



Certain Parameterized Inequalities Arising from Fractional Integral Operators with Exponential Kernels

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Abstract. We utilize the definition of a fractional integral operators, which was presented by Ahmad et al., to investigate a general fractional-type identity with a parameter. We establish certain parameterized fractional integral inequalities based on this identity, and provide two examples to illustrate the obtained results. Also, these results derived in this paper are applied to the estimations of q -digamma function, divergence measures and cumulative distribution function, respectively.

1. Introduction

In the whole paper, let $I \subseteq \mathbb{R}$ be a interval of real numbers and I° be the interior of I . Let $f : I \rightarrow \mathbb{R}$ be a convex mapping on the interval I , for any $a, b \in I$ along with $a < b$. Then one has

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(t)dt \leq \frac{f(a)+f(b)}{2}, \quad (1)$$

which is referred to as a Hermite–Hadamard inequality. This prominent inequality gives estimations for the mean value of a continuous convex mapping $f : [a, b] \rightarrow \mathbb{R}$.

The inequality (1) has been extensively generalised and improved in recent studies. For example, see [6, 16, 19, 20, 23, 30, 31, 34] and the references therein.

Another classical inequality of equal significance, which is named Simpson’s inequality, is stated as follows:

$$\left| \frac{1}{6} \left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right] - \frac{1}{b-a} \int_a^b f(t)dt \right| \leq \frac{1}{2880} \|f^{(4)}\|_\infty (b-a)^4, \quad (2)$$

where $f : I \rightarrow \mathbb{R}$ is a four-order continuously differentiable mapping on I° with $\|f^{(4)}\|_\infty = \sup_{t \in I^\circ} |f^{(4)}(t)| < \infty$.

A variety of inequations have been discovered based on inequality (2) involving mappings of different classes, such as convex mappings [10], extended (s, m) -convex mappings [8], p -quasi-convex mappings [12], geometrically relative convex mappings [25], strongly preinvex mappings [5], and h -convex mappings [21].

In [14], Kirmaci demonstrated the succeeding lemma concerning the left-hand part of the inequality (1).

2010 *Mathematics Subject Classification.* Primary 26A33; Secondary 41A55, 26D15, 26E60

Keywords. Hermite–Hadamard type inequalities, Simpson’s inequality, Fractional integral operators

Received: 30 April 2020; Accepted: 25 November 2020

Communicated by Miodrag Spalević

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Lemma 1.1. Let $f : I^\circ \rightarrow \mathbb{R}$ be a differentiable mapping on I° , $a, b \in I^\circ$ along with $a < b$. If $f' \in L^1([a, b])$, then the coming equality is valid:

$$\frac{1}{b-a} \int_a^b f(t)dt - f\left(\frac{a+b}{2}\right) = (b-a) \left[\int_0^{\frac{1}{2}} \xi f'(\xi a + (1-\xi)b) d\xi + \int_{\frac{1}{2}}^1 (\xi-1) f'(\xi a + (1-\xi)b) d\xi \right]. \quad (3)$$

Also, the author provided the following theorem on the basis of the above lemma.

Theorem 1.2. Let $f : I^\circ \rightarrow \mathbb{R}$ be a differentiable mapping on I° , $a, b \in I^\circ$ together with $a < b$. If $|f'|$ is convex on $[a, b]$, then the subsequent inequality is valid:

$$\left| \frac{1}{b-a} \int_a^b f(t)dt - f\left(\frac{a+b}{2}\right) \right| \leq \frac{b-a}{8} (|f'(a)| + |f'(b)|). \quad (4)$$

In [7], using mappings whose first derivative’s absolute values are convex, Dragomir and Agarwal presented several Hadamard’s inequalities on the basis of the following lemma.

Lemma 1.3. Under all hypotheses of Lemma 1.1, we have that

$$\frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(t)dt = \frac{b-a}{2} \int_0^1 (1-2\xi) f'(\xi a + (1-\xi)b) d\xi. \quad (5)$$

Theorem 1.4. Under all hypotheses of Theorem 1.2, we have that

$$\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(t)dt \right| \leq \frac{b-a}{8} (|f'(a)| + |f'(b)|). \quad (6)$$

Fractional calculus, as a very practical tool, has become a fascinating field of mathematics. This field has attracted many mathematicians to take into account this issue. In consequence, certain well-known integral inequalities through the method of fractional calculus have been carried out by many authors, including Chen [3] and Mohammed [24] in the research of the Hermite–Hadamard inequality, and Set et al. [28] in the Simpson type integral inequality for Riemann–Liouville fractional integral operators, Wang et al. [32] in the Ostrowski type inequality for Hadamard fractional integral operators, Chen and Katugampola [4] in the Fejér–Hermite–Hadamard type inequality for Katugampola fractional integral operators, Khan et al. [15] in the Hermite–Hadamard inequality for conformable fractional integral operators, and Du et al. [9] in the extensions of trapezium inequalities for k -fractional integral operators. With respect to more results in connection with the fractional integral operators, we refer the interested readers to [1, 11, 13, 17, 18, 22, 26, 27, 29] and the related references therein.

In 2019, Ahmad et al. [2] proposed a fractional integrals with exponential kernels as follows.

Definition 1.5. Let $g \in L^1([a, b])$. The fractional integral operators $\mathcal{I}_{a^+}^\vartheta g$ and $\mathcal{I}_{b^-}^\vartheta g$ of order $\vartheta \in (0, 1)$ are, respectively, defined by

$$\mathcal{I}_{a^+}^\vartheta g(s) = \frac{1}{\vartheta} \int_a^s e^{(-\frac{1-\vartheta}{\vartheta}(s-t))} g(t)dt, \quad s > a$$

and

$$\mathcal{I}_{b^-}^\vartheta g(s) = \frac{1}{\vartheta} \int_s^b e^{(-\frac{1-\vartheta}{\vartheta}(t-s))} g(t)dt, \quad s < b.$$

Note that,

$$\lim_{\vartheta \rightarrow 1} \mathcal{I}_{a^+}^\vartheta g(s) = \int_a^s g(t)dt, \quad \lim_{\vartheta \rightarrow 1} \mathcal{I}_{b^-}^\vartheta g(s) = \int_s^b g(t)dt.$$

In the same article, they established a fractional version of Hermite–Hadamard type below, which had some connection with exponential kernels.

Theorem 1.6. Let $g : [a, b] \rightarrow \mathbb{R}$ be a positive convex mapping with $0 \leq a < b$. If $g \in L^1([a, b])$, then the coming inequality for fractional integrals with an exponential kernel is valid:

$$g\left(\frac{a+b}{2}\right) \leq \frac{1-\vartheta}{2(1-e^{-\rho})} \left[\mathcal{I}_{a^+}^{\vartheta} g(b) + \mathcal{I}_{b^-}^{\vartheta} g(a) \right] \leq \frac{g(a) + g(b)}{2}, \quad (7)$$

where

$$\rho = \frac{1-\vartheta}{\vartheta}(b-a).$$

Also, Ahmad et al. [2] set up a fractional integral operators identity below.

Lemma 1.7. Let $g : [a, b] \rightarrow \mathbb{R}$ be a differentiable mapping on (a, b) together with $a < b$. If $g' \in L^1([a, b])$, then the coming equation is valid:

$$\begin{aligned} & \frac{1-\vartheta}{2(1-e^{-\rho})} \left[\mathcal{I}_{a^+}^{\vartheta} g(b) + \mathcal{I}_{b^-}^{\vartheta} g(a) \right] - \frac{g(a) + g(b)}{2} \\ &= \frac{b-a}{2(1-e^{-\rho})} \int_0^1 (e^{-\rho(1-u)} - e^{-\rho u}) g'(ua + (1-u)b) du. \end{aligned} \quad (8)$$

Another integral equation with respect to the first differentiable mapping was presented by Wu et al. [33] as follows.

Lemma 1.8. Under all hypotheses of Lemma 1.7, we have that

$$\begin{aligned} & \frac{1-\vartheta}{2(1-e^{-\rho})} \left[\mathcal{I}_{a^+}^{\vartheta} g(b) + \mathcal{I}_{b^-}^{\vartheta} g(a) \right] - g\left(\frac{a+b}{2}\right) \\ &= \frac{b-a}{2(1-e^{-\rho})} \int_0^1 \left(\eta(u)(1-e^{-\rho}) - e^{-\rho u} + e^{-\rho(1-u)} \right) g'(ua + (1-u)b) du, \end{aligned} \quad (9)$$

where

$$\eta(u) = \begin{cases} 1, & u \in \left[0, \frac{1}{2}\right], \\ -1, & u \in \left(\frac{1}{2}, 1\right]. \end{cases}$$

The objective of this article is to investigate the parameterized inequalities for fractional integrals with exponential kernels. For this purpose, we will establish a general fractional-type identity with a parameter. Using this identity, we present certain fractional integrals inequalities for the first differentiable mappings, whose absolute value is convex. In addition, we acquire some estimation-type results for fractional integrals inequalities by considering the boundedness and Lipschitz condition. We also point out certain relevant relationship between the derived results in this article and preceding ones.

2. Main Results

To prove our primary theorems, we put forward the subsequent lemma.

Lemma 2.1. Assume that $g : [a, b] \rightarrow \mathbb{R}$ is a differentiable mapping on (a, b) together with $a < b$. If $g' \in L^1([a, b])$ and $0 \leq \lambda \leq 1$, then the coming identity for fractional integral operators is valid:

$$\begin{aligned} & \frac{1 - \vartheta}{2(1 - e^{-\rho})} \left[\mathcal{I}_{a^+}^{\vartheta} g(b) + \mathcal{I}_{b^-}^{\vartheta} g(a) \right] - (1 - \lambda)g\left(\frac{a + b}{2}\right) - \lambda \frac{g(a) + g(b)}{2} \\ &= \frac{b - a}{2(1 - e^{-\rho})} \int_0^1 k(u)g'(ua + (1 - u)b)du, \end{aligned} \tag{10}$$

where

$$k(u) = \begin{cases} (1 - \lambda)(1 - e^{-\rho}) + e^{-\rho(1-u)} - e^{-\rho u}, & u \in \left[0, \frac{1}{2}\right], \\ (1 - \lambda)(e^{-\rho} - 1) + e^{-\rho(1-u)} - e^{-\rho u}, & u \in \left(\frac{1}{2}, 1\right]. \end{cases}$$

Proof. Considering the right part of the equality (10), we have that

$$\begin{aligned} & \frac{b - a}{2(1 - e^{-\rho})} \int_0^1 k(u)g'(ua + (1 - u)b)du \\ &= \frac{b - a}{2(1 - e^{-\rho})} \left[\int_0^{\frac{1}{2}} \left((1 - \lambda)(1 - e^{-\rho}) + e^{-\rho(1-u)} - e^{-\rho u} \right) g'(ua + (1 - u)b)du \right. \\ & \quad \left. + \int_{\frac{1}{2}}^1 \left((1 - \lambda)(e^{-\rho} - 1) + e^{-\rho(1-u)} - e^{-\rho u} \right) g'(ua + (1 - u)b)du \right] \\ &= \frac{b - a}{2(1 - e^{-\rho})} \left[\int_0^{\frac{1}{2}} (1 - \lambda)(1 - e^{-\rho})g'(ua + (1 - u)b)du \right. \\ & \quad \left. + \int_{\frac{1}{2}}^1 (1 - \lambda)(e^{-\rho} - 1)g'(ua + (1 - u)b)du + \int_0^1 (e^{-\rho(1-u)} - e^{-\rho u})g'(ua + (1 - u)b)du \right]. \end{aligned} \tag{11}$$

Directly computation yields that

$$\begin{aligned} & \frac{b - a}{2(1 - e^{-\rho})} \left[\int_0^{\frac{1}{2}} (1 - \lambda)(1 - e^{-\rho})g'(ua + (1 - u)b)du + \int_{\frac{1}{2}}^1 (1 - \lambda)(e^{-\rho} - 1)g'(ua + (1 - u)b)du \right] \\ &= -\frac{(1 - \lambda)}{2} \left(g(ua + (1 - u)b) \Big|_0^{\frac{1}{2}} - g(ua + (1 - u)b) \Big|_{\frac{1}{2}}^1 \right) \\ &= (1 - \lambda) \frac{g(a) + g(b)}{2} - (1 - \lambda)g\left(\frac{a + b}{2}\right). \end{aligned} \tag{12}$$

Integrating by parts and changing the variables, we get that

$$\begin{aligned} & \frac{b - a}{2(1 - e^{-\rho})} \int_0^1 (e^{-\rho(1-u)} - e^{-\rho u})g'(ua + (1 - u)b)du \\ &= \frac{b - a}{2(1 - e^{-\rho})} \left(\frac{1}{a - b} g(ua + (1 - u)b)(e^{-\rho(1-u)} - e^{-\rho u}) \Big|_0^1 \right. \\ & \quad \left. - \int_0^1 \frac{\rho}{a - b} (e^{-\rho(1-u)} + e^{-\rho u})g(ua + (1 - u)b)du \right) \\ &= -\frac{1}{2(1 - e^{-\rho})} \left((1 - e^{-\rho})(g(a) + g(b)) + \frac{\rho}{a - b} \int_a^b \left(e^{-\frac{1-\vartheta}{\vartheta}(\omega-a)} + e^{-\frac{1-\vartheta}{\vartheta}(b-\omega)} \right) g(\omega)d\omega \right) \\ &= -\frac{g(a) + g(b)}{2} + \frac{1 - \vartheta}{2(1 - e^{-\rho})} \left[\mathcal{I}_{a^+}^{\vartheta} g(b) + \mathcal{I}_{b^-}^{\vartheta} g(a) \right]. \end{aligned} \tag{13}$$

Applying equation (12) and (13) to (11), we derive results that we expect in (10). Thus, the proof is concluded. \square

Before giving our first main result, we recall that hyperbolic tangent function is defined by

$$\tanh(x) = \frac{\sinh(x)}{\cosh(x)} = \frac{e^x - e^{-x}}{e^x + e^{-x}}.$$

Theorem 2.2. Assume that $g : [a, b] \rightarrow \mathbb{R}$ is a differentiable mapping on (a, b) with $a < b$ satisfying $g' \in L^1([a, b])$ and $0 \leq \lambda \leq 1$. If $|g'|$ is convex on $[a, b]$, then the subsequent inequality holds:

$$\begin{aligned} & \left| \frac{1 - \vartheta}{2(1 - e^{-\rho})} [\mathcal{I}_{a^+}^{\vartheta} g(b) + \mathcal{I}_{b^-}^{\vartheta} g(a)] - (1 - \lambda)g\left(\frac{a + b}{2}\right) - \lambda \frac{g(a) + g(b)}{2} \right| \\ & \leq \frac{b - a}{2} \left(\frac{1 - \lambda}{2} + \frac{\tanh(\frac{\rho}{4})}{\rho} \right) (|g'(a)| + |g'(b)|). \end{aligned} \tag{14}$$

Proof. By means of Lemma 2.1 and the definition of $k(u)$, we have that

$$\begin{aligned} & \left| \frac{1 - \vartheta}{2(1 - e^{-\rho})} [\mathcal{I}_{a^+}^{\vartheta} g(b) + \mathcal{I}_{b^-}^{\vartheta} g(a)] - (1 - \lambda)g\left(\frac{a + b}{2}\right) - \lambda \frac{g(a) + g(b)}{2} \right| \\ & \leq \frac{b - a}{2(1 - e^{-\rho})} \left[\int_0^{\frac{1}{2}} |(1 - \lambda)(1 - e^{-\rho}) + e^{-\rho(1-u)} - e^{-\rho u}| |g'(ua + (1 - u)b)| \, du \right. \\ & \quad \left. + \int_{\frac{1}{2}}^1 |(1 - \lambda)(e^{-\rho} - 1) + e^{-\rho(1-u)} - e^{-\rho u}| |g'(ua + (1 - u)b)| \, du \right] \\ & \leq \frac{b - a}{2(1 - e^{-\rho})} \left[\int_0^{\frac{1}{2}} |e^{-\rho(1-u)} - e^{-\rho u}| |g'(ua + (1 - u)b)| \, du \right. \\ & \quad \left. + \int_{\frac{1}{2}}^1 |e^{-\rho(1-u)} - e^{-\rho u}| |g'(ua + (1 - u)b)| \, du \right. \\ & \quad \left. + \int_0^1 (1 - \lambda)(1 - e^{-\rho}) |g'(ua + (1 - u)b)| \, du \right]. \end{aligned}$$

Since $e^{-\rho} - 1 \leq e^{-\rho(1-u)} - e^{-\rho u} \leq 0$ for any $u \in [0, \frac{1}{2}]$, $0 \leq e^{-\rho(1-u)} - e^{-\rho u} \leq 1 - e^{-\rho}$ for any $u \in [\frac{1}{2}, 1]$ and $|g'|$ is convex on $[a, b]$, we obtain that

$$\begin{aligned} & \left| \frac{1 - \vartheta}{2(1 - e^{-\rho})} [\mathcal{I}_{a^+}^{\vartheta} g(b) + \mathcal{I}_{b^-}^{\vartheta} g(a)] - (1 - \lambda)g\left(\frac{a + b}{2}\right) - \lambda \frac{g(a) + g(b)}{2} \right| \\ & \leq \frac{b - a}{2(1 - e^{-\rho})} \left[\int_0^{\frac{1}{2}} (e^{-\rho u} - e^{-\rho(1-u)})(u|g'(a)| + (1 - u)|g'(b)|) \, du \right. \\ & \quad \left. + \int_{\frac{1}{2}}^1 (e^{-\rho(1-u)} - e^{-\rho u})(u|g'(a)| + (1 - u)|g'(b)|) \, du \right. \\ & \quad \left. + \int_0^1 (1 - \lambda)(1 - e^{-\rho})(u|g'(a)| + (1 - u)|g'(b)|) \, du \right] \\ & = \frac{b - a}{2(1 - e^{-\rho})} \left[(E_1 + E_3)|g'(a)| + (E_2 + E_4)|g'(b)| + \frac{(1 - \lambda)(1 - e^{-\rho})}{2} (|g'(a)| + |g'(b)|) \right], \end{aligned} \tag{15}$$

where

$$\begin{aligned}
 E_1 &= \int_0^{\frac{1}{2}} (e^{-\rho u} - e^{-\rho(1-u)})u \, du = -\frac{e^{-\frac{\rho}{2}}}{\rho} + \frac{1 - e^{-\rho}}{\rho^2}, \\
 E_2 &= \int_0^{\frac{1}{2}} (e^{-\rho u} - e^{-\rho(1-u)})(1-u) \, du = \frac{1 - e^{-\frac{\rho}{2}} + e^{-\rho}}{\rho} - \frac{1 - e^{-\rho}}{\rho^2}, \\
 E_3 &= \int_{\frac{1}{2}}^1 (e^{-\rho(1-u)} - e^{-\rho u})u \, du = \frac{1 - e^{-\frac{\rho}{2}} + e^{-\rho}}{\rho} - \frac{1 - e^{-\rho}}{\rho^2}, \\
 E_4 &= \int_{\frac{1}{2}}^1 (e^{-\rho(1-u)} - e^{-\rho u})(1-u) \, du = -\frac{e^{-\frac{\rho}{2}}}{\rho} + \frac{1 - e^{-\rho}}{\rho^2}.
 \end{aligned}$$

Plugging these four equalities into the inequality (15) deduces the desired inequality. Thus, this ends the proof. \square

Corollary 2.3. *Under all hypotheses of Theorem 2.2, if $|g'(t)| \leq M$ on $[a, b]$, then we have that*

$$\begin{aligned}
 &\left| \frac{1 - \vartheta}{2(1 - e^{-\rho})} \left[\mathcal{I}_{a^+}^{\vartheta} g(b) + \mathcal{I}_{b^-}^{\vartheta} g(a) \right] - (1 - \lambda)g\left(\frac{a+b}{2}\right) - \lambda \frac{g(a) + g(b)}{2} \right| \\
 &\leq M(b-a) \left(\frac{1 - \lambda}{2} + \frac{\tanh(\frac{\rho}{4})}{\rho} \right).
 \end{aligned}$$

Corollary 2.4. *Consider Theorem 2.2.*

(1) For $\lambda = 0$, we have the coming midpoint inequality:

$$\begin{aligned}
 &\left| \frac{1 - \vartheta}{2(1 - e^{-\rho})} \left[\mathcal{I}_{a^+}^{\vartheta} g(b) + \mathcal{I}_{b^-}^{\vartheta} g(a) \right] - g\left(\frac{a+b}{2}\right) \right| \\
 &\leq \frac{b-a}{2} \left(\frac{1}{2} + \frac{\tanh(\frac{\rho}{4})}{\rho} \right) (|g'(a)| + |g'(b)|).
 \end{aligned}$$

(2) For $\lambda = \frac{1}{3}$, we have the following Simpson’s inequality:

$$\begin{aligned}
 &\left| \frac{1}{6} \left[g(a) + 4g\left(\frac{a+b}{2}\right) + g(b) \right] - \frac{1 - \vartheta}{2(1 - e^{-\rho})} \left[\mathcal{I}_{a^+}^{\vartheta} g(b) + \mathcal{I}_{b^-}^{\vartheta} g(a) \right] \right| \\
 &\leq \frac{b-a}{2} \left(\frac{1}{3} + \frac{\tanh(\frac{\rho}{4})}{\rho} \right) (|g'(a)| + |g'(b)|).
 \end{aligned}$$

(3) For $\lambda = \frac{1}{2}$, we have the succeeding averaged midpoint-trapezoid integral inequality:

$$\begin{aligned}
 &\left| \frac{1}{4} \left[g(a) + 2g\left(\frac{a+b}{2}\right) + g(b) \right] - \frac{1 - \vartheta}{2(1 - e^{-\rho})} \left[\mathcal{I}_{a^+}^{\vartheta} g(b) + \mathcal{I}_{b^-}^{\vartheta} g(a) \right] \right| \\
 &\leq \frac{b-a}{2} \left(\frac{1}{4} + \frac{\tanh(\frac{\rho}{4})}{\rho} \right) (|g'(a)| + |g'(b)|).
 \end{aligned}$$

(4) For $\lambda = 1$, we obtain the trapezoid inequality:

$$\begin{aligned}
 &\left| \frac{1 - \vartheta}{2(1 - e^{-\rho})} \left[\mathcal{I}_{a^+}^{\vartheta} g(b) + \mathcal{I}_{b^-}^{\vartheta} g(a) \right] - \frac{g(a) + g(b)}{2} \right| \\
 &\leq \frac{(b-a)\tanh(\frac{\rho}{4})}{2\rho} (|g'(a)| + |g'(b)|),
 \end{aligned}$$

which is set up by Ahmad et al. in [2, Theorem 3].

Remark 2.5. In (14) of Theorem 2.2, if we take $\vartheta \rightarrow 1$, i.e. $\rho = \frac{1-\vartheta}{\vartheta}(b-a) \rightarrow 0$, then we have

$$\lim_{\vartheta \rightarrow 1} \frac{1-\vartheta}{2(1-e^{-\rho})} = \frac{1}{2(b-a)} \quad (16)$$

and

$$\lim_{\vartheta \rightarrow 1} \frac{(1-e^{-\frac{\rho}{2}})^2}{\rho(1-e^{-\rho})} = \frac{1}{4}. \quad (17)$$

Thus, Theorem 2.2 is transformed to

$$\begin{aligned} & \left| \frac{1}{b-a} \int_a^b g(t) dt - (1-\lambda)g\left(\frac{a+b}{2}\right) - \lambda \frac{g(a)+g(b)}{2} \right| \\ & \leq \frac{b-a}{2} \left(\frac{1-\lambda}{2} + \frac{1}{4} \right) (|g'(a)| + |g'(b)|). \end{aligned} \quad (18)$$

Remark 2.6. (1) For $\lambda = 0$ and $\vartheta \rightarrow 1$, we get the subsequent inequality:

$$\left| \frac{1}{b-a} \int_a^b g(t) dt - g\left(\frac{a+b}{2}\right) \right| \leq \frac{3(b-a)}{8} (|g'(a)| + |g'(b)|). \quad (19)$$

Therefore, the upper bound in the inequality (4) is smaller than that in the inequality (19), i.e. when $\vartheta \rightarrow 1$ and $\lambda = 0$, Theorem 2.2 cannot reduce to Theorem 1.2.

(2) For $\lambda = \frac{1}{3}$ and $\vartheta \rightarrow 1$, we obtain the coming Simpson's inequality:

$$\begin{aligned} & \left| \frac{1}{6} \left[g(a) + 4g\left(\frac{a+b}{2}\right) + g(b) \right] - \frac{1}{b-a} \int_a^b g(t) dt \right| \\ & \leq \frac{7(b-a)}{24} (|g'(a)| + |g'(b)|). \end{aligned}$$

(3) For $\lambda = \frac{1}{2}$ and $\vartheta \rightarrow 1$, we have the averaged midpoint-trapezoid integral inequality:

$$\begin{aligned} & \left| \frac{1}{4} \left[g(a) + 2g\left(\frac{a+b}{2}\right) + g(b) \right] - \frac{1}{b-a} \int_a^b g(t) dt \right| \\ & \leq \frac{b-a}{4} (|g'(a)| + |g'(b)|). \end{aligned}$$

(4) For $\lambda = 1$ and $\vartheta \rightarrow 1$, we have the inequality (6).

Another similar result is obtained in the following theorem.

Theorem 2.7. Assume that $g : [a, b] \rightarrow \mathbb{R}$ is a differentiable mapping on (a, b) with $a < b$ satisfying $g' \in L^1([a, b])$ and $0 \leq \lambda \leq 1$. For $q > 1$ with $p^{-1} + q^{-1} = 1$, if $|g'|^q$ is convex on $[a, b]$, then the succeeding inequality for fractional integrals is valid:

$$\begin{aligned} & \left| \frac{1-\vartheta}{2(1-e^{-\rho})} \left[\mathcal{I}_{a^+}^{\vartheta} g(b) + \mathcal{I}_{b^-}^{\vartheta} g(a) \right] - (1-\lambda)g\left(\frac{a+b}{2}\right) - \lambda \frac{g(a)+g(b)}{2} \right| \\ & \leq \frac{(b-a)(2-\lambda)}{2} \left(\frac{|g'(a)|^q + |g'(b)|^q}{2} \right)^{\frac{1}{q}}. \end{aligned} \quad (20)$$

Proof. Utilizing Lemma 2.1, the Hölder inequality, and the definition of $k(u)$, we obtain that

$$\begin{aligned} & \left| \frac{1 - \vartheta}{2(1 - e^{-\rho})} [\mathcal{I}_{a^+}^{\vartheta} g(b) + \mathcal{I}_{b^-}^{\vartheta} g(a)] - (1 - \lambda)g\left(\frac{a + b}{2}\right) - \lambda \frac{g(a) + g(b)}{2} \right| \\ & \leq \frac{b - a}{2(1 - e^{-\rho})} \int_0^1 |k(u)| |g'(ua + (1 - u)b)| du \\ & \leq \frac{b - a}{2(1 - e^{-\rho})} \left(\int_0^1 |k_1(u)|^p du \right)^{\frac{1}{p}} \left(\int_0^1 |g'(ua + (1 - u)b)|^q du \right)^{\frac{1}{q}} \\ & = \frac{b - a}{2(1 - e^{-\rho})} \left(\int_0^{\frac{1}{2}} |k_1(u)|^p du + \int_{\frac{1}{2}}^1 |k_2(u)|^p du \right)^{\frac{1}{p}} \left(\int_0^1 |g'(ua + (1 - u)b)|^q du \right)^{\frac{1}{q}}, \end{aligned} \tag{21}$$

where

$$k_1(u) = (1 - \lambda)(1 - e^{-\rho}) + e^{-\rho(1-u)} - e^{-\rho u}, \quad u \in \left[0, \frac{1}{2}\right],$$

and

$$k_2(u) = (1 - \lambda)(e^{-\rho} - 1) + e^{-\rho(1-u)} - e^{-\rho u}, \quad u \in \left[\frac{1}{2}, 1\right].$$

Owing to $e^{-\rho} - 1 \leq e^{-\rho(1-u)} - e^{-\rho u} \leq 0$ for any $u \in [0, \frac{1}{2}]$ and $0 \leq e^{-\rho(1-u)} - e^{-\rho u} \leq 1 - e^{-\rho}$ for any $u \in [\frac{1}{2}, 1]$, we have that

$$\begin{aligned} \int_{\frac{1}{2}}^1 |k_2(u)|^p du &= \int_0^{\frac{1}{2}} |k_1(u)|^p du \\ &\leq \int_0^{\frac{1}{2}} \left((1 - \lambda)(1 - e^{-\rho}) + e^{-\rho u} - e^{-\rho(1-u)} \right)^p du \\ &\leq \int_0^{\frac{1}{2}} \left((2 - \lambda)(1 - e^{-\rho}) \right)^p du \\ &= \frac{1}{2} \left((2 - \lambda)(1 - e^{-\rho}) \right)^p. \end{aligned} \tag{22}$$

On account of the convexity of $|g'|^q$ on $[a, b]$, we get that

$$\int_0^1 |g'(ua + (1 - u)b)|^q du \leq \frac{|g'(a)|^q + |g'(b)|^q}{2}. \tag{23}$$

An assembly of (21)-(23) fulfills the required result. Thus, we accomplish the proof of Theorem 2.7. \square

Corollary 2.8. Under all hypotheses of Theorem 2.7, if $|g'(t)| \leq M$ on $[a, b]$, then the coming inequality is effective:

$$\begin{aligned} & \left| \frac{1 - \vartheta}{2(1 - e^{-\rho})} [\mathcal{I}_{a^+}^{\vartheta} g(b) + \mathcal{I}_{b^-}^{\vartheta} g(a)] - (1 - \lambda)g\left(\frac{a + b}{2}\right) - \lambda \frac{g(a) + g(b)}{2} \right| \\ & \leq \frac{M(b - a)(2 - \lambda)}{2}. \end{aligned}$$

Corollary 2.9. Consider Theorem 2.7.

(1) For $\lambda = 0$, we derive the subsequent midpoint inequality:

$$\left| \frac{1 - \vartheta}{2(1 - e^{-\rho})} [\mathcal{I}_{a^+}^{\vartheta} g(b) + \mathcal{I}_{b^-}^{\vartheta} g(a)] - g\left(\frac{a+b}{2}\right) \right| \leq (b-a) \left(\frac{|g'(a)|^q + |g'(b)|^q}{2} \right)^{\frac{1}{q}}.$$

(2) For $\lambda = \frac{1}{3}$, we gain the succeeding Simpson's inequality:

$$\left| \frac{1}{6} \left[g(a) + 4g\left(\frac{a+b}{2}\right) + g(b) \right] - \frac{1 - \vartheta}{2(1 - e^{-\rho})} [\mathcal{I}_{a^+}^{\vartheta} g(b) + \mathcal{I}_{b^-}^{\vartheta} g(a)] \right| \leq \frac{5(b-a)}{6} \left(\frac{|g'(a)|^q + |g'(b)|^q}{2} \right)^{\frac{1}{q}}.$$

(3) For $\lambda = \frac{1}{2}$, we have the averaged midpoint-trapezoid integral inequality:

$$\left| \frac{1}{4} \left[g(a) + 2g\left(\frac{a+b}{2}\right) + g(b) \right] - \frac{1 - \vartheta}{2(1 - e^{-\rho})} [\mathcal{I}_{a^+}^{\vartheta} g(b) + \mathcal{I}_{b^-}^{\vartheta} g(a)] \right| \leq \frac{3(b-a)}{4} \left(\frac{|g'(a)|^q + |g'(b)|^q}{2} \right)^{\frac{1}{q}}.$$

(4) For $\lambda = 1$, we obtain the trapezoid inequality:

$$\left| \frac{1 - \vartheta}{2(1 - e^{-\rho})} [\mathcal{I}_{a^+}^{\vartheta} g(b) + \mathcal{I}_{b^-}^{\vartheta} g(a)] - \frac{g(a) + g(b)}{2} \right| \leq \frac{b-a}{2} \left(\frac{|g'(a)|^q + |g'(b)|^q}{2} \right)^{\frac{1}{q}}.$$

A different approach leads to the following result.

Theorem 2.10. Assume that $g : [a, b] \rightarrow \mathbb{R}$ is a differentiable mapping on (a, b) with $a < b$ satisfying $g' \in L^1([a, b])$ and $0 \leq \lambda \leq 1$. If $|g'|^q$ is convex on $[a, b]$ together with $q > 1$, then the coming inequality is effective:

$$\left| \frac{1 - \vartheta}{2(1 - e^{-\rho})} [\mathcal{I}_{a^+}^{\vartheta} g(b) + \mathcal{I}_{b^-}^{\vartheta} g(a)] - (1 - \lambda)g\left(\frac{a+b}{2}\right) - \lambda \frac{g(a) + g(b)}{2} \right| \leq \frac{b-a}{2} \left(1 - \lambda + \frac{2 \tanh(\frac{\rho}{4})}{\rho} \right) \left(\frac{|g'(a)|^q + |g'(b)|^q}{2} \right)^{\frac{1}{q}}. \tag{24}$$

Proof. Utilizing Lemma 2.1, the power-mean integral inequality, and the definition of $k(u)$, we derive that

$$\begin{aligned} & \left| \frac{1 - \vartheta}{2(1 - e^{-\rho})} [\mathcal{I}_{a^+}^{\vartheta} g(b) + \mathcal{I}_{b^-}^{\vartheta} g(a)] - (1 - \lambda)g\left(\frac{a+b}{2}\right) - \lambda \frac{g(a) + g(b)}{2} \right| \\ & \leq \frac{b-a}{2(1 - e^{-\rho})} \int_0^1 |k(u)| |g'(ua + (1-u)b)| \, du \\ & \leq \frac{b-a}{2(1 - e^{-\rho})} \left(\int_0^1 |k(u)| \, du \right)^{1-\frac{1}{q}} \left(\int_0^1 |k(u)| |g'(ua + (1-u)b)|^q \, du \right)^{\frac{1}{q}} \\ & \leq \frac{b-a}{2(1 - e^{-\rho})} \left(\int_0^1 |k(u)| \, du \right)^{1-\frac{1}{q}} \left(|g'(a)|^q \int_0^1 u |k(u)| \, du + |g'(b)|^q \int_0^1 (1-u) |k(u)| \, du \right)^{\frac{1}{q}}. \end{aligned} \tag{25}$$

Using the properties of the module and simple calculation, we obtain that

$$\begin{aligned}
 \int_0^1 |k(u)|du &= \int_0^{\frac{1}{2}} |k_1(u)|du + \int_{\frac{1}{2}}^1 |k_2(u)|du \\
 &\leq \int_0^{\frac{1}{2}} \left((1-\lambda)(1-e^{-\rho}) + e^{-\rho u} - e^{-\rho(1-u)} \right) du \\
 &\quad + \int_{\frac{1}{2}}^1 \left((1-\lambda)(1-e^{-\rho}) + e^{-\rho(1-u)} - e^{-\rho u} \right) du \\
 &= (1-\lambda)(1-e^{-\rho}) + \frac{2(1-e^{-\frac{\rho}{2}})^2}{\rho}
 \end{aligned} \tag{26}$$

and

$$\begin{aligned}
 \int_0^1 (1-u)|k(u)|du &= \int_0^1 u|k(u)|du \\
 &= \int_0^{\frac{1}{2}} u|k_1(u)|du + \int_{\frac{1}{2}}^1 u|k_2(u)|du \\
 &\leq \int_0^{\frac{1}{2}} u \left((1-\lambda)(1-e^{-\rho}) + e^{-\rho u} - e^{-\rho(1-u)} \right) du \\
 &\quad + \int_{\frac{1}{2}}^1 u \left((1-\lambda)(1-e^{-\rho}) + e^{-\rho(1-u)} - e^{-\rho u} \right) du \\
 &= \frac{(1-\lambda)(1-e^{-\rho})}{2} + \frac{(1-e^{-\frac{\rho}{2}})^2}{\rho}.
 \end{aligned} \tag{27}$$

Making use of (26) and (27) in (25), we obtain the expected result in (24). Thus, the proof is completed. \square

Corollary 2.11. *Under all assumptions of Theorem 2.10, if $|g'(t)| \leq M$ on $[a, b]$, then the coming inequality is effective:*

$$\begin{aligned}
 &\left| \frac{1-\vartheta}{2(1-e^{-\rho})} \left[\mathcal{I}_{a^+}^{\vartheta} g(b) + \mathcal{I}_{b^-}^{\vartheta} g(a) \right] - (1-\lambda)g\left(\frac{a+b}{2}\right) - \lambda \frac{g(a)+g(b)}{2} \right| \\
 &\leq \frac{M(b-a)}{2} \left(1-\lambda + \frac{2 \tanh\left(\frac{\rho}{4}\right)}{\rho} \right).
 \end{aligned}$$

Corollary 2.12. *Consider Theorem 2.10.*

(1) For $\lambda = 0$, we have the succeeding midpoint inequality:

$$\begin{aligned}
 &\left| \frac{1-\vartheta}{2(1-e^{-\rho})} \left[\mathcal{I}_{a^+}^{\vartheta} g(b) + \mathcal{I}_{b^-}^{\vartheta} g(a) \right] - g\left(\frac{a+b}{2}\right) \right| \\
 &\leq \frac{b-a}{2} \left(1 + \frac{2 \tanh\left(\frac{\rho}{4}\right)}{\rho} \right) \left(\frac{|g'(a)|^q + |g'(b)|^q}{2} \right)^{\frac{1}{q}}.
 \end{aligned}$$

(2) For $\lambda = \frac{1}{3}$, we have the subsequent Simpson’s inequality:

$$\begin{aligned}
 &\left| \frac{1}{6} \left[g(a) + 4g\left(\frac{a+b}{2}\right) + g(b) \right] - \frac{1-\vartheta}{2(1-e^{-\rho})} \left[\mathcal{I}_{a^+}^{\vartheta} g(b) + \mathcal{I}_{b^-}^{\vartheta} g(a) \right] \right| \\
 &\leq \frac{b-a}{2} \left(\frac{2}{3} + \frac{2 \tanh\left(\frac{\rho}{4}\right)}{\rho} \right) \left(\frac{|g'(a)|^q + |g'(b)|^q}{2} \right)^{\frac{1}{q}}.
 \end{aligned}$$

(3) For $\lambda = \frac{1}{2}$, we have the averaged midpoint-trapezoid integral inequality:

$$\left| \frac{1}{4} \left[g(a) + 2g\left(\frac{a+b}{2}\right) + g(b) \right] - \frac{1-\vartheta}{2(1-e^{-\rho})} \left[\mathcal{I}_{a^+}^{\vartheta} g(b) + \mathcal{I}_{b^-}^{\vartheta} g(a) \right] \right| \leq \frac{b-a}{2} \left(\frac{1}{2} + \frac{2 \tanh(\frac{\rho}{4})}{\rho} \right) \left(\frac{|g'(a)|^q + |g'(b)|^q}{2} \right)^{\frac{1}{q}}.$$

(4) For $\lambda = 1$, we have the trapezoid inequality:

$$\left| \frac{1-\vartheta}{2(1-e^{-\rho})} \left[\mathcal{I}_{a^+}^{\vartheta} g(b) + \mathcal{I}_{b^-}^{\vartheta} g(a) \right] - \frac{g(a) + g(b)}{2} \right| \leq \frac{(b-a) \tanh(\frac{\rho}{4})}{\rho} \left(\frac{|g'(a)|^q + |g'(b)|^q}{2} \right)^{\frac{1}{q}}.$$

Remark 2.13. Using (16) and (17) in (24), Theorem 2.10 is transformed to

$$\left| \frac{1}{b-a} \int_a^b g(t) dt - (1-\lambda)g\left(\frac{a+b}{2}\right) - \lambda \frac{g(a) + g(b)}{2} \right| \leq \frac{b-a}{2} \left((1-\lambda) + \frac{1}{2} \right) \left(\frac{|g'(a)|^q + |g'(b)|^q}{2} \right)^{\frac{1}{q}}. \tag{28}$$

Corollary 2.14. Under all hypotheses of Theorem 2.2–Theorem 2.10 with $0 \leq \lambda \leq 1$, we have that

$$\left| \frac{1-\vartheta}{2(1-e^{-\rho})} \left[\mathcal{I}_{a^+}^{\vartheta} g(b) + \mathcal{I}_{b^-}^{\vartheta} g(a) \right] - (1-\lambda)g\left(\frac{a+b}{2}\right) - \lambda \frac{g(a) + g(b)}{2} \right| \leq \min\{\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3\},$$

where

$$\mathcal{L}_1 = M(b-a) \left(\frac{1-\lambda}{2} + \frac{\tanh(\frac{\rho}{4})}{\rho} \right),$$

$$\mathcal{L}_2 = \frac{M(b-a)(2-\lambda)}{2},$$

and

$$\mathcal{L}_3 = \frac{M(b-a)}{2} \left(1-\lambda + \frac{2 \tanh(\frac{\rho}{4})}{\rho} \right).$$

Our next result is about an estimation of the upper bound of fractional integral inequality.

Theorem 2.15. If there exist constants $\zeta < \Upsilon$ satisfying that $-\infty < \zeta \leq g' \leq \Upsilon < \infty$ on $[a, b]$, then the following inequality

$$\left| \frac{1-\vartheta}{2(1-e^{-\rho})} \left[\mathcal{I}_{a^+}^{\vartheta} g(b) + \mathcal{I}_{b^-}^{\vartheta} g(a) \right] - (1-\lambda)g\left(\frac{a+b}{2}\right) - \lambda \frac{g(a) + g(b)}{2} \right| \leq \frac{(b-a)(\Upsilon - \zeta)}{4} \left(1-\lambda + \frac{2 \tanh(\frac{\rho}{4})}{\rho} \right) \tag{29}$$

holds with $0 \leq \lambda \leq 1$ and $\vartheta \in (0, 1)$.

Proof. Using Lemma 2.1, we have that

$$\begin{aligned} & \frac{1 - \vartheta}{2(1 - e^{-\rho})} \left[\mathcal{I}_{a^+}^{\vartheta} g(b) + \mathcal{I}_{b^-}^{\vartheta} g(a) \right] - (1 - \lambda)g\left(\frac{a + b}{2}\right) - \lambda \frac{g(a) + g(b)}{2} \\ &= \frac{b - a}{2(1 - e^{-\rho})} \left[\int_0^{\frac{1}{2}} \left((1 - \lambda)(1 - e^{-\rho}) + e^{-\rho(1-u)} - e^{-\rho u} \right) \left(g'(ua + (1 - u)b) - \frac{\zeta + \Upsilon}{2} \right) du \right. \\ & \quad \left. + \int_{\frac{1}{2}}^1 \left((1 - \lambda)(e^{-\rho} - 1) + e^{-\rho(1-u)} - e^{-\rho u} \right) \left(g'(ua + (1 - u)b) - \frac{\zeta + \Upsilon}{2} \right) du \right]. \end{aligned}$$

Utilizing the fact that

$$\zeta - \frac{\zeta + \Upsilon}{2} \leq g'(ua + (1 - u)b) - \frac{\zeta + \Upsilon}{2} \leq \Upsilon - \frac{\zeta + \Upsilon}{2},$$

one has

$$\left| g'(ua + (1 - u)b) - \frac{\zeta + \Upsilon}{2} \right| \leq \frac{\Upsilon - \zeta}{2}.$$

Therefore,

$$\begin{aligned} & \left| \frac{1 - \vartheta}{2(1 - e^{-\rho})} \left[\mathcal{I}_{a^+}^{\vartheta} g(b) + \mathcal{I}_{b^-}^{\vartheta} g(a) \right] - (1 - \lambda)g\left(\frac{a + b}{2}\right) - \lambda \frac{g(a) + g(b)}{2} \right| \\ & \leq \frac{b - a}{2(1 - e^{-\rho})} \left[\int_0^{\frac{1}{2}} \left| (1 - \lambda)(1 - e^{-\rho}) + e^{-\rho(1-u)} - e^{-\rho u} \right| \left| g'(ua + (1 - u)b) - \frac{\zeta + \Upsilon}{2} \right| du \right. \\ & \quad \left. + \int_{\frac{1}{2}}^1 \left| (1 - \lambda)(e^{-\rho} - 1) + e^{-\rho(1-u)} - e^{-\rho u} \right| \left| g'(ua + (1 - u)b) - \frac{\zeta + \Upsilon}{2} \right| du \right] \\ & \leq \frac{(b - a)(\Upsilon - \zeta)}{4(1 - e^{-\rho})} \left[\int_0^{\frac{1}{2}} \left| (1 - \lambda)(1 - e^{-\rho}) + e^{-\rho(1-u)} - e^{-\rho u} \right| du \right. \\ & \quad \left. + \int_{\frac{1}{2}}^1 \left| (1 - \lambda)(e^{-\rho} - 1) + e^{-\rho(1-u)} - e^{-\rho u} \right| du \right] \\ & \leq \frac{(b - a)(\Upsilon - \zeta)}{4(1 - e^{-\rho})} \left[\int_0^{\frac{1}{2}} \left| e^{-\rho(1-u)} - e^{-\rho u} \right| du + \int_{\frac{1}{2}}^1 \left| e^{-\rho(1-u)} - e^{-\rho u} \right| du + \int_0^1 (1 - \lambda)(1 - e^{-\rho}) du \right]. \end{aligned} \tag{30}$$

Directly computation yields that

$$\int_0^{\frac{1}{2}} \left| e^{-\rho(1-u)} - e^{-\rho u} \right| du = \int_{\frac{1}{2}}^1 \left| e^{-\rho(1-u)} - e^{-\rho u} \right| du = \frac{(1 - e^{-\frac{\rho}{2}})^2}{\rho}.$$

Plugging these equalities into the inequality (30) deduces the desired inequality. Thus, this ends the proof. \square

Remark 2.16. Using (16) and (17) in (29), Theorem 2.15 is transformed to

$$\begin{aligned} & \left| \frac{1}{b - a} \int_a^b g(t) dt - (1 - \lambda)g\left(\frac{a + b}{2}\right) - \lambda \frac{g(a) + g(b)}{2} \right| \\ & \leq \frac{(b - a)(\Upsilon - \zeta)}{4} \left((1 - \lambda) + \frac{1}{2} \right). \end{aligned}$$

Finally, if we consider that g' satisfies Lipschitz condition, then we have the following result.

Theorem 2.17. For certain $L > 0$, if g' meets Lipschitz condition on $[a, b]$, then the succeeding inequality

$$\begin{aligned} & \left| \frac{1 - \vartheta}{2(1 - e^{-\rho})} \left[\mathcal{I}_{a^+}^{\vartheta} g(b) + \mathcal{I}_{b^-}^{\vartheta} g(a) \right] - (1 - \lambda) g\left(\frac{a + b}{2}\right) - \lambda \frac{g(a) + g(b)}{2} \right| \\ & \leq \frac{L(b - a)^2}{2(1 - e^{-\rho})} \left(\frac{1}{4}(1 - \lambda)(1 - e^{-\rho}) + \frac{\rho - 2 + \rho e^{-\rho} + 2e^{-\rho}}{\rho^2} \right) \end{aligned} \tag{31}$$

holds with $0 \leq \lambda \leq 1$ and $\vartheta \in (0, 1)$.

Proof. From Lemma 2.1, we get that

$$\begin{aligned} & \frac{1 - \vartheta}{2(1 - e^{-\rho})} \left[\mathcal{I}_{a^+}^{\vartheta} g(b) + \mathcal{I}_{b^-}^{\vartheta} g(a) \right] - (1 - \lambda) g\left(\frac{a + b}{2}\right) - \lambda \frac{g(a) + g(b)}{2} \\ & = \frac{b - a}{2(1 - e^{-\rho})} \left[\int_0^{\frac{1}{2}} \left((1 - \lambda)(1 - e^{-\rho}) + e^{-\rho(1-u)} - e^{-\rho u} \right) g'(ua + (1 - u)b) du \right. \\ & \quad \left. - \int_0^{\frac{1}{2}} \left((1 - \lambda)(1 - e^{-\rho}) + e^{-\rho(1-u)} - e^{-\rho u} \right) g'((1 - u)a + ub) du \right] \\ & = \frac{b - a}{2(1 - e^{-\rho})} \int_0^{\frac{1}{2}} \left((1 - \lambda)(1 - e^{-\rho}) + e^{-\rho(1-u)} - e^{-\rho u} \right) (g'(ua + (1 - u)b) - g'((1 - u)a + ub)) du. \end{aligned}$$

Since g' satisfies Lipschitz condition on $[a, b]$, for all $u \in [0, \frac{1}{2}]$ and certain $L > 0$, we have that

$$|g'(ua + (1 - u)b) - g'((1 - u)a + ub)| \leq L(b - a)(1 - 2u).$$

Therefore,

$$\begin{aligned} & \left| \frac{1 - \vartheta}{2(1 - e^{-\rho})} \left[\mathcal{I}_{a^+}^{\vartheta} g(b) + \mathcal{I}_{b^-}^{\vartheta} g(a) \right] - (1 - \lambda) g\left(\frac{a + b}{2}\right) - \lambda \frac{g(a) + g(b)}{2} \right| \\ & \leq \frac{b - a}{2(1 - e^{-\rho})} \int_0^{\frac{1}{2}} \left| (1 - \lambda)(1 - e^{-\rho}) + e^{-\rho(1-u)} - e^{-\rho u} \right| |g'(ua + (1 - u)b) - g'((1 - u)a + ub)| du \\ & \leq \frac{L(b - a)^2}{2(1 - e^{-\rho})} \int_0^{\frac{1}{2}} \left| (1 - \lambda)(1 - e^{-\rho}) + e^{-\rho(1-u)} - e^{-\rho u} \right| (1 - 2u) du \\ & \leq \frac{L(b - a)^2}{2(1 - e^{-\rho})} \left(\int_0^{\frac{1}{2}} (1 - \lambda)(1 - e^{-\rho})(1 - 2u) du + \int_0^{\frac{1}{2}} (e^{-\rho u} - e^{-\rho(1-u)}) (1 - 2u) du \right) \\ & = \frac{L(b - a)^2}{2(1 - e^{-\rho})} \left(\frac{1}{4}(1 - \lambda)(1 - e^{-\rho}) + \frac{\rho - 2 + \rho e^{-\rho} + 2e^{-\rho}}{\rho^2} \right). \end{aligned}$$

The proof is done. \square

Remark 2.18. In (31) of Theorem 2.17, if we take $\vartheta \rightarrow 1$, i.e. $\rho = \frac{1-\vartheta}{\vartheta}(b - a) \rightarrow 0$, then we have that

$$\lim_{\vartheta \rightarrow 1} \frac{\rho - 2 + \rho e^{-\rho} + 2e^{-\rho}}{\rho^2(1 - e^{-\rho})} = \frac{1}{6}. \tag{32}$$

Using (16) and (32) in (31), Theorem 2.17 is transformed to

$$\begin{aligned} & \left| \frac{1}{b - a} \int_a^b g(t) dt - (1 - \lambda) g\left(\frac{a + b}{2}\right) - \lambda \frac{g(a) + g(b)}{2} \right| \\ & \leq \frac{L(b - a)^2}{2} \left(\frac{1}{4}(1 - \lambda) + \frac{1}{6} \right). \end{aligned}$$

3. Examples

Example 3.1. Let $g(s) = s^2$, for $s \in (-\infty, \infty)$. Then $|g'|$ is convex on $(-\infty, \infty)$. If we take $a = 0, b = 1, \vartheta = \frac{1}{2}$ and $\lambda = \frac{3}{4}$, then all hypotheses in Theorem 2.2 are fulfilled.

Distinctly, $\rho = \frac{1-\vartheta}{\vartheta}(b-a) = 1$. The left-hand side term of inequality (14) is:

$$\begin{aligned} & \left| \frac{1-\vartheta}{2(1-e^{-\rho})} \left[I_{a^+}^{\vartheta} g(b) + I_{b^-}^{\vartheta} g(a) \right] - (1-\lambda)g\left(\frac{a+b}{2}\right) - \lambda \frac{g(a)+g(b)}{2} \right| \\ &= \left| \frac{1}{2(1-e^{-1})} \left(\int_0^1 e^{s-1} s^2 ds + \int_0^1 e^{-s} s^2 ds \right) - \frac{7}{16} \right| \\ &= \left| \frac{1}{2(1-e^{-1})} \left((1-2e^{-1}) + (2-5e^{-1}) \right) - \frac{7}{16} \right| \approx 0.1015. \end{aligned}$$

The right-hand side term of inequality (14) is:

$$\frac{(b-a)}{2} \left(\frac{1-\lambda}{2} + \frac{\tanh\left(\frac{\rho}{4}\right)}{\rho} \right) (|g'(a)| + |g'(b)|) = \left(\frac{1}{8} + \tanh\left(\frac{1}{4}\right) \right) \approx 0.3699.$$

It is evident to find that $0.1015 < 0.3699$, which manifests the result described in Theorem 2.2.

Example 3.2. Theorem 2.2–Theorem 2.10 provide an upper bound for the estimation of the fractional integrals $\frac{1-\vartheta}{2(1-e^{-\rho})} \left[I_{a^+}^{\vartheta} g(b) + I_{b^-}^{\vartheta} g(a) \right]$. There exist some integral functions that can not be expressed by elementary functions. So Theorem 2.2–Theorem 2.10 are of importance to deal with such integral functions.

For example, let $g(r) = e^{-r^2+r}$, for $r \in [2, \infty)$. Then $|g'|^q$ for $q \geq 1$ is convex on $[2, \infty)$. If we take $a = 2, b = 3, \vartheta = \frac{1}{2}$ and $\lambda = \frac{1}{2}$, then all hypotheses in Theorem 2.7 are fulfilled. Overtly, $\rho = \frac{1-\vartheta}{\vartheta}(b-a) = 1$. The left-hand side term of inequality (20) is:

$$\left| \frac{1}{2(1-e^{-1})} \left(e^{-2} \int_2^3 e^{-(r-1)^2} dr + e^2 \int_2^3 e^{-r^2} dr \right) - \frac{1}{2} e^{-\frac{15}{4}} - \frac{e^{-2} + e^{-6}}{4} \right|. \tag{33}$$

Obviously, the term $\int_2^3 e^{-(r-1)^2} dr$ and $\int_2^3 e^{-r^2} dr$ can not solved directly because $\int e^{-r^2} dr$ can not be expressed by elementary functions. However, if we apply Theorem 2.7 with $q = 2$, then we obtain an upper bound for (33), i.e.

$$\frac{(b-a)(2-\lambda)}{2} \left(\frac{|g'(a)|^q + |g'(b)|^q}{2} \right)^{\frac{1}{q}} = \frac{3}{4} \left(\frac{9e^{-4} + 25e^{-12}}{2} \right)^{\frac{1}{2}} \approx 0.2154.$$

4. Applications

4.1. q -digamma function

Let $0 < q < 1$, the q -digamma mapping Φ_q is defined by

$$\Phi_q(s) = -\ln(1-q) + \ln q \sum_{n=0}^{\infty} \frac{q^{n+s}}{1-q^{n+s}} = -\ln(1-q) + \ln q \sum_{n=1}^{\infty} \frac{q^{ns}}{1-q^n}.$$

Proposition 4.1. Let $a, b \in \mathbb{R}$ along with $0 < a < b$ and $0 < q < 1$. Then, we have that

$$\begin{aligned} & \left| \frac{\Phi_q(b) - \Phi_q(a)}{b-a} - (1-\lambda)\Phi_q\left(\frac{a+b}{2}\right) - \lambda \frac{\Phi_q'(a) + \Phi_q'(b)}{2} \right| \\ & \leq \frac{b-a}{2} \left(\frac{1-\lambda}{2} + \frac{1}{4} \right) (|\Phi_q''(a)| + |\Phi_q''(b)|). \end{aligned} \tag{34}$$

Proof. According to the definition of q -digamma mapping, we have that

$$\Phi_q^{(k)}(s) = (\ln q)^{k+1} \sum_{n=1}^{\infty} \frac{n^k q^{ns}}{1 - q^n}.$$

Note that the mappings $\Phi_q''(s) < 0$ and $\Phi_q^{(4)}(s) < 0$ on $(0, \infty)$. In consequence, the mapping $s \rightarrow |\Phi_q''(s)|$ is convex on $(0, \infty)$. Applying the mapping $g(s) = \Phi_q'(s)$ to Remark 2.5, we obtain the desired inequality (34). \square

4.2. Divergence measures

Assume that the set of all probability densities on ν is defined on

$$\Omega := \{m|n : T \rightarrow \mathbb{R}, m(s) > 0, \int_T m(s) d\nu(s) = 1\},$$

where the σ -finite measure ν and the set T are offered.

Let $g : (0, \infty) \rightarrow \mathbb{R}$ be a mapping and $G_g(m, n)$ be expressed as

$$G_g(m, n) := \int_T m(s) g\left(\frac{n(s)}{m(s)}\right) d\nu(s), m, n \in \Omega. \tag{35}$$

If g is convex, then (35) is named as the Csiszar g -divergence.

Consider the coming Hermite–Hadamard (HH) divergence

$$G_{HH}^g(m, n) := \int_T m(s) \frac{\int_1^{\frac{n(s)}{m(s)}} g(\xi) d\xi}{\frac{n(s)}{m(s)} - 1} d\nu(s), m, n \in \Omega,$$

where g is convex on $(0, \infty)$ together with $g(1) = 0$. Notice that $G_{HH}^g(m, n) \geq 0$ with the identity is effective if and only if $m = n$.

Proposition 4.2. *Let all hypotheses of Remark 2.13 hold with $g(1) = 0$, if $m, n \in \Omega$ and $0 \leq \lambda \leq 1$, then the subsequent inequality is valid:*

$$\begin{aligned} & \left| G_{HH}^g(m, n) - (1 - \lambda) \int_T m(s) g\left(\frac{m(s) + n(s)}{2m(s)}\right) d\nu(s) - \frac{\lambda}{2} G_g(m, n) \right| \\ & \leq \left(\frac{1 - \lambda}{2} + \frac{1}{4} \right) \int_T |n(s) - m(s)| \left(\frac{|g'(1)|^q + |g'(\frac{n(s)}{m(s)})|^q}{2} \right)^{\frac{1}{q}} d\nu(s). \end{aligned} \tag{36}$$

Proof. Let $T_1 = \{s \in T : n(s) > m(s)\}$, $T_2 = \{s \in T : n(s) < m(s)\}$ and $T_3 = \{s \in T : n(s) = m(s)\}$.

Obviously, if $s \in T_3$, then the identity is effective in (36). Now if $s \in T_1$, then utilizing Remark 2.13 for $a = 1, b = \frac{n(s)}{m(s)}$, multiplying both sides of (28) by $m(s)$ and then integrating on T_1 , we deduce that

$$\begin{aligned} & \left| \int_{T_1} m(s) \frac{\int_1^{\frac{n(s)}{m(s)}} g(\xi) d\xi}{\frac{n(s)}{m(s)} - 1} d\nu(s) - (1 - \lambda) \int_{T_1} m(s) g\left(\frac{m(s) + n(s)}{2m(s)}\right) d\nu(s) - \frac{\lambda}{2} \int_{T_1} m(s) g\left(\frac{n(s)}{m(s)}\right) d\nu(s) \right| \\ & \leq \left(\frac{1 - \lambda}{2} + \frac{1}{4} \right) \int_{T_1} (n(s) - m(s)) \left(\frac{|g'(1)|^q + |g'(\frac{n(s)}{m(s)})|^q}{2} \right)^{\frac{1}{q}} d\nu(s). \end{aligned} \tag{37}$$

Analogously, if $s \in T_2$, then taking advantage of Remark 2.13 for $a = \frac{n(s)}{m(s)}$, $b = 1$, multiplying both sides of (28) by $m(s)$ and then integrating on T_2 , we derive that

$$\left| \int_{T_2} m(s) \frac{\int_1^{\frac{n(s)}{m(s)}} g(\xi) d\xi}{\frac{n(s)}{m(s)} - 1} d\nu(s) - (1 - \lambda) \int_{T_2} m(s) g\left(\frac{m(s) + n(s)}{2m(s)}\right) d\nu(s) - \frac{\lambda}{2} \int_{T_2} m(s) g\left(\frac{n(s)}{m(s)}\right) d\nu(s) \right| \quad (38)$$

$$\leq \left(\frac{1 - \lambda}{2} + \frac{1}{4}\right) \int_{T_2} (m(s) - n(s)) \left(\frac{|g'(1)|^q + |g'(\frac{n(s)}{m(s)})|^q}{2}\right)^{\frac{1}{q}} d\nu(s).$$

Adding inequalities (37) and (38), and making use of triangular inequality, we obtain the desired result. \square

4.3. Probability density mappings

Suppose that $h : [a, b] \rightarrow [0, 1]$ is the probability density mapping of a continuous random variable X with the cumulative distribution function

$$H(s) = \Pr(X \leq s) = \int_a^s h(t) dt.$$

Choosing $g = H$ in (18) and (28), and taking into account $H(a) = 0, H(b) = 1$ and

$$E(X) = \int_a^b t dH(t) = b - \int_a^b H(t) dt,$$

we yield the coming results.

Proposition 4.3. *Under all assumptions of Remark 2.5, the following inequality holds*

$$\left| \frac{1}{b-a} (b - E(X)) - (1 - \lambda) \Pr\left(X \leq \frac{a+b}{2}\right) - \frac{1}{2} \lambda \right| \quad (39)$$

$$\leq \frac{b-a}{2} \left(\frac{1-\lambda}{2} + \frac{1}{4}\right) (|h(a)| + |h(b)|).$$

Proposition 4.4. *Under all hypotheses of Remark 2.13, the coming inequality holds*

$$\left| \frac{1}{b-a} (b - E(X)) - (1 - \lambda) \Pr\left(X \leq \frac{a+b}{2}\right) - \frac{1}{2} \lambda \right| \quad (40)$$

$$\leq \frac{b-a}{2} \left(\frac{3}{2} - \lambda\right) \left(\frac{|h(a)|^q + |h(b)|^q}{2}\right)^{\frac{1}{q}}.$$

5. Conclusion

Utilizing the discovered general fractional-type identity with exponential kernels, the parameterized inequalities for fractional integrals are established. Some estimation-type results in connection with fractional integral inequalities involving the boundedness and Lipschitz condition are obtained. These inequalities generalize and extend parts of the results provided by Ahmad et al. in [2] and Wu et al. in [33]. Applications of the derived results to the estimations of q -digamma function, divergence measures and cumulative distribution function are also presented. With these ideas and techniques stated in our study, it is likely to discover further estimations of other form integral inequalities for the fractional integral operators with exponential kernels which involve other related classes of functions.

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