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Soft Set-Valued Mappings and their Application in Decision Making Problems

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Abstract. In this paper, we introduce the notion of a set-valued mapping on soft classes and study several properties of images and inverse images of soft sets supported by examples and counterexamples. Finally, these notions have been applied in decision making problems.

1. Introduction

It is known that classical mathematics methods are inadequate in modeling the problem in cases of uncertainty and ambiguity. In order to overcome such situations, researchers have begun new searches and introduced new theories such as theory of probability, fuzzy set theory [27], intuitionistic fuzzy sets [5], vague sets [13], theory of interval mathematics [14], rough set theory [24], etc. to model uncertainty situations.

One of the most important of these theories is the theory of fuzzy sets introduced by Zadeh [27]. This theory tries to digitize the uncertainties in human thoughts and perceptions and offers concepts and methods that bring certainty to uncertain situations and eliminate problems in solution. On the other hand, since the definition of membership function required for a fuzzy set depends on the person defining the function, fuzzy set operations can be far from reality. The difficulty of defining this membership function causes the fuzzy set theory to be insufficient in some cases. Molodtsov [23], who argued that the reason for similar problems existing in also other theories is that the elements of the sets cannot be adequately parameterized, put forward soft set theory, which is a novel theory alternative to these set theories to model uncertainties. The absence of any limitation in defining objects in soft set theory, that is, choosing any number, word or phrase can be selected as a parameter, enables much more suitable models for real-life problems by minimizing information loss.

Therefore, researchers has shown great interest to this new theory and studied its applications in different disciplines such as decision-makings [20], Perron integration, Riemann-integration, smoothness of functions, Theory of Probability, Theory of Measurement, the smoothness of functions [23], Game Theory, Optimization Theory, Operations Research [23], algebraic structures [1, 3, 11, 15, 18] and topological structures [9, 25, 26, 28, 29].

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The first application of soft sets to decision making problems was done by Maji et al. [20]. Later, many researchers developed new decision making methods with the help of soft sets [4, 6–8, 10, 12, 16, 17]. One of the most important of these methods is the $uni - int$ decision making method put forward by Çağman and Enginoğlu [7]. This method aims to obtain a suitable subset of the set of alternatives according to the given parameters determined by the decision maker. Thus, the decision maker is provided to work on fewer alternatives rather than a large number of alternatives. However, it should be noted that there are some cases where the this decision making method could not work successfully. Because two soft sets in the same universe are needed to decide by the $uni - int$ method, which may not always happen. In other words, there may be parameters that are not directly related to the decision universe but affect the decision.

In this study, we first introduce the notion of set-valued mapping on soft set classes. We also define and study the properties of upper and lower images and upper and lower inverse images of soft sets, and support them with examples and counterexamples.Finally, these notions have been applied to a decision making problem in which the $uni - int$ decision making scheme cannot work successfully and this will be demonstrated on an example.

2. Preliminaries

Throughout the work, any universe of objects will be denoted by U, a set of parameters suitable for the elements in U will be denoted by E, and the power set of U will be also denoted by $P(U)$.

Definition 2.1. ([23]) The pair (F, E) is called a soft set on U where $F : E \to P(U)$ is a map.

Thus a soft set is a parameterized family of subsets of U and for each $e \in E$, the set $F(e)$ can be considered as the set of e-elements or e-approximations of the soft set (F, E)

According to Majumdar and Samanta [22], any (F, A) soft set can be extended to a soft set (F, E), where $F(e) \neq \emptyset$ when $e \in A$ and $F(e) = \emptyset$ when $e \in E \backslash A$. Based on this idea, Çağman and Enginoğlu [7] revised the algebraic operations of soft sets in [21] as follows. From now on, the soft set defined by a map F with $F(e) \neq \emptyset$ when $e \in A \subseteq E$ and $F(e) = \emptyset$ when $e \in E - A$ be denoted by F_A and this soft set will also be considered as the map $F_A : E \to P(U)$. Also, the family of all of soft sets on U will be denoted by $S(U, E)$.

Definition 2.2. ([7]) Let $F_A, F_B \in S(U, E)$. Then:

(1) if $F_A(e) \subseteq F_B(e)$ for all $e \in E$, then F_A is a soft subset of F_B, denoted by $F_A \widetilde{\subseteq} F_B$.

(2) union of F_A and F_B, denoted by $F_A \widetilde{\cup} F_B$, is a soft set defined by $(F_A \widetilde{\cup} F_B)(e) = F_A(e) \cup F_B(e)$ for all $e \in E$.

(3) intersection of F_A and F_B, denoted by $F_A \widetilde{\cap} F_B$, is a soft set defined by $(F_A \widetilde{\cap} F_B)(e) = F_A(e) \cap F_B(e)$ for all $e \in E$.

(4) if $F_A(e) = \emptyset$ for all $e \in E$, then F_A is called a empty soft set, denoted by $F_\emptyset$. $F_A(e) = \emptyset$ means that there is no element in U related to the parameter $e \in E$.

(5) if $F_A(e) = U$ for all $e \in E$, then F_A is called a universal soft set, denoted by F_E.

Definition 2.3. ([2]) Let $F_A \in S(U, E)$. Then complement of F_A, denoted by F_A^c, is a soft set defined by $F_A^c(e) = U - F_A(e)$ for all $e \in E$.

It is noted in [7] that $(F_A^c)^c = F_A$, $F_E^c = F_\emptyset$ and $F_\emptyset^c = F_E$.

Now let us express the $uni - int$ decision making method of Çağman and Enginoğlu [7]. For this, we will first give the necessary definitions.

Definition 2.4. ([7]) If $F_A, F_B \in S(E, U)$, then $\wedge$-product of soft sets F_A and F_B, denoted by $F_A \wedge F_B$, is a soft set defined by

$$F_A \wedge F_B : E \times E \longrightarrow P(U),\ (F_A \wedge F_B)(x, y) = F_A(x) \cap F_B(y)$$

Definition 2.5. ([7]) Let F_A, $F_B \in S(E,U)$ and let $\wedge(U)$ be the set of all $\wedge$-products of the soft sets over U. Then $uni-int$ operators for the $\wedge$-products, denoted by $uni_x int_y$ and $uni_y int_x$, are defined, respectively,

$$uni_x int_y : \wedge(U) \rightharpoonup P(U),\ uni_x int_y(F_A \wedge F_B) = \cup_{x\in A}(\cap_{y\in B}(F_A \wedge F_B)(x,y)))$$

$$uni_y int_x : \wedge(U) \rightharpoonup P(U),\ uni_y int_x(F_A \wedge F_B) = \cup_{y\in B}(\cap_{x\in A}(F_A \wedge F_B)(x,y)))$$

Each of them transforms the $\wedge$-product $F_A \wedge F_B$ into a subset of the universe U.

Definition 2.6. ([7]) Let $F_A \wedge F_B \in \wedge(U)$. Then uni–int decision function for the $\wedge$-products, denoted by $uni-int$, is defined by, $uni-int : \wedge(U) \rightharpoonup P(U)$

$$uni-int(F_A \wedge F_B) = uni_x int_y(F_A \wedge F_B) \cup uni_y int_x(F_A \wedge F_B)$$

that reduces the size of the universe U. Hence, the values $uni-int(F_A \wedge F_B)$ is a subset of U called uni–int decision set of $F_A \wedge F_B$.

For details, reference [7] can be examined. Now, let us give the algorithm of the $uni-int$ decision making method. According to the problem,
Step 1: Choose feasible subsets of the set of parameters,
Step 2: Construct the soft sets for each set of parameters,
Step 3: Find the $\wedge$-product of the soft sets,
Step 4: Compute the $uni-int$ decision set of the product.
Note that obtained uni–int decision set is not small enough to work on it, subset of the decision set can be reached by the method.

Definition 2.7. Let X and Y be two sets. An F relation that corresponds to each element of X with a non-null subset of Y, is called a set-valued mapping from X to Y and is denoted by $F : X \rightsquigarrow Y$. The subset corresponding to $x \in X$ is indicated by $F(x)$.

Definition 2.8. For a set set-valued mapping $F : X \rightsquigarrow Y$, the upper and lower inverse of any subset B of Y, denoted by $F^+(B)$ and $F^-(B)$ respectively, are the subsets $F^+(B) = \{x \in X : F(x) \subseteq B\}$ and $F^-(B) = \{x \in X : F(x) \cap B \neq \emptyset\}$. In particular, $F^-(y) = \{x \in X : y \in F(x)\}$ for each $y \in Y$, and the image of an $A \subseteq X$ under F is $F(A) = \cup\{F(x) : x \in A\}$.

Theorem 2.9. *Let X and Y be two sets and $F : X \rightsquigarrow Y$ be a set-valued mapping. Then $X - F^+(B) = F^-(Y - B)$ for each $B \subseteq Y$.*

3. Soft Set-Valued Mappings

In this section, a new mapping between two soft set families will be defined with help of set-valued mappings between classical sets. Then, an example of this mapping will be given and basic properties of it will be proven.

Definition 3.1. Let $\mathfrak{u} : U \rightsquigarrow V$ and $p : E \rightsquigarrow K$ be two set-valued mappings. Then a soft set-valued mapping $\mathfrak{u}_p : S(U,E) \rightsquigarrow S(V,K)$ is defined as below:

(1) Let $F_A \in S(U,E)$. The upper and lower images of F_A under $\mathfrak{u}_p$, denoted by $\mathfrak{u}_{p^+}(F_A)$ and $\mathfrak{u}_{p^-}(F_A)$ respectively, are defined as

$$\mathfrak{u}_{p^+}(F_A)(k) = \begin{cases} \bigcup\limits_{e\in p^+(k)} \mathfrak{u}(F_A(e)) & ; p^+(k) \neq \emptyset \\ \emptyset & ; p^+(k) = \emptyset \end{cases}$$

and

$$\mathfrak{u}_{p^-}(F_A)(k) = \begin{cases} \bigcup_{e\in p^-(k)} \mathfrak{u}(F_A(e)) & ; p^-(k) \neq \varnothing \\ \varnothing & ; p^-(k) = \varnothing \end{cases}$$

for all $k \in K$.

(2) Let $G_B \in S(V,K)$. The upper and lower invers images of G_B under $\mathfrak{u}_p$, written as $\mathfrak{u}_p^+(G_B)$ and $\mathfrak{u}_p^-(G_B)$ respectively, are defined as

$$\mathfrak{u}_p^+(G_B)(e) = \mathfrak{u}^+(\bigcup_{k\in p(e)} G_B(k))$$

and

$$\mathfrak{u}_p^-(G_B)(e) = \mathfrak{u}^-(\bigcup_{k\in p(e)} G_B(k))$$

for all $e \in E$.

Example 3.2. Let $E = \{e_1, e_2, e_3, e_4\}$, $K = \{k_1, k_2, k_3\}$, $U = \{u_1, u_2, u_3, u_4\}$ and $V = \{v_1, v_2, v_3, v_4\}$. Let $\mathfrak{u} : U \rightsquigarrow V$ and $p : E \rightsquigarrow K$ be set-valued mappings defined as $\mathfrak{u}(u_1) = \{v_1\}$, $\mathfrak{u}(u_2) = \{v_2, v_3\}$, $\mathfrak{u}(u_3) = \{v_4\}$, $\mathfrak{u}(u_4) = \{v_1, v_4\}$, $p(e_1) = \{k_1, k_2, k_3\}$, $p(e_2) = \{k_1, k_2\}$, $p(e_3) = \{k_2\}$, $p(e_4) = \{k_3\}$. Choose the soft set in $S(U,E)$ and $S(V,K)$,respectively, $F_A = \{(e_1, \{u_1, u_2\}), (e_2, \{u_1, u_3\}), (e_3, \{u_4\}), (e_4, \{u_2\})\}$ and $G_B = \{(k_1, \{v_1, v_3\}), (k_2, \{v_3, v_4\}), (k_3, \{v_2, v_4\})\}$. Then we have
$\mathfrak{u}_{p^+}(F_A)(k_1) = \bigcup_{e\in p^+(k_1)} \mathfrak{u}(F_A(e)) = \varnothing$, $\mathfrak{u}_{p^-}(F_A)(k_1) = \bigcup_{e\in p^-(k_1)} \mathfrak{u}(F_A(e)) = \mathfrak{u}(F_A(e_1)) \cup \mathfrak{u}(F_A(e_2)) = \mathfrak{u}(\{u_1, u_2\}) \cup \mathfrak{u}(\{u_1, u_3\}) = \{v_1, v_2, v_3, v_4\}$. In the same way, we can find that $\mathfrak{u}_{p^+}(F_A)(k_2) = \{v_1, v_4\}$, $\mathfrak{u}_{p^-}(F_A)(k_2) = \{v_1, v_2, v_3, v_4\}$, $\mathfrak{u}_{p^+}(F_A)(k_3) = \{v_2, v_3\}$, $\mathfrak{u}_{p^-}(F_A)(k_3) = \{v_1, v_2, v_3\}$ and so we obtain that $\mathfrak{u}_{p^+}(F_A) = \{(k_2, \{v_1, v_4\}), (k_3, \{v_2, v_3\})\}$ and $\mathfrak{u}_{p^-}(F_A) = \{(k_1, V), (k_2, V), (k_3, \{v_1, v_2, v_3\})\}$.

Again, $\mathfrak{u}_p^+(G_B)(e_1) = \mathfrak{u}^+(\bigcup_{k\in p(e_1)} G_B(k)) = U$, $\mathfrak{u}_p^-(G_B)(e_1) = \mathfrak{u}^-(\bigcup_{k\in p(e_1)} G_B(k)) = U$, $\mathfrak{u}_p^+(G_B)(e_2) = \{u_1, u_3, u_4\}$, $\mathfrak{u}_p^-(G_B)(e_2) = U$, $\mathfrak{u}_p^+(G_B)(e_3) = \{u_3\}$, $\mathfrak{u}_p^-(G_B)(e_3) = \{u_2, u_3, u_4\}$, $\mathfrak{u}_p^+(G_B)(e_4) = \{u_3\}$, $\mathfrak{u}_p^-(G_B)(e_4) = \{u_2, u_3, u_4\}$ and so we obtain that $\mathfrak{u}_p^+(G_B) = \{(e_1, U), (e_2, \{u_1, u_3, u_4\}), (e_3, \{u_3\}), (e_4, \{u_3\})\}$ and $\mathfrak{u}_p^-(G_B) = \{(e_1, U), (e_2, U), (e_3, \{u_2, u_3, u_4\}), (e_4, \{u_2, u_3, u_4\})\}$.

Theorem 3.3. *Let $\mathfrak{u}_p : S(U,E) \rightsquigarrow S(V,K)$ be a soft set-valued mapping and $F_A, G_B \in S(V,K)$. Then the following are true:*

(1) $\mathfrak{u}_p^+(F_\varnothing) = F_\varnothing$ *and* $\mathfrak{u}_p^-(F_\varnothing) = F_\varnothing$

(2) $\mathfrak{u}_p^+(F_K) = F_E$ *and* $\mathfrak{u}_p^-(F_K) = F_E$

(3) $\mathfrak{u}_p^+(F_A \widetilde{\cup} G_B) \widetilde{\supseteq} \mathfrak{u}_p^+(F_A) \widetilde{\cup} \mathfrak{u}_p^+(G_B)$

(4) $\mathfrak{u}_p^-(F_A \widetilde{\cup} G_B) = \mathfrak{u}_p^-(F_A) \widetilde{\cup} \mathfrak{u}_p^-(G_B)$

(5) $\mathfrak{u}_p^+(F_A \widetilde{\cap} G_B) = \mathfrak{u}_p^+(F_A) \widetilde{\cap} \mathfrak{u}_p^+(G_B)$

(6) $\mathfrak{u}_p^-(F_A \widetilde{\cap} G_B) \widetilde{\subseteq} \mathfrak{u}_p^-(F_A) \widetilde{\cap} \mathfrak{u}_p^-(G_B)$

(7) *If* $F_A \widetilde{\subseteq} G_B$, *then* $\mathfrak{u}_p^+(F_A) \widetilde{\subseteq} \mathfrak{u}_p^+(G_B)$ *and* $\mathfrak{u}_p^-(F_A) \widetilde{\subseteq} \mathfrak{u}_p^-(G_B)$.

Proof. (1) Let us prove $\mathfrak{u}_p^+(F_\varnothing) = F_\varnothing$. The other can be done in a similar way. For all $e \in E$, we have that

$$\mathfrak{u}_p^+(F_\varnothing)(e) = \mathfrak{u}^+(\bigcup_{k\in p(e)} F_\varnothing(k)) = \mathfrak{u}^+(\bigcup_{k\in p(e)} \varnothing) = \varnothing$$

This shows that $\mathfrak{u}_p^+(F_\varnothing) = F_\varnothing$

(2) Let us prove $\mathfrak{u}_p^-(F_K) = F_E$. For all $e \in E$, we have that

$$\mathfrak{u}_p^-(F_K)(e) = \mathfrak{u}^-(\bigcup_{k\in p(e)} F_K(k)) = \mathfrak{u}^-(\bigcup_{k\in p(e)} V) = U$$

This shows that $\mathfrak{u}_p^-(F_K) = F_E$.

Let us prove (3) and (4). (5) and (6) can be proved similarly.

(3) For all $e \in E$, we have that

$$\begin{aligned}\mathfrak{u}_p^+(F_A\widetilde{\cup}G_B)(e) &= \mathfrak{u}^+(\bigcup_{k\in p(e)}(F_A\widetilde{\cup}G_B)(k)) \\ &= \mathfrak{u}^+(\bigcup_{k\in p(e)}(F_A(k)\cup G_B(k))) \\ &= \mathfrak{u}^+((\bigcup_{k\in p(e)}F_A(k))\cup(\bigcup_{k\in p(e)}G_B(k))) \\ &\supset \mathfrak{u}^+(\bigcup_{k\in p(e)}F_A(k))\cup\mathfrak{u}^+(\bigcup_{k\in p(e)}G_B(k)) \\ &= \mathfrak{u}_p^+(F_A)(e)\cup\mathfrak{u}_p^+(G_B)(e) \\ &= (\mathfrak{u}_p^+(F_A)\widetilde{\cup}\mathfrak{u}_p^+(G_B))(e)\end{aligned}$$

This shows that $\mathfrak{u}_p^+(F_A\widetilde{\cup}G_B)\widetilde{\supset}\mathfrak{u}_p^+(F_A)\widetilde{\cup}\mathfrak{u}_p^+(G_B)$.

(4) For all $e \in E$, we have that

$$\begin{aligned}\mathfrak{u}_p^-(F_A\widetilde{\cup}G_B)(e) &= \mathfrak{u}^-(\bigcup_{k\in p(e)}(F_A\widetilde{\cup}G_B)(k)) \\ &= \mathfrak{u}^-(\bigcup_{k\in p(e)}(F_A(k)\cup G_B(k))) \\ &= \mathfrak{u}^-((\bigcup_{k\in p(e)}F_A(k))\cup(\bigcup_{k\in p(e)}G_B(k))) \\ &= \mathfrak{u}^-(\bigcup_{k\in p(e)}F_A(k))\cup\mathfrak{u}^-(\bigcup_{k\in p(e)}G_B(k)) \\ &= \mathfrak{u}_p^-(F_A)(e)\cup\mathfrak{u}_p^-(G_B)(e) \\ &= (\mathfrak{u}_p^-(F_A)\widetilde{\cup}\mathfrak{u}_p^-(G_B))(e)\end{aligned}$$

This shows that $\mathfrak{u}_p^-(F_A\widetilde{\cup}G_B) = \mathfrak{u}_p^-(F_A)\widetilde{\cup}\mathfrak{u}_p^-(G_B)$.

(7) Let $F_A\widetilde{\subseteq}G_B$. Then for all $k \in K$, we have that

$$\mathfrak{u}_p^+(F_A)(k) = \mathfrak{u}^+(\bigcup_{k\in p(e)}F_A(k)) \subseteq \mathfrak{u}^+(\bigcup_{k\in p(e)}G_B(k)) = \mathfrak{u}_{p^+}(G_B)(k)$$

This shows that $\mathfrak{u}_p^+(F_A)\widetilde{\subseteq}\mathfrak{u}_p^+(G_B)$. The other is similar. □

Theorem 3.4. *Let $\mathfrak{u}_p : S(U,E) \rightsquigarrow S(V,K)$ be a soft set-valued mapping and $F_A, G_B \in S(U,E)$. Then the following are true:*

(1) $\mathfrak{u}_{p^+}(F_\varnothing) = F_\varnothing$ *and* $\mathfrak{u}_{p^-}(F_\varnothing) = F_\varnothing$.

(2) *If p and $\mathfrak{u}$ are surjective, then* $\mathfrak{u}_{p^-}(F_E) = F_K$.

(3) $\mathfrak{u}_{p^+}(F_A\widetilde{\cup}G_B) = \mathfrak{u}_{p^+}(F_A)\widetilde{\cup}\mathfrak{u}_{p^+}(G_B)$.

(4) $\mathfrak{u}_{p^-}(F_A\widetilde{\cup}G_B) = \mathfrak{u}_{p^-}(F_A)\widetilde{\cup}\mathfrak{u}_p^-(G_B)$.

(5) $\mathfrak{u}_{p^+}(F_A\widetilde{\cap}G_B)\widetilde{\subseteq}\mathfrak{u}_{p^+}(F_A)\widetilde{\cap}\mathfrak{u}_{p^+}(G_B)$.

(6) $\mathfrak{u}_{p^-}(F_A\widetilde{\cap}G_B)\widetilde{\subseteq}\mathfrak{u}_{p^-}(F_A)\widetilde{\cap}\mathfrak{u}_{p^-}(G_B)$.

(7) *If $F_A\widetilde{\subseteq}G_B$, then* $\mathfrak{u}_{p^+}(F_A)\widetilde{\subseteq}\mathfrak{u}_{p^+}(G_B)$ *and* $\mathfrak{u}_{p^-}(F_A)\widetilde{\subseteq}\mathfrak{u}_{p^-}(G_B)$.

Proof. (1) Let us prove $\mathfrak{u}_{p^-}(F_\varnothing) = F_\varnothing$. The other can be done in a similar way. For all $k \in K$, we have that

$$\mathfrak{u}_{p^-}(F_\varnothing)(k) = \begin{cases}\bigcup_{e\in p^-(k)}\mathfrak{u}(F_\varnothing(e)) & ;p^-(k)\neq\varnothing \\ \varnothing & ;p^-(k)=\varnothing\end{cases} = \begin{cases}\bigcup_{e\in p^-(k)}\mathfrak{u}(\varnothing) & ;p^-(k)\neq\varnothing \\ \varnothing & ;p^-(k)=\varnothing\end{cases} = \varnothing$$

This shows that $\mathfrak{u}_{p^-}(F_\varnothing) = F_\varnothing$.

(2) Let p and $\mathfrak{u}$ be surjective. Then for all $k \in K$, we have that

$$\mathfrak{u}_{p^-}(F_E)(k) = \begin{cases}\bigcup_{e\in p^-(k)}\mathfrak{u}(F_E(e)) & ;p^-(k)\neq\varnothing \\ \varnothing & ;p^-(k)=\varnothing\end{cases} = \bigcup_{e\in p^-(k)}\mathfrak{u}(U) = V$$

This shows that $\mathfrak{u}_{p^-}(F_E) = F_K$.

Here we will provide proofs of (3) and (4). (5) and (6) can be proved similarly.

(3) For all $k \in K$, we have that

$$\begin{aligned}
\mathfrak{u}_{p^+}(F_A \widetilde{\cup} G_B)(k) &= \begin{cases} \bigcup\limits_{e \in p^+(k)} \mathfrak{u}(F_A(e) \cup G_B(e)) & ;p^+(k) \neq \emptyset \\ \emptyset & ;p^+(k) = \emptyset \end{cases} \\
&= \begin{cases} \bigcup\limits_{e \in p^+(k)} \mathfrak{u}(F_A(e)) \cup \mathfrak{u}(G_B(e)) & ;p^+(k) \neq \emptyset \\ \emptyset & ;p^+(k) = \emptyset \end{cases} \\
&= \begin{cases} \bigcup\limits_{e \in p^+(k)} \mathfrak{u}(F_A(e)) & ;p^+(k) \neq \emptyset \\ \emptyset & ;p^+(k) = \emptyset \end{cases} \cup \begin{cases} \bigcup\limits_{e \in p^+(k)} \mathfrak{u}(G_B(e)) & ;p^+(k) \neq \emptyset \\ \emptyset & ;p^+(k) = \emptyset \end{cases} \\
&= \mathfrak{u}_{p^+}(F_A)(k) \cup \mathfrak{u}_{p^+}(G_B)(k)
\end{aligned}$$

This shows that $\mathfrak{u}_{p^+}(F_A \widetilde{\cup} G_B) = \mathfrak{u}_{p^+}(F_A) \widetilde{\cup} \mathfrak{u}_{p^+}(G_B)$.

(4) For all $k \in K$, we have that

$$\begin{aligned}
\mathfrak{u}_{p^-}(F_A \widetilde{\cup} G_B)(k) &= \begin{cases} \bigcup\limits_{e \in p^-(k)} \mathfrak{u}(F_A(e) \cup G_B(e)) & ;p^-(k) \neq \emptyset \\ \emptyset & ;p^-(k) = \emptyset \end{cases} \\
&= \begin{cases} \bigcup\limits_{e \in p^-(k)} \mathfrak{u}(F_A(e)) \cup \mathfrak{u}(G_B(e)) & ;p^-(k) \neq \emptyset \\ \emptyset & ;p^-(k) = \emptyset \end{cases} \\
&= \begin{cases} \bigcup\limits_{e \in p^-(k)} \mathfrak{u}(F_A(e)) & ;p^-(k) \neq \emptyset \\ \emptyset & ;p^-(k) = \emptyset \end{cases} \cup \begin{cases} \bigcup\limits_{e \in p^-(k)} \mathfrak{u}(G_B(e)) & ;p^-(k) \neq \emptyset \\ \emptyset & ;p^-(k) = \emptyset \end{cases} \\
&= \mathfrak{u}_{p^-}(F_A)(k) \cup \mathfrak{u}_{p^-}(G_B)(k)
\end{aligned}$$

This shows that $\mathfrak{u}_{p^-}(F_A \widetilde{\cup} G_B) = \mathfrak{u}_{p^-}(F_A) \widetilde{\cup} \mathfrak{u}_p^-(G_B)$.

(7) Let $F_A \widetilde{\subseteq} G_B$ Then for all $k \in K$, we have that

$$\begin{aligned}
\mathfrak{u}_{p^-}(F_A)(k) &= \begin{cases} \bigcup\limits_{e \in p^-(k)} \mathfrak{u}(F_A(e)) & ;p^-(k) \neq \emptyset \\ \emptyset & ;p^-(k) = \emptyset \end{cases} \\
&\subseteq \begin{cases} \bigcup\limits_{e \in p^-(k)} \mathfrak{u}(G_B(e)) & ;p^-(k) \neq \emptyset \\ \emptyset & ;p^-(k) = \emptyset \end{cases} \\
&= \mathfrak{u}_{p^-}(G_B)(k)
\end{aligned}$$

This shows that $\mathfrak{u}_{p^-}(F_A) \widetilde{\subseteq} \mathfrak{u}_{p^-}(G_B)$. The proof that $F_A \widetilde{\subseteq} G_B$ requires $\mathfrak{u}_{p^+}(F_A) \widetilde{\subseteq} \mathfrak{u}_{p^+}(G_B)$ is similar. □

Remark 3.5. Even if $p : E \rightsquigarrow K$ and $\mathfrak{u} : U \rightsquigarrow V$ are surjective, $\mathfrak{u}_{p^+}(F_E) = F_K$ may not for $\mathfrak{u}_p : S(U, E) \rightsquigarrow S(V, K)$. Consider Example 3.2. Then since

$\mathfrak{u}_{p^+}(F_E)(k_1) = \bigcup\limits_{e \in p^+(k_1)} \mathfrak{u}(F_E(e)) = \emptyset$

$\mathfrak{u}_{p^+}(F_E)(k_2) = \bigcup\limits_{e \in p^+(k_2)} \mathfrak{u}(F_E(e)) = \mathfrak{u}(F_E(e_3)) = \mathfrak{u}(U) = V$

$\mathfrak{u}_{p^+}(F_E)(k_3) = \bigcup\limits_{e \in p^+(k_3)} \mathfrak{u}(F_E(e)) = \mathfrak{u}(F_E(e_4)) = \mathfrak{u}(U) = V$

we have $\mathfrak{u}_{p^+}(F_E) = \{(k_2, V), (k_3, V)\} \neq F_K = \{((k_1, V), (k_2, V), (k_3, V)\}$.

Theorem 3.6. *Let $\mathfrak{u}_p : S(U, E) \rightsquigarrow S(V, K)$ be a soft set-valued mapping and $G_B \in S(V, K)$. Then the following are true:*

(1) $\mathfrak{u}_p^+(G_B^c) = (\mathfrak{u}_p^-(G_B))^c$

(2) $\mathfrak{u}_p^-(G_B^c) = (\mathfrak{u}_p^+(G_B))^c$

Proof. (1) For all $e \in E$, we have that

$$\begin{aligned}
\mathfrak{u}_p^+(G_B^c)(e) &= \mathfrak{u}^+(\bigcup_{k\in p(e)} G_B^c(k)) \\
&= \mathfrak{u}^+(\bigcup_{k\in p(e)} (V\setminus G_B(k))) \\
&= \mathfrak{u}^+(V\setminus(\bigcap_{k\in p(e)} G_B(k))) \\
&= U\setminus(\mathfrak{u}^-(\bigcup_{k\in p(e)} G_B(k))) \\
&= U\setminus \mathfrak{u}_p^-(G_B)(e)
\end{aligned}$$

This shows that $\mathfrak{u}_p^+(G_B^c) = (\mathfrak{u}_p^-(G_B))^c$.

(2) It is similar to that of (1). □

4. An Application

Decision making is the process of choosing the best one among some alternatives based on some criteria. However, in the developing world, this processes in many fields such as Engineering, Economy, Management, Medicine and Social Sciences are often encountered as very complex systems due to uncertain and inaccurate data. In fact, the human mind has the ability to make decisions in many of such situations. However, if the selection criteria are too large to be held in human memory, and complex relationships exist, mathematical methods are needed.

For example, If we want to decide which sector is advantageous for the investor who wants to open a branch of one of the store chains in different branches of the retail industry, in this decision process, consumers, consumers' needs and the features of the product that the consumers are interested such as quality, well-known brand, cheapness etc. affect the decision. Therefore, there are many parameters that will affect the decision, but are not directly related to the objects in the universe to be selected, and in a different universe. So it does not seem possible to implement $uni - int$ decision making method, and therefore decision making appears to be a difficult process. The following example shows the solution to the above problem using soft set-valued mappings

Let us assume that an investor wants to open one of the stores $U = \{u_1, u_2, u_3.u_4\}$ which sells from the sectors $E = \{e_1 =$ clothing, $e_2 =$ sports, $e_3 =$ cosmetics, $e_4=$ toys$\}$. Let us assume that, u_1 shop sells clothing and sports equipment, u_2 shop sells clothing and toys, u_3 shop sells sports equipment and u_4 shop sells cosmetics. Then we can write the soft set

$$F_A = \{(e_1, \{u_1, u_2\}), (e_2, \{u_1, u_3\}), (e_3, \{u_4\}), (e_4, \{u_2\})\}$$

that shows these relationships.

On the other hand, the investor wants to use the tendency of consumers of different sex and age groups $K = \{k_1 =$ male, $k_2 =$ female, $k_3 =$ children$\}$ to the qualifications of the stores $V = \{v_1 =$ reasonable price, $v_2 =$ well-known brand, $v_3 =$quality, $v_4 =$plentiful$\}$ in decision making. Let us assume that the above soft set G_B gives these trends.

$$G_B = \{(k_1, \{v_1, v_3\}), (k_2, \{v_3, v_4\}), (k_3, \{v_2, v_4\})\}$$

Assume that the relationship of sectors with gender and age groups gives a set-valued mapping $p : E \rightsquigarrow K$ defined by $p(e_1) = \{k_1, k_2, k_3\}$, $p(e_2) = \{k_1, k_2\}$, $p(e_3) = \{k_2\}$, $p(e_4) = \{k_3\}$. Again relationships between stores and qualifications of their gives a set-valued mapping $\mathfrak{u} : U \rightsquigarrow V$ defined by $\mathfrak{u}(u_1) = \{v_1\}$, $\mathfrak{u}(u_2) = \{v_2, v_3\}$, $\mathfrak{u}(u_3) = \{v_4\}$, $\mathfrak{u}(u_4) = \{v_1, v_4\}$. Then we have that

$$\mathfrak{u}_p^+(G_B) = \{(e_1, U), (e_2, \{u_1, u_3, u_4\}), (e_3, \{u_3\}), (e_4, \{u_3\})\}$$

and

$$\mathfrak{u}_p^-(G_B) = \{(e_1, U), (e_2, U), (e_3, \{u_2, u_3, u_4\}), (e_4, \{u_2, u_3, u_4\})\}$$

The investor can use these two soft sets in the $uni-int$ decision making method. The investor who wants to have more control over the decision can apply the $uni-int$ decision method to the big set $\text{ɥ}_p^-(G_B)$ and F_A. Because there is a possibility that a larger set of suitable alternatives can be obtained as a result of the application of the decision-making method.

First, let us apply the $uni-int$ decision method for sets F_A and $\text{ɥ}_p^+(G_B)$. For ease of operation, assume that $\text{ɥ}_p^+(G_B) = F_C$. In this case, we have

$$\begin{aligned} F_A \wedge F_C = \quad & \{((e_1,e_1),\{u_1,u_2\}),((e_1,e_2),\{u_1\}),((e_1,e_3),\emptyset),((e_1,e_4),\emptyset),\\ & ((e_2,e_1),\{u_1,u_3\}),((e_2,e_2),\{u_1,u_3\}),((e_2,e_3),\{u_3\}),((e_2,e_4),\{u_3\}),\\ & ((e_3,e_1),\{u_4\}),((e_3,e_2),\{u_4\}),((e_3,e_3),\emptyset),((e_3,e_4),\emptyset),\\ & ((e_4,e_1),\{u_2\}),((e_4,e_2),\emptyset),((e_4,e_3),\emptyset),((e_4,e_4),\emptyset)\} \end{aligned}$$

and then we calculate that

$$\begin{aligned} uni_x int_y(F_A \wedge F_C) &= \cup_{x\in A}\left(\cap_{y\in C}\left((F_A \wedge F_C)(x,y)\right)\right)\\ &= \bigcup \begin{cases} \cap\{ & \{u_1,u_2\},\{u_1\},\emptyset,\emptyset\} \\ \cap\{ & \{u_1,u_3\},\{u_1,u_3\},\{u_3\},\{u_3\}\} \\ \cap\{ & \{u_4\},\{u_4\},\emptyset,\emptyset\} \\ \cap\{ & \{u_2\},\emptyset,\emptyset,\emptyset\} \end{cases}\\ &= \{u_3\} \end{aligned}$$

and

$$\begin{aligned} uni_y int_x(F_A \wedge F_C) &= \cup_{y\in C}\left(\cap_{x\in A}\left((F_A \wedge F_C)(x,y)\right)\right)\\ &= \bigcup \begin{cases} \cap\{ & \{u_1,u_2\},\{u_1,u_3\}\{u_4\},\{u_2\} \\ \cap\{ & \{u_1\},\{u_1,u_3\},\{u_4\},\emptyset \\ \cap\{ & \emptyset,\{u_3\},\emptyset,\emptyset \\ \cap\{ & \emptyset,\{u_3\},\emptyset,\emptyset \end{cases}\\ &= \emptyset \end{aligned}$$

Therefore we obtain that

$$uni-int(F_A \wedge \text{ɥ}_p^+(G_B)) = uni-int(F_A \wedge F_C) = \{u_3\} \cup \emptyset = \{u_3\}$$

This means that when using soft sets F_A and $\text{ɥ}_p^+(G_B)$ in the $uni-int$ decision making method, the store u_3 is the best result for investment

Now let us apply the $uni-int$ decision method for sets F_A and $\text{ɥ}_p^-(G_B)$ and assume that $\text{ɥ}_p^-(G_B) = F_D$. Then we have

$$\begin{aligned} F_A \wedge F_D = \quad & \{((e_1,e_1),\{u_1,u_2\}),((e_1,e_2),\{u_1,u_2\}),((e_1,e_3),\{u_2\}),((e_1,e_4),\{u_2\}),\\ & ((e_2,e_1),\{u_1,u_3\}),((e_2,e_2),\{u_1,u_3\}),((e_2,e_3),\{u_3\}),((e_2,e_4),\{u_3\}),\\ & ((e_3,e_1),\{u_4\}),((e_3,e_2),\{u_4\}),((e_3,e_3),\{u_4\}),((e_3,e_4),\{u_4\}),\\ & ((e_4,e_1),\{u_2\}),((e_4,e_2),\{u_2\}),((e_4,e_3),\{u_2\}),((e_4,e_4),\{u_2\})\} \end{aligned}$$

$$\begin{aligned} uni_x int_y(F_A \wedge F_D) &= \cup_{x\in A}\left(\cap_{y\in D}\left((F_A \wedge F_D)(x,y)\right)\right)\\ &= \bigcup \begin{cases} \cap\{ & \{u_1,u_2\},\{u_1,u_2\},\{u_2\},\{u_2\} \\ \cap\{ & \{u_1,u_3\},\{u_1,u_3\},\{u_3\},\{u_3\} \\ \cap\{ & \{u_4\},\{u_4\},\{u_4\},\{u_4\} \\ \cap\{ & \{u_2\},\{u_2\},\{u_2\},\{u_2\} \end{cases}\\ &= \{u_2,u_3,u_4\} \end{aligned}$$

$$\begin{aligned} uni_y int_x(F_A \wedge F_D) &= \cup_{y\in D}\left(\cap_{x\in A}\left((F_A \wedge F_D)(x,y)\right)\right)\\ &= \bigcup \begin{cases} \cap\{ & \{u_1,u_2\},\{u_1,u_3\},\{u_4\},\{u_2\} \\ \cap\{ & \{u_1,u_2\},\{u_1,u_3\},\{u_4\},\{u_2\} \\ \cap\{ & \{u_2\},\{u_3\},\{u_4\},\{u_2\} \\ \cap\{ & \{u_2\},\{u_3\},\{u_4\},\{u_2\} \end{cases}\\ &= \emptyset \end{aligned}$$

and we conclude that

$$uni - int(F_A \wedge \text{ų}_p^-(G_B)) = uni - int(F_A \wedge F_D) = \{u_2, u_3, u_4\} \cup \emptyset = \{u_2, u_3, u_4\}$$

Accordingly, the investor can choose any of the stores u_2, u_3 or u_4 when using soft sets F_A and $\text{ų}_p^-(G_B)$ in the $uni - int$ decision making method.

Consequently, u_3 store is the most suitable store for the investor. However, the investor may also consider u_2 or u_4 stores besides the u_3 store. Another result of the decision-making method is that the shop u_1 is not suitable for investment.

5. Conclusion

The main aim of this paper is to define soft set-valued mappings, investigate basic properties of them and to expand the application areas of soft set decision making methods. Mappings introduced in this study can be used not only for the uni-int decision making but also all decision making methods created with soft sets. Therefore, I hope that this study will be a useful guide for new studies.

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