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The Existence of Solutions and Positive Solution of a First - Order Differential System With Initial and Multi-Point Boundary Conditions

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Abstract. In this paper, we consider a nonlinear differential system with initial and multi - point boundary conditions. The existence of solutions is proved by using the Banach contraction principle or the Krasnoselskii's fixed point theorem. Furthermore, the existence of positive solutions is also obtained by applying the Guo-Krasnoselskii's fixed point theorem in cones. As a consequence of the Guo-Krasnosellskii's fixed point theorem, the multiplicity of positive solutions is established.

1. Introduction

In this paper, we consider the following nonlinear system

$$\begin{cases} u'(t) = f(t, u(t), v(t)), & t \in (0, T), \\ v'(t) = g(t, u(t), v(t)), & t \in (0, T), \end{cases} \tag{1.1}$$

asscociated with the initial and multi - point boundary conditions

$$\begin{cases} u(0) = u_0, \\ v(0) = \sum\limits_{j=1}^{N} \mu_j v(T_j), \end{cases} \tag{1.2}$$

where $f, g : [0, T] \times \mathbb{R}^2 \to \mathbb{R}$ are given functions and u_0, $\mu_1, \cdots, \mu_N$, $0 < T_1 < T_2 < \cdots < T_N = T$ are given constants.

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The boundary value problems for ordinary differential equations with the initial and multi - point boundary conditions or with anti-periodic/periodic boundary conditions play an important role in different fields of science and engineering due to their applications in physics, mechanics, biology, medicine, population dynamics, biotechnology, etc, see [1] - [16] and the references given therein. Many authors have studied various aspects of boundary value problems, by using different methods and various techniques, such as the Leray-Schauder continuation Theorem, Nonlinear alternatives of Leray-Schauder, the fixed point theory (the fixed point theorems of Banach or Krasnoselskii, or Schaefer and related fixed point theorems, the fixed point theorem in cones, etc), the coincidence degree theory, monotone iterative techniques. We refer the reader to the references in this article and the references therein for the results of the boundary-value problems.

In [14], by applying the Banach's fixed point theorem and the Schaefer's fixed point theorem, Mardanov et al. proved the existence and uniqueness theorems for the system of ordinary differential equations with three-point boundary conditions as follows

$$\begin{cases} y' = f(t, y), \ t \in (0, T), \\ Ay(0) + By(\tau) + Cy(T) = d, \end{cases} \tag{1.3}$$

where A, B, C were constant square matrices of order n such that $\det(A + B + C) \neq 0$, $f : [0, T] \times \mathbb{R}^n \to \mathbb{R}^n$ was a given function, $d \in \mathbb{R}^n$ was a given vector, τ satisfied the condition of $0 < \tau < T$, and $y : [0, T] \to \mathbb{R}^n$ was unknown.

In [3], H. H. Alsulami et al. introduced a class of α-admissible contractions defined via altering distance functions. The existence and uniqueness conditions for fixed points of such maps on complete metric spaces were also investigated and, furthermore, related fixed point theorems were presented. These results were reconsidered in the context of partially ordered metric spaces and applied to the following first-order periodic boundary value problem

$$\begin{cases} u'(t) = f(t, u(t)), \ t \in [0, T], \\ u(0) = u(T), \end{cases} \tag{1.4}$$

where $T > 0$ and $f : [0, T] \times \mathbb{R} \to \mathbb{R}$ was a continuous function.

In [4], Ü. Aksoy et al. established the existence and uniqueness of fixed points for a general class of contractive and nonexpansive mappings on modular metric spaces. As an application of the theoretical results, the authors considered the existence of a solution of the following anti-periodic boundary value problems for nonlinear first order differential equations

$$\begin{cases} u'(t) = f(t, u(t)), \ a.e. \ t \in [0, L], \\ u(0) = -u(L), \end{cases} \tag{1.5}$$

in the context of modular metric spaces, where $f : [0, L] \times \mathbb{R} \to \mathbb{R}$ was a Caratheodory's type function satisfying suitable conditions.

In [12], E. Karapınar et al. established some new fixed point theorems and applied the obtained results to show the existence and uniqueness of solutions to some fractional and integer order differential equations. For an example, the authors proved the existence of solutions for the following two-point boundary value problem of a second order differential equation

$$\begin{cases} -u''(t) - f(t, u(t)) = 0, \ \ t \in [0, 1], \\ u(0) = u(1) = 0, \end{cases} \tag{1.6}$$

where $f : [0, 1] \times \mathbb{R} \to \mathbb{R}$ was a continuous function.

In [8], Han considered the second-order three-point boundary value problem in the form

$$\begin{cases} x''(t) = f(t, x(t)), \ t \in (0, 1), \\ x'(0) = 0, \ x(\eta) = x(1), \end{cases} \tag{1.7}$$

with $\eta \in (0,1)$. By means of the fixed point theorem in cones, the existence and multiplicity of positive solutions were proved.

In [16], Truong et al. studied the following m-point boundary value problem

$$\begin{cases} x''(t) = f(t, x(t)),\ t \in (0,1), \\ x'(0) = 0,\ x(1) = \sum_{j=1}^{m-2} \alpha_j x(\eta_j), \end{cases} \tag{1.8}$$

where $m \geq 3$, $\eta_j \in (0,1)$ and $\alpha_j \geq 0$, for all $j = 1, \cdots, m-2$ such that $\sum_{j=1}^{m-2} \alpha_j < 1$. By applying well-know Guo-Krasnoselskii's fixed point theorem and applying the monotone iterative technique, the results obtained in [16] were the existence and multiplicity of positive solutions. Furthermore, the compactness of the set of positive solutions was proved.

In [1], R. P. Agarwal et al. formulated existence results for solutions to discrete equations which approximate three-point boundary value problems for second-order ordinary differential equations. The proofs of these results were finished based on extending the notion of discrete compatibility, which was a degree-based relationship between the given boundary conditions and the lower or upper solutions chosen, to three-point boundary conditions. On the other hand, the invariance of the degree under the homotopy of the degree theory was also applied in the above proofs.

In [10], J. Henderson, R. Luca investigated the following multi-point boundary value problem for the system of nonlinear higher-order ordinary differential equations of the type

$$\begin{cases} u^{(n)}(t) = f(t, v(t)), \quad t \in (0,T),\ n \in \mathbb{N},\ n \geq 2, \\ v^{(m)}(t) = g(t, u(t)), \quad t \in (0,T),\ m \in \mathbb{N},\ m \geq 2, \end{cases} \tag{1.9}$$

with the multi-point boundary conditions

$$\begin{cases} u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0, \quad u(T) = \sum\limits_{i=1}^{p-2} a_i u(\xi_i),\ p \in \mathbb{N},\ p \geq 3, \\ v(0) = v'(0) = \cdots = v^{(m-2)}(0) = 0, \quad v(T) = \sum\limits_{i=1}^{q-2} b_i v(\eta_i),\ q \in \mathbb{N},\ q \geq 3. \end{cases} \tag{1.10}$$

Under sufficient assumptions on f and g, the authors proved the existence and multiplicity of positive solutions of the above problem by applying the fixed point index theory.

Motivated by the above-mentioned works, we study the initial and multi-point boundary problem (1.1) - (1.2). This paper is organized as follows. In Section 2, we present some necessary preliminaries which are the key tools for our main result. Here, at first, the fixed point theorems used in our proof are recalled. Next, by constructing the Green function for the multi-point boundary problem (1.1) - (1.2), the considered problem is reduced to the equivalent integral sytem. Section 3 is devoted to the existence and uniqueness of solutions via the Banach's fixed point theorem and the Krasnoselskii's fixed point theorem. In Section 4, we prove the existence of positive solutions by using the Guo-Krasnoselskii's fixed point theorem in a cone. In Section 5, we observe some multiplicity results for positive solutions. Finally, a remark is also given in Section 6 for a system of multiple differential equations.

2. Preliminaries

We consider the Banach spaces $C([0,T])$ and $C^1([0,T])$ with normal norms, respectively, as follows

$$\|u\|_{C([0,T])} = \max_{t \in [0,T]} |u(t)|,$$
$$\|u\|_{C^1([0,T])} = \|u\|_{C([0,T])} + \|u'\|_{C([0,T])}.$$

First, for the convenience of the reader, we shall recall the following fixed point theorems.

Theorem 2.1 (Krasnosellskii) [17]. *Let M be a nonempty bounded closed convex subset of a Banach space $(X, \|\cdot\|)$. Suppose that $U : M \to X$ is a contraction and $C : M \to X$ is a compact operator such that*

$$U(x) + C(y) \in M, \ \forall x, y \in M.$$

Then, $U + C$ has a fixed point in M.

Theorem 2.2 (Guo-Krasnoselskii) [7]. *Let X be a Banach space and let $P \subset X$ be a cone. Assume that Ω_1, Ω_2 are two open bounded subsets of X with $0 \in \Omega_1$, $\overline{\Omega_1} \subset \Omega_2$ and let $T : P \cap (\overline{\Omega_2} \setminus \Omega_1) \to P$ be a completely continuous operator satisfying one of the following conditions*

(i) $\|Tu\| \leq \|u\|$, $u \in P \cap \partial\Omega_1$ *and* $\|Tu\| \geq \|u\|$, $u \in P \cap \partial\Omega_2$;

or

(ii) $\|Tu\| \geq \|u\|$, $u \in P \cap \partial\Omega_1$ *and* $\|Tu\| \leq \|u\|$, $u \in P \cap \partial\Omega_2$.

Then, the operator T has a fixed point in $P \cap (\overline{\Omega_2} \setminus \Omega_1)$.

Next, we shall construct an expression for the solution of the problem (1.1) - (1.2), with $f, g \in C([0, T] \times \mathbb{R}^2; \mathbb{R})$. For details, in the following, we shall construct the Green function for the multi-point boundary problem (1.1) - (1.2). Then, it is clear that the considered problem can be reduced to the equivalent integral system.

We begin by establishing the Lemma 2.3 as below, in which we denote $f_\alpha(t, u, v) = f(t, u, v) + \alpha u$ and $g_\beta(t, u, v) = g(t, u, v) + \beta v$, for each $\alpha, \beta > 0$.

Lemma 2.3. *Assume that $f, g \in C([0, T] \times \mathbb{R}^2; \mathbb{R})$ and $\sum_{j=1}^{N} \mu_j e^{-\beta T_j} \neq 1$.*

The pair of functions $(u, v) \in C^1([0, T]) \times C^1([0, T])$ is a solution of the problem (1.1) - (1.2) *if and only if $(u, v) \in C([0, T]) \times C([0, T])$ is a solution of the following integral system*

$$u(t) = u_0 e^{-\alpha t} + \int_0^t e^{-\alpha(t-s)} f_\alpha(s, u(s), v(s))ds, \ t \in [0, T], \tag{2.1}$$

$$\begin{aligned} v(t) = & \int_0^t e^{-\beta(t-s)} g_\beta(s, u(s), v(s))ds \\ & + \frac{e^{-\beta t}}{1 - \sum_{j=1}^{N} \mu_j e^{-\beta T_j}} \sum_{j=1}^{N} \mu_j \int_0^{T_j} e^{-\beta(T_j - s)} g_\beta(s, u(s), v(s))ds, \ t \in [0, T]. \end{aligned} \tag{2.2}$$

Proof of Lemma 2.3. Let $(u, v) \in C^1([0, T]) \times C^1([0, T])$ be a solution of the problem (1.1) - (1.2). For each α, $\beta > 0$, the system (1.1) can be transformed into an equivalent form as

$$\begin{cases} u'(t) + \alpha u(t) = f_\alpha(t, u(t), v(t)), & t \in (0, T), \\ v'(t) + \beta v(t) = g_\beta(t, u(t), v(t)), & t \in (0, T). \end{cases} \tag{2.3}$$

It is also equivalent to the following system

$$\begin{cases} u'(t)e^{\alpha t} + \alpha u(t)e^{\alpha t} = e^{\alpha t} f_\alpha(t, u(t), v(t)), & t \in (0, T), \\ v'(t)e^{\beta t} + \beta v(t)e^{\beta t} = e^{\beta t} g_\beta(t, u(t), v(t)), & t \in (0, T). \end{cases}$$

By integrating from 0 to t, respectively, we obtain

$$u(t) = u_0 e^{-\alpha t} + \int_0^t e^{-\alpha(t-s)} f_\alpha(s, u(s), v(s))ds, \ t \in [0, T], \tag{2.4}$$

$$v(t) = v(0)e^{-\beta t} + \int_0^t e^{-\beta(t-s)} g_\beta(s, u(s), v(s))ds, \ t \in [0, T]. \tag{2.5}$$

By (2.5), we also have

$$v(T_j) - e^{-\beta T_j} v(0) = \int_0^{T_j} e^{-\beta(T_j - s)} g_\beta(s, u(s), v(s))ds, \ j = 1, \cdots, N,$$

it gives

$$\sum_{j=1}^{N}\mu_j v(T_j)-v(0)\sum_{j=1}^{N}\mu_j e^{-\beta T_j}=\sum_{j=1}^{N}\mu_j\int_0^{T_j}e^{-\beta(T_j-s)}g_\beta(s,u(s),v(s))ds,$$

hence

$$v(0)\left(1-\sum_{j=1}^{N}\mu_j e^{-\beta T_j}\right)=\sum_{j=1}^{N}\mu_j\int_0^{T_j}e^{-\beta(T_j-s)}g_\beta(s,u(s),v(s))ds, \tag{2.6}$$

since $v(0)=\sum_{j=1}^{N}\mu_j v(T_j)$. Combining (2.4), (2.5) and (2.6), we infer that (u,v) is a solution of the nonlinear integral system (2.1), (2.2).

Otherwise, let $(u,v)\in C([0,T])\times C([0,T])$ be a solution of the nonlinear integral equations (2.1), (2.2). It is not difficult to prove that $(u,v)\in C^1([0,T])\times C^1([0,T])$ and (u,v) satisfies the boundary value problem (1.1) - (1.2).

Indeed, by the fact that $(u,v)\in C([0,T])\times C([0,T])$ is a solution of the nonlinear integral equations (2.1), (2.2), it leads to

$$\begin{cases} u(t)e^{\alpha t}=u_0+\int_0^t e^{\alpha s}f_\alpha(s,u(s),v(s))ds,\ t\in[0,T],\\ v(t)e^{\beta t}=v_*+\int_0^t e^{\beta s}g_\beta(s,u(s),v(s))ds,\ t\in[0,T],\\ u(0)=u_0,\\ v(0)=v_*=\dfrac{1}{1-\sum_{j=1}^{N}\mu_j e^{-\beta T_j}}\sum\limits_{j=1}^{N}\mu_j\int_0^{T_j}e^{-\beta(T_j-s)}g_\beta(s,u(s),v(s))ds,\\ \sum\limits_{j=1}^{N}\mu_j v(T_j)=\sum\limits_{j=1}^{N}\mu_j e^{-\beta T_j}\left[v_*+\int_0^{T_j}e^{\beta s}g_\beta(s,u(s),v(s))ds\right]\\ \qquad=\sum\limits_{j=1}^{N}\mu_j e^{-\beta T_j}v_*+\sum\limits_{j=1}^{N}\mu_j\int_0^{T_j}e^{-\beta(T_j-s)}g_\beta(s,u(s),v(s))ds\\ \qquad=\sum\limits_{j=1}^{N}\mu_j e^{-\beta T_j}v_*+\left(1-\sum\limits_{j=1}^{N}\mu_j e^{-\beta T_j}\right)v_*=v_*=v(0). \end{cases}$$

Then, we have $(u,v)\in C^1([0,T])\times C^1([0,T])$ and we also have

$$\begin{cases} u'(t)e^{\alpha t}+u(t)\alpha e^{\alpha t}=e^{\alpha t}f_\alpha(t,u(t),v(t)),\\ v'(t)e^{\beta t}+\beta v(t)e^{\beta t}=e^{\beta t}g_\beta(t,u(t),v(t)). \end{cases}$$

It implies that $u'(t)+\alpha u(t)=f_\alpha(t,u(t),v(t))$, $v'(t)+\beta v(t)=g_\beta(t,u(t),v(t))$, $\forall t\in(0,T)$ and $u(0)=u_0$, $v(0)=\sum_{j=1}^{N}\mu_j v(T_j)$. Therefore, (u,v) satisfies the boundary value problem (1.1) - (1.2). Lemma 2.3 is proved. □

We note that, the integral equation (2.2) can be written in the form

$$v(t)=\int_0^T G(t,s)g_\beta(s,u(s),v(s))ds, \tag{2.7}$$

where $G(t,s)$ is defined as follows

$$G(t,s) = \begin{cases} e^{-\beta(t-s)}, & 0 \le s \le t \le T, \\ 0, & 0 \le t \le s \le T, \end{cases}$$
$$+ \frac{e^{-\beta t}}{1-\sum_{j=1}^{N}\mu_j e^{-\beta T_j}} \begin{cases} \sum_{j=1}^{N}\mu_j e^{-\beta(T_j-s)}, & 0 \le s \le T_1, \\ \vdots & \vdots \\ \sum_{j=k+1}^{N}\mu_j e^{-\beta(T_j-s)}, & T_k < s \le T_{k+1}, \\ \vdots & \vdots \\ \mu_N e^{-\beta(T-s)} & T_{N-1} < s \le T. \end{cases} \tag{2.8}$$

then $G(t,s)$ is the Green's function for the boundary value problem (1.1) - (1.2). The next Lemma will introduce a property of the Green's function $G(t,s)$.

Lemma 2.4. *Suppose that $\mu_j \ge 0$, for $j = \overline{1, N-1}$, $\mu_N > 0$, such that $1 - \sum_{j=1}^{N}\mu_j > 0$. Then, there exist positive constants $\hat{g}_0$, $\hat{g}_1$ such that*

$$\hat{g}_0 \le G(t,s) \le \hat{g}_1, \ \forall (t,s) \in [0,T]\times[0,T].$$

Proof of Lemma 2.4. By direct computations, we have

$$G(t,s) \ge \mu_N e^{-2\beta T}\left(1-\sum_{j=1}^{N}\mu_j e^{-\beta T_j}\right)^{-1} \equiv \hat{g}_0, \tag{2.9}$$

and

$$G(t,s) \le 1 + \sum_{j=1}^{N}\mu_j\left(1-\sum_{j=1}^{N}\mu_j e^{-\beta T_j}\right)^{-1} \equiv \hat{g}_1 \tag{2.10}$$

Lemma 2.4 is completely proved. □

We also note that if the sign of μ_j can not be determined, there exists a constant G_{max} such that

$$|G(t,s)| \le G_{max}, \ \forall (t,s) \in [0,T]\times[0,T]. \tag{2.11}$$

3. Existence and Uniqueness

Based on the preliminaries, this section is devoted to the proofs of two existence results of solutions for the problem (1.1) - (1.2), in which $f,\ g \in C([0,T]\times\mathbb{R}^2;\mathbb{R})$. The first result (Theorem 3.1) is the unique existence of a solution by applying the Banach's fixed point theorem. Under weaker conditions, we obtain the second result (Theorem 3.5) by using the Krasnoselskii's fixed point theorem.

We consider the space $X = C([0,T])\times C([0,T])$ equipped with the norm

$$\|(u,v)\|_X = \|u\|_{C([0,T])} + \|v\|_{C([0,T])}, \tag{3.1}$$

and we define an operator $\mathcal{P}: X \longrightarrow X$ as follows

$$\begin{array}{llll} \mathcal{P}: & X & \longrightarrow & X \\ & (u,v) & \longmapsto & (\mathcal{P}_1(u,v), \mathcal{P}_2(u,v)), \end{array}$$

where

$$\mathcal{P}_1(u,v)(t) = u_0 e^{-\alpha t} + \int_0^t e^{-\alpha(t-s)} f_\alpha(s,u(s),v(s))ds, \tag{3.2}$$
$$\mathcal{P}_2(u,v)(t) = \int_0^T G(t,s) g_\beta(s,u(s),v(s))ds.$$

We make the following assumptions

(H_1) $0 < \mu = \sum\limits_{j=1}^{N} \left|\mu_j\right| < 1$;

(H_2) There exists a positive function $\Lambda_f \in L^1(0,T)$ such that

$$\left|f(t,u,v) - f(t,\bar{u},\bar{v})\right| \le \Lambda_f(t)\left(|u-\bar{u}| + |v-\bar{v}|\right), \tag{3.3}$$

for all (t,u,v), $(t,\bar{u},\bar{v}) \in [0,T]\times\mathbb{R}^2$;

(H_3) There exists a positive function $\Lambda_g \in L^1(0,T)$ such that

$$\left|g(t,u,v) - g(t,\bar{u},\bar{v})\right| \le \Lambda_g(t)\left(|u-\bar{u}| + |v-\bar{v}|\right), \tag{3.4}$$

for all (t,u,v), $(t,\bar{u},\bar{v}) \in [0,T]\times\mathbb{R}^2$.

Theorem 3.1. *Suppose that* (H_1)–(H_3) *are satisfied. Additionally, assume that there exist two positive constants* α *and* β *small enough such that*

$$L = \alpha T + \left\|\Lambda_f\right\|_{L^1(0,T)} + \left(\beta T + \left\|\Lambda_g\right\|_{L^1(0,T)}\right)\left[1 + \mu\left(1 - \sum_{j=1}^{N}\mu_j e^{-\beta T_j}\right)^{-1}\right] < 1.$$

Then, the problem (1.1) - (1.2) *has a unique solution.*

Proof of Theorem 3.1.

First, we put $m_f = \max\limits_{0\le t\le T}\left|f(t,0,0)\right|$, $m_g = \max\limits_{0\le t\le T}\left|g(t,0,0)\right|$ and choose $R > 0$ large enough such that

$$R > \frac{1}{1-L}\left\{|u_0| + Tm_f + Tm_g\left[1 + \mu\left(1 - \sum_{j=1}^{N}\mu_j e^{-\beta T_j}\right)^{-1}\right]\right\}. \tag{3.5}$$

Next, we will finish the proof of this theorem through a process with two steps as follows.

Step 1. Let $B_R = \{(u,v) \in X : \|(u,v)\|_X \le R\}$. We show that $\mathcal{P}(B_R) \subset B_R$.

Indeed, for $(u,v) \in B_R$ and for all $t \in [0,T]$, we have the following estimates

$$\begin{aligned} |\mathcal{P}_1(u,v)(t)| &\le |u_0| + \int_0^t \left|f_\alpha(s,u(s),v(s)) - f_\alpha(s,0,0)\right| ds + \int_0^t \left|f_\alpha(s,0,0)\right| ds \\ &\le |u_0| + R\left(\alpha T + \left\|\Lambda_f\right\|_{L^1(0,T)}\right) + Tm_f, \end{aligned} \tag{3.6}$$

and

$$\begin{aligned} |\mathcal{P}_2(u,v)(t)| &\le \int_0^T |G(t,s)|\,|g_\beta(s,u(s),v(s))| ds \\ &\le \sum_{j=1}^{N}\mu_j\left(1 - \sum_{j=1}^{N}\mu_j e^{-\beta T_j}\right)^{-1}\int_0^T \left|g_\beta(s,u(s),v(s))\right| ds + \int_0^t \left|g_\beta(s,u(s),v(s))\right| ds \\ &\le \mu\left(1 - \sum_{j=1}^{N}\mu_j e^{-\beta T_j}\right)^{-1}\left(\int_0^T \left|g_\beta(s,u(s),v(s)) - g_\beta(s,0,0)\right| ds + \int_0^T \left|g_\beta(s,0,0)\right| ds\right) \\ &\quad + \int_0^t \left|g_\beta(s,u(s),v(s)) - g_\beta(s,0,0)\right| ds + \int_0^t \left|g_\beta(s,0,0)\right| ds \\ &\le \mu\left(1 - \sum_{j=1}^{N}\mu_j e^{-\beta T_j}\right)^{-1}\left[R\left(\beta T + \left\|\Lambda_g\right\|_{L^1(0,T)}\right) + m_g T\right] \\ &\quad + R\left(\beta T + \left\|\Lambda_g\right\|_{L^1(0,T)}\right) + m_g T. \end{aligned} \tag{3.7}$$

Combining (3.6) - (3.7) and the choice of R as in (3.5), we infer that $\mathcal{P}(B_R) \subset B_R$, it means that the operator $\mathcal{P} : B_R \to B_R$ is defined.

Step 2. We prove that the operator $\mathcal{P}$ is a contraction mapping.

Indeed, let (u, v) and $(\bar{u}, \bar{v})$ be arbitrary elements in B_R. We have

$$\begin{aligned} |\mathcal{P}_1(u,v)(t) - \mathcal{P}_1(\bar{u},\bar{v})(t)| &\le \int_0^t \left| f_\alpha(s,u(s),v(s)) - f_\alpha(s,\bar{u}(s),\bar{v}(s)) \right| ds \\ &\le \left(\alpha T + \left\| \Lambda_f \right\|_{L^1(0,T)} \right) \|(u,v) - (\bar{u},\bar{v})\|_X , \end{aligned} \tag{3.8}$$

and

$$\begin{aligned} &|\mathcal{P}_2(u,v)(t) - \mathcal{P}_2(\bar{u},\bar{v})(t)| \\ &\le \mu \left(1 - \sum_{j=1}^N \mu_j e^{-\beta T_j} \right)^{-1} \int_0^T \left| g_\beta(s,u(s),v(s)) - g_\beta(s,\bar{u}(s),\bar{v}(s)) \right| ds \\ &+ \int_0^t \left| g_\beta(s,u(s),v(s)) - g_\beta(s,\bar{u}(s),\bar{v}(s)) \right| ds \\ &\le \left(\beta T + \left\| \Lambda_g \right\|_{L^1(0,T)} \right) \left[1 + \mu \left(1 - \sum_{j=1}^N \mu_j e^{-\beta T_j} \right)^{-1} \right] \|(u,v) - (\bar{u},\bar{v})\|_X . \end{aligned} \tag{3.9}$$

It follows from (3.8) - (3.9) and the assumption in Theorem 3.1 that $\mathcal{P} : B_R \to B_R$ is a contraction mapping. Applying the Banach's fixed point theorem, we verify that the problem (1.1) - (1.2) has a unique solution (u, v). Theorem 3.1 is proved. □

In what follows, under weaker conditions, the second result is given without the Lipschitzian condition on g as in (H_3). The main tool is the Krasnoselskii's fixed point theorem. We make the assumption (H_4) as below.

(H_4) $g : [0, T] \times \mathbb{R}^2$ is a continuous function and there exist two positive functions $g_1, g_2 \in L^1(0, T)$ such that

$$\left| g(t,u,v) \right| \le g_1(t) \left(|u| + |v| \right) + g_2(t), \ \forall (t,u,v) \in [0,T] \times \mathbb{R}^2. \tag{3.10}$$

We now define two operators $U, C : X \to X$ as follows

$$\begin{array}{llll} U: & X & \longrightarrow & X \\ & (u,v) & \longmapsto & (\mathcal{P}_1(u,v), 0) , \end{array} \tag{3.11}$$

with

$$\mathcal{P}_1(u,v)(t) := u_0 e^{-\alpha t} + \int_0^t e^{-\alpha(t-s)} f_\alpha(s,u(s),v(s)) ds, \tag{3.12}$$

and

$$\begin{array}{llll} C: & X & \longrightarrow & X \\ & (u,v) & \longmapsto & (0, \mathcal{P}_2(u,v)) , \end{array} \tag{3.13}$$

with

$$\begin{aligned} \mathcal{P}_2(u,v)(t) = &\frac{e^{-\beta t}}{1 - \sum_{j=1}^N \mu_j e^{-\beta T_j}} \sum_{j=1}^N \mu_j \int_0^{T_j} e^{-\beta(T_j - s)} g_\beta(s,u(s),v(s)) ds \\ &+ \int_0^t e^{-\beta(t-s)} g_\beta(s,u(s),v(s)) ds. \end{aligned} \tag{3.14}$$

It is easy to check that $\mathcal{P} = U + C$.

Lemma 3.2. *Suppose that* (H_1), (H_2) *and* (H_4) *are satisfied. Additionally, assume that there exist two positive constants* α *and* β *small enough such that*

$$\begin{cases} \alpha T + \left\|\Lambda_f\right\|_{L^1(0,T)} \le \dfrac{1}{4}, \\ \left(\beta T + \left\|g_1\right\|_{L^1(0,T)}\right)\left[1 + \mu\left(1 - \sum_{j=1}^N \mu_j e^{-\beta T_j}\right)^{-1}\right] \le \dfrac{1}{4}. \end{cases}$$

Then, there exists a positive constant $R > 0$ *such that*

$$\begin{aligned} &\|\mathcal{P}_1(u,v)\|_{C([0,T])} \le \frac{R}{2}, \\ &\|\mathcal{P}_2(u,v)\|_{C([0,T])} \le \frac{R}{2}, \end{aligned} \tag{3.15}$$

for all $(u,v) \in B_R = \{(u,v) \in X : \|(u,v)\|_X \le R\}$.

Proof of Lemma 3.2. Let (u,v) be an arbitrary element in B_R. We have the following estimate

$$\begin{aligned} |\mathcal{P}_1(u,v)(t)| &\le |u_0| + \int_0^t \left|f_\alpha(s,u(s),v(s))\right| ds \\ &\le |u_0| + Tm_f + R\left(\alpha T + \left\|\Lambda_f\right\|_{L^1(0,T)}\right). \end{aligned} \tag{3.16}$$

Besides, we also have an estimate for $\mathcal{P}_2(u,v)$ as follows

$$\begin{aligned} |\mathcal{P}_2(u,v)(t)| &\le \mu\left(1 - \sum_{j=1}^N \mu_j e^{-\beta T_j}\right)^{-1} \int_0^T \left|g_\beta(s,u(s),v(s))\right| dt \\ &\quad + \int_0^t \left|g_\beta(s,u(s),v(s))\right| ds \\ &\le R\left(\beta T + \left\|g_1\right\|_{L^1(0,T)}\right)\left[1 + \mu\left(1 - \sum_{j=1}^N \mu_j e^{-\beta T_j}\right)^{-1}\right] \\ &\quad + \left[1 + \mu\left(1 - \sum_{j=1}^N \mu_j e^{-\beta T_j}\right)^{-1}\right]\left\|g_2\right\|_{L^1(0,T)}. \end{aligned} \tag{3.17}$$

Choosing $R > 0$ large enough such that

$$R \ge 4\max\left\{|u_0| + Tm_f,\ \left[1 + \mu\left(1 - \sum_{j=1}^N \mu_j e^{-\beta T_j}\right)^{-1}\right]\left\|g_2\right\|_{L^1(0,T)}\right\}. \tag{3.18}$$

Combining (3.16) - (3.18) and doing some direct calculations, we obtain an estimate as in (3.15). Lemma 3.2 is proved. □

Lemma 3.3. *Suppose that the conditions in the Lemma 3.2 are satisfied. Then, the operator* $U : X \to X$ *is a contraction.*

Proof of Lemma 3.3. Let (u,v) and $(\bar{u},\bar{v})$ be arbitrary elements in X. We have

$$\begin{aligned} |\mathcal{P}_1(u,v)(t) - \mathcal{P}_1(\bar{u},\bar{v})(t)| &\le \int_0^t e^{-\alpha(t-s)}\left|f_\alpha(s,u(s),v(s)) - f_\alpha(s,\bar{u}(s),\bar{v}(s))\right| ds \\ &\le \left(\alpha T + \left\|\Lambda_f\right\|_{L^1(0,T)}\right)\|(u,v) - (\bar{u},\bar{v})\|_X. \end{aligned} \tag{3.19}$$

By the assumption $\rho = \alpha T + \left\|\Lambda_f\right\|_{L^1(0,T)} < 1$, we infer that $\mathcal{P}_1 : X \to C([0,T])$ is a contraction mapping, so is the operator $U : X \to X$. Lemma 3.3 is proved. □

Lemma 3.4. *Suppose that the conditions in the Lemma 3.2 are satisfied. Then, the operator $C : B_R \to X$ is continuous and compact.*

Proof of Lemma 3.4.

Step 1: *$\mathcal{P}_2$ is continuous.* Let $\{(u_n, v_n)\} \subset B_R$ and $(u,v) \in B_R$ such that

$$\|(u_n, v_n) - (u,v)\|_X \to 0, \text{ as } n \to +\infty. \tag{3.20}$$

By the continuity of g_β and the Lebesgue's dominated convergence theorem, we get

$$\int_0^T g_\beta(t, u_n(t), v_n(t))dt \to \int_0^T g(t, u(t), v(t))dt, \text{ as } n \to +\infty. \tag{3.21}$$

Using (3.21), we infer that

$$\begin{aligned} &\sup_{t\in[0,T]} |\mathcal{P}_2(u_n,v_n)(t) - \mathcal{P}_2(u,v)(t)| \qquad (3.22)\\ &\le \mu\left(1 - \sum_{j=1}^N \mu_j e^{-\beta T_j}\right)^{-1} \int_0^T \left|g_\beta(t,u_n(t),v_n(t)) - g_\beta(t,u(t),v(t))\right| dt \\ &+ \int_0^T \left|g_\beta(t,u_n(t),v_n(t)) - g_\beta(t,u(t),v(t))\right| dt \ \to 0, \text{ as } n \to +\infty. \end{aligned}$$

Step 2: *$\mathcal{P}_2(B_R)$ is relatively compact.* It follows from the continuity of g_β that there exists $M_R > 0$ such that $\left|g_\beta(t,u(t),v(t))\right| \le M_R$ for all $(u,v) \in B_R$. Hence, the set $\mathcal{P}_2(B_R)$ is bounded in $C([0,T])$.

Taking arbitrary $(u,v) \in B_R$ and t_1, $t_2 \in [0,T]$, $t_1 < t_2$, we obtain

$$\begin{aligned} |\mathcal{P}_2(u,v)(t_1) - \mathcal{P}_2(u,v)(t_2)| &\le M_R T\mu\left(1 - \sum_{j=1}^N \mu_j e^{-\beta T_j}\right)^{-1} \left|e^{-\beta t_1} - e^{-\beta t_2}\right| \\ &+ \int_{t_1}^{t_2} \left|g_\beta(t,u(t),v(t))\right| dt, \end{aligned} \tag{3.23}$$

it leads to $\mathcal{P}_2(B_R)$ is equicontinuous. Therefore, the set $\mathcal{P}_2(B_R)$ is relatively compact in $C([0,T])$ due to the Arzelà-Ascoli's theorem. Lemma 3.4 is proved. □

Theorem 3.5. *Suppose that the conditions in the Lemma 3.2 are satisfied. Then, the problem* (1.1) - (1.2) *has a solution.*

Proof of Theorem 3.5. Combining Lemmas 3.2, 3.3, 3.4 and applying the Krasnoselskii's fixed point theorem, it is clear to see that $\mathcal{P} = U + C$ has a fixed point.

Theorem 3.5 is proved. □

Example 3.1. We consider the following problem

$$\begin{cases} u'(t) = \varepsilon \sin(v(t)), \ t \in (0,1), \\ v'(t) = \dfrac{\tilde{\varepsilon}\,|u(t)|}{1 + |u(t)|}, \ t \in (0,1), \\ u(0) = u_0 \in \mathbb{R}, \ v(0) = \dfrac{1}{2}v(1/2) + \dfrac{1}{4}v(1), \end{cases} \tag{3.24}$$

where $\varepsilon \le \dfrac{1}{6}$ and $\tilde{\varepsilon} \le \dfrac{1}{12}$.

By some calculations, we can check that the conditions of Theorem 3.1 are satisfied. Indeed, in Example 3.1, we have $f(t,u,v) = \varepsilon \sin v$, $g(t,u,v) = \dfrac{\tilde{\varepsilon}\,|u|}{1+|u|}$, $\mu_1 = T_1 = \dfrac{1}{2}$, $\mu_2 = \dfrac{1}{4}$, $T_2 = T = 1$.

It is clear to see that $\mu = \sum_{j=1}^{2} |\mu_j| = \frac{3}{4} < 1$ satisfying (H_1) and $f(t,u,v)$, $g(t,u,v)$ satisfy (H_2), (H_3) with $\Lambda_f(t) = \varepsilon$, $\Lambda_g(t) = \tilde{\varepsilon}$. We have $\|\Lambda_f\|_{L^1(0,T)} = \varepsilon$, $\|\Lambda_g\|_{L^1(0,T)} = \tilde{\varepsilon}$, so

$$\begin{aligned}
L &= \alpha T + \|\Lambda_f\|_{L^1(0,T)} + \left(\beta T + \|\Lambda_g\|_{L^1(0,T)}\right)\left[1 + \mu\left(1 - \sum_{j=1}^{N} \mu_j e^{-\beta T_j}\right)^{-1}\right] \\
&= \alpha + \varepsilon + (\beta + \tilde{\varepsilon})\left[1 + \frac{3}{4}\left(1 - \frac{1}{2}e^{-\beta/2} - \frac{1}{4}e^{-\beta}\right)^{-1}\right] \\
&\le \alpha + \frac{1}{6} + \left(\beta + \frac{1}{12}\right)\left[1 + \frac{3}{4}\left(1 - \frac{1}{2}e^{-\beta/2} - \frac{1}{4}e^{-\beta}\right)^{-1}\right].
\end{aligned}$$

We note that

$$\begin{aligned}
\lim_{\alpha\to 0_+,\, \beta\to 0_+} L &= \lim_{\alpha\to 0_+,\, \beta\to 0_+} \left\{\alpha + \frac{1}{6} + \left(\beta + \frac{1}{12}\right)\left[1 + \frac{3}{4}\left(1 - \frac{1}{2}e^{-\beta/2} - \frac{1}{4}e^{-\beta}\right)^{-1}\right]\right\} \\
&= \frac{1}{2} < 1,
\end{aligned}$$

Hence, we can choose $\alpha > 0$, $\beta > 0$ small enough such that $L < 1$. Thus, we deduce that the problem (3.24) has a unique solution. □

Example 3.2. Let us consider the following system

$$\begin{cases}
u'(t) = \dfrac{\varepsilon u(t)v(t)}{1 + |u(t)| + |v(t)|},\ t \in (0,1), \\
v'(t) = \tilde{\varepsilon} u(t)\sin(v(t)),\ t \in (0,1), \\
u(0) = u_0 \in \mathbb{R},\ v(0) = \dfrac{1}{3}v(1/4) - \dfrac{1}{7}v(1),
\end{cases} \tag{3.25}$$

with $\varepsilon < \frac{1}{9}$ and $\tilde{\varepsilon} \le \frac{1}{9}$. We also imply that the conditions of Theorem 3.5 are satisfied, where $f(t,u,v) = \frac{\varepsilon uv}{1+|u|+|v|}$, $g(t,u,v) = \tilde{\varepsilon} u \sin v$, $\mu_1 = \frac{1}{3}$, $T_1 = \frac{1}{4}$, $\mu_2 = \frac{-1}{7}$, $T_2 = T = 1$.

Indeed, by some calculations, we obtain (H_1), (H_2) and (H_4), in which

$$\mu = \sum_{j=1}^{N} |\mu_j| = \frac{10}{21} < 1;$$

$$\Lambda_f(t) = 2\varepsilon,\ g_1(t) = \tilde{\varepsilon},\ g_2(t) = 0,\ \|\Lambda_f\|_{L^1(0,T)} = 2\varepsilon;$$

On the other hand, we also have

$$\alpha T + \|\Lambda_f\|_{L^1(0,T)} = \alpha + 2\varepsilon \le \alpha + \frac{2}{9} \le \frac{1}{4},$$

if $0 < \alpha \le \frac{1}{36}$;

$$\begin{aligned}
&\left(\beta T + \|g_1\|_{L^1(0,T)}\right)\left[1 + \mu\left(1 - \sum_{j=1}^{N} \mu_j e^{-\beta T_j}\right)^{-1}\right] \\
&= (\beta + \tilde{\varepsilon})\left[1 + \frac{10}{21}\left(1 - \frac{1}{3}e^{-\beta/4} + \frac{1}{7}e^{-\beta}\right)^{-1}\right] \\
&\le \left(\beta + \frac{1}{9}\right)\left[1 + \frac{10}{21}\left(1 - \frac{1}{3}e^{-\beta/4} + \frac{1}{7}e^{-\beta}\right)^{-1}\right] \to \frac{3}{17} < \frac{1}{4},\ \text{as } \beta \to 0_+.
\end{aligned}$$

Consequently, we can choose $0 < \alpha \leq \frac{1}{36}$, $\beta > 0$ small enough such that

$$\alpha T + \left\|\Lambda_f\right\|_{L^1(0,T)} \leq \frac{1}{4} \text{ and } \left(\beta T + \left\|g_1\right\|_{L^1(0,T)}\right)\left[1 + \mu\left(1 - \sum_{j=1}^{N} \mu_j e^{-\beta T_j}\right)^{-1}\right] \leq \frac{1}{4}.$$

Applying Theorem 3.5, we also verify that the system (3.25) has a solution. □

4. Positive Solutions

The main purpose of this section is to prove the existence of positive solutions for the problem (1.1)-(1.2), in which $f, g \in C([0,T] \times \mathbb{R}^2; \mathbb{R})$. The main tool is the Guo-Krasnoselskii's fixed point theorem in a cone.

First, for the sake of simplicity, we consider the case $u_0 = 0$. Then, based on the preliminaries, the integral system (2.1), (2.2) can be written as follows

$$\begin{cases} u(t) = \int_0^t e^{-\alpha(t-s)} f_\alpha(s, u(s), v(s))ds, \\ v(t) = \int_0^T G(t,s) g_\beta(s, u(s), v(s))ds. \end{cases} \tag{4.1}$$

We make the following assumptions.

$(\tilde{H}_0)$ $\{\mu_j,\ j = \overline{1, N-1}\}$ satisfies (H_1) and $\mu_j \geq 0$, $\forall j = \overline{1, N-1}$, $\mu_N > 0$;

$(\tilde{H}_1)$ $f : [0,T] \times \mathbb{R}^2 \to \mathbb{R}$ is a continuous function and there exists a positive constant α such that

$$f(t,u,v) \geq -\alpha u, \text{ for all } (t,u,v) \in [0,T] \times \mathbb{R}_+ \times \mathbb{R}_+;$$

$(\tilde{H}_2)$ $g : [0,T] \times \mathbb{R}^2 \to \mathbb{R}$ is a continuous function and there exists a positive constant β such that

$$g(t,u,v) \geq -\beta v, \text{ for all } (t,u,v) \in [0,T] \times \mathbb{R}_+ \times \mathbb{R}_+.$$

We consider the space X and define the operator $\mathcal{P} : X \longrightarrow X$ as in (3.1), (3.2). We have the following simple lemmas.

Lemma 4.1. *Suppose that $(\tilde{H}_0)$–$(\tilde{H}_2)$ are satisfied. Then, for each $(u,v) \in X$ such that $u(t) \geq 0$, $v(t) \geq 0$, for all $t \in [0,T]$, we have $\mathcal{P}_1(u,v)(t) \geq 0$, $\mathcal{P}_2(u,v)(t) \geq 0$, for all $t \in [0,T]$.*

Proof of Lemma 4.1. The proof is easy, so we omit it. □

Lemma 4.2. *There exist a constant $\gamma \in (0,1)$ and a function $\Phi : [0,T] \to [0,+\infty)$ such that*

$$\gamma\Phi(s) \leq G(t,s) \leq \Phi(s), \ (s,t) \in [0,T] \times [0,T]. \tag{4.2}$$

Proof of Lemma 4.2. By the definition of the Green's function $G(t,s)$, we obtain that

$$\begin{aligned} G(t,s) &\leq e^{\beta s} + \left(1 - \sum_{j=1}^{N} \mu_j e^{-\beta T_j}\right)^{-1} \sum_{j=1}^{N} \mu_j e^{-\beta(T_j - s)} \\ &= \left[1 + \left(1 - \sum_{j=1}^{N} \mu_j e^{-\beta T_j}\right)^{-1} \sum_{j=1}^{N} \mu_j e^{-\beta T_j}\right] e^{\beta s} \\ &\leq \left[1 + \left(1 - \sum_{j=1}^{N} \mu_j e^{-\beta T_j}\right)^{-1}\right] e^{\beta s} := \Phi(s). \end{aligned} \tag{4.3}$$

On the other hand, if we choose $\gamma > 0$ defined as follows

$$\gamma = \frac{\mu_N e^{-2\beta T}}{2 - \sum_{j=1}^{N} \mu_j e^{-\beta T_j}}, \tag{4.4}$$

then $\gamma \leq \sum_{j=1}^{N} \mu_j < 1$ and we immediately obtain that $\gamma\Phi(s) \leq G(t,s)$, for all $(t,s) \in [0,T] \times [0,T]$.

Lemma 4.2 is completely proved. □

Now, we define the cone K in X as follows

$$K = \{(u,v) \in X : u(t) \geq 0,\ v(t) \geq \gamma \|(u,v)\|_X,\ \forall t \in [0,T]\}. \tag{4.5}$$

Lemma 4.3. *Suppose that the following conditions are fulfilled*

(i) $\alpha \leq \beta$;

(ii) $f_\alpha(t,u,v) \leq g_\beta(t,u,v)$ *for all* $(t,u,v) \in [0,T] \times \mathbb{R}_+ \times \mathbb{R}_+$.

Then, $\mathcal{P} : K \to K$.

Proof of Lemma 4.3. Let (u,v) be an arbitrary element in K. We have

$$\begin{aligned} \|\mathcal{P}(u,v)\|_X &= \max_{t\in[0,T]} \int_0^t e^{-\alpha(t-s)} \left| f_\alpha(s,u(s),v(s)) \right| ds + \max_{t\in[0,T]} \int_0^T G(t,s) \left| g_\beta(s,u(s),v(s)) \right| ds \\ &\leq \int_0^T e^{\alpha s} f_\alpha(s,u(s),v(s)) ds \\ &+ \left[1 + \left(1 - \sum_{j=1}^{N} \mu_j e^{-\beta T_j} \right)^{-1} \sum_{j=1}^{N} \mu_j e^{-\beta T_j} \right] \int_0^T e^{\beta s} g_\beta(s,u(s),v(s)) ds \\ &\leq \int_0^T \Phi(s) g_\beta(s,u(s),v(s)) ds. \end{aligned} \tag{4.6}$$

On the other hand, we also have

$$\begin{aligned} \mathcal{P}_2(u,v)(t) &= \int_0^T G(t,s) g_\beta(s,u(s),v(s)) ds \\ &\geq \gamma \int_0^T \Phi(s) g_\beta(s,u(s),v(s)) ds. \end{aligned} \tag{4.7}$$

It follows from (4.6), (4.7) that $\mathcal{P}_2(u,v)(t) \geq \gamma \|\mathcal{P}(u,v)\|_X$, so $\mathcal{P} : K \to K$. Lemma 4.3 is proved. □

Theorem 4.4. *Suppose that* $(\tilde{H}_0)$–$(\tilde{H}_2)$ *and the conditions in Lemma 4.3 are satisfied. Additionally, the following assertions are fulfilled*

(i) *There is a constant* $0 < \theta < T/2$ *such that*

$$\left| f_\alpha(t,u,v) \right| \leq \theta \left(|u| + |v| \right),\ \forall (t,u,v) \in [0,T] \times \mathbb{R}^2; \tag{4.8}$$

(ii) *There exist two positive constants* r, R, $r < R$, *such that*

$$\begin{aligned} &\text{(i)} \quad g_\beta(t,u,v) \leq \frac{r}{2T\hat{g}_1},\ \forall (t,u,v) \in [0,T] \times [0,r] \times [\gamma r, r], \\ &\text{(ii)} \quad g_\beta(t,u,v) \geq \frac{R}{2T\hat{g}_0},\ \forall (t,u,v) \in [0,T] \times [0,R] \times [\gamma R, R], \end{aligned} \tag{4.9}$$

or

$$\begin{aligned} &\text{(i)} \quad g_\beta(t,u,v) \geq \frac{r}{2T\hat{g}_0},\ \forall (t,u,v) \in [0,T] \times [0,r] \times [\gamma r, r], \\ &\text{(ii)} \quad g_\beta(t,u,v) \leq \frac{R}{2T\hat{g}_1},\ \forall (t,u,v) \in [0,T] \times [0,R] \times [\gamma R, R]. \end{aligned} \tag{4.10}$$

Then, the boundary value problem (1.1) - (1.2) *has a solution* (u,v) *with* $u(t) \geq 0$, $v(t) \geq 0$, *for all* $t \in [0,T]$.

Proof of Theorem 4.4. Using similar calculations and arguments as in Lemma 3.4, we obtain that the operator $\mathcal{P}$ is completely continuous. Let us consider two bounded sets as follows

$$\begin{aligned} \Omega_r &= \{(u,v) \in X : \|(u,v)\|_X < r\}, \\ \Omega_R &= \{(u,v) \in X : \|(u,v)\|_X < R\}. \end{aligned} \tag{4.11}$$

It is easy to see that Ω_r and Ω_R are open subsets of X with $0 \in \Omega_r$ and $\overline{\Omega_r} \subset \Omega_R$. We shall consider two cases.

Case 1. The (4.9) is true.

Take an arbitrary element $(u,v) \in K$ with $\|(u,v)\|_X = r$. We have the following estimates

$$\mathcal{P}_1(u,v)(t) \leq \theta \int_0^t (|u(s)| + |v(s)|)\, ds \leq T\theta \, \|(u,v)\|_X \,, \tag{4.12}$$

and

$$\mathcal{P}_2(u,v)(t) \leq \hat{g}_1 \int_0^T \left| g_\beta(s,u(s),v(s)) \right| ds \leq \frac{r}{2} = \frac{1}{2} \|(u,v)\|_X \,. \tag{4.13}$$

It follows from (4.12), (4.13) and (4.9, (i)) that

$$\|\mathcal{P}(u,v)\|_X \leq \|(u,v)\|_X \,, \ \forall (u,v) \in K \cap \partial\Omega_r. \tag{4.14}$$

On the other hand, for each $(u,v) \in K \cap \partial\Omega_R$, we have

$$\begin{aligned} \mathcal{P}_1(u,v)(t) + \mathcal{P}_2(u,v)(t) &\geq \int_0^T G(t,s) g_\beta(s,u(s),v(s)) ds \\ &\geq \hat{g}_0 \int_0^T g_\beta(s,u(s),v(s)) ds \geq R = \|(u,v)\|_X \,. \end{aligned} \tag{4.15}$$

Combining (4.14), (4.15) and applying the first part of Theorem 2.2, we deduce that there exists $(u^*,v^*) \in P \cap (\overline{\Omega_R} \setminus \Omega_r)$ such that $\mathcal{P}(u^*,v^*) = (u^*,v^*)$. It means that the problem (1.1) - (1.2), with $u_0 = 0$, has positive solutions.

Case 2. The (4.10) is true.

Using the same method as in Case 1, by applying the second part of Theorem 2.2, we obtain the similar result.

Theorem 4.4 is proved. □

Remark 4.1. In order to show the existence of positive solutions of the problem (1.1) - (1.2) with $u_0 > 0$, we put $\bar{u}(t) = u(t) - u_0$. Then, the pair of functions $(\bar{u}, v)$ is the solution of the following problem

$$\begin{cases} \bar{u}' = \bar{f}(t,\bar{u},v), \ t \in (0,T), \\ v' = \bar{g}(t,\bar{u},v), \ t \in (0,T), \\ \bar{u}(0) = 0, \ v(0) = \sum_{j=1}^N \mu_j v(T_j), \end{cases} \tag{4.16}$$

where

$$\begin{aligned} \bar{f}(t,u,v) &= f(t,u+u_0,v), \\ \bar{g}(t,u,v) &= g(t,u+u_0,v). \end{aligned}$$

Applying results in Theorem 4.4 for the system (4.16), we can obtain the existence of a solution (u,v) such that $u(t) \geq u_0$, $v(t) \geq 0$ for all $t \in [0,T]$. □

We will provide an example for Theorem 4.4 below.

Example 4.1. Let $\hat{g}_0$, $\hat{g}_1$ be denoted as in Lemma 2.4, γ be defined by (4.4) such that $r < \gamma R$. Let us consider the nonlinear first-order ordinary differential system as follows

$$\begin{cases} u'(t) = f(u(t), v(t)),\ t \in (0,T), \\ v'(t) = g(u(t), v(t)),\ t \in (0,T), \\ u(0) = u_0,\ v(0) = \sum\limits_{j=1}^{N} \mu_j v(T_j), \end{cases} \tag{4.17}$$

where $\{\mu_j,\ j = \overline{1, N-1}\}$ satisfies (H_1) and $\mu_j \geq 0$, $j = \overline{1, N-1}$, $\mu_N > 0$, it means that $(\tilde{H}_0)$ holds, and

$$f(u,v) = \varepsilon\left(g(u,v) + \beta v\right) - \alpha\left(u - \varepsilon\,|v| \sin^2(\sqrt[3]{v})\right), \tag{4.18}$$

$$g(u,v) + \beta v = \begin{cases} c_1 Q(u,v), & (u,v) \in \mathbb{R} \times (-\infty, r], \\ \dfrac{c_1(v - \gamma R)}{r - \gamma R} Q(u,r) + \dfrac{c_2(u + \gamma R)(v - r)}{\gamma R - r}, & (u,v) \in \mathbb{R} \times [r, \gamma R], \\ c_2(u + v), & (u,v) \in \mathbb{R} \times [\gamma R, +\infty), \end{cases}$$

for $0 \leq c_1 \leq \dfrac{1}{2T\hat{g}_1}$, $c_2 \geq \dfrac{1}{\gamma T \hat{g}_0}$ and $Q(u,v) = \dfrac{vu^2}{1 + u^2}$.

Now, we verify that $(\tilde{H}_1)$, $(\tilde{H}_2)$ are satisfied.

It is clear to see that $g \in C(\mathbb{R}^2; \mathbb{R})$ and for all $(u,v) \in \mathbb{R}_+ \times \mathbb{R}_+$, we have

(i) $(u,v) \in \mathbb{R}_+ \times [0,r] : g(u,v) + \beta v = c_1 Q(u,v) = \dfrac{c_1 v u^2}{1 + u^2} \geq 0$;

(ii) $(u,v) \in \mathbb{R}_+ \times [r, \gamma R]$:

$$\begin{aligned} g(u,v) + \beta v &= \frac{c_1(v - \gamma R)}{r - \gamma R} Q(u,r) + \frac{c_2(u + \gamma R)(v - r)}{\gamma R - r} \\ &= \left(1 - \frac{v - r}{\gamma R - r}\right) c_1 Q(u,r) + \frac{v - r}{\gamma R - r} c_2 (u + \gamma R) \\ &= (1 - \lambda)\, c_1 Q(u,r) + \lambda c_2 (u + \gamma R) \geq 0, \text{ with } \lambda = \frac{v - r}{\gamma R - r} \in [0,1]; \end{aligned}$$

(iii) $(u,v) \in \mathbb{R}_+ \times [\gamma R, +\infty) : g(u,v) + \beta v = c_2(u + v) \geq 0$.

Thus $g(u,v) + \beta v \geq 0$, $\forall (u,v) \in \mathbb{R}_+ \times \mathbb{R}_+$. It implies that $(\tilde{H}_2)$ holds.

On the other hand, f satisfies $(\tilde{H}_1)$. Indeed, by $f(t,u,v) = \varepsilon\left(g(u,v) + \beta v\right) - \alpha\left(u - \varepsilon\,|v| \sin^2(\sqrt[3]{v})\right)$, we have $f \in C(\mathbb{R}^2; \mathbb{R})$ and for all $(u,v) \in \mathbb{R}_+ \times \mathbb{R}_+$,

$$f(u,v) + \alpha u = \varepsilon\left[g(u,v) + \beta v + \alpha\,|v| \sin^2(\sqrt[3]{v})\right] \geq \varepsilon\left(g(u,v) + \beta v\right) \geq 0.$$

Next, the conditions in Lemma 4.3 are satisfied. We need prove that if $\alpha \leq \beta$, then

$$f(u,v) + \alpha u \leq g(u,v) + \beta v,\ \forall (u,v) \in \mathbb{R}_+ \times \mathbb{R}_+.$$

We have

$$\begin{aligned} f_\alpha(u,v) &= \varepsilon\left[g(u,v) + \beta v + \alpha\,|v| \sin^2(\sqrt[3]{v})\right] \leq \varepsilon\left[g(u,v) + \beta v + \alpha v\right] \\ &= \varepsilon\left[g(u,v) + \left(1 + \frac{\alpha}{\beta}\right)\beta v\right] \leq \varepsilon\left(1 + \frac{\alpha}{\beta}\right) g_\beta(u,v) \\ &\leq g_\beta(u,v),\ \forall (u,v) \in \mathbb{R}_+ \times \mathbb{R}_+, \end{aligned}$$

with $\varepsilon > 0$ small enough, such that $0 < \varepsilon\left(1 + \dfrac{\alpha}{\beta}\right) \leq 1$.

Finally, (4.8) and (4.9) are true.

Indeed, f satisfies the condition (4.8), ie., $\exists\theta \in (0, \frac{1}{2T}] : \left|f_\alpha(u,v)\right| = \left|f(u,v) + \alpha u\right| \le \theta\left(|u| + |v|\right), \forall u, v \in \mathbb{R}$.
For all $(u, v) \in \mathbb{R}^2$, we have

(i) $(u,v) \in \mathbb{R}\times(-\infty, r] : \left|g(u,v) + \beta v\right| = c_1\left|Q(u,v)\right| = \dfrac{c_1 |v|\, u^2}{1+u^2} \le c_1 |v| \le c_1\left(|u| + |v|\right);$

(ii) $(u,v) \in \mathbb{R}\times[r, \gamma R] :$ with $\lambda = \dfrac{v-r}{\gamma R - r} \in [0,1]$, we get

$$\begin{aligned}\left|g(u,v)+\beta v\right| &= \left|\left(1 - \frac{v-r}{\gamma R - r}\right)c_1 Q(u,r) + \frac{v-r}{\gamma R-r}c_2\left(u+\gamma R\right)\right| \\ &\le c_1 Q(u,r) + c_2(|u| + \gamma R) \\ &\le c_1|v| + c_2(|u| + \frac{\gamma R}{r}|v|) = c_2|u| + \left(c_1 + \frac{c_2\gamma R}{r}\right)|v| \\ &\le \left(c_1 + \frac{c_2\gamma R}{r}\right)\left(|u| + |v|\right);\end{aligned}$$

(iii) $(u,v) \in \mathbb{R}\times[\gamma R, +\infty) : \left|g(u,v) + \beta v\right| = |c_2(u+v)| \le c_2\left(|u| + |v|\right).$

It implies from (i)-(iii) that

$$\left|g(u,v) + \beta v\right| \le \left(c_1 + \frac{c_2\gamma R}{r}\right)\left(|u| + |v|\right),\ \forall (u,v) \in \mathbb{R}^2.$$

Hence

$$\begin{aligned}\left|f(u,v) + \alpha u\right| &= \varepsilon\left|\left[g(u,v) + \beta v + \alpha|v|\sin^2(\sqrt[3]{v})\right]\right| \\ &\le \varepsilon\left[\left|g(u,v) + \beta v\right| + \alpha|v|\right] \\ &\le \varepsilon\left[\left(c_1 + \frac{c_2\gamma R}{r}\right)\left(|u| + |v|\right) + \alpha\left(|u|+|v|\right)\right] \\ &\le \left[\varepsilon\left(c_1 + \frac{c_2\gamma R}{r}\right) + \alpha\right]\left(|u| + |v|\right) \equiv \theta\left(|u| + |v|\right),\ \forall u, v \in \mathbb{R},\end{aligned}$$

where $\theta = \left[\varepsilon\left(c_1 + \dfrac{c_2\gamma R}{r}\right) + \alpha\right] \le \dfrac{1}{2T}$, with $\varepsilon > 0$, $\alpha > 0$ small enough.

The function g_β satisfies the condition(4.9), because

(i) $(u,v) \in [0,r]\times[\gamma r, r] : g_\beta(u,v) = c_1 Q(u,v) = \dfrac{c_1 v u^2}{1+u^2} \le c_1 v \le \dfrac{r}{2T\hat{g}_1};$

(ii) $(u,v) \in [0,R]\times[\gamma R, R] : g_\beta(u,v) = c_2(u+v) \ge c_2 v \ge \dfrac{1}{\gamma T\hat{g}_0}\gamma R = \dfrac{R}{T\hat{g}_0}.$

We deduce that the assumptions and the conditions in Theorem 4.4 are satisfied, so we also verify that the system (4.17) has a positive solution. □

5. Multiplicity of positive solutions

In this section, we will show that the problem (1.1) - (1.2) can have two distinct solutions or even finitely many distinct solutions. The multiplicity of positive solutions depends strongly in the nonlinear term in (1.1). For the sake of simplicity, we just consider the case $u_0 = 0$.

First, in order to prove the multiplicity result, we assume that there exists $R_1 < \gamma R_2 < \gamma^2 R_3$ such that, for $j = \overline{1,2}$,

$(\hat{H}_1)$ $g_\beta(t,u,v) \le \dfrac{R_j}{2T\hat{g}_1}$ for all $(t,u,v) \in [0,T]\times[0,R_j]\times[\gamma R_j, R_j];$

$(\hat{H}_2)$ $g_\beta(t,u,v) \ge \dfrac{R_{j+1}}{T\hat{g}_0}$ for all $(t,u,v) \in [0,T]\times[0,R_{j+1}]\times[\gamma R_{j+1}, R_{j+1}].$

Theorem 5.1. *Assume that* $(\tilde{H}_0)$–$(\tilde{H}_1)$, (4.8) *and* $(\hat{H}_1)$–$(\hat{H}_2)$ *are satisfied. Then, the boundary value problem* (1.1) - (1.2) *has two solution* (u_1, v_1) *and* (u_2, v_2) *such that*

$$\begin{aligned} &R_1 < \|(u_1, v_1)\|_X \le R_2, \\ &R_2 < \|(u_2, v_2)\|_X \le R_3. \end{aligned} \tag{5.1}$$

Proof of Theorem 5.1. We denote the sets

$$\Omega_j = \left\{(u, v) \in X : \|(u, v)\|_X < R_j\right\}, \ j = \overline{1, 3}. \tag{5.2}$$

For $(u, v) \in K \cap \partial\Omega_1$, we have

$$\begin{aligned} &u(t) \le \|(u, v)\|_X = R_1, \\ &\gamma R_1 = \gamma \|(u, v)\|_X \le v(t) \le \|(u, v)\|_X = R_1. \end{aligned} \tag{5.3}$$

It follows from (5.3) and $(\hat{H}_1)$ that

$$g_\beta(t, u(t), v(t)) \le \frac{R_1}{2T\hat{g}_1}. \tag{5.4}$$

Combining (5.4) and (4.10), we obtain the following estimate

$$\begin{aligned} \|\mathcal{P}(u, v)\|_X &= \max_{0\le t\le T} \int_0^t \left|f_\alpha(s, u(s), v(s)\right| ds + \max_{0\le t\le T} \int_0^T \left|G(t, s)g_\beta(s, u(s), v(s))\right| ds \\ &\le \frac{1}{2} \|(u, v)\|_X + \frac{R_1}{2} = \|(u, v)\|_X . \end{aligned} \tag{5.5}$$

If $(u, v) \in K \cap \partial\Omega_2$, we have

$$\begin{aligned} &u(t) \le \|(u, v)\|_X = R_2, \\ &\gamma R_1 = \gamma\|(u, v)\|_X \le v(t) \le \|(u, v)\| = R_2. \end{aligned} \tag{5.6}$$

It follows from (5.6) and the assumption $(\hat{H}_2)$ that

$$\|\mathcal{P}(u, v)\|_X \ge \|(u, v)\|_X . \tag{5.7}$$

Applying the Guo-Krasnoselskii's fixed point theorem, we verify that there exists a pair of functions $(u_1, v_1) \in K \cap (\overline{\Omega_2} \setminus \Omega_1)$ such that $\mathcal{P}(u_1, v_1) = (u_1, v_1)$.

By using the similar calculations and arguments as the previous part, we also deduce that there exists a pair of functions $(u_2, v_2) \in K \cap (\overline{\Omega_3} \setminus \Omega_2)$ which is a fixed point of the operator $\mathcal{P}$.

Theorem 5.1 is proved. □

Next, we shall generalize results obtained in Theorem 5.1 to have the existence of finitely many distinct solutions. For this purpose, we assume that there exists $\{R_j\}_{j=1}^p$ such that $R_{j-1} < \gamma R_j$. We make the following assumptions

$(\hat{H}_3)$ $g_\beta(t, u, v) \le \dfrac{R_j}{2T\hat{g}_1}$ for all $(t, u, v) \in [0, T] \times [0, R_j] \times [\gamma R_j, R_j]$, $j = \overline{1, p-1}$,

$(\hat{H}_4)$ $g_\beta(t, u, v) \ge \dfrac{R_{j+1}}{T\hat{g}_0}$ for all $(t, u, v) \in [0, T] \times [0, R_{j+1}] \times [\gamma R_{j+1}, R_{j+1}]$, $j = \overline{1, p-1}$;

or

$(\hat{H}_5)$ $g_\beta(t, u, v) \ge \dfrac{R_j}{T\hat{g}_0}$ for all $(t, u, v) \in [0, T] \times [0, R_j] \times [\gamma R_j, R_j]$, $j = \overline{1, p-1}$,

$(\hat{H}_6)$ $g_\beta(t, u, v) \le \dfrac{R_{j+1}}{2T\hat{g}_1}$ for all $(t, u, v) \in [0, T] \times [0, R_{j+1}] \times [\gamma R_{j+1}, R_{j+1}]$, $j = \overline{1, p-1}$.

Theorem 5.2. *Assume that* $(\tilde{H}_0)$–$(\tilde{H}_1)$, (4.8) *and* $(\hat{H}_3)$–$(\hat{H}_4)$ (or $(\hat{H}_5)$–$(\hat{H}_6)$) *are satisfied. Then, the boundary value problem* (1.1) - (1.2) *has at least* $p-1$ *solutions* (u_j, v_j), $1 \le j \le p-1$ *such that*

$$R_j < \|(u,v)\|_X \le R_{j+1}, \ j = \overline{1, p-1}. \tag{5.8}$$

Finally, assume that we have a positive sequence $\{R_j\}$ such that $\frac{R_j}{R_{j+1}} < \gamma < 1$ such that for each $j \in \mathbb{N}$,

$(\hat{H}_7)$ $g_\beta(t,u,v) \le \frac{R_j}{2T\hat{g}_1}$ for all $(t,u,v) \in [0,T] \times [0,R_j] \times [\gamma R_j, R_j]$,

$(\hat{H}_8)$ $g_\beta(t,u,v) \ge \frac{R_{j+1}}{T\hat{g}_0}$ for all $(t,u,v) \in [0,T] \times [0,R_{j+1}] \times [\gamma R_{j+1}, R_{j+1}]$;

or

$(\hat{H}_9)$ $g_\beta(t,u,v) \ge \frac{R_j}{T\hat{g}_0}$ for all $(t,u,v) \in [0,T] \times [0,R_j] \times [\gamma R_j, R_j]$,

$(\hat{H}_{10})$ $g_\beta(t,u,v) \le \frac{R_{j+1}}{2T\hat{g}_1}$ for all $(t,u,v) \in [0,T] \times [0,R_{j+1}] \times [\gamma R_{j+1}, R_{j+1}]$.

Then, we have the following theorem

Theorem 5.3. *Assume that* $(\tilde{H}_0)$–$(\tilde{H}_1)$, (4.8) *and* $(\hat{H}_7)$–$(\hat{H}_8)$ (or $(\hat{H}_9)$–$(\hat{H}_{10})$) *are satisfied. Then, the boundary value problem* (1.1) - (1.2) *has infinitely many solutions* $\{(u_j, v_j)\}$, $j \in \mathbb{N}$ *such that*

$$R_j < \|(u,v)\|_X \le R_{j+1}, \ \forall j \in \mathbb{N}. \tag{5.9}$$

6. Conclusions

We have applied suitable fixed point theorems such as Banach contraction principle, the Krasnoselskii's fixed point theorem, the Guo-Krasnoselskii's fixed point theorem in cones, to prove the existence of solutions/positive solutions/the multiplicity of positive solutions for a nonlinear differential system with initial and multi-point boundary conditions. In future works, we shall apply the fixed point techniques as above to consider the existence and the properties of the solutions for some problems in more general forms, for example

$$u_k'(t) = f_k(t, u_1, \cdots, u_m, u_{m+1}, \cdots, u_{m+n}), \ t \in (0,T), \ k = \overline{1, m+n}, \tag{6.1}$$

asscociated with the initial and multipoint conditions

$$\begin{cases} u_k(0) = u_{0k}, \ k = \overline{1,m}, \\ u_k(0) = \sum\limits_{j=1}^{N} \mu_{kj} u_k(T_{kj}), \ k = \overline{m+1, m+n}, \end{cases} \tag{6.2}$$

where $f_k : [0,T] \times \mathbb{R}^{m+n} \to \mathbb{R}$ $(k = \overline{1, m+n})$ are given functions; and u_{0k} $(k = \overline{1,m})$, $0 < T_{k1} < T_{k2} < \cdots < T_{kN} = T$ $(k = \overline{m+1, m+n})$, μ_{kj} are given constants, with $\sum\limits_{j=1}^{N} |\mu_{kj}| \le 1$.

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