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Some Versions of Supercyclicity for a Set of Operators

Mohamed Amouch[a], **Otmane Benchiheb**[a]

[a] *University Chouaib Doukkali. Department of Mathematics, Faculty of science Eljadida, Morocco*

Abstract. Let X be a complex topological vector space and $L(X)$ the set of all continuous linear operators on X. An operator $T \in L(X)$ is supercyclic if there is $x \in X$ such that; $\mathbb{C}Orb(T,x) = \{\alpha T^n x : \alpha \in \mathbb{C}, n \geq 0\}$, is dense in X. In this paper, we extend this notion from a single operator $T \in L(X)$ to a subset of operators $\Gamma \subseteq L(X)$. We prove that most of related proprieties to supercyclicity in the case of a single operator T remains true for subset of operators Γ. This leads us to obtain some results for C-regularized groups of operators.

1. Introduction and Preliminary

Let X be a complex topological vector space and $L(X)$ the space of all continuous linear operators on X. By an operator, we always mean a continuous linear operator.

The most important and studied notion in the linear dynamics is that of hypercyclicity: an operator T acting on X is said to be *hypercyclic* if there exists some vector x whose orbit under T;

$$Orb(T,x) := \{T^n x : n \geq 0\},$$

is dense in X. Such a vector x is called a *hypercyclic vector* for T, and the set of all *hypercyclic vectors* for T is denoted by $HC(T)$. The first examples of *hypercyclic* operators on a Banach space were given by Rolewicz in 1969 in [13]. He proved that if B is a backward shift on the Banach space $\ell^p(\mathbb{N})$; $1 \leq p < \infty$, then λB is *hypercyclic* for any complex number λ such that $|\lambda| > 1$.

Another important notion in the linear dynamics is that of supercyclicity: we say that $T \in L(X)$ is a *supercyclic* operator if there is some vector $x \in X$ such that the cone generated by $Orb(T,x)$;

$$\mathbb{C}Orb(T,x) = \{\alpha T^n x : \alpha \in \mathbb{C}, n \geq 0\},$$

is dense in X. Such a vector x is called a *supercyclic vector* for T, and the set of all *supercyclic vectors* for T is denoted by $SC(T)$, see [11]. In the context of separable Banach spaces, Feldman [9] proved that an operator T is *supercyclic* if and only if it is *supercyclic transitive*, that is; for each pair (U,V) of nonempty open subsets of X there exist $\alpha \in \mathbb{C}$ and $n \geq 0$ such that

$$\alpha T^n(U) \cap V \neq \emptyset.$$

Another important notion that implies the supercyclicity is the supercyclicity criterion [14]. It provides several sufficient conditions that ensure supercyclicity. We say that an operator $T \in L(X)$ satisfies the

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Email addresses: amouch.m@ucd.ac.ma (Mohamed Amouch), otmane.benchiheb@gmail.com (Otmane Benchiheb)

supercyclicity criterion if there exist an increasing sequence of integers (n_k), a sequence (α_{n_k}) of nonzero complex numbers, two dense sets $X_0, Y_0 \subset X$ and a sequence of maps $S_{n_k} : Y_0 \longrightarrow X$ such that:

- $\alpha_{n_k} T^{n_k} x \longrightarrow 0$ for any $x \in X_0$;
- $\alpha_{n_k}^{-1} S_{n_k} y \longrightarrow 0$ for any $y \in Y_0$;
- $T^{n_k} S_{n_k} y \longrightarrow y$ for any $y \in Y_0$.

For a general overview of the hypercyclicity and supercyclicity see [6, 10].

An operator $T \in L(X)$ is called quasi-conjugate or quasi-similar to an operator $S \in L(Y)$ if there exists an operator $\phi : X \longrightarrow Y$ with dense range such that $\phi \circ T = S \circ \phi$. If ϕ can be chosen to be a homeomorphism, then T and S are called conjugate or similar, see [10, Definition 1.5]. A property $\mathcal{P}$ is said to be preserved under quasi-similarity if the following holds: if an operator $T \in L(X)$ has property $\mathcal{P}$, then every operator $S \in L(Y)$ that is quasi-similar to T has also property $\mathcal{P}$, see [10, Definition 1.7].

A set Γ of operators is called *hypercyclic* if there exists a vector x in X such that its orbit under Γ;

$$Orb(\Gamma, x) = \{Tx : T \in \Gamma\},$$

is dense in X. Such a vector x is called a *hypercyclic vector* for Γ. The set of all *hypercyclic vectors* for Γ is denoted by $HC(\Gamma)$, see [2, 4]. If the space generated by $Orb(\Gamma, x)$;

$$\text{span}\{Orb(\Gamma, x)\} = \text{span}(\{Tx : T \in \Gamma\}),$$

is dense in X for some vector x, then Γ is cyclic. The vector x is called a cyclic vector for Γ. The set of all cyclic vector for Γ is denoted by $C(\Gamma)$, see [1, 5].

In this work, we introduce and study the supercyclicity for a set of operators.

In Section 2, we introduce the notion of supercyclicity for a subset $\Gamma \subseteq L(X)$ and we prove most of related results to supercyclicity for Γ. We show that the set of *supercyclic vectors* for a set Γ is G_δ type and that the supercyclicity for a set Γ is preserved under quasi-similarity.

In Section 3, we introduce the notions of supercylic transitivity, strictly supertransitivity, supertransitivity and the supercyclicity criterion for a set Γ of operators. Also, we give the relationship between these notions and the supercyclicity.

In Sections 4, we apply previous results to prove some results for C-regularized groups of operators.

2. Supercyclic sets of operators

In the following definition, we introduce the notion of the supercyclicity of a set of operators. This definition generalizes the notion of the supercyclicity of a single operator.

Definition 2.1. *Let $\Gamma \subset L(X)$. We say that Γ is a supercyclic set of operators if there exists $x \in X$ such that the cone generated by $Orb(\Gamma, x)$;*

$$\mathbb{C}Orb(\Gamma, x) := \{\alpha Tx : \alpha \in \mathbb{C}, T \in \Gamma\},$$

is dense in X. Such a vector x is called a supercyclic vector for Γ. The set of all supercyclic vectors for Γ is denoted by $SC(\Gamma)$.

The following example shows the existence of supercyclic sets of operators on the field of complex numbers.

Example 2.2. *Let $X = \mathbb{C}$ and T be a nonzero operator on $\mathbb{C}$, then there exists $x \in \mathbb{C}$ such that $Tx \neq 0$. Let $\Gamma = \{T\}$, then*

$$\mathbb{C}Orb(\Gamma, x) = \mathbb{C}\{Tx\} = \mathbb{C}.$$

This means that Γ is supercyclic and x is a supercyclic vector for Γ.

Remark 2.3. *Let Γ be a subset of $L(X)$. Since for all $x \in X$ we have*

$$Orb(\Gamma, x) \subset \mathbb{C}Orb(\Gamma, x),$$

if Γ is hypercyclic, then it is supercyclic. The converse does not hold in general. Indeed, let Γ be the set defined as in Example 2.2, then Γ is supercyclic, but it is not hypercyclic.

It has been shown in [11] that X supports supercyclic operators if and only if $\dim(X) = 1$ or $\dim(X) = +\infty$. This result does not hold in general in the case of a set of operators. Moreover, the supercyclicity of a set of operators exists in each topological vector space X.

Example 2.4. *Let f be a nonzero linear form on a locally convex space X and D be a subset of X such that the set*

$$\mathbb{C}D := \{\alpha x : \alpha \in \mathbb{C}, x \in D\}$$

is a dense subset of X. For all $x \in X$, let T_x be an operator defined by:

$$\begin{array}{rcccc} T_x & : & X & \longrightarrow & X \\ & & y & \longmapsto & f(y)x. \end{array}$$

We consider $\Gamma_f = \{T_x : x \in D\}$ and let y be a vector of X such that $f(y) \neq 0$. Then

$$\mathbb{C}Orb(\Gamma_f, y) = \{\alpha T_x y : x \in D, \alpha \in \mathbb{C}\} = \{\alpha f(y)x : x \in D, \alpha \in \mathbb{C}\} = \mathbb{C}D.$$

Hence, $\overline{\mathbb{C}Orb(\Gamma_f, y)} = X$, which means that Γ_f is supercyclic and y is a supercyclic vector for Γ_f.

Remark 2.5. *Let T be an operator acting on a complex separable Banach space X such that $\dim(X) \geq 1$. By Ansari's theorem, if T is supercyclic, then for any $n \geq 2$, the operator T^n is supercyclic. Moreover, T and T^n share the same supercyclic vectors, see [3]. This result does not hold in general in the case of a set of operators. Indeed, let Γ_f be the set of operators defined as in Example 2.4, then Γ_f is supercyclic. However, every single operator T_x is not supercyclic since supercyclic operators are of dense range, see [7].*

Let $\Gamma \subset L(X)$. We denote by $\{\Gamma\}'$ the set of all elements of $L(X)$ which commute with every element of Γ.

Proposition 2.6. *Let T be an operator with dense range. If $T \in \{\Gamma\}'$, then $Tx \in SC(\Gamma)$, for all $x \in SC(\Gamma)$.*

Proof. Let O be a nonempty and open subset of X. Since T is of dense range, $T^{-1}(O)$ is nonempty and open. Let $x \in SC(\Gamma)$, then there exist $\alpha \in \mathbb{C}$ and $S \in \Gamma$ such that $\alpha Sx \in T^{-1}(O)$, that is $\alpha T(Sx) \in O$. Since $T \in \{\Gamma\}'$, it follows that $\alpha S(Tx) = \alpha T(Sx) \in O$. Hence, $Tx \in SC(\Gamma)$. □

Corollary 2.7. *Let Γ be a supercyclic set of operators. If $x \in SC(\Gamma)$, then $\alpha x \in SC(\Gamma)$, for all $\alpha \in \mathbb{C} \setminus \{0\}$.*

Let X and Y be topological vector spaces and let $\Gamma \subset L(X)$ and $\Gamma_1 \subset L(Y)$. Recall from [1], that Γ and Γ_1 are called quasi-similar if there exists an operator $\phi : X \longrightarrow Y$ with dense range such that for all $T \in \Gamma$, there exists $S \in \Gamma_1$ satisfying $S \circ \phi = \phi \circ T$. If ϕ is a homeomorphism, then Γ and Γ_1 are called similar.

It has been shown in [10] that the supercyclicity of a single operator is stable under quasi-similarity. In the following, we prove that the same result holds for sets of operators.

Proposition 2.8. *If $\Gamma \subset L(X)$ and $\Gamma_1 \subset L(Y)$ are quasi-similar, then Γ is supercyclic in X implies that Γ_1 is supercyclic in Y. Moreover, $\phi(SC(\Gamma) \subset SC(\Gamma_1)$.*

Proof. Let O be a nonempty open subset of Y, then $\phi^{-1}(O)$ is a nonempty open subset of X. If $x \in SC(\Gamma)$, then there exist $\alpha \in \mathbb{C}$ and $T \in \Gamma$ such that $\alpha Tx \in \phi^{-1}(O)$, that is $\alpha\phi(Tx) \in O$. Let $S \in \Gamma_1$ such that $S \circ \phi = \phi \circ T$. Hence, $\alpha S(\phi x) = \alpha\phi(Tx) \in O$. Hence $\phi x \in SC(\Gamma_1)$. □

Corollary 2.9. *Assume that $\Gamma \subset L(X)$ and $\Gamma_1 \subset L(Y)$ are similar. Then Γ is supercyclic in X if and only if Γ_1 is supercyclic in Y. Moreover,*

$$\phi(SC(\Gamma) = SC(\Gamma_1).$$

Proposition 2.10. *Let $\{\alpha_T\}_{T\in\Gamma}$ be a net of nonzero complex numbers. Then, Γ is supercyclic if and only if $\Gamma_1 := \{\alpha_T T : T \in \Gamma\}$ is supercyclic. Moreover, Γ and Γ_1 share the same supercyclic vectors.*

Proof. This is since $\mathbb{C}Orb(\Gamma, x) = \mathbb{C}Orb(\Gamma_1, x)$ for all $x \in X$. □

Proposition 2.11. *Let $\{X_i\}_{i=1}^n$ be a family of complex topological vector spaces and Γ_i be a subset of $L(X_i)$, for all $1 \le i \le n$. If $\oplus_{i=1}^n \Gamma_i$ is a supercyclic set in $\oplus_{i=1}^n X_i$, then Γ_i is a supercyclic set in X_i, for all $1 \le i \le n$. Moreover, if $(x_1, x_2, \dots, x_n) \in SC(\oplus_{i=1}^n \Gamma_i)$, then $x_i \in SC(\Gamma_i)$, for all $1 \le i \le n$. That is $SC(\oplus_{i=1}^n \Gamma_i) \subset \oplus_{i=1}^n SC(\Gamma_i)$.*

Proof. Let $(x_1, x_2, \dots, x_n) \in SC(\oplus_{i=1}^n \Gamma_i)$. For all $1 \le i \le n$, let O_i be a nonempty open subset of X_i, then $O_1 \times O_2 \times \cdots \times O_n$ is a nonempty open subset of $\oplus_{i=1}^n X_i$. Since $Orb(\oplus_{i=1}^n \Gamma_i, \oplus_{i=1}^n x_i)$ is dense in $\oplus_{i=1}^n X_i$, it follows that there exist $\alpha \in \mathbb{C}$ and $T_i \in \Gamma_i$; $1 \le i \le n$ such that

$$(\alpha T_1 x_1, \alpha T_2 x_2, \dots, \alpha T_n x_n) = \alpha(T_1 \times T_2 \times \cdots \times T_n)(x_1, x_2, \dots, x_n) \in O_1 \times O_2 \times \cdots \times O_n,$$

that is $\alpha T_i x_i \in O_i$, for all $1 \le i \le n$. Hence, Γ_i is a supercyclic set in X_i and $x_i \in SC(\Gamma_i)$, for all $1 \le i \le n$. □

A subset of X is said to be G_δ type if it is an intersection of a countable collection of open sets.

Using a countable basis of the topology of X, we can prove that the set of all supercyclic vectors for a set Γ is G_δ type as shows the next proposition.

Proposition 2.12. *Let X be a second countable topological vector space and $\Gamma \subset L(X)$ a supercyclic set. Then,*

$$SC(\Gamma) = \bigcap_{n\ge1}\left(\bigcup_{\beta\in\mathbb{C}\setminus\{0\}}\bigcup_{T\in\Gamma} T^{-1}(\beta U_n)\right),$$

where $(U_n)_{n\ge1}$ is a countable basis of the topology of X. As a consequence, $SC(\Gamma)$ is a G_δ type set.

Proof. Suppose that Γ is a supercyclic set. Then, $x \in SC(\Gamma)$ if and only if $\overline{\mathbb{C}Orb(\Gamma, x)} = X$. Equivalently, for all $n \ge 1$ we have $U_n \cap \mathbb{C}Orb(\Gamma, x) \ne \emptyset$. That is, for all $n \ge 1$ there exist $\alpha \in \mathbb{C}$ and $T \in \Gamma$ such that $x \in \alpha T^{-1}(U_n)$. This is equivalent to the fact that $x \in \bigcap_{n\ge1}\left(\bigcup_{\beta\in\mathbb{C}\setminus\{0\}}\bigcup_{T\in\Gamma} T^{-1}(\beta U_n)\right)$. Hence, $SC(\Gamma) = \bigcap_{n\ge1}\left(\bigcup_{\beta\in\mathbb{C}\setminus\{0\}}\bigcup_{T\in\Gamma} T^{-1}(\beta U_n)\right)$.

Since $\bigcup_{\beta\in\mathbb{C}\setminus\{0\}}\bigcup_{T\in\Gamma} T^{-1}(\beta U_n)$ is an open subset of X, for all $n \ge 1$, it follows that $SC(\Gamma)$ is a G_δ type. □

3. Density and Transitivity of Sets of Operators

The supercyclic transitivity of a single operator was introduced in [9]. In the following definition, we extend this notion to sets of operators.

Definition 3.1. *We say that Γ is a supercyclic transitive set of operators if for each pair of nonempty open subsets (U, V) of X, there exist $\alpha \in \mathbb{C} \setminus \{0\}$ and $T \in \Gamma$ such that*

$$T(\alpha U) \cap V \ne \emptyset.$$

The following example shows that each topological vector space X supports supercyclic transitive sets of operators.

Example 3.2. *Assume that X is a locally convex space. Let $x, y \in X$ and let f_y be a linear form on X such that $f_y(y) \ne 0$. We define an operator $T_{f_y,x}$ by*

$$\begin{array}{rccc} T_{f_y,x} & : & X & \longrightarrow & X \\ & & z & \longmapsto & f_y(z)x. \end{array}$$

Let Γ be a set of operators on X defined by $\Gamma = \{T_{f_y,x} : x, y \in X$ such that $f_y(y) \neq 0\}$. Then Γ is a supercyclic transitive set of operators. Indeed, let U and V be two nonempty open subsets of X. There exist $x, y \in X$ such that $x \in U$ and $y \in V$. We have

$$T_{f_y,x}(y) = f_y(y)x.$$

Since $f_y(y) \neq 0$, it follows that $x = \frac{1}{f_y(y)}T_{f_y,x}(y)$. Hence $x \in U$ and $x \in \frac{1}{f_y(y)}T_{f_y,x}(V)$, which implies that $U \cap \frac{1}{f_y(y)}T_{f_y,x}(V) \neq \emptyset$. Thus Γ is a supercyclic transitive set of operators.

The supercyclic transitivity of a single operators is preserved under quasi-similarity, see [10, Proposition 1.13]. The following proposition proves that this result holds for sets of operators.

Proposition 3.3. *Assume that $\Gamma \subset L(X)$ and $\Gamma_1 \subset L(Y)$ are quasi-similar. If Γ is supercyclic transitive in Y, then Γ_1 is supercyclic transitive in Y.*

Proof. Let U, V be nonempty open and subsets of Y. Since ϕ is of dense range, $\phi^{-1}(U)$ and $\phi^{-1}(V)$ are nonempty and open subsets of X. Since Γ is supercyclic transitive in X, there exist $y \in \phi^{-1}(U)$ and $\alpha \in \mathbb{C}$, $T \in \Gamma$ with $\alpha Ty \in \phi^{-1}(V)$, which implies that $\phi(y) \in U$ and $\alpha\phi(Ty) \in V$. Let $S \in \Gamma$ such that $S \circ \phi = \phi \circ T$. Then, $\phi(y) \in U$ and $\alpha S\phi(y) \in V$. Thus, $\alpha S(U) \cap V \neq \emptyset$. Hence, Γ_1 is supercyclic transitive in Y. □

Corollary 3.4. *Assume that $\Gamma \subset L(X)$ and $\Gamma_1 \subset L(Y)$ are similar. Then, Γ is supercyclic transitive in X if and only if Γ_1 is supercyclic transitive in Y.*

In the following result, we give necessary and sufficient conditions for a set of operators to be supercyclic transitive.

Theorem 3.5. *Let X be a normed space and $\Gamma \subset L(X)$. The following assertions are equivalent:*

(*i*) *Γ is supercyclic transitive;*

(*ii*) *For each $x, y \in X$, there exist sequences $\{k\}$ in $\mathbb{N}$, $\{x_k\}$ in X, $\{\alpha_k\}$ in $\mathbb{C}$ and $\{T_k\}$ in Γ such that $x_k \longrightarrow x$ and $T_k(\alpha_k x_k) \longrightarrow y$;*

(*iii*) *For each $x, y \in X$ and for W a neighborhood of zero, there exist $z \in X$, $\alpha \in \mathbb{C}$ and $T \in \Gamma$ such that $x - z \in W$ and $T(\alpha z) - y \in W$.*

Proof. (*i*) ⇒ (*ii*) Let $x, y \in X$. For all $k \geq 1$, let $U_k = B(x, \frac{1}{k})$ and $V_k = B(y, \frac{1}{k})$. Then U_k and V_k are nonempty open subsets of X. Since Γ is supercyclic transitive, there exist $\alpha_k \in \mathbb{C}$ and $T_k \in \Gamma$ such that $T_k(\alpha_k U_k) \cap V_k \neq \emptyset$. For all $k \geq 1$, let $x_k \in U_k$ such that $T_k(\alpha_k x_k) \in V_k$, then $\|x_k - x\| < \frac{1}{k}$ and $\|T_k(\alpha_k x_k) - y\| < \frac{1}{k}$ which implies that $x_k \longrightarrow x$ and $T_k(\alpha_k x_k) \longrightarrow y$.

(*ii*) ⇒ (*iii*) Clear;

(*iii*) ⇒ (*i*) Let U and V be two nonempty open subsets of X. There exist $x, y \in X$ such that $x \in U$ and $y \in V$. Since for all $k \geq 1$, $W_k = B(0, \frac{1}{k})$ is a neighborhood of 0, there exist $z_k \in X$, $\alpha_k \in \mathbb{C}$ and $T_k \in \Gamma$ such that $\|x - z_k\| < \frac{1}{k}$ and $\|T_k(\alpha_k z_k) - y\| < \frac{1}{k}$. This implies that $z_k \longrightarrow x$ and $T_k(\alpha_k z_k) \longrightarrow y$. There exists $N \in \mathbb{N}$ such that $z_k \in U$ and $T_k(\alpha_k z_k) \in V$, for all $k \geq N$. This implies that Γ is supercyclic transitive. □

Theorem 3.6. *Let X be a second countable Baire topological vector space and Γ a subset of $L(X)$. The following assertions are equivalent*:

(*i*) *$SC(\Gamma)$ is dense in X;*

(*ii*) *Γ is supercylic transitive.*

As a consequence, a supercyclic transitive set is supercyclic.

Proof. Since X is a second countable topological vector space, we can consider $(U_m)_{m\geq 1}$ a countable basis of the topology of X.
$(i) \Rightarrow (ii)$: Assume that $SC(\Gamma) = \bigcap_{n\geq 1}\left(\bigcup_{\beta\in\mathbb{C}\setminus\{0\}}\bigcup_{T\in\Gamma} T^{-1}(\beta U_n)\right)$ is dense in X. Hence, for all $n \geq 1$ the set $A_n = \bigcup_{\beta\in\mathbb{C}\setminus\{0\}}\bigcup_{T\in\Gamma} T^{-1}(\beta U_n)$ is dense in X. Thus, for all $n, m \geq 1$, we have $A_n \cap U_m \neq \emptyset$ which implies that for all $n, m \geq 1$ there exist $\beta \in \mathbb{C} \setminus \{0\}$ and $T \in \Gamma$, such that $T(\beta U_m) \cap U_n \neq \emptyset$, which implies that Γ supercyclic transitive.
$(ii) \Rightarrow (i)$: Let $n, m \geq 1$, then there exist $\beta \in \mathbb{C} \setminus \{0\}$ and $T \in \Gamma$ such that $T(\beta U_m) \cap U_n \neq \emptyset$ which implies that $T^{-1}(\frac{1}{\beta}U_n) \cap U_m \neq \emptyset$. Hence, for all $n \geq 1$ the set $\bigcup_{T\in\Gamma}\bigcup_{\beta\in\mathbb{C}\setminus\{0\}} T^{-1}(\beta U_n)$ is dense in X. □

In the following, we prove that the converse of Theorem 3.6 holds with some additional assumptions.

Theorem 3.7. *Let $\Gamma \subset L(X)$ such that for all $T, S \in \Gamma$ with $T \neq S$, there exists $A \in \Gamma$ such that $T = AS$. Then Γ is supercyclic implies that Γ is supercyclic transitive.*

Proof. Since Γ is supercyclic, there exists $x \in X$ such that $\mathbb{C}Orb(\Gamma, x)$ is a dense subset of X. Let U, V be nonempty and open subsets of X, then there exist $\alpha, \beta \in \mathbb{C} \setminus \{0\}$, and $T, S \in \Gamma$ such that

$$\alpha Tx \in U \qquad \text{and} \qquad \beta Sx \in V. \tag{1}$$

There exists $A \in \Gamma$ such that $T = AS$. By (1), we have $\alpha A(Sx) \in U$ and $\beta A(Sx) \in A(V)$ which implies that $U \cap A(\frac{\alpha}{\beta}V) \neq \emptyset$. Hence, Γ is supercyclic transitive. □

Remark 3.8. *Let Γ be a set of mutually commuting operators, that is; for each T, and S in Γ, we have $TS = ST$. Assume that each operator of Γ is of dense range. Then Γ is supercyclic implies that Γ is supercyclic transitive.*

Definition 3.9. *We say that $\Gamma \subset L(X)$ is strictly supertransitive if for each pair of nonzero elements x, y in X, there exist $\alpha \in \mathbb{C}$ and $T \in \Gamma$ such that $\alpha Tx = y$.*

Example 3.10. *Let X be a locally convex space and f a nonzero linear form on X. Let D be a subset of X such*

$$\mathbb{C}D := \{\alpha x : \alpha \in \mathbb{C}, x \in D\}$$

is dense in X. Let Γ_f be the set of operators defined as in Example 2.4. Let x and y be two elements of X, then

$$T_x(y) = f(y)x = \alpha x.$$

Hence Γ_f is strictly supertransitive.

Proposition 3.11. *A strictly supertransitive set is supercyclic transitive. As a consequence, a strictly supertransitive set is supercyclic.*

Proof. Let $\Gamma \subset L(X)$ be a strictly supertransitive set. If U and V are two nonempty open subsets of X, there exist $x, y \in X$ such that $x \in U$ and $y \in V$. Since Γ is strictly supertransitive, there exist $\alpha \in \mathbb{C}$ and $T \in \Gamma$ such that $\alpha Tx = y$. Hence, $\alpha Tx \in \alpha T(U)$ and $\alpha Tx \in V$. Thus, $\alpha T(U) \cap V \neq \emptyset$, which implies that Γ is supercyclic transitive. □

Proposition 3.12. *Assume $\Gamma \subset L(X)$ and $\Gamma_1 \subset L(Y)$ are similar. Then Γ is strictly supertransitive in X if and only Γ_1 is strictly supertransitive in Y.*

Proof. Let $x, y \in Y$. There exist $a, b \in X$ such that $\phi(a) = x$ and $\phi(b) = y$. Since Γ is strictly supertransitive in X, there exist $\alpha \in \mathbb{C}$ and $T \in \Gamma$ such that $\alpha Ta = b$. Let $S \in \Gamma_1$ such that $S \circ \phi = \phi \circ T$, this implies that $\alpha Sx = y$. Hence Γ_1 is strictly supertransitive in Y. □

The strong operator topology (SOT for short) on $L(X)$ is the topology for which a neighborhood of $T \in L(X)$ is given by

$$\Omega = \{S \in L(X) : Se_i - Te_i \in U, i = 1, 2, \dots, k\},$$

where $k \in \mathbb{N}$, $e_1, e_2, \dots e_k \in X$ are linearly independent and U is a neighborhood of zero in X.

In the following theorem, the proof is true for norm-density if X is assumed to be a normed linear space.

Theorem 3.13. *Let X be a topological vector space. Then for each pair of nonzero linearly independent vectors x, $y \in X$ there exists a SOT-dense set $\Gamma_{xy} \subset L(X)$ which is not strictly supertransitive. Furthermore, $\Gamma \subset L(X)$ is a dense nonstrictly transitive set if and only if Γ is a dense subset of Γ_{xy} for some $x, y \in X$.*

Proof. Fix nonzero linearly independent vectors $x, y \in X$ and put

$$\Gamma_{xy} = \{T \in L(X) : y \text{ and } Tx \text{ are linearly independent}\}.$$

It is clear that Γ_{xy} is not strictly supertransitive. Let Ω be a nonempty open subset of $L(X)$ and $S \in \Omega$. If Sx and y are linearly independent, then $S \in \Omega \cap \Gamma_{xy}$. Otherwise, putting $S_n = S + \frac{1}{n}I$, we see that $S_k \in \Omega$ for some k, but $S_k x$ and y are linearly independent. Hence, $\Omega \cap \Gamma_{xy} \neq \emptyset$ and the proof of the first part is complete.

We prove the second part of the theorem. Suppose that Γ is a dense subset of $L(X)$ that is not strictly supertransitive. Then there exist nonzero vectors $x, y \in X$ such that Tx and y are linearly independent for all $T \in \Gamma$ and hence $\Gamma \subset \Gamma_{xy}$. To show that Γ is dense in Γ_{xy}, assume that Ω_0 is an open subset of Γ_{xy}. Thus, $\Omega_0 = \Gamma_{xy} \cap \Omega$ for some open set Ω in $L(X)$. Then $\Gamma \cap \Omega_0 = \Gamma \cap \Omega \neq \emptyset$.

For the converse, let Γ be a dense subset of Γ_{xy} for some $x, y \in X$. Then Γ is not strictly supertransitive. Also, since Γ_{xy} is a dense open subset of $L(X)$, we conclude that Γ is also dense in $L(X)$. Indeed, if Ω is any open set in $L(X)$ then $\Omega \cap \Gamma_{xy} \neq \emptyset$ since Γ_{xy} is dense in $L(X)$. On the other hand, $\Omega \cap \Gamma_{xy}$ is open in Γ_{xy} and so it must intersect Γ since Γ is dense in Γ_{xy}. Thus, $\Omega \cap \Gamma \neq \emptyset$ and so Γ is dense in LX). □

Corollary 3.14. *Let X be a topological vector space and Γ be a SOT-dense subset of $L(X)$. Then there is a subset Γ_1 of Γ such that $\overline{\Gamma_1}^{SOT} = L(X)$ and Γ_1 is not strictly supertransitive.*

Proof. For nonzero linearly independent vectors x, y put $\Gamma_1 = \Gamma \cap \Gamma_{xy}$. □

Definition 3.15. *A set $\Gamma \subset L(X)$ is said to be supertransitive if $SC(\Gamma) = X \setminus \{0\}$.*

Remarks 3.16. *Let $T \in L(X)$.*

(*i*) *If T is supertransitive, then it is injective of dense range.*

(*ii*) *T is supertransitive if and only if T^p is supertransitive, for all $p \geq 2$.*

The next proposition shows that supertransitivity implies supercyclic transitivity.

Proposition 3.17. *If Γ is supertransitive, then it is supercyclic transitive.*

Proof. Let U, V be nonempty and open subsets of X. There exists $x \in X \setminus \{0\}$ such that $x \in U$. Since Γ is supertransitive, there exist $\alpha \in \mathbb{C}$ and $T \in \Gamma$ such that $\alpha Tx \in V$, it follows that $\alpha T(U) \cap V \neq \emptyset$. Hence, Γ is supercyclic transitive. □

Proposition 3.18. *Assume that $\Gamma \subset L(X)$ and $\Gamma_1 \subset L(Y)$ are similar. Then, Γ is supertransitive on X if and only if Γ_1 is supertransitive on Y.*

Proof. It suffices to use Proposition 2.8 and verify that $\phi(X \setminus \{0\}) = X \setminus \{0\}$. □

The following result shows that the SOT-closure of Γ is not large enough to have more supercyclic vectors than Γ.

Proposition 3.19. *Let $\Gamma \subset L(X)$. Then $SC(\Gamma) = SC\left(\overline{\Gamma}^{SOT}\right)$.*

Proof. Let $x \in SC\left(\overline{\Gamma}^{SOT}\right)$. If U is an open set of X, then there is some $\alpha \in \mathbb{C}$ and $T \in \overline{\Gamma}^{SOT}$ such that $\alpha Tx \in U$. Since $\Omega = \{S \in L(X) : \alpha Sx \in U\}$ is a SOT-neighborhood of T, there is some $S \in \Gamma$ such that $\alpha Sx \in U$ and this shows that $x \in SC(\Gamma)$. □

Corollary 3.20. *Let $\Gamma \subset L(X)$. Then, Γ is supertransitive if and only if $\overline{\Gamma}^{SOT}$ is supertransitive.*

In the following definition, we introduce the notion of the supercyclicity criterion for a set of operators.

Definition 3.21. *We say that Γ satisfies the criterion of supercyclicity if there exist two dense subsets X_0, Y_0 in X, and sequences $\{k\} \subset \mathbb{N}$, $\{\alpha_k\} \subset \mathbb{C} \setminus \{0\}$, $\{T_k\} \subset \Gamma$, and maps $S_k : Y_0 \longrightarrow X$ such that*:

(*i*) $\alpha_k T_k x \longrightarrow 0$ *for all* $x \in X_0$;

(*ii*) $\alpha_k^{-1} S_k y \longrightarrow 0$ *for all* $y \in Y_0$;

(*iii*) $T_k S_k y \longrightarrow y$ *for all* $y \in Y_0$.

Theorem 3.22. *Let X be a second countable Baire topological vector space and Γ a subset of $L(X)$. If Γ satisfies the criterion of supercyclicity, then it is supercyclic.*

Proof. Let U, V be nonempty open subsets of X. There exist x_0, y_0 in X such that $x_0 \in X_0 \cap U$ and $y_0 \in Y_0 \cap V$. For all $k \geq 1$, let $z_k = x_0 + \alpha_k^{-1} S_k y$. It follows that $z_k \longrightarrow x_0$, and $\alpha_k T_k z_k \longrightarrow y_0$. Hence, there exists k such that $\alpha_k T_k(U) \cap V \neq \emptyset$. □

4. Supercyclicity of *C*-Regularized Groups

In this section, we study the particular case where Γ stands for a C-regularized group. Recall from [8], that an entire C-regularized group is an operator family $(S(z))_{z\in\mathbb{C}}$ on $L(X)$ that satisfies:

(1) $S(0) = C$;

(2) $S(z + w)C = S(z)S(w)$ for every $z, w \in \mathbb{C}$,

(3) The mapping $z \mapsto S(z)x$, with $z \in \mathbb{C}$, is entire for every $x \in X$.

Example 4.1. *Let $X = \mathbb{C}$. For all $z \in \mathbb{C}$, let $S(z)x = \exp(z)x$, for all $x \in \mathbb{C}$. $(S(z))_{z\in\mathbb{C}}$ is a C-regularized group of operators and we have $\overline{\mathbb{C}Orb((S(z))_{z\in\mathbb{C}}, x)} = \mathbb{C}$, for all $x \in \mathbb{C} \setminus \{0\}$. Hence $(S(z))_{z\in\mathbb{C}}$ is supercyclic and $SC((S(z))_{z\in\mathbb{C}}) = \mathbb{C} \setminus \{0\}$.*

By Theorem 3.6, every supercyclic transitive C-regularized group is supercyclic. In the following, we prove that the converse holds.

Theorem 4.2. *Assume that C is of dense range. If $(S(z))_{z\in\mathbb{C}})$ is supercyclic, then it is supercyclic transitive.*

Proof. Let $x \in SC((S(z))_{z\in\mathbb{C}})$. If U and V are two nonempty open subsets of X, then there exist $\alpha, \beta, z_1, z_2 \in \mathbb{C}$ such that $\alpha S(z_1)x \in C^{-1}(U)$ and $\beta S(z_2)x \in V$. Let $z_3 = z_1 - z_2$, then $U \cap \frac{\alpha}{\beta} S(z_3)(V) \neq \emptyset$. Hence, $(S(z))_{z\in\mathbb{C}}$ is a supercyclic transitive C-regularized group. □

Theorem 4.3. *Let $(S(z))_{z\in\mathbb{C}}$ be a supercyclic C-regularized group on a Banach infinite-dimensional space X. Assume that C is of dense range. If $x \in X$ is a supercyclic vector of $(S(z))_{z\in\mathbb{C}}$, then the following assertions hold:*

(1) *$S(z)x \neq 0$ for all $z \in \mathbb{C}$;*

(2) *The set $\{\alpha S(z)x : \alpha, z \in \mathbb{C}, |z| > |\omega_0|\}$ is dense in X for all $\omega_0 \in \mathbb{C}$.*

Proof. (1) Clear.
(2) Let $\omega_0 \in \mathbb{C}$ such that the set $A := \{\alpha S(z)x : \alpha, z \in \mathbb{C}, |z| > |\omega_0|\}$ is not dense in X. Hence there exists a bounded open set U such that $U \cap \overline{A} = \emptyset$. Therefore we have

$$U \subset \overline{\{\alpha S(z)x : \alpha, z \in \mathbb{C}, |z| \leq |\omega_0|\}}$$

by using the relation

$$X = \overline{\{\alpha S(z)x : \alpha, z \in \mathbb{C}\}} = \overline{\{\alpha S(z)x : \alpha, z \in \mathbb{C}, |z| > |\omega_0|\}} \cup \overline{\{\alpha S(z)x : \alpha, z \in \mathbb{C}, |z| \leq |\omega_0|\}}.$$

Since $S(z)x$ is continuous with z and $S(z)x \neq 0$ holds for all $z \in \mathbb{C}$ by (1), there exist $m_1, m_2 > 0$ such that $0 < m_1 \leq \|S(z)x\| < m_2$ for $z \in \mathbb{C}$ with $|z| \leq |\omega_0|$. There exists $M > 0$ such that $\|y\| \leq M$ for any $y \in U$ because U is bounded. So we have

$$U \subset \overline{\left\{\alpha S(z)x : |z| \leq |\omega_0|, |\alpha| \leq \frac{M}{m_1}\right\}},$$

which means that $\overline{U}$ is compact. Hence X is finite dimensional, which contradicts that X is infinite dimensional. □

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