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Gaussian Pell and Gaussian Pell-Lucas Quaternions

Hasan Arslan[a]

[a]*Erciyes University, Faculty of Science, Department of Mathematics, 38039, Kayseri, Turkey*

Abstract. The main aim of this work is to introduce the Gaussian Pell quaternion QGp_n and Gaussian Pell-Lucas quaternion QGq_n, where the components of QGp_n and QGq_n are Pell numbers p_n and Pell-Lucas numbers q_n, respectively. Firstly, we obtain the recurrence relations and Binet formulas for QGp_n and QGq_n. We use Binet formulas to prove Cassini's identity for these quaternions. Furthermore, we give some basic identities for QGp_n and QGq_n such as some summation formulas, the terms with negative indices and the generating functions for these complex quaternions.

1. PRELIMINARIES AND INTRODUCTION

The quaternions, which are a members of a noncommutative division algebra, were first invented by W. R. Hamilton in 1843 as an extension of the set of complex numbers. The set of real quaternions is denoted by **H**. A quaternion k is represented in the form:

$$k = k_0e_0 + k_1e_1 + k_2e_2 + k_3e_3 = (k_0, k_1, k_2, k_3),$$

where k_0, k_1, k_2 and k_3 are real numbers and e_0, e_1, e_2, and e_3 are the fundamental quaternionic units such that

$$e_0^2 = 1,\ e_0e_i = e_ie_0 = e_i,\ i = 1, 2, 3,\ e_1^2 = e_2^2 = e_3^2 = e_1e_2e_3 = -1. \tag{1}$$

Quaternions find uses in pure and applied mathematics, quantum physics, the special theory of relativity and analysis, see for example [1], [8], [9], [15], [16]. In the literature, it can be found many researchers working on the structure of Fibonacci sequences and their generalizations see [2], [3], [6], [10], [17], [21], [25], [26], [27], [28]. Due to [27], the generalized Gaussian Fibonacci sequence $Gf_n(p, q; a, b)$ is defined by in the following way:

$$Gf_{n+1} = pGf_n + qGf_{n-1}, Gf_0 = a, Gf_1 = b \tag{2}$$

where a and b are initial values. If we take $p = 2, q = 1, a = i, b = 1$ in the equation (2) then we get the Gaussian Pell sequence

$$\{Gp_n\} = \{i, 1, 2 + i, 5 + 2i, \cdots\}.$$

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Email address: hasanarslan@erciyes.edu.tr (Hasan Arslan)

In the equation (2) if we substitute $p = 2, q = 1, a = 1 - i, b = 1 + i$ then we obtain the Gaussian Pell-Lucas sequence

$$\{Gq_n\} = \{1 - i, 1 + i, 3 + i, 7 + 3i, \cdots\}.$$

Also we have $Gp_n = p_n + ip_{n-1}$ and $Gq_n = q_n + iq_{n-1}$, where p_n and q_n are *nth* Pell and Pell-Lucas numbers, respectively.

In [11], Djordjevic and Srivastava introduced the generalized incomplete Fibonacci polynomials and the generalized incomplete Lucas polynomials in a systematic way and also they investigated their structures. In [12], Djordjevic and Srivastava defined incomplete generalized Jacobsthal and Jacobsthal-Lucas numbers and derived their generating functions. In [13], Djordjevic and Srivastava established two different sequences of numbers, which are generalizations of the classical Fibonacci numbers, and obtained many important combinatorial properties of these general sequences of numbers. In [14], Srivastava, Tuglu and Cetin defined new families of the q-Fibonacci and q-Lucas polynomials providing the q-analogues of the incomplete Fibonacci and Lucas numbers, respectively. They proved some properties of these q-polynomial families such as recurrence relations, summation formulas and generating functions. In [22], Raina and Srivastava constructed a new class of numbers involving the familiar Lucas sequences and they deduced a number of results for this class of numbers such as hypergeometric representations, recurrence relations, generating functions and summation formulas.

Horadam [3] introduced the quaternions with the *nth* Fibonacci and Lucas numbers coefficients as follows:

$$Q_n = f_n e_0 + f_{n+1} e_{n+1} + f_{n+2} e_{n+2} + f_{n+3} e_3$$

$$K_n = l_n e_0 + l_{n+1} e_1 + l_{n+2} e_2 + l_{n+3} e_3$$

respectively, where f_n and l_n are the usual *nth* Fibonacci and Lucas numbers.

Quaternions whose coeficients consist of Fibonacci-like numbers have been studying by many researchers, recently, see [3], [5], [7], [19], [20], [23], [24]. In [23], Halici dealt with the Fibonacci quaternions and obtained their some combinatorial properties. In [4], Horadam studied on the Pell and Pell-Lucas sequences and he presented some identities for them as follows:

$$p_{n+1}p_{n-1} - p_n^2 = (-1)^n \quad \text{(Cassini-like formula)}$$

$$p_n(p_{n+1} - p_{n-1}) = p_{2n}$$

$$p_r p_{n+1} - p_{r-1} p_n = p_{n+r}$$

$$p_n^2 + p_{n+1}^2 = p_{2n+1}$$

$$p_{2n+1}p_{2n} = 2p_{2n+1}^2 - 2p_{2n}^2 - (-1)^n$$

$$(-1)^n p_a p_b = p_{n+a}p_{n+b} - p_n p_{n+a+b}$$

$$p_{-n} = (-1)^{n+1} p_n.$$

It was firstly suggested by Horadam in [5] the idea to consider Pell quaternions. In [7], Cimen and Ipek introduced the Pell and Pell-Lucas quaternions, respectively, as follows:

$$Qp_n = p_n e_0 + p_{n+1} e_1 + p_{n+2} e_2 + p_{n+3} e_3, \tag{3}$$

$$Qq_n = q_n e_0 + q_{n+1} e_1 + q_{n+2} e_2 + q_{n+3} e_3. \tag{4}$$

They then investigated the structures of Pell and Pell-Lucas quaternions by the methods which depend more on the properties of Pell and Pell-Lucas numbers in [7].

In this paper, we introduce Gaussian Pell and Gaussian Pell-Lucas quaternions and derive their some combinatorial properties such as Binet formulas, Cassini identities, negatively subscripted terms and the generating functions.

2. GAUSSIAN PELL AND GAUSSIAN PELL-LUCAS QUATERNIONS

Any complex quaternion Ψ is defined in the following form

$$\Psi = \Psi_0 e_0 + \Psi_1 e_1 + \Psi_2 e_2 + \Psi_3 e_3,$$

where each $\Psi_i, i = 0, 1, 2, 3$ is complex numbers and e_0, e_1, e_2, e_3 are defined as in (1). The set of all complex quaternions is denoted by $\mathbf{H}_{\mathbb{C}}$. We can rewrite the complex quaternion Ψ as

$$\Psi = k + ik', i^2 = -1$$

where k and k' are real quaternions.
Halici [24] introduced the complex Fibonacci quaternions and gave their some properties. In the similar way, we can define Gaussian Pell and Gaussian Pell-Lucas quaternions as follows:

$$QGp_n = Gp_n e_0 + Gp_{n+1} e_1 + Gp_{n+2} e_2 + Gp_{n+3} e_3 \quad (5)$$

$$QGq_n = Gq_n e_0 + Gq_{n+1} e_1 + Gq_{n+2} e_2 + Gq_{n+3} e_3 \quad (6)$$

where Gp_n and Gq_n stand for *nth* Gaussian Pell and Gaussian Pell-Lucas numbers. Since $Gp_n = p_n + ip_{n-1}$, we get $QGp_n = Qp_n + iQp_{n-1}$, where Qp_n and Qp_{n-1} are *nth* and *(n-1) th* Pell quaternions as in (3). Similarly, because of $Gq_n = q_n + iq_{n-1}$, we have $QGq_n = Qq_n + iQq_{n-1}$, where Qq_n and Qq_{n-1} are *nth* and *(n-1) th* Pell-Lucas quaternions as in (4).

Basic operations on Gaussian Pell quaternions such as addition, substraction, multiplication are defined just as in real quaternions. The quaternion conjugate of QGp_n is defined as

$$QGp_n^* = Gp_n e_0 - Gp_{n+1} e_1 - Gp_{n+2} e_2 - Gp_{n+3} e_3.$$

The complex conjugate of QGp_n is given by

$$\overline{QGp_n} = \overline{Gp_n} e_0 + \overline{Gp_{n+1}} e_1 + \overline{Gp_{n+2}} e_2 + \overline{Gp_{n+3}} e_3.$$

For any complex quaternion $\Psi = \Psi_0 e_0 + \Psi_1 e_1 + \Psi_2 e_2 + \Psi_3 e_3$, the quaternion norm of Ψ is defined by $\|\Psi\| = \Psi_0^2 + \Psi_1^2 + \Psi_2^2 + \Psi_3^2$. Since each component of Ψ is a complex number, then the norm of Ψ is a complex number. Thus, we give the norms of the Gaussian Pell quaternion QGp_n and the Gaussian Pell-Lucas quaternion QGq_n in the following form:

$$N_{QGp_n} = QGp_n QGp_n^* = Gp_n^2 + Gp_{n+1}^2 + Gp_{n+2}^2 + Gp_{n+3}^2$$

and

$$N_{QGq_n} = QGq_n QGq_n^* = Gq_n^2 + Gq_{n+1}^2 + Gq_{n+2}^2 + Gq_{n+3}^2,$$

respectively.

Proposition 2.1. *For the Gaussian Pell quaternion QGp_n and the Gaussian Pell-Lucas quaternion QGq_n, we have the following identities;*

$$N_{QGp_n} = 12(1 + i)p_{2n+2}.$$

$$N_{QGq_n} = 24(1 + i)p_{2n+2}.$$

Proof. From the definition of the norm of a Gaussian Pell quaternion, we can write

$$\begin{aligned} N_{QGp_n} &= QGp_n QGp_n^* = Gp_n^2 + Gp_{n+1}^2 + Gp_{n+2}^2 + Gp_{n+3}^2 \\ &= (p_n + ip_{n-1})^2 + (p_{n+1} + ip_n)^2 + (p_{n+2} + ip_{n+1})^2 + (p_{n+3} + ip_{n+2})^2 \\ &= p_{n+3}^2 - p_{n-1}^2 + 2i(p_n(p_{n-1} + p_{n+1}) + p_{n+2}(p_{n+1} + p_{n+3})). \end{aligned}$$

Since $p_{n-1} + p_{n+1} = 2q_n$ and $2p_nq_n = p_{2n}$, then we have $N_{QGp_n} = p^2_{n+3} - p^2_{n-1} + 12ip_{2n+2}$. Moreover, from Page 148 of Koshy [29] , we get $p^2_{n+3} - p^2_{n-1} = 12p_{2n+2}$. Therefore, we get $N_{QGp_n} = 12(1 + i)p_{2n+2}$.

As for the norm of N_{QGq_n}, since the well-known identities $q_{n+1} + q_{n-1} = 4p_n$, $2p_nq_n = p_{2n}$, $p_{n+2} + p_{n-2} = 6p_n$ and $q^2_{n+3} - q^2_{n-1} = 24p_{2n+2}$, then we obtain

$$\begin{aligned}
N_{QGq_n} &= Gq_n^2 + Gq_{n+1}^2 + Gq_{n+2}^2 + Gq_{n+3}^2 \\
&= q^2_{n+3} - q^2_{n-1} + 2i(q_n(q_{n-1} + q_{n+1}) + q_{n+2}(q_{n+1} + q_{n+3})) \\
&= = 24(1 + i)p_{2n+2}.
\end{aligned}$$

Hence the proof is completed. □

The inverse of any complex quaternion Ψ is given by $\Psi^{-1} = \frac{\Psi^*}{N_\Psi}$, $N_\Psi \neq 0$, see [18]. The next corollary is clearly seen by the definition of Gaussian Pell quaternion.

Corollary 2.2. *For the QGp_n, QGp_n^* and $\overline{QGp_n}$, we have the following identities;*

$$QGp_n + QGp_n^* = 2Gp_ne_0.$$

$$QGp_n + \overline{QGp_n} = 2Qp_n.$$

$$QGp_n^2 + QGp_nQGp_n^* = 2QGp_nGp_n.$$

In the following lemma, we give the second-order linear recurrence relations for Gaussian Pell and Gaussian Pell-Lucas quaternions.

Lemma 2.3. *Let n be a positive integer. Then we have the following identities:*

$$QGp_n + 2QGp_{n+1} = QGp_{n+2}, \tag{7}$$

$$QGq_n + 2QGq_{n+1} = QGq_{n+2}, \tag{8}$$

$$QGp_n - QGp_{n+1}e_1 - QGp_{n+2}e_2 - QGp_{n+3}e_3 = 12Gq_{n+3}.$$

Proof. Using the equation $QGp_n = Qp_n + iQp_{n-1}$ and the relation $Qp_n = 2Qp_{n-1} + Qp_{n-2}$ given in Proposition 2 of [7] , we conclude that

$$\begin{aligned}
QGp_n + 2QGp_{n+1} &= Qp_n + iQp_{n-1} + 2(Qp_{n+1} + iQp_n) \\
&= (Qp_n + 2Qp_{n+1}) + i(Qp_{n-1} + 2Qp_n) \\
&= Qp_{n+2} + iQp_{n+1} = QGp_{n+2}.
\end{aligned}$$

The second-order recurrence relation for Gaussian Pell-Lucas quaternions is obtained in the similar way. To prove the last assertion, we need the relations $p_{n+1} + p_{n-1} = 2q_n$ and $q_{n+2} + q_{n-2} = 6q_n$ given in [29]. Thus we get

$$\begin{aligned}
QGp_n - QGp_{n+1}e_1 - QGp_{n+2}e_2 - QGp_{n+3}e_3 &= Gp_n + Gp_{n+2} + Gp_{n+4} + Gp_{n+6} \\
&= 2Gq_{n+1} + 2Gq_{n+5} \\
&= 12Gq_{n+3}.
\end{aligned}$$

We hence complete the proof. □

The next corollary immediately follows from the definitions (5) and (6) and the identities $q_{n+1} = p_{n+1} + p_n$, $q_n = p_{n+1} - p_n$, $p_{n+1} + p_{n-1} = 2q_n$ and $2p_n + q_n = q_{n+1}$ given in [29].

Corollary 2.4. *Let n be a positive integer. Then we have*

$$QGp_n + QGp_{n+1} = QGq_{n+1}, \tag{9}$$

$$QGp_{n+1} - QGp_n = QGq_n,$$

$$QGp_{n+1} + QGp_{n-1} = 2QGq_n,$$

$$2QGp_n + QGq_n = QGq_{n+1},$$

Taking into the relations (7) and (8) account, we deduce the explicit formulas for Gausssian Pell and Gaussian Pell-Lucas quaternions. From [29], Binet formulas for the Pell and Pell-Lucas numbers are

$$p_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \text{ and } q_n = \frac{\alpha^n + \beta^n}{2}$$

respectively, where $\alpha = 1 + \sqrt{2}$ and $\beta = 1 - \sqrt{2}$. Here we note that $\alpha + \beta = 2$, $\alpha - \beta = 2\sqrt{2}$ and $\alpha\beta = -1$.

Before we prove Binet formulas for Gaussian Pell and Gaussian Pell-Lucas quaternions we will give the following useful lemma.

Lemma 2.5. *For $n \geq 1$ we have*

$$\alpha QGp_n + QGp_{n-1} = \alpha^n A$$

$$\beta QGp_n + QGp_{n-1} = \beta^n B,$$

where $A = \sum_{s=0}^{3}(\alpha^s + i\alpha^{s-1})e_s$ and $B = \sum_{s=0}^{3}(\beta^s + i\beta^{s-1})e_s$.

Proof. Let $n \geq 1$. For the Gaussian Pell quaternions αQGp_n and QGp_{n-1}, we obtain

$$\alpha QGp_n + QGp_{n-1} = \sum_{s=0}^{3}(\alpha Gp_{n+s} + Gp_{n-1+s})e_s \tag{10}$$

If we take into account $\alpha Gp_{n+s} + Gp_{n-1+s}$, thus from the relation $Gp_n = p_n + ip_{n-1}$ and by the identity $\alpha^n = \alpha p_n + p_{n-1}$ we have

$$\begin{aligned}\alpha Gp_{n+s} + Gp_{n-1+s} &= \alpha(p_{n+s} + ip_{n-1+s}) + (p_{n-1+s} + ip_{n-2+s})\\ &= (\alpha p_{n+s} + p_{n-1+s}) + i(\alpha p_{n-1+s} + p_{n-2+s})\\ &= \alpha^{n+s} + i\alpha^{n+s-1}\\ &= \alpha^n(\alpha^s + i\alpha^{s-1}).\end{aligned}$$

Therefore, we get

$$\alpha QGp_n + QGp_{n-1} = \alpha^n A, \tag{11}$$

where $A = \sum_{s=0}^{3}(\alpha^s + i\alpha^{s-1})e_s$. In a similar way to the equation (10), by considering the identity $\beta^n = \beta p_n + p_{n-1}$, we get

$$\beta QGp_n + QGp_{n-1} = \beta^n B, \tag{12}$$

where $B = \sum_{s=0}^{3}(\beta^s + i\beta^{s-1})e_s$ □

Now we are in a position to give the Binet formula for Gaussian Pell and Gaussian Pell-Lucas quaternions.

Theorem 2.6 (Binet Formula). *For any positive integer n, the Binet formula for the Gaussian Pell quaternion QGp_n is*

$$QGp_n = \frac{\alpha^n A - \beta^n B}{\alpha - \beta} \tag{13}$$

and for the Gaussian Pell-Lucas quaternion QGq_n is

$$QGq_n = \frac{\alpha^n A + \beta^n B}{2}, \tag{14}$$

where $A = \sum_{s=0}^{3}(\alpha^s + i\alpha^{s-1})e_s$ *and* $B = \sum_{s=0}^{3}(\beta^s + i\beta^{s-1})e_s$.

Proof. By substracting the equation (12) from the equation (11), we obtain

$$QGp_n = \frac{\alpha^n A - \beta^n B}{\alpha - \beta}.$$

By adding the equation (11) to the equation (12), we have

$$\alpha^n A + \beta^n B = (\alpha + \beta)QGp_n + 2QGp_{n-1}.$$

Taking into account $\alpha + \beta = 2$ and the identity $QGp_n + QGp_{n-1} = QGq_n$ from the equation (9), we get

$$QGq_n = \frac{\alpha^n A + \beta^n B}{2}.$$

□

Theorem 2.7 (Cassini identities). *For any positive integer n, the following identities are hold:*

$$QGp_{n+1}QGp_{n-1} - QGp_n^2 = (-1)^n \frac{(\alpha^2 + 2)AB + \beta^2 BA}{(\alpha - \beta)^2},$$

and

$$QGq_{n+1}QGq_{n-1} - QGq_n^2 = (-1)^n \frac{(\alpha^2 + 2)AB + \beta^2 BA}{4},$$

where $A = \sum_{s=0}^{3}(\alpha^s + i\alpha^{s-1})e_s$ *and* $B = \sum_{s=0}^{3}(\beta^s + i\beta^{s-1})e_s$.

Proof. The proof follows immediately from the Theorem 2.6. □

The following corollary analogous to Theorem 8 in [7] is obtained by using Binet formulas for Gaussian Pell and Gaussian Pell-Lucas quaternions.

Corollary 2.8. *For $n \geq 0$, the following equality hold:*

$$QGq_n^2 - 2QGp_n^2 = (-1)^n AB,$$

where $A = \sum_{s=0}^{3}(\alpha^s + i\alpha^{s-1})e_s$ *and* $B = \sum_{s=0}^{3}(\beta^s + i\beta^{s-1})e_s$.

We will give the next lemma analogous to the identity $p_{m+n} = p_m p_{n+1} + p_{m-1} p_n$ given in [4].

Lemma 2.9. *For $m, n \geq 0$, we have*

$$Gp_{m+n} = p_m Gp_{n+1} + p_{m-1} Gp_n.$$

Proof. By the definition of Gaussian Pell number and the identity $p_{m+n} = p_m p_{n+1} + p_{m-1} p_n$, we get

$$\begin{aligned} Gp_{m+n} &= p_{m+n} + ip_{m+n-1} \\ &= p_m p_{n+1} + p_{m-1} p_n + i(p_m p_n + p_{m-1} p_{n-1}) \\ &= p_m (p_{n+1} + ip_n) + p_{m-1}(p_n + ip_{n-1}) \\ &= p_m Gp_{n+1} + p_{m-1} Gp_n. \end{aligned}$$

□

As a result of Lemma 2.9, we can restate Gp_n as $Gp_n = p_{n-1}Gp_2 + p_{n-2}Gp_1$ by putting $n-1$ and 1 instead of m and n, respectively. By using the identity $p_{-n} = (-1)^{n+1} p_n$ and $Gp_n = p_{n-1}Gp_2 + p_{n-2}Gp_1$, we define the negatively subscripted terms for Gp_n as

$$Gp_{-n} = (-1)^n (p_{n+1} Gp_2 - p_{n+2} Gp_1)$$

Before obtaining the negatively subscripted terms of Gaussian Pell quaternions, we will give the following theorem.

Theorem 2.10. *For $m, n \geq 0$, we have the following identity:*

$$QGp_{m+n} = p_m QGp_{n+1} + p_{m-1} QGp_n.$$

Proof. Using Lemma 2.9 and the definition of Gaussian Pell quaternions, then we get

$$\begin{aligned} QGp_{m+n} =& Gp_{m+n}e_0 + Gp_{m+n+1}e_1 + Gp_{m+n+2}e_2 + Gp_{m+n+3}e_3 \\ =& (p_m Gp_{n+1} + p_{m-1}Gp_n)e_0 + (p_m Gp_{n+2} + p_{m-1}Gp_{n+1})e_1 + \\ & (p_m Gp_{n+3} + p_{m-1}Gp_{n+2})e_2 + (p_m Gp_{n+4} + p_{m-1}Gp_{n+3})e_3 \\ =& p_m(Gp_{n+1}e_0 + Gp_{n+2}e_1 + Gp_{n+3}e_2 + Gp_{n+4}e_3) + \\ & p_{m-1}(Gp_n e_0 + Gp_{n+1}e_1 + Gp_{n+2}e_2 + Gp_{n+3}e_3) \\ =& p_m QGp_{n+1} + p_{m-1} QGp_n. \end{aligned}$$

□

If we take $m \to n-1$ and $n \to 1$, then we deduce alternative formula for QGp_n as follows:

$$QGp_n = p_{n-1} QGp_2 + p_{n-2} QGp_1. \tag{15}$$

In the same way, we write another formula for QGq_n as

$$QGq_n = p_{n-1} QGq_2 + p_{n-2} QGq_1. \tag{16}$$

Therefore, from equation (15) and the identity $p_{-n} = (-1)^{n+1} p_n$ we can define Gaussian Pell quaternion with negative indices in the following way:

$$QGp_{-n} = (-1)^n (p_{n+1} QGp_2 - p_{n+2} QGp_1).$$

Similarly, by means of the equation (16) one can obtain Gaussian Pell-Lucas quaternion with negative indices

$$QGq_{-n} = (-1)^n (p_{n+1} QGq_2 - p_{n+2} QGp_1).$$

Theorem 2.11. *For $n \geq 0$, we have the following summation formulas:*

$$\sum_{i=0}^{n} \binom{n}{i} 2^i QGp_i = QGp_{2n},$$

$$\sum_{i=0}^{n} \binom{n}{i} (-1)^i QGp_i = (\frac{\alpha - \beta}{2})^{(n-3)} [B - (-1)^n A].$$

Proof. From the Binet Formula of QGp_n, we get

$$\begin{aligned}\sum_{i=0}^{n}\binom{n}{i}2^iQGp_i &= \sum_{i=0}^{n}\binom{n}{i}2^i(\frac{\alpha^iA-\beta^iB}{\alpha-\beta})\\ &= \frac{A}{\alpha-\beta}\sum_{i=0}^{n}\binom{n}{i}(2\alpha)^i - \frac{B}{\alpha-\beta}\sum_{i=0}^{n}\binom{n}{i}(2\beta)^i\\ &= \frac{A}{\alpha-\beta}[(1+2\alpha)^n] - \frac{B}{\alpha-\beta}[(1+2\beta)^n]\\ &= \frac{A\alpha^{2n}-B\beta^{2n}}{\alpha-\beta} = QGp_{2n},\end{aligned}$$

and

$$\begin{aligned}\sum_{i=0}^{n}\binom{n}{i}(-1)^iQGp_i &= \sum_{i=0}^{n}\binom{n}{i}(-1)^i(\frac{\alpha^iA-\beta^iB}{\alpha-\beta})\\ &= \frac{A}{\alpha-\beta}\sum_{i=0}^{n}\binom{n}{i}(-\alpha)^i - \frac{B}{\alpha-\beta}\sum_{i=0}^{n}\binom{n}{i}(-\beta)^i\\ &= \frac{A}{\alpha-\beta}[(1-\alpha)^n] - \frac{B}{\alpha-\beta}[(1-\beta)^n]\\ &= (\frac{\alpha-\beta}{2})^{(n-3)}[B-(-1)^nA],\end{aligned}$$

where $A = QGp_1 - \beta QGp_0$ and $B = QGp_1 - \alpha QGp_0$ from the equations (13) and (14). □

The proof of the next corollary is easily seen by using the equations (7) and (9), the relation $QGp_n = Qp_n + iQp_{n-1}$ and Theorem 6 in [7] and Theorem 5 in [25].

Corollary 2.12. *For the Gaussian Pell quaternion QGp_n, the following identities hold:*

$$\sum_{i=1}^{n} QGp_i = \frac{1}{2}[QGq_{n+1} - QGq_1].$$

$$\sum_{i=1}^{n} QGp_{2i} = \frac{1}{2}[QGp_{2n+1} - QGp_1].$$

$$\sum_{i=1}^{n} QGp_{2i-1} = \frac{1}{2}[QGp_{2n} - QGp_1].$$

Since Gaussian Pell and Gaussian Pell-Lucas quaternions also satisfy second-order linear recurrence relation, then we can derive the generating functions for these quaternions. Thus we can give the following theorem.

Theorem 2.13. *The generating function for the nth Gaussian Pell quaternion QGp_n is*

$$G(x,t) = \frac{(1-2t)QGp_0 + QGp_1t}{1-2t-t^2},$$

and the generating function for the nth Gaussian Pell-Lucas quaternion QGq_n is

$$H(x,t) = \frac{(1-2t)QGq_0 + QGq_1t}{1-2t-t^2}.$$

Proof. Let $G(x,t)=\sum_{n=0}^{\infty} QGp_n(x)t^n = QGp_0 + QGp_1 t + QGp_2 t^2 + QGp_3 t^3 + \cdots + QGp_n t^n + \cdots$ be the generating function for the *nth* Gaussian Pell quaternion QGp_n. Then we derive

$$tG(x,t) = QGp_0 t + QGp_1 t^2 + QGp_2 t^3 + QGp_3 t^4 + \cdots + QGp_n t^{n+1} + \cdots$$

and

$$t^2 G(x,t) = QGp_0 t^2 + QGp_1 t^3 + QGp_2 t^4 + QGp_3 t^5 + \cdots + QGp_n t^{n+2} + \cdots .$$

After simple computations, we get $G(x,t) = \frac{(1-2t)QGp_0 + QGp_1 t}{1-2t-t^2}$ due to the fact that $QGp_n = 2QGp_{n-1} + QGp_{n-2}$. In as similar manner, we obtain the generating function of Gaussian Pell-Lucas quaternions. □

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