphi)\Delta\varphi dx. \end{aligned} \tag{44}$$

Using Young's, for  $\varepsilon_1 > 0$ , and the replacement of (44) into (43), yields (41).  $\square$

**Lemma 3.7.** *The functional*

$$D_2(t) := -\tau \int_{\mathbb{R}^n} \nabla\varphi\nabla(\varphi + \tau\psi)dx,$$

satisfies, for any  $\varepsilon_2, \varepsilon_3 > 0$

$$\begin{aligned} D'_2(t) \leq & -(1 - \varepsilon_2) \int_{\mathbb{R}^n} |\nabla(\varphi + \tau\psi)|^2 dx + 2\varepsilon_3 \int_{\mathbb{R}^n} |\Delta(u + \tau\varphi)|^2 dx \\ & + c(1 + \frac{1}{\varepsilon_2}) \int_{\mathbb{R}^n} |\nabla\varphi|^2 dx + c(1 + \frac{1}{\varepsilon_3}) \int_{\mathbb{R}^n} |\Delta\varphi|^2 dx \\ & + c \int_{\mathbb{R}^n} |\Delta u|^2 dx + ch \circ \Delta u + |R_4|, \end{aligned} \tag{45}$$

with

$$R_4 = \tau \int_{\mathbb{R}^n} \Delta\varphi \left( \frac{B}{A} \psi\varphi + 2\nabla\varphi\nabla u \right) dx. \tag{46}$$

*Proof.* Direct computations give

$$D'_2(t) = \tau \int_{\mathbb{R}^n} \psi \Delta(\varphi + \tau\psi) dx + \tau \int_{\mathbb{R}^n} \Delta\varphi(\varphi + \tau\psi)_t dx.$$

Multiplying equation (15)<sub>3</sub> by  $\Delta\varphi$ , we get.

$$\begin{aligned} \tau \int_{\mathbb{R}^n} \Delta\varphi(\varphi + \tau\psi)_t dx &= \tau \int_{\mathbb{R}^n} \Delta\varphi(\Delta u + \beta\Delta\varphi) dx \\ &+ \tau \int_{\mathbb{R}^n} \Delta\varphi \int_0^t h(t-s)\Delta u(s) ds dx \\ &+ \tau \int_{\mathbb{R}^n} \Delta\varphi \left( \frac{B}{A} \psi\varphi + 2\nabla\varphi\nabla u \right) dx, \end{aligned}$$

and we have

$$\begin{aligned} \tau \int_{\mathbb{R}^n} \Delta\varphi(\Delta u + \beta\Delta\varphi) dx &= \int_{\mathbb{R}^n} \Delta\varphi \left( \tau\Delta u + \beta\tau\Delta\varphi + \tau^2\Delta\varphi - \tau^2\Delta\varphi \right. \\ &\left. + (\varphi + \tau\psi) - (\varphi + \tau\psi) \right) dx, \end{aligned}$$

we get

$$\begin{aligned} D'_2(t) &= - \int_{\mathbb{R}^n} |\nabla(\varphi + \tau\psi)|^2 dx + \tau(\beta - \tau) \int_{\mathbb{R}^n} |\nabla\varphi|^2 dx \\ &+ \tau \int_{\mathbb{R}^n} \Delta(u + \tau\varphi)\Delta\varphi dx + \int_{\mathbb{R}^n} \nabla(\varphi + \tau\psi)\nabla\varphi dx \\ &+ \tau \int_{\mathbb{R}^n} \Delta\varphi \int_0^t h(t-s)\Delta u(s) ds dx \\ &+ \tau \int_{\mathbb{R}^n} \Delta\varphi \left( \frac{B}{A} \psi\varphi + 2\nabla\varphi\nabla u \right) dx. \end{aligned}$$

Using Young's inequality, we obtain (45) for all  $\varepsilon_2, \varepsilon_3 > 0$ .  $\square$

*Proof.* We define a Lyapunov functional

$$\mathcal{L}(t) := N(E_1(t) + E_2(t)) + D_1(t) + N_2 D_2(t), \quad (47)$$

where  $N$  and  $N_2$  are positive constants to be selected later.

By differentiating (82) and using (16), (69), (73), (77), (41) and (45), we have

$$\begin{aligned} \mathcal{L}'(t) &\leq - \left[ N(\beta - \tau) - cN_2 \left( 1 + \frac{1}{\varepsilon_2} \right) \right] \|\nabla\varphi\|_{L^2}^2 \\ &- \left[ N(\beta - \tau) - cN_2 \left( 1 + \frac{1}{\varepsilon_3} \right) - c \left( 1 + \frac{1}{\varepsilon_1} \right) \right] \|\Delta\varphi\|^2 \\ &- \left[ (1 - 3\varepsilon_1) - N_2\varepsilon_3 \right] \|\Delta(u + \tau\varphi)\|^2 \\ &- \left[ N_2(1 - \varepsilon_2) - 1 \right] \|\nabla(u + \tau\varphi)\|^2 \\ &+ \left[ \frac{c}{\varepsilon_1} + cN_2 \right] h \circ \Delta u \\ &+ \frac{N}{2} \left( (h' - \tau h'') \circ \nabla u + (h' - \tau h'') \circ \Delta u \right) \\ &+ N(|R_1| + |R_2|) + |R_3| + N_2|R_4|. \end{aligned}$$

We pick  $\varepsilon_1$  and  $\varepsilon_2$  small enough such that

$$\varepsilon_1 < \frac{1}{3}, \quad \varepsilon_2 < 1.$$

After that, we take  $N_2$  large enough such that

$$N_2(1 - \varepsilon_2) - 1 > 0.$$

Once  $N_2$  and  $\varepsilon_1$  are fixed, we select  $\varepsilon_3$  small enough such that

$$\varepsilon_3 < \frac{(1 - 3\varepsilon_1)}{N_2},$$

thus, we arrive at

$$\begin{aligned} \mathcal{L}'(t) \leq & -\gamma_1(\|\nabla(u + \tau\varphi)\|^2 + \|\Delta(u + \tau\varphi)\|^2) \\ & -[(\beta - \tau)N - c](\|\nabla\varphi\|_{L^2}^2 + \|\Delta\varphi\|_{L^2}^2) \\ & + ch \circ \Delta u + \frac{N}{2}((h' - \tau h'') \circ \nabla u + (h' - \tau h'') \circ \Delta u) \\ & + N(|R_1| + |R_2|) + |R_3| + N_2|R_4|. \end{aligned} \tag{48}$$

Where  $\gamma_1 > 0$ . On the other hand, if we let

$$\mathfrak{Q}(t) = D_1(t) + N_2 D_2(t),$$

then

$$|\mathfrak{Q}(t)| \leq \int_{\mathbb{R}^n} |\nabla(u + \tau\varphi)\nabla(\varphi + \tau\psi)|dx + N_2\tau \int_{\mathbb{R}^n} |\nabla\varphi\nabla(\varphi + \tau\psi)|dx.$$

Exploiting (73),(77), Young’s and Poincaré inequalities, we obtain

$$|\mathfrak{Q}(t)| \leq c(F_1(t) + F_2(t)).$$

Consequently, we obtain

$$|\mathfrak{Q}(t)| = |\mathcal{L}(t) - N(F_1(t) + F_2(t))| \leq c(F_1(t) + F_2(t)),$$

that is

$$(N - c)(F_1(t) + F_2(t)) \leq \mathcal{L}(t) \leq (N + c)(F_1(t) + F_2(t)). \tag{49}$$

Now, by choosing  $N$  large enough and (4) such that

$$N - c > 0, N(\beta - \tau) - c > 0,$$

and exploiting (16),(69), estimates (48) and (49), respectively and (2), gives

$$\mathcal{L}'(t) \leq -k_1(F_1(t) + F_2(t)) + k_2 h \circ \Delta u + c \sum_{i=1}^{i=4} |R_i|, \forall t \geq 0, \tag{50}$$

and

$$c_2(F_1(t) + F_2(t)) \leq \mathcal{L}(t) \leq c_3(F_1(t) + F_2(t)), \forall t > 0, \tag{51}$$

for some  $k_1, k_2, c_2, c_3 > 0$ .

By multiplying (50) by  $\vartheta(t)$ , we obtain

$$\begin{aligned} \vartheta(t) \mathcal{L}'(t) &\leq -k_1 \vartheta(t) (F_1(t) + F_2(t)) + k_2 \vartheta(t) h \circ \Delta u \\ &\quad + c \vartheta(t) \sum_{i=1}^{i=4} |R_i|, \forall t \geq 0. \end{aligned} \tag{52}$$

Using (2)-(4), we have

$$\begin{aligned} \vartheta(t) h \circ \Delta u &\leq - \int_{\mathbb{R}^n} \int_0^t h'(t-s) (\Delta u(t) - \Delta u(s))^2 ds dx \\ &= -h' \circ \Delta u \\ &\leq -(h' - \tau h'') \circ \Delta u \\ &\leq -2F_2'(t) + 2|R_2|. \end{aligned}$$

Thus, (52) and by  $\vartheta$  is decreasing function, we get

$$\begin{aligned} \vartheta(t) \mathcal{L}'(t) &\leq -k_1 \vartheta(t) (F_1(t) + F_2(t)) - 2k_2 F_2'(t) \\ &\quad + c \sum_{i=1}^{i=4} |R_i|, \forall t \geq 0, \end{aligned}$$

which can be rewritten as

$$\begin{aligned} (\vartheta(t) \mathcal{L}(t) + 2k_2 F_2(t))' - \vartheta'(t) \mathcal{L}(t) &\leq -k_1 \vartheta(t) (F_1(t) + F_2(t)) \\ &\quad + c \sum_{i=1}^{i=4} |R_i|, \end{aligned}$$

using the fact that  $\vartheta'(t) \leq 0, \forall t \geq 0$ , we have

$$(\vartheta(t) \mathcal{L}(t) + 2k_2 F_2(t))' \leq -k_1 \vartheta(t) (F_1(t) + F_2(t)) + c \sum_{i=1}^{i=4} |R_i|.$$

By exploiting (51), we notice that

$$\mathcal{R}(t) = \vartheta(t) \mathcal{L}(t) + 2k_2 F_2(t) \sim (F_1(t) + F_2(t)). \tag{53}$$

Consequently, for some positive constant  $\lambda$ , we obtain

$$\mathcal{R}'(t) \leq -\lambda \mathcal{R}(t) \vartheta(t) + c \sum_{i=1}^{i=4} |R_i|. \tag{54}$$

Integrating (55) over  $(0, t)$ , we get

$$\mathcal{R}(t) + \int_0^t \lambda \mathcal{R}(\sigma) \vartheta(\sigma) d\sigma \leq \mathcal{R}(0) + c \sum_{i=1}^{i=4} \int_0^t |R_i| d\sigma. \tag{55}$$

Now we estimate the remaining terms  $|R_i|$ ,  $i = 1, \dots, 4$  in (55). First we have

$$\begin{aligned}
 |R_1| &= \left| \int_{\mathbb{R}^n} \left( \frac{B}{A} \psi \varphi + 2\nabla \varphi \nabla u \right) (\varphi + \tau \psi) dx \right| \\
 &= \left| \int_{\mathbb{R}^n} \left( \frac{B}{A\tau} \varphi (\varphi + \tau \psi - \varphi) + 2\nabla \varphi \nabla (u + \tau \varphi - \tau \varphi) \right) (\varphi + \tau \psi) dx \right| \\
 &\leq c \left( \|\varphi\|_{L^\infty} \|\varphi\|_{L^2} + \|\varphi\|_{L^\infty} \|(\tau \psi + \varphi)\|_{L^2} \right) \|(\tau \psi + \varphi)\|_{L^2} \\
 &\quad + c \left( \|\nabla \varphi\|_{L^\infty} \|\nabla (u + \tau \varphi)\|_{L^2} + \|\nabla \varphi\|_{L^\infty} \|\nabla \varphi\|_{L^2} \right) \|(\tau \psi + \varphi)\|_{L^2}.
 \end{aligned} \tag{56}$$

Using integration by parts, we have

$$\begin{aligned}
 |R_2| &= - \int_{\mathbb{R}^n} \left( \frac{B}{A} \psi \varphi + 2\nabla \varphi \nabla u \right) \Delta (\varphi + \tau \psi) dx \\
 &= \int_{\mathbb{R}^n} \nabla \left( \frac{B}{A\tau} \varphi (\varphi + \tau \psi - \varphi) + 2\nabla \varphi \nabla (u + \tau \varphi - \tau \varphi) \right) \nabla (\varphi + \tau \psi) dx \\
 &= \int_{\mathbb{R}^n} \left( \frac{B}{A\tau} \varphi \nabla (\varphi + \tau \psi) + \frac{B}{A\tau} \nabla \varphi (\varphi + \tau \psi) + \nabla |\varphi|^2 \right) \nabla (\varphi + \tau \psi) dx \\
 &\quad + \int_{\mathbb{R}^n} \left( 2H(u + \tau \varphi) \varphi \nabla + 2H(\varphi) (u + \tau \varphi) \nabla \varphi \right. \\
 &\quad \left. - 4\tau H(\varphi) \nabla \varphi \right) \nabla (\varphi + \tau \psi) dx,
 \end{aligned} \tag{57}$$

where  $H(f)(\partial_{x_i} \partial_{x_j} f)$ , is the Hessian matrix of  $f$ .

Using the fact that  $\|H(f)\| = \|\Delta f\|_{L^2}$ , and by Holder's inequality, we get

$$\begin{aligned}
 |R_2| &\leq c \left\{ \|\varphi\|_{L^\infty} \left( \|\nabla (u + \tau \varphi)\|_{L^2} + \|\nabla \varphi\|_{L^2} \right) \right. \\
 &\quad \left. + \|(\tau \psi + \varphi)\|_{L^2} + \|\nabla \varphi\|_{L^\infty} \right\} \|\nabla (\tau \psi + \varphi)\|_{L^2} \\
 &\quad + c \left\{ \|\nabla \varphi\|_{L^\infty} \left( \|\Delta (u + \tau \varphi)\|_{L^2} + \|\Delta \varphi\|_{L^2} \right) \right. \\
 &\quad \left. + \|\nabla (\tau \psi + \varphi)\|_{L^2} + \|\Delta \varphi\|_{L^2} \right\} \|\nabla (\tau \psi + \varphi)\|_{L^2}.
 \end{aligned} \tag{58}$$

Similarly, we have

$$\begin{aligned}
 |R_3| &\leq c \left\{ \|\varphi\|_{L^\infty} \left( \|\nabla (u + \tau \varphi)\|_{L^2} + \|\nabla \varphi\|_{L^2} \right) \right. \\
 &\quad \left. + \|(\tau \psi + \varphi)\|_{L^2} + \|\nabla \varphi\|_{L^\infty} \right\} \|\nabla (\tau \psi + \varphi)\|_{L^2} \\
 &\quad + c \left\{ \|\nabla \varphi\|_{L^\infty} \left( \|\Delta (u + \tau \varphi)\|_{L^2} + \|\Delta \varphi\|_{L^2} \right) \right. \\
 &\quad \left. + \|\nabla (\tau \psi + \varphi)\|_{L^2} + \|\Delta \varphi\|_{L^2} \right\} \|\nabla (\tau \psi + \varphi)\|_{L^2}.
 \end{aligned} \tag{59}$$

For  $R_4$ , we have, as in  $R_2$

$$\begin{aligned}
 |R_4| \leq & c \left\{ \|\varphi\|_{L^\infty} \left( \|\nabla(u + \tau\varphi)\|_{L^2} + \|\nabla\varphi\|_{L^2} \right) \right. \\
 & \left. + \|(\tau\psi + \varphi)\|_{L^2} + \|\nabla\varphi\|_{L^\infty} \right\} \|\nabla(\tau\psi + \varphi)\|_{L^2} \\
 & + c \left\{ \|\nabla\varphi\|_{L^\infty} \left( \|\Delta(u + \tau\varphi)\|_{L^2} + \|\Delta\varphi\|_{L^2} \right) \right. \\
 & \left. + \|\nabla(\tau\psi + \varphi)\|_{L^2} + \|\Delta\varphi\|_{L^2} \right\} \|\nabla(\tau\psi + \varphi)\|_{L^2}.
 \end{aligned} \tag{60}$$

Collecting the above estimates, we have

$$\begin{aligned}
 & \mathcal{R}(t) + \int_0^t \lambda \mathcal{R}(\sigma) \vartheta(\sigma) d\sigma \\
 & \leq \mathcal{R}(0) + c \int_0^t \left( \|\varphi(s)\|_{L^\infty} \|\varphi(s)\|_{L^2} \|(\varphi + \tau\psi)(s)\|_{L^2} \right) ds \\
 & \quad + c \int_0^t \left( \|\nabla\varphi(s)\|_{L^\infty} + \|\varphi(s)\|_{L^\infty} + \|\nabla(\varphi + \tau\psi)(s)\|_{L^\infty} \right) \mathcal{R}(s) ds
 \end{aligned} \tag{61}$$

Now, multiplying the equation (15)<sub>2</sub> by  $\varphi$  and integrating over  $\mathbb{R}^n$ , we get

$$\begin{aligned}
 \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^n} |\varphi|^2 dx &= \int_{\mathbb{R}^n} \psi \varphi dx \\
 &= \int_{\mathbb{R}^n} \frac{1}{\tau} \varphi (\tau\psi + \varphi - \varphi) dx \\
 &\leq c \|\varphi(t)\|_{L^2} \|(\tau\psi + \varphi)(t)\|_{L^2}.
 \end{aligned} \tag{62}$$

Integrating (62) with respect to  $t$ , we get

$$\|\varphi(t)\|_{L^2}^2 \leq \|\varphi(0)\|_{L^2}^2 + c \int_0^t \|\varphi(s)\|_{L^2} \|(\tau\psi + \varphi)(s)\|_{L^2} ds. \tag{63}$$

Collecting (61) and (63), we get

$$\begin{aligned}
 & \mathcal{K}(t) + \int_0^t \lambda \mathcal{R}(\sigma) \vartheta(\sigma) d\sigma \\
 & \leq \mathcal{R}(0) + c \int_0^t \left( \|\varphi(s)\|_{L^\infty} \|\varphi(s)\|_{L^2} \|(\varphi + \tau\psi)(s)\|_{L^2} \right) ds \\
 & \quad + c \int_0^t \left( \|\nabla\varphi(s)\|_{L^\infty} + \|\varphi(s)\|_{L^\infty} + \|\nabla(\varphi + \tau\psi)(s)\|_{L^\infty} \right) \mathcal{R}(s) ds \\
 & \quad + c \int_0^t \|\varphi(s)\|_{L^2} \|(\tau\psi + \varphi)(s)\|_{L^2} ds,
 \end{aligned} \tag{64}$$

where

$$\mathcal{K}(t) = \mathcal{R}(t) + \|\varphi(t)\|_{L^2}^2. \tag{65}$$

From (64), we get

$$\begin{aligned} & \mathcal{K}(t) + \int_0^t \lambda \mathcal{R}(\sigma) \vartheta(\sigma) d\sigma \\ & \leq \mathcal{K}(0) + c \int_0^t \left(1 + \|\varphi(s)\|_{W^{1,\infty}} + \|\nabla u(s)\|_{L^\infty}\right) \mathcal{K}(s) ds. \end{aligned} \tag{66}$$

By using Gronwall’s inequality, we get

$$\mathcal{K}(t) \leq \mathcal{K}(0) e^{\int_0^t (\lambda + c(\|\varphi(s)\|_{W^{1,\infty}} + \|\nabla u(s)\|_{L^\infty})) ds}. \tag{67}$$

□

### 3.2. Higher-order energy estimates

Applying the operator  $\nabla^k, k \geq 1$ , into (15), we get for  $U = \nabla^k u, V = \nabla^k \varphi$ , and  $W = \nabla^k \psi$

$$\begin{cases} U_t = V \\ V_t = W \\ \tau W_t = -W + \Delta U + \beta \Delta V + \int_0^t h(t-s) \Delta U(s) ds \\ \quad + \frac{B}{A} [\nabla^k, \varphi] \psi + \frac{B}{A} \varphi W + 2[\nabla^k, \nabla u] \nabla \varphi + 2 \nabla u \nabla V, \end{cases} \tag{68}$$

where  $[A, B] = AB - BA$ .

**Lemma 3.8.** *The energy functional of order  $k$ , defined by*

$$\begin{aligned} E_1^{(k)}(t) &= \frac{1}{2} \int_{\mathbb{R}^n} \left( |\nabla^k \varphi + \tau \nabla^k \psi|^2 + \tau(\beta - \tau) |\nabla^{k+1} \varphi|^2 + |\nabla^{k+1}(u + \tau \varphi)|^2 \right) dx \\ &\quad + \frac{1}{2} \left\{ (h - \tau h') \circ \nabla^{k+1} u + (\tau h(t) - G(t)) \int_{\mathbb{R}^n} |\nabla^{k+1} u|^2 dx \right\} \\ &\quad + \tau \int_{\mathbb{R}^n} \nabla^{k+1} \varphi \int_0^t h(t-s) \nabla^{k+1} u(s) ds dx \\ &= \frac{1}{2} \int_{\mathbb{R}^n} \left( |V + \tau W|^2 + \tau(\beta - \tau) |\nabla V|^2 + |\nabla(U + \tau V)|^2 \right) dx \\ &\quad + \frac{1}{2} \left\{ (h - \tau h') \circ \nabla U + (\tau h(t) - G(t)) \int_{\mathbb{R}^n} |\nabla U|^2 dx \right\} \\ &\quad + \tau \int_{\mathbb{R}^n} \nabla V \int_0^t h(t-s) \nabla U(s) ds dx, . \end{aligned}$$

satisfies

$$\begin{aligned} \frac{d}{dt} E_1^{(k)}(t) &= \frac{1}{2} (h' - \tau h'') \circ \nabla U - \frac{1}{2} (h(t) - \tau h'(t)) \int_{\mathbb{R}^n} |\nabla U|^2 dx \\ &\quad - (\beta - \tau) \int_{\mathbb{R}^n} |\nabla V|^2 dx + \widehat{R}_1^{(k)}, \end{aligned} \tag{69}$$

where

$$\widehat{R}_1^{(k)} = \int_{\mathbb{R}^n} R_1^{(k)} (V + \tau W) dx, \tag{70}$$

and

$$R_1^{(k)} = \left( \frac{B}{A} [\nabla^k, \varphi] \psi + \frac{B}{A} \varphi W + 2[\nabla^k, \nabla u] \nabla \varphi + 2\nabla u \nabla V \right). \quad (71)$$

Now we denote the functional of order  $k$

$$F_1^{(k)}(t) = \int_{\mathbb{R}^n} \left( |V + \tau W|^2 + |\nabla V|^2 + |\nabla(U + \tau V)|^2 + |\nabla U|^2 \right) dx - h' \circ \nabla U. \quad (72)$$

**Lemma 3.9.**

$$E_1^{(k)}(t) \sim F_1^{(k)}(t). \quad (73)$$

Similarly to (20), we define the energy of second order

**Lemma 3.10.** *The energy functional  $E_2^{(k)}$ , defined by*

$$\begin{aligned} E_2^{(k)}(t) &= \frac{1}{2} \int_{\mathbb{R}^n} \left( |\nabla(V + \tau W)|^2 + \tau(\beta - \tau) |\Delta V|^2 + |\Delta(U + \tau V)|^2 \right) dx \\ &\quad + \frac{1}{2} \left\{ (h - \tau h') \circ \Delta U + (\tau h(t) - G(t)) \int_{\mathbb{R}^n} |\Delta U|^2 dx \right\} \\ &\quad + \tau \int_{\mathbb{R}^n} \Delta V \int_0^t h(t-s) \Delta U(s) ds dx, \end{aligned} \quad (74)$$

satisfies

$$\begin{aligned} \frac{d}{dt} E_2(t) &= \frac{1}{2} (h' - \tau h'') \circ \Delta U - \frac{1}{2} (h(t) - \tau h'(t)) \int_{\mathbb{R}^n} |\Delta U|^2 dx \\ &\quad - (\beta - \tau) \int_{\mathbb{R}^n} |\Delta V|^2 dx + \widehat{R}_2^{(k)}, \end{aligned} \quad (75)$$

where

$$\widehat{R}_2^{(k)} = - \int_{\mathbb{R}^n} R_1^{(k)} \Delta(V + \tau W). \quad (76)$$

Similarly to  $E_1^{(k)}(t)$ , we have the following lemma

**Lemma 3.11.**

$$E_2^{(k)}(t) \sim F_2^{(k)}(t), \quad (77)$$

where

$$F_2^{(k)}(t) = \int_{\mathbb{R}^n} \left( |\nabla(V + \tau W)|^2 + |\Delta V|^2 + |\Delta(U + \tau V)|^2 + |\Delta U|^2 \right) dx - h' \circ \Delta U.$$

Then, we have the following lemmas, where its proof can be done following the same strategy as in first part.

**Lemma 3.12.** *The functional*

$$D_1^{(k)}(t) := \int_{\mathbb{R}^n} \nabla(U + \tau V) \nabla(V + \tau W) dx,$$



satisfies, for any  $\varepsilon_1 > 0$

$$\begin{aligned} \frac{d}{dt} D_1^{(k)}(t) &\leq -(1 - 3\varepsilon_1) \int_{\mathbb{R}^n} |\Delta(U + \tau V)|^2 dx + \int_{\mathbb{R}^n} |\nabla(V + \tau W)|^2 dx \\ &\quad + c(1 + \frac{1}{\varepsilon_1}) \int_{\mathbb{R}^n} |\Delta V|^2 dx + \frac{c}{\varepsilon_1} \int_{\mathbb{R}^n} |\Delta U|^2 dx + \frac{c}{\varepsilon_1} h \circ \Delta U + |\widehat{R}_3^{(k)}|, \end{aligned} \quad (78)$$

with

$$\widehat{R}_3^{(k)} = \int_{\mathbb{R}^n} \Delta(U + \tau V) \widehat{R}_1^{(k)} dx. \quad (79)$$

**Lemma 3.13.** *The functional*

$$D_2^{(k)}(t) := -\tau \int_{\mathbb{R}^n} \nabla V \nabla(V + \tau W) dx,$$

satisfies, for any  $\varepsilon_2, \varepsilon_3 > 0$

$$\begin{aligned} \frac{d}{dt} D_2^{(k)}(t) &\leq -(1 - \varepsilon_2) \int_{\mathbb{R}^n} |\nabla(V + \tau W)|^2 dx + 2\varepsilon_3 \int_{\mathbb{R}^n} |\Delta(U + \tau V)|^2 dx \\ &\quad + c(1 + \frac{1}{\varepsilon_2}) \int_{\mathbb{R}^n} |\nabla V|^2 dx + c(1 + \frac{1}{\varepsilon_3}) \int_{\mathbb{R}^n} |\Delta V|^2 dx \\ &\quad + c \int_{\mathbb{R}^n} |\Delta U|^2 dx + ch \circ \Delta U + |\widehat{R}_4^{(k)}|, \end{aligned} \quad (80)$$

with

$$\widehat{R}_4^{(k)} = \tau \int_{\mathbb{R}^n} \widehat{R}_1^{(k)} \Delta V dx. \quad (81)$$

We define a Lyapunov functional

$$\mathcal{L}^{(k)}(t) := N_1'^{(k)}(t) + E_2^{(k)}(t) + D_1^{(k)}(t) + N_2'D_2^{(k)}(t), \quad (82)$$

where  $N'$  and  $N_2'$  are positive constants, and we proceed exactly as in the first stage to get the following estimate,

$$\mathcal{R}^{(k)}(t) + \int_0^t \lambda_1 \mathcal{R}^{(k)}(\sigma) \vartheta(\sigma) d\sigma \leq \mathcal{R}^{(k)}(0) + c \sum_{i=1}^{i=4} \int_0^t |\widehat{R}_i^{(k)}| d\sigma. \quad (83)$$

Now we estimate the remaining terms  $|\widehat{R}_i^{(k)}|, i = 1, \dots, 4$  in (83). First we have by using Holder's inequality

$$\begin{aligned} |\widehat{R}_1^{(k)}| &\leq \|R_1^{(k)}\|_{L^2} \|(V + \tau W)\|_{L^2} \\ |\widehat{R}_2^{(k)}| &\leq \|\nabla R_1^{(k)}\|_{L^2} \|\nabla(V + \tau W)\|_{L^2} \\ |\widehat{R}_3^{(k)}| &\leq \|\nabla R_1^{(k)}\|_{L^2} \|\nabla(U + \tau V)\|_{L^2} \\ |\widehat{R}_4^{(k)}| &\leq \tau \|\nabla R_1^{(k)}\|_{L^2} \|\nabla V\|_{L^2} \end{aligned} \quad (84)$$

**Lemma 3.14.** [18] For  $k > 0$ , it hold that

$$\begin{aligned} \|R_1^{(k)}\|_{L^2} &\leq c\Gamma(t)\left(\|W\|_{L^2} + \|V\|_{H^1} + \|\nabla(U + \tau V)\|_{L^2}\right) \\ \|\nabla R_1^{(k)}\|_{L^2} &\leq c\Gamma(t)\left(\|\nabla V\|_{L^2} + \|\nabla W\|_{L^2} + \|\Delta V\|_{L^2} + \|\Delta(U + \tau V)\|_{L^2}\right), \end{aligned} \tag{85}$$

where

$$\Gamma(t) = \|\nabla v\|_{W^{1,\infty}} + \|W\|_{L^\infty} + \|\nabla u\|_{L^\infty}. \tag{86}$$

*Proof.* From (71), we have

$$\begin{aligned} \|R_1^{(k)}\|_{L^2} &\leq c\left(\|[\nabla^{(k)}, v]w\|_{L^2} + \|v\|_{L^\infty}\|W\|_{L^2}\right) \\ &\quad + c\left(\|[\nabla^{(k)}, \nabla u]\nabla v\|_{L^2} + \|\nabla u\|_{L^\infty}\|\nabla v\|_{L^2}\right). \end{aligned} \tag{87}$$

Now applying the commutator estimate (2.1), we get

$$\begin{aligned} \|[\nabla^{(k)}, v]w\|_{L^2} &\leq c\left(\|v\|_{L^\infty}\|\nabla^{(k)}w\|_{L^\infty} + \|w\|_{L^\infty}\|\nabla^{(k)}v\|_{L^2}\right) \\ \|[\nabla^{(k)}, \nabla u]\nabla v\|_{L^2} &\leq c\left(\|\nabla v\|_{L^\infty}\|\nabla^{(k+1)}u\|_{L^2} + \|\nabla u\|_{L^\infty}\|\nabla^{(k+1)}v\|_{L^\infty}\right) \\ &\leq c\left(\|\nabla v\|_{L^\infty}\left\{\|\nabla^{(k+1)}(u + \tau v)\|_{L^2} + \|\nabla^{(k+1)}v\|_{L^2}\right\}\right. \\ &\quad \left. + \|\nabla u\|_{L^\infty}\|\nabla^{(k+1)}v\|_{L^\infty}\right). \end{aligned} \tag{88}$$

Plugging the last two estimates into (87), then (85)<sub>1</sub> holds.

Now, to get (85)<sub>2</sub>, we have

$$\nabla R_1^{(k)} = \nabla^{k+1}\left(\frac{B}{A}\varphi\psi + 2\nabla u\nabla\varphi\right). \tag{89}$$

Thus, applying (5), we get

$$\begin{aligned} \|\nabla R_1^{(k)}\|_{L^2} &\leq c\left(\|w\|_{L^\infty}\|\nabla^{(k+1)}v\|_{L^2} + \|v\|_{L^\infty}\|\nabla^{(k+1)}w\|_{L^2}\right) \\ &\quad + c\left(\|\nabla u\|_{L^\infty}\|\nabla^{(k+2)}v\|_{L^2} + \|\nabla v\|_{L^\infty}\|\nabla^{(k+2)}u\|_{L^\infty}\right). \end{aligned} \tag{90}$$

Now, we estimate the last terms in of the above formula as

$$\begin{aligned} &\|\nabla u\|_{L^\infty}\|\nabla^{(k+2)}v\|_{L^2} + \|\nabla v\|_{L^\infty}\|\nabla^{(k+2)}u\|_{L^\infty} \\ &\leq c\left(\|\nabla u\|_{L^\infty}\|\Delta\nabla^{(k)}v\|_{L^2} + \|\nabla v\|_{L^\infty}\|\Delta\nabla^{(k)}u\|_{L^\infty}\right) \\ &\leq c\left(\|\nabla u\|_{L^\infty}\|\Delta\nabla^{(k)}v\|_{L^2} + \|\nabla v\|_{L^\infty}\left(\|\Delta\nabla^{(k)}(u + \tau\varphi)\|_{L^\infty} + \|\Delta\nabla^{(k)}\varphi\|_{L^\infty}\right)\right). \end{aligned} \tag{91}$$

Inserting the above estimates into (90), we get (85)<sub>2</sub>. Now, taking (85) into account, then (83) yields

$$\mathcal{R}^{(k)}(t) + \int_0^t \lambda_1 \mathcal{R}^{(k)}(\sigma) \vartheta(\sigma) d\sigma \leq \mathcal{R}^{(k)}(0) + c \int_0^t \Gamma(s)\left(\Psi_1^{(k)}(s) + \Psi_2^{(k)}(s)\right) ds, \tag{92}$$

where

$$\begin{aligned} \Psi_1^{(k)}(t) &:= \left( \|W(t)\|_{L^2} + \|V(t)\|_{H^2} + \|\nabla(U + \tau V)\|_{L^2} \right) \|\nabla(V + \tau W)\|_{L^2} \\ \Psi_1^{(k)}(t) &:= \left\{ \left( \|\nabla V(t)\|_{L^2} + \|\Delta \nabla W(t)\|_{L^2} + \|\Delta V(t)\|_{L^2} + \|\Delta(U + \tau w(t))\|_{L^2} \right) \right. \\ &\quad \left. \times \left( \|\nabla(U + \tau V)(t)\|_{L^2} + \|\Delta \nabla V(t)\|_{L^2} \right) \right\}. \end{aligned} \tag{93}$$

Now, summing up (94) for  $k$  with  $1 \leq k \leq s - 1$  and adding the result to (66), we get

$$\begin{aligned} \Theta_s(t) + \int_0^t \mathcal{Y}^{(s)}(\sigma) d\sigma &\leq \Theta_s(0) + c \int_0^t \Gamma(s) \left( \Psi_1^{(s)}(\sigma) + \Psi_2^{(s)}(\sigma) \right) d\sigma \\ &\quad + c \int_0^t (1 + \Gamma(\sigma)) \Theta_s(\sigma) d\sigma, \end{aligned} \tag{94}$$

where

$$\begin{aligned} \Theta_s(t) &= \|(\varphi + \tau \psi)(t)\|_{H^s}^2 + \|\Delta \varphi(t)\|_{H^{s-1}}^2 + \|\varphi(t)\|_{H^{s-1}}^2 \\ &\quad + \|\Delta(u + \tau \varphi)(t)\|_{H^{s-1}}^2 + \|\nabla(u + \tau \varphi)(t)\|_{H^{s-1}}^2 \\ &\quad + \|\Delta u(t)\|_{H^{s-1}}^2 + \|u(t)\|_{H^{s-1}}^2 + \|\nabla u\|_{h,s-1}^2 + \|\Delta u\|_{h,s-1}^2, \end{aligned} \tag{95}$$

and

$$\mathcal{Y}^{(s)}(t) = \lambda_1 \mathcal{R}^{(s)}(t) \vartheta(t) \tag{96}$$

$$\|\nabla u\|_{h,s-1}^2 = - \int_0^t h'(t - \sigma) \|\nabla u(t) - \nabla u(\sigma)\|_{H^{s-1}}^2 d\sigma.$$

Hence, (94) takes the form

$$\Theta_s(t) + \int_0^t \mathcal{Y}^{(s)}(\sigma) d\sigma \leq \Theta_s(0) + c \int_0^t (1 + \Gamma(\sigma)) \Theta_s(\sigma) d\sigma. \tag{97}$$

By Gronwall’s inequality, (97) gives

$$\Theta_s(t) \leq c \Theta_s(0) e^{\int_0^t (1 + \sup_{0 < t < T} \Gamma(t)) T ds}, \tag{98}$$

where  $T = T(\Theta_s(0))$  is the maximal time of the local existence. Hence, if we had that

$$\sup_{0 < t < T} \Gamma(t),$$

was bounded then our result holds. One way to show that the above quantity is bounded is by assuming smallness assumption on the initial data in some regular spaces and using the bootstrap argument. Indeed, we assume the following a priori assumption

$$\sup_{0 < t < T} \Theta_s(t) \leq \delta, \tag{99}$$

for a constant  $\delta > 0$ , small enough. Due to the embedding  $H^s(\mathbb{R}^n) \hookrightarrow W^{m,\infty}(\mathbb{R}^n)$ , for  $s_1 > \frac{n}{2}, s_2 > \frac{n}{2} - 1$  and  $s_3 > \frac{n}{2}$ , we have

$$\begin{aligned} \sup_{0 < t < T} \|\varphi(t)\|_{W^{1,\infty}} &\leq \sup_{0 < t < T} \|\varphi(t)\|_{H^{s_2}} \leq c\delta \\ \sup_{0 < t < T} \|\nabla u(t)\|_{L^\infty} &\leq \sup_{0 < t < T} \|\nabla u(t)\|_{H^{s_1}} \leq c\delta \\ \sup_{0 < t < T} \|\psi(t)\|_{L^\infty} &\leq \sup_{0 < t < T} \|\psi(t)\|_{H^{s_3}} \leq c\delta. \end{aligned} \tag{100}$$

Hence, from (98) and (100), we deduce that if (99) holds, (98) yields

$$\Theta_s(t) \leq c_1 \Theta_s(0) e^{(1+c_2\delta)T}, \tag{101}$$

with  $c_1, c_2 > 0$  not depending on  $T$  and  $\delta$ . Now, we fix  $\delta > 0$  and choose  $\alpha$  small enough such that

$$\Theta_s(0) \leq \alpha \leq \alpha_0 := \frac{\delta}{2c_1(0)e^{(1+c_2\delta)}} \leq \frac{\delta}{2}. \tag{102}$$

Then, there is  $T = T(\Theta_s(0)) = T(\alpha) = T(\delta)$ , where (99) holds and hence by (101), we obtain

$$\Theta_s(t) \leq \frac{\delta}{2}, \tag{103}$$

for all  $0 \leq t \leq T$ . In particular

$$\Theta_s(T) \leq \frac{\delta}{2} \leq \delta. \tag{104}$$

Consequently, the bootstrap argument together with the local existence implies that we can continue the local solution to  $T(\delta) + T(\delta) = 2T(\delta)$  and we get analogously, (101), now for  $0 \leq t \leq 2T = 2T(\delta)$ , then  $\Theta_s(2T) < \delta$  and so on, we can continue the solution to  $T = \infty$ .  $\square$

#### 4. A Locale Existence Theorem

In this section, using the contraction mapping theorem to show the following local existence theorem.

**Theorem 4.1.** *Let  $s > n/2 + 1$ . Let  $U_0 = (u_0, u_1, u_2)^T = (u_0, \varphi_0, \psi_0)^T$  be such that  $\Theta_s(0) \leq \widehat{\delta}_0$ , for some  $\widehat{\delta}_0 > 0$ . Assume (2)- (4) hold. Then, there exists a small time  $T(\Theta_s(0)) > 0$ , such that problem (1) has a unique solution  $u$  on  $[0, T) \times \mathbb{R}^n$  satisfying*

$$\sup_{0 \leq t \leq T} \Theta_s(t) + \int_0^T \mathcal{Y}^{(s)}(\sigma) d\sigma \leq C_{\widehat{\delta}_0} \tag{105}$$

where  $\Theta_s(t)$  and  $\mathcal{Y}^{(s)}(t)$  are given in (95) and (96), respectively, determining the regularity of  $u$ , and  $C_{\widehat{\delta}_0}$  is a positive constant depending on  $\widehat{\delta}_0$ .

*Proof.* First, let  $U(t) = (u, \varphi, \psi)^T = (u, u_t, u_{tt})^T, U_0 = (u_0, u_1, u_2)^T$  Therefore, problem (1) takes the form

$$\begin{cases} u_t = \varphi \\ \varphi_t = \psi \\ \tau \psi_t = -\psi + \Delta u + \beta \Delta \varphi + \int_0^t h(t-s) \Delta u(x, s) ds + \frac{B}{A} \psi \varphi + 2 \nabla \varphi \nabla u. \end{cases} \tag{106}$$

Then

$$\begin{cases} \frac{d}{dt}U = \mathcal{A}U(t) + \mathcal{B}(U, \nabla U)(t) \\ U(0) = U_0 \end{cases} \quad (107)$$

Here, the operator  $\mathcal{A}$  is given by

$$\mathcal{A} \begin{pmatrix} u \\ \varphi \\ \psi \end{pmatrix} = \begin{pmatrix} \varphi \\ \psi \\ \frac{1}{\tau}(\Delta u + \beta\varphi) + \frac{1}{\tau} \int_0^t h(t-s)\Delta u(s)ds - \frac{1}{\tau}\psi \end{pmatrix}, \quad (108)$$

and  $\mathcal{B}$  is the nonlinear term

$$\mathcal{B}(U, \nabla U) = \begin{pmatrix} 0 \\ 0 \\ \frac{1}{\tau} \left( \frac{B}{A} \psi\varphi + 2\nabla\varphi\nabla u \right) \end{pmatrix}. \quad (109)$$

If  $U$  is a smooth solution of (1), then

$$U(t) = \Phi(U)(t) = e^{t\mathcal{A}}U_0 + \int_0^t e^{t-r}\mathcal{B}(U, \nabla U)(r)dr. \quad (110)$$

We define for  $s > s_0 + n/2$ ,

$$\begin{aligned} \mathcal{X} = \{ & U = (u, \varphi, \psi, \phi) / \Delta u, \nabla u \in C^0([0, T], H^{s-1}), u \in W, \\ & \varphi \in C^0([0, T], H^{s+1}), \phi \in C^0([0, T], H^{s-1}([0, T], H^s)), \\ & \psi \in C^0([0, T], H^s), \|U\|_{\mathcal{X}} := \sup_{0 < t < T} \Theta_s(t) < \infty \}, \end{aligned} \quad (111)$$

where  $W$  is the completion of  $C_0^\infty$  under the seminorm  $\|\nabla \cdot\|_2$ .

We also define

$$\mathcal{Z} = \left\{ U / \|U\|_{\mathcal{Z}} := \int_0^t \mathcal{Y}^{(s)}(\sigma)d\sigma < \infty \right\}, \quad (112)$$

where  $\Theta_s(t)$  and  $\mathcal{Y}^{(s)}(t)$  are defined in (95) and (96), respectively. It is clear that

$$\|U\|_{\mathcal{Z}} \leq cT\|U\|_{\mathcal{X}},$$

for some  $c > 0$ . Define

$$\mathcal{F} = \left\{ \mathcal{X} \cap \mathcal{Z}, U(x, 0) = U_0(x) \right\}. \quad (113)$$

Hence, from the above computation, we deduce that  $\Phi(U)$  is well defined and it maps  $\mathcal{F}$  into  $\mathcal{X} \cap \mathcal{Z}$ . We define the ball  $D_R$  as

$$D_R = \left\{ U \in \mathcal{F}, \|U\|_{\mathcal{X} \cap \mathcal{Z}} < R \right\}. \quad (114)$$

It is clear that  $D_R$  is closed subset of the space  $\mathcal{F}$  and non-empty for all  $R > R_0$  with  $R_0^2 < \Theta_s(0)$ . Our goal is to show that:

- (1)  $\Phi$  maps the ball  $D_R$  into itself,
- (2)  $\Phi$  is a contraction in  $D_R$ .

As we will see properties (1) and (2) are valid for  $R$  large enough, depending on the initial data, and for  $T$

sufficiently small and its choice is given later. Once the properties (1)-(2) are verified, the application of the contraction mapping theorem gives the existence of a unique solution of (107).

We write

$$\Phi(U) = U^0 + \mathcal{J}(U), \quad (115)$$

where  $U^0 = e^{t\mathcal{A}}U_0$ ,  $\mathcal{J}(U) = \int_0^t e^{t-r}\mathcal{B}(U, \nabla U)(r)dr$ .

It is clear that  $U^0$  satisfies the linear equation

$$U_t^0 + \mathcal{A}U^0 = 0, \quad U^0(x) = U_0(x), \quad (116)$$

and  $\mathcal{J}(U)$  satisfies the nonlinear equation with zero initial data, that is

$$\partial_t \mathcal{J}(U) + \mathcal{A}\mathcal{J}(U) = \mathcal{B}(U, \nabla U), \quad \mathcal{J}(U)(x, 0) = 0. \quad (117)$$

As in the proof of Theorem 3.1, we have for all  $t \in [0, T]$ ;

$$\|U^0(t)\|_X^2 + \|U^0(t)\|_Z^2 \leq c\Theta_s(0). \quad (118)$$

Now, to bound  $\mathcal{J}$  in  $D_R$ , we have by applying once again the proof of Theorem 3.1, (especially the estimate (97)) for all  $0 \leq t \leq T$ ,

$$\|\mathcal{J}(U)(t)\|_X^2 + \|\mathcal{J}(U)(t)\|_Z^2 \leq c \int_0^t (1 + \Gamma(\sigma))\Theta_s(\sigma)d\sigma. \quad (119)$$

Now, using the Sobolev embedding  $H^s(\mathbb{R}^n) \hookrightarrow W^{1,1}(\mathbb{R}^n)$ , for  $s > 1 + n/2$ , we get

$$\Gamma(t) \leq c \sqrt{\Theta_s(t)} \leq c\|U(t)\|_X. \quad (120)$$

Hence, (119) yields

$$\begin{aligned} \|\mathcal{J}(U)(t)\|_X^2 + \|\mathcal{J}(U)(t)\|_Z^2 &\leq cT\|U(t)\|_X^2 + cT\|U(t)\|_X^{3/2} \\ &\leq cT(R^2 + R^{3/2}). \end{aligned} \quad (121)$$

Collecting (100) and (105), we obtain

$$\|\Phi(U)\|_{X \cap Z}^2 \leq c\Theta_s(0) + cT(R^2 + R^{3/2}). \quad (122)$$

Choosing  $R$  sufficiently large and  $T$  sufficiently small such that

$$c\Theta_s(0) + cT(R^2 + R^{3/2}) \leq R^2.$$

That is

$$T \leq \frac{R^2 - c\Theta_s(0)}{c(R^2 + R^{3/2})},$$

provided that  $R^2 > c\Theta_s(0)$ , then, we obtain

$$\|\Phi(U)\|_{X \cap Z}^2 \leq R,$$

hence, we have prove that  $\Phi(D_R) \subset D_R$ .

Now, we need to prove that  $\Phi$  is contractive. We have, as above for  $U$  and  $V$  in  $D_R$ ,  $\mathcal{J}(U)$  and  $\mathcal{J}(V)$  solve the equation, hence, we obtain (117).

$$\begin{aligned} \partial_t(\mathcal{J}(U) - \mathcal{J}(V)) + \mathcal{A}(\mathcal{J}(U) - \mathcal{J}(V)) &= \mathcal{B}(U, \nabla U) - \mathcal{B}(V, \nabla V) \\ \mathcal{J}(U)(x, 0) = \mathcal{J}(V)(x, 0) &= 0. \end{aligned} \quad (123)$$

We put  $\mathcal{W}(t) = \mathcal{J}(U(t)) - \mathcal{J}(V(t))$ . Then, we obtain from above

$$\begin{aligned} \partial_t \mathcal{W}(t) + \mathcal{A}\mathcal{W}(t) &= \mathcal{B}(U, \nabla U) - \mathcal{B}(V, \nabla V) \\ \mathcal{W}(U)(x, 0) &= 0. \end{aligned} \tag{124}$$

Let  $U = (u, \varphi, \psi)^T$  and  $V = (\widehat{u}, \widehat{\varphi}, \widehat{\psi})^T$ , then we have

$$\begin{aligned} &\mathcal{B}(U, \nabla U) \\ &= \begin{pmatrix} 0 \\ 0 \\ \frac{1}{\tau} \left( \frac{B}{A} (\psi\varphi - \widehat{\varphi}\widehat{\psi}) + 2(\nabla\varphi\nabla u - \nabla\widehat{\varphi}\nabla\widehat{u}) \right) \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ 0 \\ \frac{1}{\tau} \left( \frac{B}{A} ((\varphi - \widehat{\varphi})\psi + \varphi(\psi - \widehat{\psi})) + 2((\nabla\widehat{\varphi}(\nabla u - \nabla\widehat{u}) + \nabla u(\nabla\widehat{\varphi} - \nabla\varphi)) \right) \\ 0 \end{pmatrix}. \end{aligned} \tag{125}$$

Following the same steps as before, we obtain (instead of (94)), for all  $0 \leq t \leq T$ ,

$$\begin{aligned} &\Theta_s(\mathcal{W}) + \int_0^t \mathcal{Y}^{(s)}(\mathcal{W})(\sigma) d\sigma \\ &\leq c \int_0^t [\Gamma(U(\sigma)) + \Gamma(U(\sigma))] \sqrt{\Theta(\mathcal{W}(\sigma))} \sqrt{\Theta(U(\sigma) - V(\sigma))} d\sigma \\ &\quad + c \int_0^t (1 + \Gamma(U(\sigma)) + \Gamma(U(\sigma))) \sqrt{\Theta(\mathcal{W}(\sigma))} \sqrt{\Theta(U(\sigma) - V(\sigma))} d\sigma. \end{aligned} \tag{126}$$

Applying (120), we deduce that

$$\|\mathcal{W}\|_{X \cap Z} \leq cT \|U - V\|_{X} (1 + \|U\|_X + \|V\|_X),$$

this implies

$$\|\mathcal{J}(U - V)\|_{X \cap Z} \leq cT \|U - V\|_{X \cap Z} (1 + 2R).$$

Now, we fix  $T$  small enough such that  $cT(1 + 2R) = \kappa < 1$ . Hence, we deduce that

$$\|\Phi(U) - \Phi(V)\|_{X \cap Z} \leq \kappa \|U - V\|_{X \cap Z},$$

thus, we conclude that  $\Phi$  is a contraction in  $D_R$ . The application of the contraction mapping principle shows that there exists a unique solution  $U \in \mathcal{F}$  of (1). This finishes the proof of Theorem 4.1.  $\square$

### 5. Decay rate

In this section, we prove decay rates for solutions to (1). Let

$$V := (\varphi + \tau\psi, \nabla(u + \tau\varphi), \nabla\varphi, \nabla u),$$

be the global solution according to Theorem 3.1, with  $\varphi = u_t$  and  $\psi = u_{tt}$ . Let

$$\|(u, \varphi, \psi)\|_{\mathcal{H}} = \|\varphi + \tau\psi\|_{L^2}^2 + \|\nabla(u + \tau\varphi)\|_{L^2}^2 + \tau(\beta - \tau)\|\nabla\varphi\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 + \|\nabla u\|_{H^1}^2. \tag{127}$$

It is clear that the norm above is equivalent to the norm  $\|V\|_{L^2}^2$ .

**Theorem 5.1.** Assume that (2)- (4) and  $s > 1 + n/2$ . Let  $u$  be the global solution of (1). Let  $\varphi_0 = u_t(t = 0)$ ;  $\varphi_1 = u_{tt}(t = 0)$  and  $\varphi_2 = u_{ttt}(t = 0)$  satisfying  $\varphi_0, \varphi_1, \varphi_2 \in L^1(\mathbb{R}^n) \times H^s(\mathbb{R}^n)$  and  $\varphi_1, \varphi_2 \in L^{1,1}(\mathbb{R}^n)$ , with  $\int_{\mathbb{R}^n} \varphi_i(x)dx = 0, i = 1, 2$ . Assume that  $\|V_0\|_{L^1 \cap H^s}^2$  is small enough. Then, the following decay estimates hold:

$$\|\nabla^j V(t)\|_{L^2}^2 + \|\nabla^j u_t(t)\|_{L^2}^2 \leq \begin{cases} c(1+t)^{-n/4-j/2} & , \text{if } n > 2 \\ c(1+t)^{-1/4-j/2} \log(1+t) & , \text{if } n = 2 \end{cases} \tag{128}$$

for all  $0 \leq j \leq s$ .

*Proof.* First, inspired by the decay estimates of the linear problem (see Propositions 2.4 and 2.5), we define

$$\mathcal{M} := \sup_{0 \leq \sigma \leq t} \sum_{j=0}^s (1+\tau)^{n/4+j/2} (\|\nabla^j V(\sigma)\|_{L^2}^2 + \|\nabla^j \varphi(\sigma)\|_{L^2}^2). \tag{129}$$

We also define the quantities

$$\begin{aligned} M_0 &:= \sup_{0 \leq \sigma \leq t} (1+\sigma)^{\frac{n}{2}} (\|V(\sigma)\|_{L^\infty}^2 + \|\varphi(\sigma)\|_{L^\infty}^2) \\ M_1 &:= \sup_{0 \leq \sigma \leq t} (1+\sigma)^{\frac{n}{2}-\frac{1}{2}} (\|V(\sigma)\|_{L^\infty}^2). \end{aligned} \tag{130}$$

So, our goal is to show that  $\mathcal{M}(t)$  is bounded uniformly in  $t$  if  $\|V_0\|_{L^1 \cap H^s}^2 = \|V_0\|_{L^1}^2 + \|V_0\|_{H^s}^2$  is small enough. From (110), we have for  $U = (u, \varphi, \psi)$ , and for  $0 \leq j \leq s$ ,

$$\begin{aligned} \|\nabla^j U(t)\|_{\mathcal{H}} &\leq \|\nabla^j e^{t\mathcal{A}} U_0\|_{\mathcal{H}} + \int_0^t \|\nabla^j e^{(t-r)\mathcal{B}}(U, \nabla U)(r)\|_{\mathcal{H}} dr \\ &= \|\nabla^j e^{t\mathcal{A}} U_0\|_{\mathcal{H}} + \int_0^{t/2} \|\nabla^j e^{(t-r)\mathcal{B}}(U, \nabla U)(r)\|_{\mathcal{H}} dr \\ &\quad + \int_{t/2}^t \|\nabla^j e^{(t-r)\mathcal{B}}(U, \nabla U)(r)\|_{\mathcal{H}} dr \\ &= \|\nabla^j e^{t\mathcal{A}} U_0\|_{\mathcal{H}} + I_1 + I_2. \end{aligned} \tag{131}$$

This gives, by using the estimate (13),

$$\|\nabla^j e^{t\mathcal{A}} U_0\|_{\mathcal{H}} \leq c(1+t)^{-n/4-j/2} (\|V_0\|_{L^1(\mathbb{R}^n)}^2 + \|\nabla^j V_0\|_{L^2(\mathbb{R}^n)}^2).$$

Now, for  $I_1$ , we have (also using the estimate (13))

$$\begin{aligned} I_1 &\leq c \int_0^{t/2} (1+t-r)^{-n/4-j/2} \|\overline{\mathcal{B}}(U, \nabla U)(r)\|_{L^1(\mathbb{R}^n)} dr \\ &\quad + c \int_0^{t/2} e^{-(t-r)} \|\nabla^j \overline{\mathcal{B}}(U, \nabla U)(r)\|_{L^2(\mathbb{R}^n)} dr \\ &= I_{11} + I_{12}, \end{aligned} \tag{132}$$

where  $\overline{\mathcal{B}}(U, \nabla U)(t) = (\frac{B}{A} \varphi \psi + 2\nabla u \nabla \varphi, 0, 0)$ .

To estimate  $I_1$  it is convenient to divide the integral into two parts  $I_{11}$  and  $I_{12}$  corresponding to  $[0, t = 2]$  and  $[t = 2, t]$  and then estimate each term separately, ( Lemma 7.4 in [19]). First, we have by using Holder’s inequality,

$$\|\overline{\mathcal{B}}(U, \nabla U)(t)\|_{L^1(\mathbb{R}^n)} \leq \|V(t)\|_{L^2}^2 + \|\varphi(t)\|_{L^2}^2. \tag{133}$$



Hence,  $I_{11}$  can be estimated as follow

$$\begin{aligned}
 I_{11} &= \int_0^{t/2} (1+t-r)^{-n/4-j/2} \|\bar{\mathcal{B}}(\mathbf{U}, \nabla \mathbf{U})(r)\|_{L^1(\mathbb{R}^n)} dr \\
 &\leq c\mathcal{M}^2(t) \int_0^{t/2} (1+t-r)^{-n/4-j/2} (1+r)^{-n/2} dr \\
 &\leq c\mathcal{M}^2(t) \int_0^{t/2} (1+t)^{-n/4-j/2} (1+r)^{-n/2} dr \\
 &\leq c\mathcal{M}^2(t) (1+t)^{-n/4-j/2} \int_0^{t/2} (1+r)^{-n/2} dr \\
 &\leq \begin{cases} c\mathcal{M}^2(t) (1+t)^{-n/4-j/2}, & n > 2 \\ c\mathcal{M}^2(t) (1+t)^{-n/4-j/2} \log(1+t), & n = 2 \end{cases}
 \end{aligned}$$

On the other hand,

$$\begin{aligned}
 \|\nabla^j \bar{\mathcal{B}}(\mathbf{U}, \nabla \mathbf{U})\|_{L^2(\mathbb{R}^n)} &\leq c\|\nabla^j \varphi \psi\|_{L^2}^2 + \|\nabla^j (\nabla u \nabla \varphi)\|_{L^2}^2 \\
 &\leq c\left(\|\nabla^j \varphi(\varphi + \tau\psi)\|_{L^2}^2 + \|\nabla^j (\varphi^2)\|_{L^2}^2 + \|\nabla^j (\nabla u \nabla \varphi)\|_{L^2}^2\right),
 \end{aligned}
 \tag{134}$$

this gives, by applying (5),

$$\begin{aligned}
 \|\nabla^j \tilde{\mathcal{B}}(\mathbf{U}, \nabla \mathbf{U})(t)\|_{L^2(\mathbb{R}^n)} &\leq C\|\varphi\|_{L^\infty} \left(\|\nabla^j (\varphi + \tau\psi)\|_{L^2} + \|\nabla^j \varphi\|_{L^2}\right) \\
 &\quad + C\|\varphi + \tau\psi\|_{L^\infty} \|\nabla^j \varphi\|_{L^2} + C\|\nabla \varphi\|_{L^\infty} \|\nabla^j \nabla \varphi\|_{L^2} \\
 &\quad + C\|\nabla \varphi\|_{L^\infty} \|\nabla^j \nabla (u + \tau\varphi)\|_{L^2} \\
 &\quad + C\|\nabla (u + \tau\varphi)\|_{L^\infty} \|\nabla^j \nabla \varphi\|_{L^2} \\
 &\leq C(1+t)^{-n/2} (1+t)^{-n/4-j/2} M_0(t) \mathcal{M}(t) \\
 &\quad + C(1+t)^{-n/2-1/2} (1+t)^{-n/4-j/2} M_1(t) \mathcal{M}(t).
 \end{aligned}
 \tag{135}$$

Consequently, using these estimates, we deduce that

$$I_{12} \leq C(1+t)^{-3n/2-j/2} (M_0(t) + M_1(t)) \mathcal{M}(t).$$

Next,  $I_2$  is estimated by applying (13) with  $j = 1$  and using  $\nabla^{j-1} \tilde{\mathcal{B}}(\mathbf{U}, \nabla \mathbf{U})(t)$  instead of  $\mathbf{V}_0$ , to obtain, for  $j \geq 1$

$$\begin{aligned}
 I_2 &= \int_{t/2}^t \|\nabla e^{(t-r)\mathcal{A}} \nabla^{j-1} \tilde{\mathcal{B}}(\mathbf{U}, \nabla \mathbf{U})(r)\|_{L^2} dr \\
 &\leq C \int_{t/2}^t (1+t-r)^{-\frac{n}{4}-\frac{1}{2}} \|\nabla^{j-1} \tilde{\mathcal{B}}(\mathbf{U}, \nabla \mathbf{U})(r)\|_{L^1} dr \\
 &\quad + C \int_{t/2}^t e^{-c(t-r)} \|\nabla^j \tilde{\mathcal{B}}(\mathbf{U}, \nabla \mathbf{U})(r)\|_{L^2} dr \\
 &\equiv I_{21} + I_{22}.
 \end{aligned}
 \tag{136}$$

On the other hand, we have by applying (5),

$$\begin{aligned}
 &\|\nabla^{j-1} \tilde{\mathcal{B}}(\mathbf{U}, \nabla \mathbf{U})(t)\|_{L^1(\mathbb{R}^n)} \\
 &\leq C(\|\mathbf{V}(t)\|_{L^2} + \|\varphi(t)\|_{L^2}) \left(\|\nabla^{j-1} \mathbf{V}(t)\|_{L^2} + \|\nabla^{j-1} \varphi(t)\|_{L^2}\right) \\
 &\leq C\mathcal{M}^2(t) (1+t)^{-\frac{n}{2}-\frac{j-1}{2}}.
 \end{aligned}
 \tag{137}$$

Thus,

$$\begin{aligned}
 I_{21} &\leq C\mathcal{M}^2(t) \int_{t/2}^t (1+t-r)^{-\frac{n}{4}-\frac{1}{2}}(1+r)^{-\frac{n}{2}-\frac{2-1}{2}} dr \\
 &\leq C\mathcal{M}^2(t)(1+t/2)^{-\frac{n}{2}-\frac{j-1}{2}} \int_{t/2}^t (1+t-r)^{-\frac{n}{4}-\frac{1}{2}} dr \\
 &\leq (1+t/2)^{-\frac{n}{2}-\frac{j-1}{2}} \int_0^{t/2} (1+r)^{-\frac{n}{4}-\frac{1}{2}} dr \\
 &\leq C(1+t)^{-\frac{n}{2}-\frac{j-1}{2}} \begin{cases} (1+t)^{-n/4-1/2+1} + 1, & \text{if } n \neq 2 \\ \log(t+1), & \text{if } n = 2 \end{cases} \\
 &\leq C \begin{cases} (1+t)^{-\frac{n}{4}-\frac{1}{2}}, & \text{if } n > 2 \\ (1+t)^{-\frac{1}{2}-\frac{1}{2}} \log(t+1), & \text{if } n = 2 \end{cases} \tag{138}
 \end{aligned}$$

For  $I_{22}$ , we have as in the estimate of  $I_{12}$

$$I_{22} \leq C(1+t)^{-3n/2-j/2} (M_0(t) + M_1(t)) \mathcal{M}(t).$$

Therefore, collecting the above estimates, we have

$$\begin{aligned}
 &\|\nabla^j \mathbf{U}(t)\|_{\mathcal{H}} \\
 &\leq C(1+t)^{-n/4-j/2} (\|\mathbf{V}_0\|_{L^1(\mathbb{R}^n)} + \|\nabla^j \mathbf{V}_0\|_{L^2(\mathbb{R}^n)}) \\
 &\quad + C\mathcal{M}^2(t)(1+t)^{-n/4-j/2} + C(1+t)^{-n/4-j/2} (M_0(t) + M_1(t)) \mathcal{M}(t). \tag{139}
 \end{aligned}$$

This yields

$$\begin{aligned}
 \mathcal{M}(t) &\leq C (\|\mathbf{V}_0\|_{L^1(\mathbb{R}^n)} + \|\nabla^j \mathbf{V}_0\|_{L^2(\mathbb{R}^n)}) \\
 &\quad + C\mathcal{M}^2(t) + C (M_0(t) + M_1(t)) \mathcal{M}(t). \tag{140}
 \end{aligned}$$

Applying Lemma 2.3 with  $\alpha = \frac{n}{2m}, q = r = 2, j = 0$  and  $p = \infty$ , we get for  $m > \frac{n}{2}$

$$\|\mathbf{V}\|_{L^\infty} \leq C \|\nabla^m \mathbf{V}\|_{L^2}^{\frac{n}{2m}} \|\mathbf{V}\|_{L^2}^{1-\frac{n}{2m}},$$

and similar estimates can be used for  $\|\varphi\|_{L^\infty}$ . This yields

$$M_0(t) \leq C\mathcal{M}(t),$$

provided that  $s \geq m > \frac{n}{2}$ . Next, to estimate  $M_1(t)$ , we apply Lemma 2.3 with  $\alpha = \frac{n+2}{2m}, q = r = 2, j = 1$  and  $p = \infty$ , we get for  $m > \frac{n+2}{2}$

$$\|\nabla \mathbf{V}\|_{L^\infty} \leq C \|\nabla^m \mathbf{V}\|_{L^2}^{\frac{n+2}{2}} \|\mathbf{V}\|_{L^2}^{1-\frac{n+2}{2}}.$$

This leads to

$$M_1(t) \leq C\mathcal{M}(t),$$

provided that  $s \geq m > \frac{n}{2} + 1$ . Hence, since  $M_0(t) + M_1(t) \leq C\mathcal{M}(t)$ , then (140) implies that

$$\mathcal{M}(t) \leq C (\|\mathbf{V}_0\|_{L^1(\mathbb{R}^n)} + \|\nabla^j \mathbf{V}_0\|_{L^2(\mathbb{R}^n)}) + C\mathcal{M}^2(t).$$

Consequently, applying Lemma 2.2 gives the desired result, provided that  $\|\mathbf{V}_0\|_{L^1(\mathbb{R}^n)} + \|\nabla^j \mathbf{V}_0\|_{L^2(\mathbb{R}^n)}$  is small enough for all  $0 \leq j \leq s$ . This the proof of Theorem 5.1 is complete.  $\square$

**6. Uniqueness of Solution**

We prove the continuous dependence and uniqueness for solutions of problem (1). Let  $\Lambda$  and  $\chi$  two different solutions of problem (1) with respect to initial data  $(\Lambda_0, \Lambda_1, \Lambda_2)$  and  $(\chi_0, \chi_1, \chi_2)$  respectively. Then  $\Sigma(t) = \Lambda(t) - \chi(t)$  verifies (1), and we have

$$\tau \Sigma_{ttt} + \Sigma_{tt} - c^2 \Delta \Sigma - \beta \Delta \Sigma_t - \int_0^t h(t-s) \Delta \Sigma(s) ds = \frac{\partial}{\partial t} \left( \frac{1}{c^2} \frac{B}{2A} (\Sigma_t)^2 + |\nabla \Sigma|^2 \right). \tag{141}$$

Multiplying (141) by  $\Sigma_t$ , and by  $\tau \Sigma_{tt}$ , then collect the results, and integrating the result over  $\mathbb{R}^n$ , we get

$$\begin{aligned} \frac{d}{dt} E_1(t) &= \frac{1}{2} (h' - \tau h'') \circ \nabla \Sigma - \frac{1}{2} (h(t) - \tau h'(t)) \int_{\mathbb{R}^n} |\nabla \Sigma|^2 dx \\ &\quad - (\beta - \tau) \int_{\mathbb{R}^n} |\nabla \Sigma_t|^2 dx + R_1, \end{aligned} \tag{142}$$

where

$$\begin{aligned} E_1(t) &= \frac{1}{2} \int_{\mathbb{R}^n} \left( |\Sigma_t + \tau \Sigma_{tt}|^2 + \tau (\beta - \tau) |\nabla \Sigma_t|^2 + |\nabla (\Sigma + \tau \Sigma_t)|^2 \right) dx \\ &\quad + \frac{1}{2} \left\{ (h - \tau h') \circ \nabla \Sigma + (\tau h(t) - G(t)) \int_{\mathbb{R}^n} |\nabla \Sigma|^2 dx \right\} \\ &\quad + \tau \int_{\mathbb{R}^n} \nabla \Sigma_t \int_0^t h(t-s) \nabla \Sigma(s) ds dx \\ &\sim F_1(t) \\ &= \|\Sigma_t + \tau \Sigma_{tt}\|_{L^2}^2 + \|\nabla (\Sigma + \tau \Sigma_t)\|_{L^2}^2 + \|\nabla \Sigma_t\|_{L^2}^2 + \|\nabla \Sigma\|_{L^2}^2 + \|\nabla \Sigma\|_h^2, \end{aligned} \tag{143}$$

and

$$R_1 = \int_{\mathbb{R}^n} \left( \frac{B}{A} \Sigma_{tt} \Sigma_t + 2 \nabla \Sigma_t \nabla \Sigma \right) (\Sigma_t + \tau \Sigma_{tt}) dx. \tag{144}$$

Using (2)-(4), and From (142), we have

$$\begin{aligned} \frac{d}{dt} E_1(t) &\leq |R_1| \\ &\leq |R_1| + c \left( \|\Sigma_t + \tau \Sigma_{tt}\|_{L^2}^2 + \|\nabla (\Sigma + \tau \Sigma_t)\|_{L^2}^2 + \|\nabla \Sigma_t\|_{L^2}^2 \right. \\ &\quad \left. + \|\nabla \Sigma\|_{L^2}^2 + \|\nabla \Sigma\|_h^2 \right), \end{aligned} \tag{145}$$

by integrating (146), we get

$$\begin{aligned} E_1(t) &\leq E_1(0) + \int_0^t |R_1(s)| ds + c \int_0^t \left( \|\Sigma_t + \tau \Sigma_{tt}\|_{L^2}^2 + \|\nabla (\Sigma + \tau \Sigma_t)\|_{L^2}^2 \right. \\ &\quad \left. + \|\nabla \Sigma_t\|_{L^2}^2 + \|\nabla \Sigma\|_{L^2}^2 + \|\nabla \Sigma\|_h^2 \right) ds, \end{aligned} \tag{146}$$

collecting (56), (63) and (146), we obtain

$$\Xi(t) \leq \Xi(0) + c \int_0^t \left( 1 + \|\Sigma_t(s)\|_{W^{1,\infty}}^2 + \|\nabla \Sigma(s)\|_{L^\infty}^2 \right) \Xi(s) ds, \tag{147}$$

where

$$\Xi(t) = E_1(t) + \|\Sigma_t(t)\|_{L^2}^2 \sim F_1(t) + \|\Sigma_t(t)\|_{L^2}^2, \quad (148)$$

on the other hand, from (143) we have

$$\begin{aligned} \Xi(t) \geq c_0 & \left( \|\Sigma_t + \tau \Sigma_{tt}\|_{L^2}^2 + \|\nabla(\Sigma + \tau \Sigma_t)\|_{L^2}^2 \right. \\ & \left. + \|\nabla \Sigma_t\|_{L^2}^2 + \|\nabla \Sigma\|_{L^2}^2 + \|\nabla \Sigma\|_h^2 + \|\Sigma_t(t)\|_{L^2}^2 \right). \end{aligned} \quad (149)$$

Applying Gronwall's inequality to (147), we get

$$\begin{aligned} & \left( \|\Sigma_t + \tau \Sigma_{tt}\|_{L^2}^2 + \|\nabla(\Sigma + \tau \Sigma_t)\|_{L^2}^2 + \|\nabla \Sigma_t\|_{L^2}^2 + \|\nabla \Sigma\|_{L^2}^2 + \|\nabla \Sigma\|_h^2 + \|\Sigma_t(t)\|_{L^2}^2 \right) \\ & \leq \Xi(0) e^{c \int_0^t \left( 1 + \|\Sigma_t(s)\|_{W^{1,\infty}}^2 + \|\nabla \Sigma(s)\|_{L^\infty}^2 \right) ds}. \end{aligned} \quad (150)$$

This shows that solution of problem (1) depends continuously on the initial data. Similarly, in higher-order energy, we obtain the same results.

## 7. Conclusion

The aim of this paper is the investigation of the Cauchy problem of a third order in time nonlinear equation known as the Jordan-Moore-Gibson-Thompson (JMGT) equation with the presence of both memory. Using the well known energy method combined with Lyapunov functionals approach, where we have proved a general decay result. JMGT is a nonlinear partial differential equation that arises in hydrodynamics and some physical applications. Recent developments in numerical schemes for solving Moore-Gibson-Thompson have placed immense interest in nonlinear dispersive wave models. In the next work, we will try to apply the same technique with Boussinesque and Hall-MHD equations which are nonlinear partial differential equation that arises in hydrodynamics and some physical applications. It was subsequently applied to problems in the percolation of water in porous subsurface strata (see for example [1]-[5],[20],[21]).

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