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Invertible Linear Relations Generated by Integral Equations with Operator Measures

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Abstract. We define a minimal relation L_0 generated by an integral equation with operators measures and give a description of the relations $L_0 - \lambda E$, $L_0^* - \lambda E$, where L_0^* is adjoint for L_0, $\lambda \in \mathbb{C}$. The obtained results are applied to a description of relations $T(\lambda)$ such that $L_0 - \lambda E \subset T(\lambda) \subset L_0^* - \lambda E$ and $T^{-1}(\lambda)$ are bounded everywhere defined operators.

1. Introduction

In this paper, we consider the integral equation

$$y(t) = x_0 - iJ\int_a^t d\mathbf{p}(s)y(s) - iJ\int_a^t d\mathbf{m}(s)f(s), \tag{1}$$

where y is an unknown function, $a \leqslant t \leqslant b$; J is an operator in a separable Hilbert space H, $J = J^*$, $J^2 = E$ (E is the identical operator); $\mathbf{p}$, $\mathbf{m}$ are operator-valued measures defined on Borel sets $\Delta \subset [a, b]$ and taking values in the set of linear bounded operators acting in H; $x_0 \in H$, $f \in L_2(H, d\mathbf{m}; a, b)$. We assume that the measures $\mathbf{p}$, $\mathbf{m}$ have bounded variations and $\mathbf{p}$ is self-adjoint, $\mathbf{m}$ is non-negative.

We define a minimal relation L_0 generated by equation (1) and give a description of the relations $L_0 - \lambda E$, $L_0^* - \lambda E$, where L_0^* is adjoint for L_0, $\lambda \in \mathbb{C}$. We apply these results to a description of relations $T(\lambda)$ such that $L_0 - \lambda E \subset T(\lambda) \subset L_0^* - \lambda E$ and $T^{-1}(\lambda)$ are bounded everywhere defined operators and give an explicit form of the operators $T^{-1}(\lambda)$.

If the measures $\mathbf{p}$, $\mathbf{m}$ are absolutely continuous (i.e., $\mathbf{p}(\Delta) = \int_\Delta p(t)dt$, $\mathbf{m}(\Delta) = \int_\Delta m(t)dt$ for all Borel sets $\Delta \subset [a, b]$, where the functions $\|p(t)\|$, $\|m(t)\|$ belong to $L_1(a, b)$), then integral equation (1) is transformed to a differential equation with a non-negative weight operator function. Linear relations and operators generated by such differential equations were considered in many works (see [14], [4], [5], further detailed bibliography can be found, for example, in [13], [3]).

The study of integral equation (1) differs essentially from the study of differential equations by the presence of the following features: i) a representation of a solution of equation (1) using an evolutional family of operators is possible if the measures $\mathbf{p}$, $\mathbf{m}$ have not common single-point atoms (see [6]); ii) the

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Lagrange formula contains summands relating to single-point atoms of the measures **p**, **m** (see [7]). Note that this work partially corrects the errors made in the article [8]. Also note that equation (1) was considered in [9], [10] under the assumption that **m** is the usual Lebesque measure on $[a,b]$. In [9], an explicit form of operators $T^{-1}(\lambda)$ is given in the case when the set of single-point atoms of the measure **p** can be arranged as an increasing sequence converging to b. In [9], L_0, L_0^* are operators. In [10], a description of $T^{-1}(\lambda)$ is given in terms of boundary values, i.e., necessary and sufficient conditions are obtained under which a boundary value problem determines relations $T(\lambda)$ such that $T^{-1}(\lambda)$ are bounded everywhere defined operators.

2. Preliminary assertions

Let H be a separable Hilbert space with a scalar product $(\cdot,\cdot)$ and a norm $\|\cdot\|$. We consider a function $\Delta\to\mathbf{P}(\Delta)$ defined on Borel sets $\Delta\subset[a,b]$ and taking values in the set of linear bounded operators acting in H. The function $\mathbf{P}$ is called an operator measure on $[a,b]$ (see, for example, [2, ch. 5]) if it is zero on the empty set and the equality $\mathbf{P}\left(\bigcup_{n=1}^{\infty}\Delta_n\right)=\sum_{n=1}^{\infty}\mathbf{P}(\Delta_n)$ holds for disjoint Borel sets Δ_n, where the series converges weakly. Further, we extend any measure $\mathbf{P}$ on $[a,b]$ to a segment $[a,b_0]$ $(b_0>b)$ letting $\mathbf{P}(\Delta)=0$ for each Borel set $\Delta\subset(b,b_0]$.

By $\mathbf{V}_\Delta(\mathbf{P})$ we denote $\mathbf{V}_\Delta(\mathbf{P})=\rho_{\mathbf{P}}(\Delta)=\sup\sum_n\|\mathbf{P}(\Delta_n)\|$, where the supremum is taken over all finite sums of disjoint Borel sets $\Delta_n\subset\Delta$. The number $\mathbf{V}_\Delta(\mathbf{P})$ is called the variation of the measure $\mathbf{P}$ on the Borel set Δ. Suppose that the measure $\mathbf{P}$ has the bounded variation on $[a,b]$. Then for $\rho_{\mathbf{P}}$-almost all $\xi\in[a,b]$ there exists an operator function $\xi\to\Psi_{\mathbf{P}}(\xi)$ such that $\Psi_{\mathbf{P}}$ possesses the values in the set of linear bounded operators acting in H, $\|\Psi_{\mathbf{P}}(\xi)\|=1$, and the equality

$$\mathbf{P}(\Delta)=\int_\Delta\Psi_{\mathbf{P}}(s)d\rho_{\mathbf{P}} \tag{2}$$

holds for each Borel set $\Delta\subset[a,b]$. The function $\Psi_{\mathbf{P}}$ is uniquely determined up to values on a set of zero $\rho_{\mathbf{P}}$-measure. Integral (2) converges with respect to the usual operator norm ([2, ch. 5]).

Further, $\int_{t_0}^{t}$ stands for $\int_{[t_0 t)}$ if $t_0<t$, for $-\int_{[t,t_0)}$ if $t_0>t$, and for 0 if $t_0=t$. This implies that $y(a)=x_0$ in equation (1). A function h is integrable with respect to the measure $\mathbf{P}$ on a set Δ if there exists the Bochner integral $\int_\Delta\Psi_{\mathbf{P}}(t)h(t)d\rho_{\mathbf{P}}=\int_\Delta(d\mathbf{P})h(t)$. Then the function $y(t)=\int_{t_0}^{t}(d\mathbf{P})h(s)$ is continuous from the left.

By $\mathcal{S}_{\mathbf{P}}$ denote a set of single-point atoms of the measure $\mathbf{P}$ (i.e., a set $t\in[a,b]$ such that $\mathbf{P}(\{t\})\neq0$). The set $\mathcal{S}_{\mathbf{P}}$ is at most countable. The measure $\mathbf{P}$ is continuous if $\mathcal{S}_{\mathbf{P}}=\emptyset$, it is self-adjoint if $(\mathbf{P}(\Delta))^*=\mathbf{P}(\Delta)$ for each Borel set $\Delta\subset[a,b]$, it is non-negative if $(\mathbf{P}(\Delta)x,x)\geqslant0$ for all Borel sets $\Delta\subset[a,b]$ and for all elements $x\in H$.

In following Lemma 2.1, $\mathbf{p}_1$, $\mathbf{p}_2$, $\mathbf{q}$ are operator measures having bounded variations on $[a,b]$ and taking values in the set of linear bounded operators acting in H. Suppose that the measure $\mathbf{q}$ is self-adjoint. We assume that these measures are extended on the segment $[a,b_0]\supset[a,b_0)\supset[a,b]$ in the manner described above.

Lemma 2.1. [7] *Let f, g be functions integrable on $[a,b_0]$ with respect to the measure $\mathbf{q}$ and $y_0,z_0\in H$. Then any functions*

$$y(t)=y_0-iJ\int_{t_0}^{t}d\mathbf{p}_1(s)y(s)-iJ\int_{t_0}^{t}d\mathbf{q}(s)f(s),\quad z(t)=z_0-iJ\int_{t_0}^{t}d\mathbf{p}_2(s)z(s)-iJ\int_{t_0}^{t}d\mathbf{q}(s)g(s)\quad(a\leqslant t_0<b_0,\ t_0\leqslant t\leqslant b_0)$$

satisfy the following formula (analogous to the Lagrange one):

$$\begin{aligned}\int_{c_1}^{c_2}(d\mathbf{q}(t)f(t),z(t))-\int_{c_1}^{c_2}(y(t),d\mathbf{q}(t)g(t))=(iJy(c_2),z(c_2))-(iJy(c_1),z(c_1))+\int_{c_1}^{c_2}(y(t),d\mathbf{p}_2(t)z(t))-\\-\int_{c_1}^{c_2}(d\mathbf{p}_1(t)y(t),z(t))-\sum_{t\in\mathcal{S}_{\mathbf{p}_1}\cap\mathcal{S}_{\mathbf{p}_2}\cap[c_1,c_2)}(iJ\mathbf{p}_1(\{t\})y(t),\mathbf{p}_2(\{t\})z(t))-\sum_{t\in\mathcal{S}_{\mathbf{q}}\cap\mathcal{S}_{\mathbf{p}_2}\cap[c_1,c_2)}(iJ\mathbf{q}(\{t\})f(t),\mathbf{p}_2(\{t\})z(t))-\\-\sum_{t\in\mathcal{S}_{\mathbf{p}_1}\cap\mathcal{S}_{\mathbf{q}}\cap[c_1,c_2)}(iJ\mathbf{p}_1(\{t\})y(t),\mathbf{q}(\{t\})g(t))-\sum_{t\in\mathcal{S}_{\mathbf{q}}\cap[c_1,c_2)}(iJ\mathbf{q}(\{t\})f(t),\mathbf{q}(\{t\})g(t)),\quad t_0\leqslant c_1<c_2\leqslant b_0.\end{aligned} \tag{3}$$

Further we assume that measures $\mathbf{p}$, $\mathbf{m}$ have bounded variations and $\mathbf{p}$ is self-adjoint, $\mathbf{m}$ is non-negative. We consider the equation

$$y(t) = x_0 - iJ\int_a^t d\mathbf{p}(s)y(s) - iJ\int_a^t d\mathbf{m}(s)f(s), \tag{4}$$

where $x_0 \in H$, f is integrable with respect to the measure $\mathbf{m}$ on $[a, b]$, $a \leqslant t \leqslant b_0$.

We construct a continuous measure $\mathbf{p}_0$ from the measure $\mathbf{p}$ in the following way. We set $\mathbf{p}_0(\{t_k\})=0$ for $t_k \in \mathcal{S}_{\mathbf{p}}$ and we set $\mathbf{p}_0(\Delta) = \mathbf{p}(\Delta)$ for all Borel sets such that $\Delta \cap \mathcal{S}_{\mathbf{p}} = \varnothing$. Similarly, we construct a continuous measure $\mathbf{m}_0$ from the measure $\mathbf{m}$. We denote $\widehat{\mathbf{p}} = \mathbf{p} - \mathbf{p}_0$, $\widehat{\mathbf{m}} = \mathbf{m} - \mathbf{m}_0$. Then $\widehat{\mathbf{p}}(\{t_k\}) = \mathbf{p}(\{t_k\})$ for all $t_k \in \mathcal{S}_{\mathbf{p}}$ and $\widehat{\mathbf{p}}(\Delta) = 0$ for all Borel sets Δ such that $\Delta \cap \mathcal{S}_{\mathbf{p}} = \varnothing$. The similar equalities hold for the measure $\widehat{\mathbf{m}}$. The measures $\mathbf{p}_0$, $\widehat{\mathbf{p}}$, $\mathbf{m}_0$, $\widehat{\mathbf{m}}$ are self-adjoint and the measures $\mathbf{m}_0$, $\widehat{\mathbf{m}}$ are non-negative.

We replace $\mathbf{p}$ by $\mathbf{p}_0$ and $\mathbf{m}$ by $\mathbf{m}_0$ in (4). Then we obtain the equation

$$y(t) = x_0 - iJ\int_a^t d\mathbf{p}_0(s)y(s) - iJ\int_a^t d\mathbf{m}_0(s)f(s). \tag{5}$$

Equations (4), (5) have unique solutions (see [6]).

By $W(t, \lambda)$ denote an operator solution of the equation

$$W(t, \lambda)x_0 = x_0 - iJ\int_a^t d\mathbf{p}_0(s)W(s, \lambda)x_0 - iJ\lambda\int_a^t d\mathbf{m}_0(s)W(s, \lambda)x_0, \tag{6}$$

where $x_0 \in H$, $\lambda \in \mathbb{C}$ ($\mathbb{C}$ is the set of complex numbers). Using Lemma 2.1, we get

$$W^*(t, \overline{\lambda})JW(t, \lambda) = J \tag{7}$$

by the standard method (see [9]). The functions $t \to W(t, \lambda)$ and $t \to W^{-1}(t, \lambda) = JW^*(t, \overline{\lambda})J$ are continuous with respect to the uniform operator topology. Consequently there exist constants $\varepsilon_1 > 0$, $\varepsilon_2 > 0$ such that the inequality

$$\varepsilon_1 \|x\|^2 \leqslant \|W(t, \lambda)x\|^2 \leqslant \varepsilon_2 \|x\|^2 \tag{8}$$

holds for all $x \in H$, $t \in [a, b_0]$, $\lambda \in C \subset \mathbb{C}$ (C is a compact set).

Lemma 2.2. *Suppose that a function f is integrable with respect to the measure $\mathbf{m}$. A function y is a solution of the equation*

$$y(t) = x_0 - iJ\int_a^t d\mathbf{p}_0(s)y(s)x - iJ\lambda\int_a^t d\mathbf{m}_0(s)y(s) - iJ\int_a^t d\mathbf{m}(s)f(s), \quad x_0 \in H, \quad a \leqslant t \leqslant b_0, \tag{9}$$

if and only if y has the form

$$y(t) = W(t, \lambda)x_0 - W(t, \lambda)iJ\int_a^t W^*(\xi, \overline{\lambda})d\mathbf{m}(\xi)f(\xi). \tag{10}$$

Proof. We denote $\widetilde{\mathbf{p}}_0 = \mathbf{p}_0 - \lambda\mathbf{m}_0$. The measure $\widetilde{\mathbf{p}}_0$ is continuous. Equation (9) has a unique solution (see [6]). It is enough to prove that if we substitute the function from the right side (10) instead y in the equation (9), then we get the identity. With this substitution, the right side (9) takes the form

$$x_0 - iJ\int_a^t d\mathbf{p}_0(s)\left(W(s, \lambda)x_0 - W(s, \lambda)iJ\int_a^s W^*(\xi, \overline{\lambda})d\mathbf{m}(\xi)f(\xi)\right) -$$
$$- iJ\lambda\int_a^t d\mathbf{m}_0(s)\left(W(s, \lambda)x_0 - W(s, \lambda)iJ\int_a^s W^*(\xi, \overline{\lambda})d\mathbf{m}(\xi)f(\xi)\right) - iJ\int_a^t d\mathbf{m}(s)f(s) =$$
$$= x_0 - iJ\int_a^t d\widetilde{\mathbf{p}}_0(s)\left(W(s, \lambda)x_0 - W(s, \lambda)iJ\int_a^s W^*(\xi, \overline{\lambda})d\mathbf{m}(\xi)f(\xi)\right) - iJ\int_a^t d\mathbf{m}(s)f(s) =$$
$$= x_0 - iJ\int_a^t d\widetilde{\mathbf{p}}_0(s)W(s, \lambda)x_0 - J\int_a^t d\widetilde{\mathbf{p}}_0(s)W(s, \lambda)J\int_a^s W^*(\xi, \overline{\lambda})d\mathbf{m}(\xi)f(\xi) - iJ\int_a^t d\mathbf{m}(s)f(s). \tag{11}$$

We change the limits of integration in the third term of the right-hand side (11). Then the third term takes the form

$$J\int_a^t d\widetilde{\mathbf{p}}_0(s)W(s,\lambda)J\int_a^s W^*(\xi,\overline{\lambda})d\mathbf{m}(\xi)f(\xi) = J\int_{[a,t)}\left(\int_{(\xi,t)} d\widetilde{\mathbf{p}}_0(s)W(s,\lambda)\right)JW^*(\xi,\overline{\lambda})d\mathbf{m}(\xi)f(\xi) =$$
$$= J\int_{[a,t)}\left(\int_{[\xi,t)} d\widetilde{\mathbf{p}}_0(s)W(s,\lambda)\right)JW^*(\xi,\overline{\lambda})d\mathbf{m}(\xi)f(\xi) - J\int_{[a,t)}\left(\int_{\{\xi\}} d\widetilde{\mathbf{p}}_0(s)W(s,\lambda)\right)JW^*(\xi,\overline{\lambda})d\mathbf{m}(\xi)f(\xi). \quad (12)$$

The last term in (12) is equal to zero since the measure $\widetilde{\mathbf{p}}_0$ is continuous. Using (6), we continue equality (11)

$$W(t,\lambda)x_0 - \int_a^t J\left(\int_\xi^t d\widetilde{\mathbf{p}}_0(s)W(s,\lambda)\right)JW^*(\xi,\overline{\lambda})d\mathbf{m}(\xi)f(\xi) - iJ\int_a^t d\mathbf{m}(s)f(s). \quad (13)$$

It follows from (6) that (13) is equal to

$$W(t,\lambda)x_0 - \int_a^t i((W(t,\lambda)-E)-(W(\xi,\lambda)-E))JW^*(\xi,\overline{\lambda})d\mathbf{m}(\xi)f(\xi) - iJ\int_a^t d\mathbf{m}(s)f(s) =$$
$$= W(t,\lambda)x_0 - i\int_a^t W(t,\lambda)JW^*(\xi,\overline{\lambda})d\mathbf{m}(\xi)f(\xi) + i\int_a^t W(\xi,\lambda)JW^*(\xi,\overline{\lambda})d\mathbf{m}(\xi)f(\xi) - iJ\int_a^t d\mathbf{m}(s)f(s).$$

Taking into account (7), we continue the last equality

$$W(t,\lambda)x_0 - iW(t,\lambda)J\int_a^t W^*(\xi,\overline{\lambda})d\mathbf{m}(\xi)f(\xi) + iJ\int_a^t d\mathbf{m}(\xi)f(\xi) - iJ\int_a^t d\mathbf{m}(s)f(s) = y(t).$$

The Lemma is proved. □

3. Linear relations generated by the integral equation

Let $\mathbf{B}$ be a Hilbert space. A linear relation T is understood as any linear manifold $T \subset \mathbf{B}\times\mathbf{B}$. The terminology on the linear relations can be found, for example, in [11], [1]. In what follows we make use of the following notations: $\{\cdot,\cdot\}$ is an ordered pair; $\mathcal{D}(T)$ is the domain of T; $\mathcal{R}(T)$ is the range of T; $\ker T$ is a set of elements $x \in \mathbf{B}$ such that $\{x, 0\} \in T$; T^{-1} is the relation inverse for T, i.e., the relation formed by the pairs $\{x', x\}$, where $\{x, x'\} \in T$. A relation T is called surjective if $\mathcal{R}(T) = \mathbf{B}$. A relation T is called invertible or injective if $\ker T = \{0\}$ (i.e., the relation T^{-1} is an operator); it is called continuously invertible if it is closed, invertible, and surjective (i.e., T^{-1} is a bounded everywhere defined operator). A relation T^* is called adjoint for T if T^* consists of all pairs $\{y_1, y_2\}$ such that equality $(x_2, y_1) = (x_1, y_2)$ holds for all pairs $\{x_1, x_2\} \in T$. A relation T is called symmetric if $T \subset T^*$.

It is known (see, for example, [12, ch.3], [11, ch.1]) that the graph of an operator $T : \mathcal{D}(T) \to \mathbf{B}$ is the set of pairs $\{x, Tx\} \in \mathbf{B}\times\mathbf{B}$, where $x \in \mathcal{D}(T) \subset \mathbf{B}$. Consequently, the linear operators can be treated as linear relations; this is why the notation $\{x_1, x_2\} \in T$ is used also for the operator T. Since all considered relations are linear, we shall often omit the word "linear".

Let $\mathbf{m}$ is a non-negative operator measure defined on Borel sets $\Delta \subset [a, b]$ and taking values in the set of linear bounded operators acting in the space H. The measure $\mathbf{m}$ is assumed to have a bounded variation on $[a, b]$. We introduce the quasi-scalar product $(x, y)_{\mathbf{m}} = \int_a^{b_0} ((d\mathbf{m})x(t), y(t))$ on a set of step-like functions with values in H defined on the segment $[a, b_0]$. Identifying with zero functions y obeying $(y, y)_{\mathbf{m}} = 0$ and making the completion, we arrive at the Hilbert space denoted by $L_2(H, d\mathbf{m}; a, b) = \mathfrak{H}$. The elements of $\mathfrak{H}$ are the classes of functions identified with respect to the norm $\|y\|_{\mathbf{m}} = (y, y)_{\mathbf{m}}^{1/2}$. In order not to complicate the terminology, the class of functions with a representative y is indicated by the same symbol and we write $y \in \mathfrak{H}$. The equality of the functions in $\mathfrak{H}$ is understood as the equality for associated equivalence classes.

Let us define a *minimal relation* L_0 in the following way. The relation L_0 consists of pairs $\{\widetilde{y}, \widetilde{f_0}\} \in \mathfrak{H} \times \mathfrak{H}$ satisfying the condition: for each pair $\{\widetilde{y}, \widetilde{f_0}\}$ there exists a pair $\{y, f_0\}$ such that the pairs $\{\widetilde{y}, \widetilde{f_0}\}$, $\{y, f_0\}$ are identical in $\mathfrak{H} \times \mathfrak{H}$ and $\{y, f_0\}$ satisfies equation (4) and the equalities

$$y(a) = y(b_0) = y(\alpha) = 0, \quad \alpha \in \mathcal{S}_{\mathbf{p}}; \quad \mathbf{m}(\{\beta\})f_0(\beta) = 0, \quad \beta \in \mathcal{S}_{\mathbf{m}}. \tag{14}$$

Further, without loss of generality it can be assumed that if $\{y, f_0\} \in L_0$, then equalities (4), (14) hold for this pair. In general, the relation L_0 is not an operator since a function y can happen to be identified with zero in $\mathfrak{H}$, while f is non-zero. It follows from Lemma 2.1 that the relation L_0 is symmetric.

Lemma 3.1. *If a pair $\{y, f\} \in L_0 - \lambda E$, then*

$$y(t) = -iJ\int_a^t d\mathbf{p}_0(s)y(s) - iJ\lambda\int_a^t d\mathbf{m}_0(s)y(s) - iJ\int_a^t d\mathbf{m}_0(s)f(s). \tag{15}$$

Proof. Let $\{y, f\} \in L_0 - \lambda E$. It follows from the definition of the relation L_0 that the pair $\{y, f\}$ satisfies the equation

$$y(t) = -iJ\int_a^t d\mathbf{p}(s)y(s) - iJ\lambda\int_a^t d\mathbf{m}(s)y(s) - iJ\int_a^t d\mathbf{m}(s)f(s). \tag{16}$$

Consequently,

$$y(t) = -iJ\int_a^t d(\mathbf{p}_0(s) + \widehat{\mathbf{p}}(s))y(s) - iJ\lambda\int_a^t d(\mathbf{m}_0(s) + \widehat{\mathbf{m}}(s))y(s) - iJ\int_a^t d(\mathbf{m}_0(s) + \widehat{\mathbf{m}}(s))f(s). \tag{17}$$

The pair $\{y, f + \lambda y\}$ belongs to L_0. Equalities (14) imply $\mathbf{m}(\{\beta\})(\lambda y(\beta) + f(\beta)) = 0$, $y(\alpha) = 0$, where $\alpha \in \mathcal{S}_{\mathbf{p}}$, $\beta \in \mathcal{S}_{\mathbf{m}}$. Using (17), we obtain (15). The Lemma is proved. □

Corollary 3.2. *Equalities* (15), (16) *hold together for any pairs $\{y, f\} \in L_0 - \lambda E$.*

Lemma 3.3. *A pair $\{\widetilde{y}, \widetilde{f}\} \in \mathfrak{H} \times \mathfrak{H}$ belongs to the relation $L_0 - \lambda E$ if and only if there exists a pair $\{y, f\}$ such that the pairs $\{\widetilde{y}, \widetilde{f}\}$, $\{y, f\}$ are identical in $\mathfrak{H} \times \mathfrak{H}$ and the equalities*

$$y(t) = -W(t, \lambda)iJ\int_a^t W^*(s, \overline{\lambda})d\mathbf{m}_0(s)f(s), \tag{18}$$

$$y(\alpha) = W(\alpha, \lambda)iJ\int_a^\alpha W^*(s, \overline{\lambda})d\mathbf{m}_0(s)f(s) = 0, \quad \alpha \in \mathcal{S}_{\mathbf{p}} \cup \{b_0\}, \tag{19}$$

$$\mathbf{m}(\{\beta\})(\lambda y(\beta) + f(\beta)) = 0, \quad \beta \in \mathcal{S}_{\mathbf{m}} \tag{20}$$

hold.

Proof. The desired assertion follows from (14) and Lemmas 2.2, 3.1 and Corollary 3.2. □

Corollary 3.4. *If $y \in \mathcal{D}(L_0)$, then y is continuous and $y(b) = 0$.*

Corollary 3.5. *Suppose a pair $\{y, f\}$ satisfies equality* (18)*. The function $f \in \mathfrak{H}$ belongs to the range $\mathcal{R}(L_0 - \lambda E)$ if and only if f satisfies the conditions*

$$\int_a^\alpha W^*(s, \overline{\lambda})d\mathbf{m}_0(s)f(s) = 0, \quad \mathbf{m}(\{\beta\})(\lambda y(\beta) + f(\beta)) = 0, \tag{21}$$

where $\alpha \in \mathcal{S}_{\mathbf{p}} \cup \{b_0\}$, $\beta \in \mathcal{S}_{\mathbf{m}}$.

Remark 3.6. The first equality in (21) is equivalent to the following

$$\int_{\alpha_1}^{\alpha_2} W^*(s,\overline{\lambda})d\mathbf{m}_0(s)f(s)=0,\quad \alpha_1,\alpha_2\in\mathcal{S}_{\mathbf{p}}\cup\{a\}\cup\{b_0\}. \tag{22}$$

Remark 3.7. It follows from Lemma 3.3, Corollary 3.4 that we can replace b_0 by b in (19), (21), (22).

Lemma 3.8. *The relation L_0 is closed.*

Proof. Suppose $\{y_n,f_n\}\in L_0$. Using (18) – (20) for $\lambda=0$, we obtain

$$y_n(t)=-W(t,0)iJ\int_a^t W^*(s,0)d\mathbf{m}_0(s)f_n(s), \tag{23}$$

$$y_n(\alpha)=W(\alpha,0)iJ\int_a^\alpha W^*(s,0)d\mathbf{m}_0(s)f_n(s)=0,\ \ \mathbf{m}(\{\beta\})f_n(\beta)=0, \tag{24}$$

where $\alpha\in\mathcal{S}_{\mathbf{p}}\cup\{b_0\}$, $\beta\in\mathcal{S}_{\mathbf{m}}$. Suppose that the sequences $\{y_n\}$, $\{f_n\}$ converge in $\mathfrak{H}$ to y,f, respectively. We note that if a sequence converges in $\mathfrak{H}=L_2(H,d\mathbf{m};a,b)$, then this sequence converges in $L_2(H,d\mathbf{m}_0;a,b)$. Moreover,

$$\left\|f_n-f\right\|_{\mathfrak{H}}^2\geqslant(\mathbf{m}(\{\beta\})(f_n(\beta)-f(\beta)),f_n(\beta)-f(\beta))=(\mathbf{m}(\{\beta\})f(\beta),f(\beta)),$$

where $\beta\in\mathcal{S}_{\mathbf{m}}$. Passing to the limit as $n\to\infty$ in (23), (24), we obtain equalities (18) – (20) for $\lambda=0$. It follows from Lemma 3.3 that the pair $\{y,f\}\in L_0$. The Lemma is proved. □

By $\mathfrak{X}_A=\mathfrak{X}_A(t)$ denote an operator characteristic function of a set A, i.e., $\mathfrak{X}_A(t)=E$ if $t\in A$ and $\mathfrak{X}_A(t)=0$ if $t\notin A$. We shall often omit the argument t in the notation $\mathfrak{X}_A$.

Remark 3.9. *Equality* (20) *means that the function $\mathfrak{X}_{\{\beta\}}(\lambda y(\beta)+f(\beta))$ is identified with zero in the space $\mathfrak{H}$.*

By $\overline{\mathcal{S}}_{\mathbf{p}}$ denote the closure of the set $\mathcal{S}_{\mathbf{p}}$. Let $\mathcal{S}_0$ be the set $t\in[a,b]$ such that $y(t)=0$ for all $y\in\mathcal{D}(L_0)$. It follows from (14) and Corollary 3.4 that $a,b\in\mathcal{S}_0$ and $\mathcal{S}_{\mathbf{p}}\subset\mathcal{S}_0$. Corollary 3.4 implies that the set $\mathcal{S}_0$ is closed. Therefore, $\overline{\mathcal{S}}_{\mathbf{p}}\cup\{a\}\cup\{b\}\subset\mathcal{S}_0$.

Lemma 3.10. *Suppose $\{y,f\}\in L_0$. Then $f(t)=0$ for $\mathbf{m}$-almost all $t\in\mathcal{S}_0$.*

Proof. Using Corollary 3.5 (for $\lambda=0$) and Remark 3.7, we get

$$\int_a^\alpha (d\mathbf{m}_0(s)f(s),W(s,0)x)=0,\qquad \mathbf{m}(\{\beta\})f(\beta)=0$$

for all $x\in H$ and for all $\alpha\in\mathcal{S}_0$, $\beta\in\mathcal{S}_{\mathbf{m}}$. Hence equality (2) implies

$$\int_a^\alpha (\Psi_{\mathbf{m}_0}(s)f(s),W(s,0)x)d\rho_{\mathbf{m}_0}(s)=0,\qquad \mathbf{m}(\{\beta\})f(\beta)=0. \tag{25}$$

We denote

$$\varphi_x(t)=(\Psi_{\mathbf{m}_0}(t)f(t),W(t,0)x),\quad \Phi_x(t)=\int_a^t\varphi_x(s)d\rho_{\mathbf{m}_0}(s).$$

The function Φ_x is continuous. Hence it follows from (25) that $\Phi_x(t)=0$ for all $t\in\mathcal{S}_0$. Therefore, $\varphi_x(t)=0$ for $\rho_{\mathbf{m}_0}$-almost all $t\in\mathcal{S}_0$.

Let $\{x_n\}$ be a countable everywhere dense set in H and let $\mathcal{X}_n$ be a set $t\in\mathcal{S}_0$ such that $\varphi_{x_n}(t)=0$. Then $\varrho_{\mathbf{m}_0}(\mathcal{X}_n)=\varrho_{\mathbf{m}_0}(\mathcal{S}_0)$. We denote $\mathcal{X}=\cap_n\mathcal{X}_n$. Then $\varrho_{\mathbf{m}_0}(\mathcal{X})=\varrho_{\mathbf{m}_0}(\mathcal{S}_0)$ and $\varphi_{x_n}(t)=0$ for all n. If a sequence

$\{z_n\}$, $z_n \in H$, converges to z in H, then the sequence $\{W(t,0)z_n\}$ converges to $W(t,0)z$ for fixed t. Therefore, $\varphi_x(t)=0$ for all $x\in H$ and for all $t\in\mathcal{X}$. The operator $W(t,0)$ has a bounded inverse for all t. This implies that $\Psi_{\mathbf{m}_0}(t)f(t)=0$ for all $t\in\mathcal{X}$. Consequently, $\Psi_{\mathbf{m}_0}(t)f(t)=0$ for $\rho_{\mathbf{m}_0}$-almost all $t\in\mathcal{S}_0$. It follows from (2) that

$$\int_a^b (d\mathbf{m}_0(t)f(t),f(t)) = \int_a^b (\Psi_{\mathbf{m}_0}(t)f(t),f(t))d\rho_{\mathbf{m}_0}(t) = 0.$$

Hence using (14), we obtain $f(t)=0$ for $\mathbf{m}$-almost all $t\in\mathcal{S}_0$. The Lemma is proved. □

By $\mathfrak{H}_0$ (by $\mathfrak{H}_1$) denote a subspace of functions that vanish on $[a,b]\setminus\mathcal{S}_0$ (on $\mathcal{S}_0$, respectively) with respect to the norm in $\mathfrak{H}$. The subspaces $\mathfrak{H}_0$, $\mathfrak{H}_1$ are orthogonal and $\mathfrak{H}=\mathfrak{H}_0\oplus\mathfrak{H}_1$. We note that $\mathfrak{H}_0=\{0\}$ if and only if $\mathbf{m}(\mathcal{S}_0)=0$. We denote $L_{10}=L_0\cap(\mathfrak{H}_1\times\mathfrak{H}_1)$. Then $\mathcal{D}(L_{10})\subset\mathfrak{H}_1$, $\mathcal{R}(L_{10})\subset\mathfrak{H}_1$. It follows from Lemma 3.10 that

$$L_0^* = (\mathfrak{H}_0\times\mathfrak{H}_0)\oplus L_{10}^*, \tag{26}$$

i.e., the relation L_0^* consists of all pairs $\{y,f\}\in\mathfrak{H}$ of the form

$$\{y,f\}=\{u,v\}+\{z,g\}=\{u+z,v+g\},$$

where $u,v\in\mathfrak{H}_0$, $\{z,g\}\in L_{10}^*$.

The set $\mathcal{T}_{\mathbf{p}}=(a,b)\setminus\mathcal{S}_0$ is open and it is the union of at most a countable number of disjoint open intervals $\mathcal{J}_k$, i.e., $\mathcal{T}_{\mathbf{p}}=\bigcup_{k=1}^{\Bbbk_1}\mathcal{J}_k$ and $\mathcal{J}_k\cap\mathcal{J}_j=\varnothing$ for $k\neq j$, where $\Bbbk_1$ is a natural number (equal to the number of intervals if this number is finite) or the symbol ∞ (if the number of intervals is infinite). By $\mathbb{J}$ denote the set of these intervals $\mathcal{J}_k$.

Remark 3.11. *The boundaries α_k, β_k of any interval $\mathcal{J}_k=(\alpha_k,\beta_k)\in\mathbb{J}$ belong to $\mathcal{S}_0$.*

We denote

$$w_k(t,\lambda)=\mathfrak{X}_{[\alpha_k,\beta_k)}W(t,\lambda)W^{-1}(\alpha_k,\lambda), \tag{27}$$

where $(\alpha_k,\beta_k)=\mathcal{J}_k\in\mathbb{J}$. Using (7), we get

$$w_k^*(t,\overline{\lambda})Jw_k(t,\lambda)=J,\quad \alpha_k\leqslant t<\beta_k. \tag{28}$$

Lemma 3.12. *Let $g\in\mathfrak{H}_1$ and let a function $G_{\mathbf{o}}$ be given by the following equality*

$$G_{\mathbf{o}}(t) = -\mathfrak{X}_{[a,b]\setminus\mathcal{S}_{\mathbf{m}}}w_k(t,\lambda)iJ\int_{\alpha_k}^t w_k^*(s,\overline{\lambda})d\mathbf{m}(s)g(s),$$

where $(\alpha_k,\beta_k)=\mathcal{J}_k\in\mathbb{J}$. Then the pair $\{G_{\mathbf{o}},g\}\in L_{10}^*-\lambda E$ if g vanishes outside of $[\alpha_k,\beta_k)$.

Proof. We denote

$$G(t) = -w_k(t,\lambda)iJ\int_{\alpha_k}^t w_k^*(s,\overline{\lambda})d\mathbf{m}(s)g(s).$$

Equalities (27), (7) imply

$$G(t) = -\mathfrak{X}_{[\alpha_k,\beta_k)}W(t,\lambda)iJ\int_{\alpha_k}^t W^*(s,\overline{\lambda})d\mathbf{m}(s)g(s).$$

It follows from Lemma 2.2 that the function G is a solution of equation (9) on the segment $[\alpha_k,\gamma]$, $\gamma<\beta_k$ (for $a=\alpha_k$, $y=G$, $f=g$, $x_0=0$).

Suppose a pair $\{y, f\} \in L_0 - \overline{\lambda}E$. The pair $\{y, f\}$ satisfies equation (16) in which λ is replaced by $\overline{\lambda}$. Therefore we can apply formula (3) to the functions y, f, G, g for $c_1 = \alpha_k$, $c_2 = \gamma$, $\mathbf{q} = \mathbf{m}$, $\mathbf{p}_1 = \mathbf{p}_0 + \overline{\lambda}\mathbf{m}$, $\mathbf{p}_2 = \mathbf{p}_0 + \lambda\mathbf{m}_0$. Since the measures $\mathbf{p}_0$, $\mathbf{m}_0$ is continuous, self-adjoint, $\mathbf{m} = \mathbf{m}_0 + \widehat{\mathbf{m}}$, and (20) holds, we obtain

$$\int_{\alpha_k}^{\gamma}(d\mathbf{m}(s)f(s), G(s)) - \int_{\alpha_k}^{\gamma}(y, d\mathbf{m}(s)g(s)) = (iJy(\gamma), G(\gamma)) - \int_{\alpha_k}^{\gamma}\overline{\lambda}(d\widehat{\mathbf{m}}(s)y(s), G(s)).$$

Using the equality $G_{\mathbf{o}}(t) = G(t) - \mathfrak{X}_{\mathcal{S}_{\mathbf{m}}}G(t)$ and (20), we get

$$\int_{\alpha_k}^{\gamma}(d\mathbf{m}(s)f(s), G_{\mathbf{o}}(s)) - \int_{\alpha_k}^{\gamma}(y, d\mathbf{m}(s)g(s)) = (iJy(\gamma), G(\gamma)) - \\ - \sum_{s\in\mathcal{S}_{\mathbf{m}}\cap[\alpha_k,\gamma)}\overline{\lambda}(\widehat{\mathbf{m}}(\{s\})y(s), G(s)) - \sum_{s\in\mathcal{S}_{\mathbf{m}}\cap[\alpha_k,\gamma)}(\widehat{\mathbf{m}}(\{s\})f(s), G(s)) = (iJy(\gamma), G(\gamma)). \quad (29)$$

The function y is continuous from the left and $y(\beta_k) = 0$ (also see Corollary 3.4). Hence passing to the limit as $\gamma \to \beta_k - 0$ in (29), we obtain

$$\int_{\alpha_k}^{\beta_k}(d\mathbf{m}(s)f(s), G_{\mathbf{o}}(s)) = \int_{\alpha_k}^{\beta_k}(y(s), d\mathbf{m}(s)g(s)).$$

This implies the desired statement. The Lemma is proved. □

By $\mathfrak{H}_{10}$ (by $\mathfrak{H}_{11}$) denote a subspace of functions that belong to $\mathfrak{H}_1$ and vanish on $\mathcal{S}_{\mathbf{m}}$ (on $[a, b] \setminus \mathcal{S}_{\mathbf{m}}$, respectively) with respect to the norm in $\mathfrak{H}$. So, $\mathfrak{H}_{10}$ ($\mathfrak{H}_{11}$) consists of functions of the form $\mathfrak{X}_{[a,b]\setminus(\mathcal{S}_0\cup\mathcal{S}_{\mathbf{m}})}h$ (of the form $\mathfrak{X}_{\mathcal{S}_{\mathbf{m}}\setminus\mathcal{S}_0}h$, respectively), where $h \in \mathfrak{H}$ is an arbitrary function. Therefore,

$$\mathfrak{H}_1 = \mathfrak{H}_{10} \oplus \mathfrak{H}_{11}, \quad \mathfrak{H} = \mathfrak{H}_0 \oplus \mathfrak{H}_{10} \oplus \mathfrak{H}_{11}.$$

Obviously, the space $\mathfrak{H}_{11}$ is the closure in $\mathfrak{H}$ of the linear span of functions that have the form $\mathfrak{X}_{\{\tau\}}(\cdot)x$, where $x \in H$, $\tau \in \mathcal{S}_{\mathbf{m}} \setminus \mathcal{S}_0$. By (14), it follows that $\mathfrak{H}_{11} \subset \ker L_{10}^*$.

Remark 3.13. *Suppose* $\tau \in \mathcal{S}_{\mathbf{m}} \cap \mathcal{S}_0$. *Then* $\mathfrak{X}_{\{\tau\}}(\cdot)x \in \mathfrak{H}_0$ *for* $x \in H$. *Hence* (26) *implies that the pair* $\{0, \mathfrak{X}_{\{\tau\}}(\cdot)x\} \in L_0^*$. *In particular, Remark 3.11 implies that this is true for* $\tau \in \mathcal{S}_{\mathbf{m}} \cap (\cup_{k=1}^{\Bbbk_1}\{\alpha_k, \beta_k\} \cup \{a, b\})$, *where* α_k, β_k *are boundaries of intervals* $(\alpha_k, \beta_k) = \mathcal{J}_k \in \mathbb{J}$.

We define an operator $\mathcal{U}_k(\lambda) : \mathfrak{H}_1 \to \mathfrak{H}_1$ by the equation

$$(\mathcal{U}_k(\lambda)f)(t) = -\mathfrak{X}_{[a,b]\setminus\mathcal{S}_{\mathbf{m}}}w_k(t, \lambda)iJ\int_a^t w_k^*(s, \overline{\lambda})d\mathbf{m}(s)\lambda f(s), \quad f \in \mathfrak{H}_1. \quad (30)$$

The operator $\mathcal{U}_k(\lambda)$ is bounded. Obviously, $\mathcal{U}_k(0) = 0$. Taking into account (27) and Lemma 3.12, we obtain that the pair $\{\mathcal{U}_k(\lambda)f, \mathfrak{X}_{[\alpha_k,\beta_k)}\lambda f\} \in L_{10}^* - \lambda E$.

Let $u_k(t, \lambda, \tau) : H \to \mathfrak{H}_1$ be an operator acting by the formula

$$u_k(t, \lambda, \tau)x = (\mathcal{U}_k(\lambda)\mathfrak{X}_{\{\tau\}}x)(t) = -\mathfrak{X}_{[a,b]\setminus\mathcal{S}_{\mathbf{m}}}w_k(t, \lambda)iJ\int_a^t w_k^*(s, \overline{\lambda})d\mathbf{m}(s)\lambda\mathfrak{X}_{\{\tau\}}(s)x, \quad (31)$$

where $x \in H$, $\tau \in (\alpha_k, \beta_k) \cap \mathcal{S}_{\mathbf{m}}$, $(\alpha_k, \beta_k) = \mathcal{J}_k \in \mathbb{J}$. Then the pair $\{u_k(\cdot, \lambda, \tau)x, \lambda\mathfrak{X}_{\{\tau\}}x\} \in L_{10}^* - \lambda E$. The definition of L_0 implies that the function $\mathfrak{X}_{\{\tau\}}x \in \ker L_0^*$. Consequently, $\{\mathfrak{X}_{\{\tau\}}x, -\lambda\mathfrak{X}_{\{\tau\}}x\} \in L_{10}^* - \lambda E$. Thus, for any $x \in H$ the function

$$u_k(\cdot, \lambda, \tau)x + \mathfrak{X}_{\{\tau\}}(\cdot)x \in \ker(L_{10}^* - \lambda E). \quad (32)$$

Using (31), we get

$$\|u_k(\cdot,\lambda,\tau)x\|_{\mathfrak{H}} \leqslant |\lambda|\,\gamma\,\left\|\mathfrak{X}_{\{\tau\}}(\cdot)x\right\|_{\mathfrak{H}} = |\lambda|\,\gamma\,\mathbf{m}^{1/2}(\{\tau\})x, \tag{33}$$

where $\gamma > 0$, $x \in H$, $\tau \in (\alpha_k, \beta_k) \cap \mathcal{S}_{\mathbf{m}}$.

The linear span of functions of the form $\mathfrak{X}_{\{\tau\}}(\cdot)x$ ($x \in H$, $\tau \in \mathcal{S}_{\mathbf{m}} \setminus \mathcal{S}_0$) is dense in the space $\mathfrak{H}_{11}$. It follows from (31), (32) that for any the function $z_1 \in \mathfrak{H}_{11}$

$$\mathcal{U}_k(\lambda)z_1 + z_1 \in \ker(L_{10}^* - \lambda E). \tag{34}$$

Lemma 3.14. *The linear span of functions of the form $\mathfrak{X}_{[a,b]\setminus\mathcal{S}_{\mathbf{m}}} w_k(\cdot,\lambda)x$ is dense in $\mathfrak{H}_{10} \cap \ker(L_{10}^* - \lambda E)$. Here $x \in H$; $k = 1, ..., \Bbbk_1$ if $\Bbbk_1$ is finite and k is any natural number if $\Bbbk_1$ is infinite.*

Proof. Suppose that $h_0 \in \mathfrak{H}_{10} \cap \ker(L_{10}^* - \lambda E)$ and

$$(h_0, \mathfrak{X}_{[a,b]\setminus\mathcal{S}_{\mathbf{m}}} w_k(\cdot,\lambda)x)_{\mathfrak{H}} = \int_a^b (d\mathbf{m}(s)h_0(s), \mathfrak{X}_{[a,b]\setminus\mathcal{S}_{\mathbf{m}}} w_k(s,\lambda)x) = 0 \tag{35}$$

for all $x \in H$ and for all k. Let us prove that $h_0(t) = 0$ $\mathbf{m}$-almost everywhere. We denote

$$y(t) = -W(t,\overline{\lambda})iJ\int_a^t W^*(s,\lambda)d\mathbf{m}_0(s)h_0(s). \tag{36}$$

We define the function h as follows. We put $h(t) = h_0(t)$ for $t \in [a,b] \setminus \mathcal{S}_{\mathbf{m}}$, and $h(t) = -\overline{\lambda}^{-1}y(t)$ for $t \in \mathcal{S}_{\mathbf{m}}$, $\lambda \neq 0$, and $h(t) = 0$ for $t \in \mathcal{S}_{\mathbf{m}}$, $\lambda = 0$. The function y will not change if h_0 is replaced by h in (36). Moreover, equality (35) will remain with this replacement. Then it follows from Lemma 3.3 and Corollary 3.5 that the pair $\{y, h\} \in L_{10} - \overline{\lambda}E$. Hence $(h_0, h)_{\mathfrak{H}} = 0$ since $h_0 \in \ker(L_{10}^* - \lambda E)$. On the other hand, $(h_0, h)_{\mathfrak{H}} = (h_0, h_0)_{\mathfrak{H}}$. This implies $h_0 = 0$. The Lemma is proved. □

Lemma 3.15. *The linear span of functions of the form $\mathfrak{X}_{[a,b]\setminus\mathcal{S}_{\mathbf{m}}} w_k(\cdot,\lambda)x_0$ and $u_k(\cdot,\lambda,\tau)x_k + \mathfrak{X}_{\{\tau\}}(\cdot)x_k$ is dense in $\ker(L_{10}^* - \lambda E)$. Here $x_k, x_0 \in H$; $\tau \in (\alpha_k, \beta_k) \cap \mathcal{S}_{\mathbf{m}}$; $k = 1, ..., \Bbbk_1$ if $\Bbbk_1$ is finite and k is any natural number if $\Bbbk_1$ is infinite.*

Proof. Let $z \in \ker(L_{10}^* - \lambda E)$. Then $z = z_0 + z_1$, where $z_0 \in \mathfrak{H}_{10}$, $z_1 \in \mathfrak{H}_{11}$. Suppose that the function z is orthogonal to the functions listed in the condition of the Lemma. We claim that $z = 0$. The pair $\{z_1, -\lambda z_1\} \in L_{10}^* - \lambda E$ since $z_1 \in \ker L_{10}^*$. Therefore, $\{z_0, \lambda z_1\} \in L_{10}^* - \lambda E$. We denote $z_k = \mathfrak{X}_{[\alpha,\beta)}z$, $z_{0k} = \mathfrak{X}_{[\alpha,\beta)}z_0$, $z_{1k} = \mathfrak{X}_{[\alpha,\beta)}z_1$. Using Lemma 3.12, we get

$$z_{0k}(t) = -\mathfrak{X}_{[a,b]\setminus\mathcal{S}_{\mathbf{m}}} w_k(t,\lambda)iJ\int_a^t w_k^*(s,\overline{\lambda})d\mathbf{m}(s)\lambda z_{1k}(s) + h_0(t), \tag{37}$$

where $h_0 \in \ker(L_{10}^* - \lambda E)$. Moreover, $h_0 \in \mathfrak{H}_{10}$ since $z_{0k} \in \mathfrak{H}_{10}$ and the first term in (37) belongs to $\mathfrak{H}_{10}$. According to Lemma 3.14, h_0 belongs to the closure of linear span of functions that have the form $\mathfrak{X}_{[\alpha_k,\beta_k)\setminus\mathcal{S}_{\mathbf{m}}} w_k(\cdot,\lambda)x'$, $x' \in H$. Using (30), (37), we obtain $z_k = \mathcal{U}_k(\lambda)z_{1k} + z_{1k} + h_0$. By assumption, $(z_k, \mathcal{U}_k(\lambda)z_{1k} + z_{1k})_{\mathfrak{H}} = 0$ and $(z_k, h_0)_{\mathfrak{H}} = 0$. Hence, $(z_k, z_k)_{\mathfrak{H}} = 0$ for all k. Therefore, $(z, z)_{\mathfrak{H}} = 0$. The Lemma is proved. □

Remark 3.16. *The Lemma 3.15 remains true if functions of the form $u_k(\cdot,\lambda,\tau)x_k + \mathfrak{X}_{\{\tau\}}(\cdot)x_k$ are replaced by functions $u_k(\cdot,\lambda,\tau)w_k(\tau,\lambda)x_k + \mathfrak{X}_{\{\tau\}}(\cdot)w_k(\tau,\lambda)x_k$. Indeed, by (8), (27), it follows that the operator $w_k(\tau,\lambda)$ is continuously invertible for $\tau \in \mathcal{J}_k = (\alpha_k, \beta_k)$. Hence the linear spans of the noted above functions coincide.*

Let $\mathbb{M}$ be a set consisting of intervals $\mathcal{J} \in \mathbb{J}$ and single-point sets $\{\tau\}$, where $\tau \in \mathcal{S}_{\mathbf{m}} \setminus \mathcal{S}_0$. The set $\mathbb{M}$ is at most countable. Let $\Bbbk$ be the number of elements in $\mathbb{M}$. We arrange the elements of $\mathbb{M}$ in the form of a finite or infinite sequence and denote these elements by $\mathcal{E}_k$, where k is any natural number if the number of elements in $\mathbb{M}$ is infinite, and $1 \leqslant k \leqslant \Bbbk$ if the number of elements in $\mathbb{M}$ is finite.

We shall assign an operator function v_k to each element $\mathcal{E}_k \in \mathbb{M}$ in the following way. If $\mathcal{E}_k$ is the interval, $\mathcal{E}_k = \mathcal{J}_k = (\alpha_k, \beta_k) \in \mathbb{J}$, then

$$v_k(t,\lambda) = \mathfrak{X}_{[\alpha_k,\beta_k)\setminus\mathcal{S}_{\mathbf{m}}} w_k(t,\lambda). \tag{38}$$

If $\mathcal{E}_k$ is a single-point set, $\mathcal{E}_k = \{\tau_k\}$, $\tau_k \in \mathcal{S}_{\mathbf{m}} \setminus \mathcal{S}_0$, and $\tau_k \in \mathcal{J}_n = (\alpha_n, \beta_n) \in \mathbb{J}$, then

$$v_k(t,\lambda) = u_n(t,\lambda,\tau_k) w_n(\tau_k,\lambda) + \mathfrak{X}_{\{\tau_k\}}(t) w_n(\tau_k,\lambda). \tag{39}$$

Remark 3.17. *It follows from* (27) *that equality* (38) *is equivalent to the following:* $v_k(t,\lambda) = \mathfrak{X}_{[a,b]\setminus\mathcal{S}_{\mathbf{m}}} w_k(t,\lambda)$.

Lemma 3.18. *The linear span of functions* $t \to v_k(t,\lambda)\xi_k$ $(\xi_k \in H)$ *is dense in* $\ker(L^*_{10} - \lambda E)$. (*Here* $k \in \mathbb{N}$ *if* $\Bbbk = \infty$, *and* $1 \leqslant k \leqslant \Bbbk$ *if* $\Bbbk$ *is finite.*)

Proof. The required statement follows from Remark 3.16 and Lemma 3.15 immediately. □

Corollary 3.19. *A function* $f \in \mathfrak{H}_1$ *belongs to the range* $\mathcal{R}(L_{10} - \lambda E)$ *if and only if the equality* $(f, v_k(\cdot,\overline{\lambda}))_{\mathfrak{H}} = 0$ *holds for all k.* (*Here* $k \in \mathbb{N}$ *if* $\Bbbk = \infty$, *and* $1 \leqslant k \leqslant \Bbbk$ *if* $\Bbbk$ *is finite.*)

Proof. The proof follows from the equality $\mathcal{R}(L_{10} - \lambda E) \oplus \ker(L^*_{10} - \overline{\lambda}E) = \mathfrak{H}_1$ and Lemma 3.18. □

Further, we denote $v_k(t,0) = v_k(t)$. We note that $u_k(t,0,\tau) = 0$ (see (31)).

Let $Q_{k,0}$ be a set $x \in H$ such that the functions $t \to v_k(t)x$ are identical with zero in $\mathfrak{H}$. We put $Q_k = H \ominus Q_{k,0}$. On the linear space Q_k we introduce a norm $\|\cdot\|_-$ by the equality

$$\|\xi_k\|_- = \|v_k(\cdot)\xi_k\|_{\mathfrak{H}}, \quad \xi_k \in Q_k. \tag{40}$$

We note that if v_k has form (38) with $\lambda = 0$, then

$$\|\xi_k\|_- = \left(\int_{[a,b]\setminus\mathcal{S}_{\mathbf{m}}} (d\mathbf{m}(s) w_k(s,0)\xi_k, w_k(s,0)\xi_k)\right)^{1/2} = \left(\int_{[a,b]} (d\mathbf{m}_0(s) w_k(s,0)\xi_k, w_k(s,0)\xi_k)\right)^{1/2}, \quad \xi_k \in Q_k.$$

If v_k has form (39) with $\lambda = 0$, then

$$\|\xi_k\|_- = (\mathbf{m}(\{\tau_k\}) w_n(\tau_k,0)\xi_k, w_n(\tau_k,0)\xi_k)^{1/2} = \left\|\mathbf{m}^{1/2}(\{\tau_k\}) w_n(\tau_k,0)\xi_k\right\|, \quad \xi_k \in Q_k.$$

By Q_k^- denote the completion of Q_k with respect to norm (40). This norm (40) is generated by the scalar product

$$(\xi_k, \eta_k)_- = (v_k(\cdot)\xi_k, v_k(\cdot)\eta_k)_{\mathfrak{H}}, \tag{41}$$

where $\xi_k, \eta_k \in Q_k$. From formula (2) in which the measure $\mathbf{P}$ is replaced by $\mathbf{m}$, it follows that

$$\|\xi_k\|_- \leqslant \gamma \|\xi_k\|, \quad \xi_k \in Q_k, \tag{42}$$

where $\gamma > 0$ is independent of $\xi_k \in Q_k$.

It follows from (42) that the space Q_k^- can be treated as a space with a negative norm with respect to Q_k ([2, ch. 1], [11, ch.2]). By Q_k^+ denote the associated space with a positive norm. The definition of spaces with positive and negative norms implies that $Q_k^+ \subset Q_k \subset Q_k^-$. By $(\cdot,\cdot)_+$ and $\|\cdot\|_+$ we denote the scalar product and the norm in Q_k^+, respectively.

Lemma 3.20. *There exist constants* $\gamma_{1k}, \gamma_{2k} > 0$ *such that the inequality*

$$\gamma_{1k} \|v_k(\cdot)x\|_{\mathfrak{H}} \leqslant \|v_k(\cdot,\lambda)x\|_{\mathfrak{H}} \leqslant \gamma_{2k} \|v_k(\cdot)x\|_{\mathfrak{H}} \tag{43}$$

holds for all $x \in H$.

Proof. Using Lemma 2.2 and (6), we get

$$W(t,\lambda)x_0 = W(t,0)x_0 - W(t,0)iJ\int_a^t W^*(s,0)d\mathbf{m}_0(s)\lambda W(s,\lambda)x_0, \quad x_0 \in H, \tag{44}$$

$$W(t,0)x_0 = W(t,\lambda)x_0 + W(t,\lambda)iJ\int_a^t W^*(\xi,\overline{\lambda})d\mathbf{m}_0(s)\lambda W(s,0)x_0, \quad x_0 \in H. \tag{45}$$

Suppose that v_k has form (38). Using (27), (44), (45), we obtain

$$v_k(t,\lambda)x_0 = v_k(t,0)x_0 - v_k(t,0)iJ\int_{\alpha_k}^t v_k^*(s,0)d\mathbf{m}_0(s)\lambda v_k(s,\lambda)x_0, \quad x_0 \in H, \tag{46}$$

$$v_k(t,0)x_0 = v_k(t,\lambda)x_0 + v_k(t,\lambda)iJ\int_{\alpha_k}^t v_k^*(\xi,\overline{\lambda})d\mathbf{m}_0(s)\lambda v_k(s,0)x_0, \quad x_0 \in H. \tag{47}$$

Equalities (8), (46), (47) imply (43) in the case when v_k has form (38). Suppose that v_k has form (39). Using (39), (31), we get

$$\|v_k(\cdot,\lambda)x\|^2_{\mathfrak{H}} = \|u_n(\cdot,\lambda,\tau_k)w_n(\tau_k,\lambda)x\|^2_{\mathfrak{H}} + \left\|\mathfrak{X}_{\{\tau_k\}}(\cdot)w_n(\tau_k,\lambda)x\right\|^2_{\mathfrak{H}} \geqslant \left\|\mathfrak{X}_{\{\tau_k\}}(\cdot)w_n(\tau_k,\lambda)x\right\|^2_{\mathfrak{H}} = \|v_k(\cdot)x\|^2_{\mathfrak{H}}.$$

On the other hand, using (31), (33), we obtain

$$\|v_k(\cdot,\lambda)x\|_{\mathfrak{H}} \leqslant \|u_n(\cdot,\lambda,\tau_k)w_n(\tau_k,\lambda)x\|_{\mathfrak{H}} + \left\|\mathfrak{X}_{\{\tau_k\}}(\cdot)w_n(\tau_k,\lambda)x\right\|_{\mathfrak{H}} \leqslant \gamma_3\left\|\mathfrak{X}_{\{\tau_k\}}(\cdot)w_n(\tau_k,\lambda)x\right\|_{\mathfrak{H}} = \gamma_3\|v_k(\cdot)x\|_{\mathfrak{H}},$$

where $\gamma_3 > 0$. The Lemma is proved. □

Remark 3.21. *By* (43), *it follows that the set* $Q_{k,0}$ *will not change if the function* $v_k(\cdot) = v_k(\cdot,0)$ *is replaced by* $v_k(\cdot,\lambda)$ *in the definition of* $Q_{k,0}$. *Moreover, with such a replacement, the space* Q_k^- *will not change in the following sense: the set* Q_k^- *will not change, and the norm in it will be replaced by the equivalent one. The similar statement holds for the space* Q_k^+.

Suppose that a sequence $\{x_{kn}\}$, $x_{kn} \in Q_k$, converges in the space Q_k^- to $x_0 \in Q_k^-$ as $n \to \infty$. It follows from Lemma 3.20 that the sequence $\{v_k(\cdot,\lambda)x_{kn}\}$ is fundamental in $\mathfrak{H}$. Therefore this sequence converges to some element in $\mathfrak{H}$. By $v_k(\cdot,\lambda)x_0$ we denote this element.

Let $\widetilde{Q}_N^- = Q_1^- \times ... \times Q_N^-$ ($\widetilde{Q}_N^+ = Q_1^+ \times ... \times Q_N^+$) be the Cartesian product of the first n sets Q_k^- (Q_k^+, respectively) and let $V_N(t,\lambda) = (v_1(t,\lambda),...,v_N(t,\lambda))$ be the operator one-row matrix. It is convenient to treat elements from $\widetilde{Q}_N^-$ as one-column matrices, and to assume that $V_N(t,\lambda)\widetilde{\xi}_N = \sum_{k=1}^N v_k(t,\lambda)\xi_k$, where we denote $\widetilde{\xi}_N = \mathrm{col}(\xi_1,...,\xi_N) \in \widetilde{Q}_N^-$, $\xi_k \in Q_k^-$.

Let $\ker_k(\lambda)$ be a linear space of functions $t \to v_k(t,\lambda)\xi_k$, $\xi_k \in Q_k^-$. By (40) and Lemma 3.20, it follows that $\ker_k(\lambda)$ is closed in $\mathfrak{H}$. The spaces $\ker_k(0)$ and $\ker_j(0)$ are orthogonal for $k \neq j$. We denote $\mathcal{K}_N(\lambda) = \ker_1(\lambda) \dot{+} ... \dot{+} \ker_N(\lambda)$. Obviously, $\mathcal{K}_{N_1}(\lambda) \subset \mathcal{K}_{N_2}(\lambda)$ for $N_1 < N_2$.

Lemma 3.22. *The set* $\cup_N \mathcal{K}_N(\lambda)$ *is dense in* $\ker(L_{10}^* - \lambda E)$.

Proof. The required statement follows from Lemma 3.18 immediately. □

By $\mathcal{V}_N(\lambda)$ denote the operator $\widetilde{\xi}_N \to V_N(\cdot,\lambda)\widetilde{\xi}_N$, where $\widetilde{\xi}_N \in \widetilde{Q}_N^-$. The operator $\mathcal{V}_N(\lambda)$ maps continuously and one-to-one $\widetilde{Q}_N^-$ onto $\mathcal{K}_N(\lambda) \subset \mathfrak{H}_1 \subset \mathfrak{H}$. Hence the adjoint operator $\mathcal{V}_N^*(\lambda)$ maps $\mathfrak{H}$ onto $\widetilde{Q}_N^+$ continuously. We find the form of the operator $\mathcal{V}_N^*$. For all $\widetilde{\xi}_N \in \widetilde{Q}_N = Q_1 \times ...Q_N$, $f \in \mathfrak{H}$, we have

$$(f,\mathcal{V}_N(\lambda)\widetilde{\xi}_N)_{\mathfrak{H}} = \int_a^{b_0}(d\mathbf{m}(s)f(s),V_N(s,\lambda)\widetilde{\xi}_N) = \int_a^{b_0}(V_N^*(s,\lambda)d\mathbf{m}(s)f(s),\widetilde{\xi}_N) = (\mathcal{V}_N^*(\lambda)f,\widetilde{\xi}_N).$$

Since $\widetilde{Q}_N$ is dense in $\widetilde{Q}_N^-$, we obtain

$$\mathcal{V}_N^*(\lambda)f=\int_a^{b_0}V_N^*(s,\lambda)d\mathbf{m}(s)f(s). \tag{48}$$

Thus, we have proved the following statement.

Lemma 3.23. *The operator $\mathcal{V}_N(\lambda)$ maps continuously and one-to-one $\widetilde{Q}_N^-$ onto $\mathcal{K}_n(\lambda)$. The adjoint operator $\mathcal{V}_N^*(\lambda)$ maps continuously $\mathfrak{H}$ onto $\widetilde{Q}_N^+$ and acts by formula* (48)*. Moreover, $\mathcal{V}_N^*(\lambda)$ maps one-to-one $\mathcal{K}_N(\lambda)$ onto $\widetilde{Q}_N^+$.*

Let $\mathcal{Q}_-,\mathcal{Q}_+,\mathcal{Q}$ be linear spaces of sequences, respectively, $\widetilde{\eta}=\{\eta_k\}$, $\widetilde{\varphi}=\{\varphi_k\}$, $\widetilde{\xi}=\{\xi_k\}$, where $\eta_k\in Q_k^-$, $\varphi_k\in Q_k^+$, $\xi_k\in Q_k$; $k\in\mathbb{N}$ if $\Bbbk=\infty$, and $1\leqslant k\leqslant\Bbbk$ if $\Bbbk$ is finite; $\Bbbk$ is the number of elements in $\mathbb{M}$. We assume that the series $\sum_{k=1}^\infty\|\eta_k\|_-^2$, $\sum_{k=1}^\infty\|\varphi_k\|_+^2$, $\sum_{k=1}^\infty\|\xi_k\|^2$ converge if $\Bbbk=\infty$. These spaces become Hilbert spaces if we introduce scalar products by the formulas

$$(\widetilde{\eta},\widetilde{\zeta})_-=\sum_{k=1}^{\Bbbk}(\eta_k,\zeta_k)_-,\ \ \widetilde{\eta},\widetilde{\zeta}\in\mathcal{Q}_-;\quad(\widetilde{\varphi},\widetilde{\psi})_+=\sum_{k=1}^{\Bbbk}(\varphi_k,\psi_k)_+,\ \ \widetilde{\varphi},\widetilde{\psi}\in\mathcal{Q}_+;\quad(\widetilde{\xi},\widetilde{\sigma})=\sum_{k=1}^{\Bbbk}(\xi_k,\sigma_k),\ \ \widetilde{\xi},\widetilde{\sigma}\in\mathcal{Q}.$$

In these spaces, the norms are defined by the equalities

$$\|\widetilde{\eta}\|_-^2=\sum_{k=1}^{\Bbbk}\|\eta_k\|_-^2,\quad\|\widetilde{\varphi}\|_+^2=\sum_{k=1}^{\Bbbk}\|\varphi_k\|_+^2,\quad\left\|\widetilde{\xi}\right\|^2=\sum_{k=1}^{\Bbbk}\|\xi_k\|^2.$$

The spaces $\mathcal{Q}_+,\mathcal{Q}_-$ can be treated as spaces with positive and negative norms with respect to $\mathcal{Q}$ ([2, ch. 1], [11, ch.2]). So $\mathcal{Q}_+\subset\mathcal{Q}\subset\mathcal{Q}_-$ and $\gamma_1\|\widetilde{\varphi}\|_-\leqslant\|\widetilde{\varphi}\|\leqslant\gamma_2\|\widetilde{\varphi}\|_+$, where $\widetilde{\varphi}\in\mathcal{Q}_+$, $\gamma_1,\gamma_2>0$. The "scalar product" $(\widetilde{\eta},\widetilde{\varphi})$ is defined for all $\widetilde{\varphi}\in\mathcal{Q}_+$, $\widetilde{\eta}\in\mathcal{Q}_-$. If $\widetilde{\eta}\in\mathcal{Q}$, then $(\widetilde{\eta},\widetilde{\varphi})$ coincides with the scalar product in $\mathcal{Q}$.

Let $\mathcal{M}\subset\mathcal{Q}_-$ be a set of sequences vanishing starting from a certain number (its own for each sequence). The set $\mathcal{M}$ is dense in the space $\mathcal{Q}_-$. The operator $\mathcal{V}_N(\lambda)$ is the restriction of $\mathcal{V}_{N+1}(\lambda)$ to $\widetilde{Q}_N^-$. By $\mathcal{V}'(\lambda)$ denote an operator in $\mathcal{M}$ such that $\mathcal{V}'(\lambda)\widetilde{\eta}=\mathcal{V}_N(\lambda)\widetilde{\eta}_N$ for all $N\in\mathbb{N}$, where $\widetilde{\eta}=(\widetilde{\eta}_N,0,...)$, $\widetilde{\eta}_N\in\widetilde{Q}_N^-$. It follows from (40), (43) that $\mathcal{V}'(\lambda)$ admits an extension by continuity to the space $\mathcal{Q}_-$. By $\mathcal{V}(\lambda)$ denote the extended operator. This operator maps continuously and one-to-one $\mathcal{Q}_-$ onto $\ker(L_{10}^*-\lambda E)\subset\mathfrak{H}_1\subset\mathfrak{H}$. Moreover, we denote $\widetilde{V}(t,\lambda)\widetilde{\eta}=(\mathcal{V}(\lambda)\widetilde{\eta})(t)$, where $\widetilde{\eta}=\{\eta_k\}\in\mathcal{Q}_-$. Using (41), we get

$$(\mathcal{V}(0)\widetilde{\eta},\mathcal{V}(0)\widetilde{\zeta})_{\mathfrak{H}}=(\widetilde{\eta},\widetilde{\zeta})_-;\quad\widetilde{\eta}=\{\eta_k\},\ \widetilde{\zeta}=\{\zeta_k\};\ \ \widetilde{\eta},\widetilde{\zeta}\in\mathcal{Q}_-.$$

The adjoint operator $\mathcal{V}^*(\lambda)$ maps continuously $\mathfrak{H}$ onto $\mathcal{Q}_+$. Let us find the form of $\mathcal{V}^*(\lambda)$. Suppose $f\in\mathfrak{H}$, $\widetilde{\eta}\in\mathcal{M}$, $\widetilde{\eta}=\{\widetilde{\eta}_N,0,...\}$. Then

$$(\widetilde{\eta},\mathcal{V}^*(\lambda)f)=(\mathcal{V}(\lambda)\widetilde{\eta},f)_{\mathfrak{H}}=\int_a^{b_0}(d\mathbf{m}(t)\widetilde{V}(t,\lambda)\widetilde{\eta},f(t))=\int_a^{b_0}(\widetilde{\eta},\widetilde{V}^*(t,\lambda)d\mathbf{m}(t)f(t)).$$

Since $\mathcal{V}^*(\lambda)f\in\mathcal{Q}_+$ and the set $\mathcal{M}$ is dense in $\mathcal{Q}_-$, we get

$$\mathcal{V}^*(\lambda)f=\int_a^{b_0}\widetilde{V}^*(t,\lambda)d\mathbf{m}(t)f(t). \tag{49}$$

Taking into account Lemmas 3.22,3.23, we obtain the following statement.

Lemma 3.24. *The operator $\mathcal{V}(\lambda)$ maps $\mathcal{Q}_-$ onto $\ker(L_{10}^*-\lambda E)$ continuously and one to one. A function z belongs to $\ker(L_{10}^*-\lambda E)$ if and only if there exists an element $\widetilde{\eta}=\{\eta_k\}\in\mathcal{Q}_-$ such that $z(t)=(\mathcal{V}(\lambda)\widetilde{\eta})(t)=\widetilde{V}(t,\lambda)\widetilde{\eta}$. The operator $\mathcal{V}^*(\lambda)$ maps $\mathfrak{H}$ onto $\mathcal{Q}_+$ continuously, and acts by formula* (49), *and* $\ker\mathcal{V}^*(\lambda)=\mathfrak{H}_0\oplus\mathcal{R}(L_{10}-\overline{\lambda}E)$*. Moreover, $\mathcal{V}^*(\lambda)$ maps $\ker(L_{10}^*-\lambda E)$ onto $\mathcal{Q}_+$ one to one.*

Theorem 3.25. *A pair* $\{\widetilde{y}, \widetilde{f}\} \in \mathfrak{H} \times \mathfrak{H}$ *belongs to* $L_0^* - \lambda E$ *if and only if there exist a pair* $\{y, f\} \in \mathfrak{H} \times \mathfrak{H}$*, functions* $y_0, y_0' \in \mathfrak{H}_0$, $\widehat{y}, \widehat{f} \in \mathfrak{H}_1$ *and an element* $\widetilde{\eta} \in Q_-$ *such that the pairs* $\{\widetilde{y}, \widetilde{f}\}$, $\{y, f\}$ *are identical in* $\mathfrak{H} \times \mathfrak{H}$ *and the equalities*

$$y = y_0 + \widehat{y}, \;\; f = y_0' + \widehat{f}, \tag{50}$$

$$\widehat{y}(t) = \widetilde{V}(t, \lambda)\widetilde{\eta} - \sum_{k=1}^{\Bbbk_1} \mathfrak{X}_{[a,b]\setminus\mathcal{S}_{\mathbf{m}}} w_k(t, \lambda) iJ \int_a^t w_k^*(s, \overline{\lambda}) d\mathbf{m}(s)\widehat{f}(s) \tag{51}$$

hold, where the series in (51) *converges in* $\mathfrak{H}$, $\Bbbk_1$ *is the number of intervals* $\mathcal{J}_k \in \mathbb{J}$.

Proof. Equalities (50) follow from (26). Let us prove that equality (51) holds. It follows from Lemma 3.24 that $\mathcal{V}(\lambda)\widetilde{\eta} \in \ker(L_{10}^* - \lambda E)$. We prove that if the functions $\widehat{y}, \widehat{f}$ satisfy equality (51), then the pair $\{\widehat{y}, \widehat{f}\} \in L_{10}^* - \lambda E$. If $\Bbbk_1$ is finite, then this statement follows from Lemmas 3.12, 3.24. We assume that $\Bbbk_1 = \infty$ and first prove that the series in (51) converges in $\mathfrak{H}$ for each function $\widehat{f} \in \mathfrak{H}_1$.

The function

$$\widehat{y_k}(t) = -\mathfrak{X}_{[a,b]\setminus\mathcal{S}_{\mathbf{m}}} w_k(t, \lambda) iJ \int_a^t w_k^*(s, \overline{\lambda}) d\mathbf{m}(s)\widehat{f}(s) = -\mathfrak{X}_{[a,b]\setminus\mathcal{S}_{\mathbf{m}}} w_k(t, \lambda) iJ \int_{\alpha_k}^t w_k^*(s, \overline{\lambda}) \Psi_{\mathbf{m}}(s)\widehat{f}(s) d\rho_{\mathbf{m}}(s) \tag{52}$$

vanishes outside the interval $[\alpha_k, \beta_k)$. (Here $\Psi_{\mathbf{m}}$, $\rho_{\mathbf{m}}$ are functions from formula (2) in which the measure $\mathbf{P}$ is replaced by $\mathbf{m}$.) We denote $\widehat{f_k}(t) = \chi_{[\alpha_k, \beta_k)}\widehat{f}(t)$. Using (52), (8), (2), we get

$$\left\|\widehat{y_k}(t)\right\| \leqslant \varepsilon_1 \left\|w_k(t, \lambda)\right\| \int_{\alpha_k}^{\beta_k} \left\|w_k^*(s, \overline{\lambda})\right\| \left\|\Psi_{\mathbf{m}}^{1/2}(s)\widehat{f_k}(s)\right\| d\rho_{\mathbf{m}}(s) \leqslant$$

$$\leqslant \varepsilon \left(\int_{\alpha_k}^{\beta_k} \left\|\Psi_{\mathbf{m}}^{1/2}(s)\widehat{f_k}(s)\right\|^2 d\rho_{\mathbf{m}}(s)\right)^{1/2} = \varepsilon \left\|\widehat{f_k}\right\|_{\mathfrak{H}}, \quad \varepsilon_1, \varepsilon > 0.$$

This implies

$$\left\|\widehat{y_k}\right\|_{\mathfrak{H}}^2 = \int_{\alpha_k}^{\beta_k} (\Psi_{\mathbf{m}}(t)\widehat{y_k}(t), \widehat{y_k}(t)) d\rho_{\mathbf{m}}(t) \leqslant \varepsilon^2 \rho_{\mathbf{m}}([\alpha_k, \beta_k)) \left\|\widehat{f_k}\right\|_{\mathfrak{H}}^2. \tag{53}$$

We denote $S_n(t) = \sum_{k=1}^n \widehat{y_k}(t)$ and prove that the sequence $\{S_n\}$ converges in $\mathfrak{H}$. From (53), we get

$$\|S_n\|_{\mathfrak{H}}^2 = \sum_{k=1}^n \left\|\widehat{y_k}\right\|_{\mathfrak{H}}^2 \leqslant \varepsilon^2 \sum_{k=1}^n \rho_{\mathbf{m}}([\alpha_k, \beta_k)) \left\|\widehat{f_k}\right\|_{\mathfrak{H}}^2 \leqslant \varepsilon^2 \rho_{\mathbf{m}}([a, b]) \left\|\widehat{f}\right\|_{\mathfrak{H}}^2.$$

Hence the sequence $\{S_n\}$ converges to some function $S \in \mathfrak{H}$ and

$$S(t) = -\sum_{k=1}^{\infty} \mathfrak{X}_{[a,b]\setminus\mathcal{S}_{\mathbf{m}}} w_k(t, \lambda) iJ \int_a^t w_k^*(s, \overline{\lambda}) d\mathbf{m}(s)\widehat{f}(s), \quad \|S\|_{\mathfrak{H}} \leqslant \varepsilon_2 \left\|\widehat{f}\right\|_{\mathfrak{H}}, \; \varepsilon_2 > 0. \tag{54}$$

It follows from Lemma 3.12 that the pair $\{S_n, \sum_{k=1}^n \widehat{f_k}\} \in L_{10}^* - \lambda E$. The relation L_{10}^* is closed. Therefore, $\{S, \widehat{f}\} \in L_{10}^* - \lambda E$ and $\{\widehat{y}, \widehat{f}\} \in L_{10}^* - \lambda E$.

Now we assume that a pair $\{\widehat{y}, \widehat{f}\} \in L_{10}^* - \lambda E$. For the function $\widehat{f}$, we find a function S by formula (54). Then $\{S, \widehat{f}\} \in L_{10}^* - \lambda E$. Hence $\widehat{y} - S \in \ker(L_{10}^* - \lambda E)$. By Lemma 3.24, it follows that there exists an element $\widetilde{\eta} \in Q_-$ such that $\widehat{y} - S = \mathcal{V}(\lambda)\widetilde{\eta}$. Therefore $\widehat{y}$ has form (51). Now (26) implies the desired assertion. The Theorem is proved. □

4. Continuously invertible extensions of the relation $L_0 - \lambda E$

We denote

$$\mathfrak{y}_k(t,\lambda) = -\mathfrak{X}_{[\alpha_k,\beta_k)\setminus(\mathcal{S}_{\mathbf{m}}\cap\mathcal{S}_0)} w_k(t,\lambda) iJ \int_a^t w_k^*(s,\overline{\lambda}) d\mathbf{m}(s) \mathfrak{X}_{[a,b]\setminus\mathcal{S}_{\mathbf{m}}} \widehat{f}(s) = \\ = -\mathfrak{X}_{[\alpha_k,\beta_k)\setminus(\mathcal{S}_{\mathbf{m}}\cap\mathcal{S}_0)} w_k(t,\lambda) iJ \int_a^t w_k^*(s,\overline{\lambda}) d\mathbf{m}_0(s) \widehat{f}(s),$$

$$\widetilde{\mathfrak{y}}_k(t,\lambda) = \mathfrak{X}_{[\alpha_k,\beta_k)\setminus(\mathcal{S}_{\mathbf{m}}\cap\mathcal{S}_0)} w_k(t,\lambda) iJ \int_t^b w_k^*(s,\overline{\lambda}) d\mathbf{m}(s) \mathfrak{X}_{[a,b]\setminus\mathcal{S}_{\mathbf{m}}} \widehat{f}(s) = \\ = \mathfrak{X}_{[\alpha_k,\beta_k)\setminus(\mathcal{S}_{\mathbf{m}}\cap\mathcal{S}_0)} w_k(t,\lambda) iJ \int_t^b w_k^*(s,\overline{\lambda}) d\mathbf{m}_0(s) \widehat{f}(s).$$

It follows from Remark 3.11 that $\mathfrak{X}_{[\alpha_k,\beta_k)\setminus(\mathcal{S}_{\mathbf{m}}\cap\mathcal{S}_0)} = \mathfrak{X}_{[\alpha_k,\beta_k)}$ if $\alpha_k \notin \mathcal{S}_{\mathbf{m}}$ and $\mathfrak{X}_{[\alpha_k,\beta_k)\setminus(\mathcal{S}_{\mathbf{m}}\cap\mathcal{S}_0)} = \mathfrak{X}_{(\alpha_k,\beta_k)}$ if $\alpha_k \in \mathcal{S}_{\mathbf{m}}$ (see also Remark 3.13).

Lemma 4.1. *Let $\lambda \neq 0$. Equality* (51) *hold if and only if*

$$\widehat{y}(t) = \widetilde{V}(t,\lambda)\widetilde{\zeta} + 2^{-1}\sum_{k=1}^{\mathbb{k}_1}[\mathfrak{y}_k(t,\lambda) - \mathfrak{X}_{\mathcal{S}_{\mathbf{m}}\cap(\alpha_k,\beta_k)}\mathfrak{y}_k(t,\lambda) - \mathfrak{X}_{\mathcal{S}_{\mathbf{m}}\cap(\alpha_k,\beta_k)}\lambda^{-1}\widehat{f}(t)] + \\ + 2^{-1}\sum_{k=1}^{\mathbb{k}_1}[\widetilde{\mathfrak{y}}_k(t,\lambda) - \mathfrak{X}_{\mathcal{S}_{\mathbf{m}}\cap(\alpha_k,\beta_k)}\widetilde{\mathfrak{y}}_k(t,\lambda) - \mathfrak{X}_{\mathcal{S}_{\mathbf{m}}\cap(\alpha_k,\beta_k)}\lambda^{-1}\widehat{f}(t)], \quad (55)$$

where $\widetilde{\zeta} \in \mathcal{Q}_-$.

Proof. By standard transformations, equality (51) is reduced to the form

$$\widehat{y}(t) = \widetilde{V}(t,\lambda)\widetilde{\vartheta} - 2^{-1}\sum_{k=1}^{\mathbb{k}_1}\mathfrak{X}_{[a,b]\setminus\mathcal{S}_{\mathbf{m}}} w_k(t,\lambda) iJ \int_a^t w_k^*(s,\overline{\lambda}) d\mathbf{m}(s)\widehat{f}(s) + \\ + 2^{-1}\sum_{k=1}^{\mathbb{k}_1}\mathfrak{X}_{[a,b]\setminus\mathcal{S}_{\mathbf{m}}} w_k(t,\lambda) iJ \int_t^b w_k^*(s,\overline{\lambda}) d\mathbf{m}(s)\widehat{f}(s), \quad (56)$$

where $\widetilde{\vartheta} = \{\vartheta_k\} \in \mathcal{Q}_-$, and $\vartheta_k = \eta_k$ if v_k has form (39), and $\vartheta_k = \eta_k - 2^{-1} iJ \int_{\alpha_k}^{\beta_k} w_k^*(s,\overline{\lambda}) d\mathbf{m}(s)\widehat{f}(s)$ if v_k has form (38).

Let us write the function

$$\mathfrak{w}_k(t,\lambda) = -\mathfrak{X}_{[a,b]\setminus\mathcal{S}_{\mathbf{m}}} w_k(t,\lambda) iJ \int_a^t w_k^*(s,\overline{\lambda}) d\mathbf{m}(s)\widehat{f}(s) \quad (57)$$

in a different form. Using (57), (30), we get

$$\mathfrak{w}_k(t,\lambda) = \mathfrak{X}_{[a,b]\setminus\mathcal{S}_{\mathbf{m}}}\mathfrak{y}_k(t,\lambda) - \mathfrak{X}_{[a,b]\setminus\mathcal{S}_{\mathbf{m}}} w_k(t,\lambda) iJ \int_a^t w_k^*(s,\overline{\lambda}) d\mathbf{m}(s)\mathfrak{X}_{\mathcal{S}_{\mathbf{m}}}\widehat{f}(s) = \\ = \mathfrak{y}_k(t,\lambda) - [\mathfrak{X}_{\mathcal{S}_{\mathbf{m}}\cap(\alpha_k,\beta_k)}\mathfrak{y}_k(t,\lambda) + \mathfrak{X}_{\mathcal{S}_{\mathbf{m}}\cap(\alpha_k,\beta_k)}\lambda^{-1}\widehat{f}(t)] + [\mathfrak{X}_{\mathcal{S}_{\mathbf{m}}\cap(\alpha_k,\beta_k)}\lambda^{-1}\widehat{f}(t) + (\mathcal{U}_k(\lambda)\lambda^{-1}\mathfrak{X}_{\mathcal{S}_{\mathbf{m}}\cap(\alpha_k,\beta_k)}\widehat{f})(t)].$$

Using (34), we get

$$\mathfrak{v}_k = \mathfrak{X}_{\mathcal{S}_{\mathbf{m}}\cap(\alpha_k,\beta_k)}\lambda^{-1} f + \mathcal{U}_k(\lambda)\lambda^{-1}\mathfrak{X}_{\mathcal{S}_{\mathbf{m}}\cap(\alpha_k,\beta_k)}\widehat{f} \in \ker(L_{10}^* - \lambda E).$$

Therefore,

$$\mathfrak{w}_k(t,\lambda) = \mathfrak{y}_k(t,\lambda) - [\mathfrak{X}_{\mathcal{S}_{\mathbf{m}}\cap(\alpha_k,\beta_k)}\mathfrak{y}_k(t,\lambda) + \mathfrak{X}_{\mathcal{S}_{\mathbf{m}}\cap(\alpha_k,\beta_k)}\lambda^{-1}\widehat{f}(t)] + \mathfrak{v}_k(t). \tag{58}$$

Similarly, we transform the function

$$\widetilde{\mathfrak{w}}_k(t,\lambda) = \mathfrak{X}_{[a,b]\setminus\mathcal{S}_{\mathbf{m}}} w_k(t,\lambda) iJ \int_t^b w_k^*(s,\overline{\lambda}) d\mathbf{m}(s)\widehat{f}(s)$$

to the form

$$\widetilde{\mathfrak{w}}_k(t,\lambda) = \widetilde{\mathfrak{y}}_k(t,\lambda) - [\mathfrak{X}_{\mathcal{S}_{\mathbf{m}}\cap(\alpha_k,\beta_k)}\widetilde{\mathfrak{y}}_k(t,\lambda) + \mathfrak{X}_{\mathcal{S}_{\mathbf{m}}\cap(\alpha_k,\beta_k)}\lambda^{-1}\widehat{f}(t)] +$$
$$+ [\mathfrak{X}_{\mathcal{S}_{\mathbf{m}}\cap(\alpha_k,\beta_k)}\lambda^{-1}f(t) + (\mathcal{U}_k(\lambda)\lambda^{-1}\mathfrak{X}_{\mathcal{S}_{\mathbf{m}}\cap(\alpha_k,\beta_k)}\widehat{f})(t)] + \mathfrak{X}_{[a,b]\setminus\mathcal{S}_{\mathbf{m}}} w_k(t,\lambda) iJ \int_a^b w_k^*(s,\overline{\lambda}) d\mathbf{m}(s)\mathfrak{X}_{\mathcal{S}_{\mathbf{m}}}\widehat{f}(s).$$

By Lemma 3.15 and (34), it follows that here the last two terms belong to $\ker(L_{10}^* - \lambda E)$. Consequently,

$$\widetilde{\mathfrak{w}}_k(t,\lambda) = \widetilde{\mathfrak{y}}_k(t,\lambda) - [\mathfrak{X}_{\mathcal{S}_{\mathbf{m}}\cap(\alpha_k,\beta_k)}\widetilde{\mathfrak{y}}_k(t) + \mathfrak{X}_{\mathcal{S}_{\mathbf{m}}\cap(\alpha_k,\beta_k)}\lambda^{-1}\widehat{f}(t)] + \widetilde{\mathfrak{v}}_k(t), \tag{59}$$

where $\widetilde{\mathfrak{v}}_k \in \ker(L_{10}^* - \lambda E)$. Now the desired statement follows from (56), (58), (59) and Lemma 3.24. The Lemma is proved. □

Lemma 4.2. *Let $\lambda = 0$. Equality (51) hold if and only if*

$$\widehat{y}(t) = \widetilde{V}(t,0)\widetilde{\zeta} + 2^{-1}\sum_{k=1}^{\mathbb{k}_1}[\mathfrak{y}_k(t,0) - \mathfrak{X}_{[a,b]\setminus\mathcal{S}_{\mathbf{m}}} w_k(t,0) iJ \int_a^t w_k^*(s,0) d\mathbf{m}(s)\mathfrak{X}_{\mathcal{S}_{\mathbf{m}}}\widehat{f}(s)] +$$
$$+ 2^{-1}\sum_{k=1}^{\mathbb{k}_1}[\widetilde{\mathfrak{y}}_k(t,0) + \mathfrak{X}_{[a,b]\setminus\mathcal{S}_{\mathbf{m}}} w_k(t,0) iJ \int_t^b w_k^*(s,0) d\mathbf{m}(s)\mathfrak{X}_{\mathcal{S}_{\mathbf{m}}}\widehat{f}(s))]. \tag{60}$$

Proof. Equality (56) holds for $\lambda = 0$. We transform the function $\mathfrak{w}_k(t,0)$ (see (57)) in the following way:

$$\mathfrak{w}_k(t,0) = -\mathfrak{X}_{[a,b]\setminus\mathcal{S}_{\mathbf{m}}} w_k(t,0) iJ \int_a^t w_k^*(s,0) d\mathbf{m}(s)\widehat{f}(s) = \mathfrak{y}_k(t,0) - \mathfrak{X}_{\mathcal{S}_{\mathbf{m}}\cap(\alpha_k,\beta_k)}\mathfrak{y}_k(t,0) -$$
$$-\mathfrak{X}_{[a,b]\setminus\mathcal{S}_{\mathbf{m}}} w_k(t,0) iJ \int_a^t w_k^*(s,0) d\mathbf{m}(s)\mathfrak{X}_{\mathcal{S}_{\mathbf{m}}}\widehat{f}(s).$$

Similarly, we transform the function $\widetilde{\mathfrak{w}}_k(t,0)$. Since $\mathfrak{X}_{\mathcal{S}_{\mathbf{m}}\cap(\alpha_k,\beta_k)}\mathfrak{y}_k(\cdot,0) \in \ker L_{10}^*$, $\mathfrak{X}_{\mathcal{S}_{\mathbf{m}}\cap(\alpha_k,\beta_k)}\widetilde{\mathfrak{y}}_k(\cdot,0) \in \ker L_{10}^*$, $\mathfrak{X}_{[a,b]\setminus\mathcal{S}_{\mathbf{m}}} w_k(t,0) iJ\int_a^b w_k^*(0,\overline{\lambda}) d\mathbf{m}(s)\mathfrak{X}_{\mathcal{S}_{\mathbf{m}}}\widehat{f}(s) \in \ker L_{10}^*$, we obtain the required statement.The Lemma is proved. □

Theorem 4.3. *Let $T(\lambda)$ be a linear relation such that $L_{10} - \lambda E \subset T(\lambda) \subset L_{10}^* - \lambda E$. The relation $T(\lambda)$ is continuously invertible in the space $\mathfrak{H}_1$ if and only if there exists a bounded operator $M(\lambda): \boldsymbol{Q}_+ \to \boldsymbol{Q}_-$ such that equalities* (61) *(for $\lambda \neq 0$) and* (62) *(for $\lambda = 0$) (see equalities below) hold for any pair $\{y, \widehat{f}\} \in T(\lambda)$*

$$\widehat{y}(t) = \int_a^b \widetilde{V}(t,\lambda)M(\lambda)\widetilde{V}^*(s,\overline{\lambda}) d\mathbf{m}(s)\widehat{f}(s) +$$
$$+ 2^{-1}\sum_{k=1}^{\mathbb{k}_1}\int_a^b \mathfrak{X}_{[\alpha_k,\beta_k)\setminus(\mathcal{S}_{\mathbf{m}}\cap\mathcal{S}_0)}(t) w_k(t,\lambda)\operatorname{sgn}(s-t) iJ w_k^*(s,\overline{\lambda}) d\mathbf{m}(s)\mathfrak{X}_{[a,b]\setminus\mathcal{S}_{\mathbf{m}}}(s)\widehat{f}(s) -$$
$$- 2^{-1}\sum_{k=1}^{\mathbb{k}_1}\int_a^b \mathfrak{X}_{\mathcal{S}_{\mathbf{m}}\cap(\alpha_k,\beta_k)}(t) w_k(t,\lambda)\operatorname{sgn}(s-t) iJ w_k^*(s,\overline{\lambda}) d\mathbf{m}(s)\mathfrak{X}_{[a,b]\setminus\mathcal{S}_{\mathbf{m}}}(s)\widehat{f}(s) - \lambda^{-1}\sum_{k=1}^{\mathbb{k}_1}\mathfrak{X}_{\mathcal{S}_{\mathbf{m}}\cap(\alpha_k,\beta_k)}(t)\widehat{f}(t), \tag{61}$$

$$\widehat{y}(t) = \int_a^b \widetilde{V}(t,0)M(0)\widetilde{V}^*(s,0)d\mathbf{m}(s)\widehat{f}(s)+$$
$$+ 2^{-1}\sum_{k=1}^{\Bbbk_1}\int_a^b \mathfrak{X}_{[\alpha_k,\beta_k)\setminus(\mathcal{S}_\mathbf{m}\cap\mathcal{S}_0)}(t)w_k(t,0)\mathrm{sgn}(s-t)iJw_k^*(s,0)d\mathbf{m}(s)\mathfrak{X}_{[a,b]\setminus\mathcal{S}_\mathbf{m}}(s)\widehat{f}(s)+$$
$$+ 2^{-1}\sum_{k=1}^{\Bbbk_1}\int_a^b \mathfrak{X}_{[a,b]\setminus\mathcal{S}_\mathbf{m}}(t)w_k(t,0)\mathrm{sgn}(s-t)iJw_k^*(s,0)d\mathbf{m}(s)\mathfrak{X}_{\mathcal{S}_\mathbf{m}}(s)\widehat{f}(s). \quad (62)$$

Proof. First note that the range $\mathcal{R}(L_{10}-\lambda E)$ is closed and $\ker(L_{10}-\lambda E)=\{0\}$. This follows from the Lemma 3.3. Suppose that the relation $T^{-1}(\lambda)$ is a boundary everywhere defined operator and $\widehat{y} = T^{-1}(\lambda)\widehat{f}$. Then $\widehat{y}$ has form (55) for $\lambda \neq 0$ and (60) for $\lambda = 0$. In this equalities, $\widetilde{\zeta} \in \mathcal{Q}_-$ is uniquely determined by $\widehat{f}$ and λ, i.e., $\widetilde{\zeta} = \widetilde{\zeta}(\widehat{f},\lambda)$. Indeed, if $\widehat{f} = 0$, then $\widetilde{V}(t,\lambda)\widetilde{\zeta} = T^{-1}(\lambda)0 = 0$. It follows from Lemma 3.24 that $\widetilde{\zeta} = 0$. Moreover, $\widetilde{\zeta}$ depends on $\widehat{f}$ linearly. Consequently, $\widetilde{\zeta} = S(\lambda)\widehat{f}$, where $S(\lambda) : \mathfrak{H}_1 \to \mathcal{Q}_-$ is a linear operator for fixed λ. We claim that the operator $S(\lambda)$ is bounded. Indeed, if a sequence $\{\widehat{f_n}\}$ converges to zero in the space $\mathfrak{H}_1$ as $n \to \infty$, then the sequence $\{\widehat{y_n}\}=\{T^{-1}(\lambda)\widehat{f_n}\}$ converges to zero in $\mathfrak{H}_1$. Hence the sequence $\{\mathcal{V}(\lambda)\widetilde{\zeta}_n\}$ (where $\widetilde{\zeta}_n = S(\lambda)\widehat{f_n}$) converges to zero in $\mathfrak{H}_1$. By Lemma 3.24, it follows that the sequence $\{S(\lambda)\widehat{f_n}\}$ converges to zero in the space $\mathcal{Q}_-$. Therefore $S(\lambda)$ is the bounded operator.

Now we prove that $\widetilde{\zeta}(\widehat{f},\lambda)$ is uniquely determined by the element $\mathcal{V}^*(\overline{\lambda})\widehat{f} \in \mathcal{Q}_+$. Suppose $\mathcal{V}^*(\overline{\lambda})\widehat{f} = 0$. The application of Lemma 3.24 yields $\widehat{f} \in \mathcal{R}(L_{10} - \lambda E)$.

Suppose $\lambda \neq 0$. Taking into account Lemma 3.3, we determine a function $\widehat{y}$ by equality (55) in which

$$\mathfrak{X}_{\mathcal{S}_\mathbf{m}\cap(\alpha_k,\beta_k)}\mathfrak{y}_k(t,\lambda) + \mathfrak{X}_{\mathcal{S}_\mathbf{m}\cap(\alpha_k,\beta_k)}\lambda^{-1}\widehat{f}(t) = 0, \quad \mathfrak{X}_{\mathcal{S}_\mathbf{m}\cap(\alpha_k,\beta_k)}\widetilde{\mathfrak{y}}_k(t,\lambda) + \mathfrak{X}_{\mathcal{S}_\mathbf{m}\cap(\alpha_k,\beta_k)}\lambda^{-1}\widehat{f}(t) = 0.$$

By Lemma 3.3 and Remark 3.9, it follows that the pairs $\{\mathfrak{y}_k, \mathfrak{X}_{(\alpha,\beta)}\widehat{f}\}, \{\widetilde{\mathfrak{y}}_k, \mathfrak{X}_{(\alpha,\beta)}\widehat{f}\} \in L_{10} - \lambda E$. This and the invertibility of $T(\lambda)$ imply that $\widetilde{\zeta}(\widehat{f},\lambda) = 0$ for $\lambda \neq 0$.

Let $\lambda = 0$. Using Lemma 3.3 (for $\lambda = 0$) and Remark 3.9, we determine a function y by equality (60) in which $\mathfrak{X}_{\{\tau\}}\widehat{f}(\tau) = 0$ for $\tau \in \mathcal{S}_\mathbf{m}$. Then equality (60) will take the form

$$\widehat{y}(t) = \widetilde{V}(t,0)\widetilde{\zeta}(\widehat{f},0) + 2^{-1}\sum_{k=1}^{\Bbbk_1}\mathfrak{y}_k(t,0) + 2^{-1}\sum_{k=1}^{\Bbbk_1}\widetilde{\mathfrak{y}}_k(t,0).$$

It follows from Lemma 3.3 and Remark 3.9 that $\{\mathfrak{y}_k, \mathfrak{X}_{[\alpha,\beta)}\widehat{f}\}, \{\widetilde{\mathfrak{y}}_k, \mathfrak{X}_{[\alpha,\beta)}\widehat{f}\} \in L_{10}$. This and the invertibility of $T(0)$ imply that $\widetilde{\zeta}(\widehat{f},0) = 0$.

Thus $S(\lambda)\widehat{f} = M(\lambda)\mathcal{V}^*(\overline{\lambda})\widehat{f}$, where $M(\lambda) : \mathcal{Q}_+ \to \mathcal{Q}_-$ is an everywhere defined operator. Let $\mathcal{V}_0^*(\overline{\lambda})$ be a restriction of $\mathcal{V}^*(\overline{\lambda})$ to $\ker(L_{10}^* - \overline{\lambda}E)$. By Lemma 3.24, it follows that $M(\lambda) = S(\lambda)(\mathcal{V}_0^*(\overline{\lambda}))^{-1}$. Hence $M(\lambda)$ is the bonded operator and equalities (61) (for $\lambda \neq 0$) and (62) (for $\lambda = 0$) hold.

Conversely, suppose that equalities (61) (for $\lambda \neq 0$) and (62) (for $\lambda = 0$) hold. Then $\widehat{y} = 0$ if $\widehat{f} = 0$ in (61), (62). Therefore, $T^{-1}(\lambda)$ is an operator. We claim that the operator $T^{-1}(\lambda)$ is bounded. Indeed, suppose that pairs $\{\widehat{y_n}, \widehat{f_n}\}$ satisfy the equality (61) or (62) and the sequence $\{\widehat{f_n}\}$ converges to zero in $\mathfrak{H}_1$. It follows from Lemma 3.24 and equalities (61), (62) that the sequence $\{\widehat{y_n}\}$ converges to zero. So, $T^{-1}(\lambda)$ is the boundary everywhere defined operator. The Theorem is proved. □

Corollary 4.4. *Let $\widetilde{T}(\lambda) \subset \mathfrak{H} \times \mathfrak{H}$ be a linear relation and $L_0 - \lambda E \subset \widetilde{T}(\lambda) \subset L_0^* - \lambda E$. Then $\widetilde{T}(\lambda)$ is continuously invertible in the space $\mathfrak{H}$ if and only if $\widetilde{T}(\lambda)$ has the form $\widetilde{T}(\lambda) = T_0 \oplus T(\lambda)$, where $T_0 \subset \mathfrak{H}_0 \times \mathfrak{H}_0$, $T(\lambda) \subset \mathfrak{H}_1 \times \mathfrak{H}_1$ are linear relations, $L_{10} - \lambda E \subset T(\lambda) \subset L_{10}^* - \lambda E$, $T(\lambda)$ is continuously invertible in $\mathfrak{H}_1$ (i.e., $T(\lambda)$ satisfies Theorem 4.3), T_0 is any continuously invertible relation in $\mathfrak{H}_0$.*

Proof. The desired statement follows from (26). □

Remark 4.5. *It follows from Lemma 3.24 that the operator $M(\lambda)$ is uniquely determined by the relation $T(\lambda)$ and by the choice of functions v_k.*

We shall write equalities (61), (62) in a short form. We denote $\widetilde{W}(t,\lambda)=\sum_{k=1}^{\Bbbk_1}\mathfrak{X}_{[\alpha_k,\beta_k)\setminus(\mathcal{S}_{\mathbf{m}}\cap\mathcal{S}_0)}w_k(t,\lambda)$, i.e., $\widetilde{W}(t,\lambda)=w_k(t,\lambda)$ for $t\in(\alpha_k,\beta_k)$, and $\widetilde{W}(\alpha_k,\lambda)=w_k(\alpha_k,\lambda)$ if $\alpha_k\notin\mathcal{S}_{\mathbf{m}}$, and $\widetilde{W}(\alpha_k,\lambda)=0$ if $\alpha_k\in\mathcal{S}_{\mathbf{m}}$. In (61), (62), the series converge in $\mathfrak{H}_1$ for any function $\widehat{f}\in\mathfrak{H}_1$. We denote

$$\mathbf{K}(t,s,\lambda)=\widetilde{V}(t,\lambda)M(\lambda)\widetilde{V}^*(s,\overline{\lambda})+2^{-1}\widetilde{W}(t,\lambda)\mathrm{sgn}(s-t)iJ\widetilde{W}^*(s,\overline{\lambda})\mathfrak{X}_{[a,b]\setminus\mathcal{S}_{\mathbf{m}}}(s)-$$
$$-2^{-1}\mathfrak{X}_{\mathcal{S}_{\mathbf{m}}}(t)\widetilde{W}(t,\lambda)\mathrm{sgn}(s-t)iJ\widetilde{W}^*(s,\overline{\lambda})\mathfrak{X}_{[a,b]\setminus\mathcal{S}_{\mathbf{m}}}(s),\quad \lambda\neq 0;$$

$$\mathbf{K}(t,s,0)=\widetilde{V}(t,0)M(0)\widetilde{V}^*(s,0)+2^{-1}\widetilde{W}(t,0)\mathrm{sgn}(s-t)iJ\widetilde{W}^*(s,0)\mathfrak{X}_{[a,b]\setminus\mathcal{S}_{\mathbf{m}}}(s)+$$
$$+2^{-1}\mathfrak{X}_{[a,b]\setminus\mathcal{S}_{\mathbf{m}}}(t)\widetilde{W}(t,0)\mathrm{sgn}(s-t)iJ\widetilde{W}^*(s,0)\mathfrak{X}_{\mathcal{S}_{\mathbf{m}}}(s).$$

Then the equalities (61), (62) can be written as

$$\widehat{y}(t)=(T^{-1}(\lambda)\widehat{f})(t)=\int_a^b\mathbf{K}(t,s,\lambda)d\mathbf{m}(s)\widehat{f}(s)-\lambda^{-1}\mathfrak{X}_{\mathcal{S}_{\mathbf{m}}\setminus\mathcal{S}_0}\widehat{f}(t),\quad \lambda\neq 0,\quad \widehat{f}\in\mathfrak{H}_1; \tag{63}$$

$$y(t)=(T^{-1}(0)\widehat{f})(t)=\int_a^b\mathbf{K}(t,s,0)d\mathbf{m}(s)\widehat{f}(s),\quad \widehat{f}\in\mathfrak{H}_1. \tag{64}$$

Let us consider some examples.

Example 4.6. *Suppose $\mathbf{p}=\mathbf{p}_0$ is a continuous measure, $\mathbf{m}=\mu$ is the usual Lebesque measure on $[a,b]$ (i.e., $\mu([\alpha,\beta))=\beta-\alpha$, where $a\leqslant\alpha<\beta\leqslant b$ (we write ds instead of $d\mu(s)$)). In this case, L_0,L_0^* are operators, $\Bbbk_1=\Bbbk=1$, $\mathfrak{H}_0=\{0\}$, $Q_{1,0}=\{0\}$, $Q_1=H=\mathcal{Q}_-=\mathcal{Q}_+$, $\widetilde{V}(t,\lambda)=W(t,\lambda)$. Equality* (51) *has the form*

$$y(t)=W(t,\lambda)\eta-W(t,\lambda)iJ\int_a^t W^*(s,\overline{\lambda})f(s)ds,\quad f=(L_0^*-\lambda E)y,\quad \eta\in H.$$

For any λ, equalities (63), (64) *take the form*

$$y(t)=(T^{-1}(\lambda)f)(t)=\int_a^b\mathbf{K}(t,s,\lambda)f(s)ds, \tag{65}$$

where $\mathbf{K}(t,s,\lambda)=W(t,\lambda)(M(\lambda)+2^{-1}\mathrm{sgn}(s-t)iJ)W^*(s,\overline{\lambda})$.

Example 4.7. *We assume that measures $\mathbf{p}$, $\mathbf{m}$ are continuous. Then L_0, L_0^* are not operators, generally. In this case, $\Bbbk_1=\Bbbk=1$, $\mathfrak{H}_0=\{0\}$. In general, $Q_1\neq H$, $Q_1\neq Q_1^-$. In this case, $\mathcal{Q}_-=Q_1^-$, $\mathcal{V}(\lambda)=\mathcal{W}(\lambda)$ is an extension of the operator $\xi\to W(\cdot,\lambda)\xi$ ($\xi\in Q_1\subset H$) to the set $\mathcal{Q}_-$, $\widetilde{V}(t,\lambda)\eta=\widetilde{W}(t,\lambda)\eta=(\mathcal{W}(\lambda)\eta)(t)$ ($\eta\in\mathcal{Q}_-$). Equality* (51) *has the form*

$$y(t)=\widetilde{W}(t,\lambda)\eta-\widetilde{W}(t,\lambda)iJ\int_a^t\widetilde{W}^*(s,\overline{\lambda})d\mathbf{m}(s)f(s),\quad \{y,f\}\in L_0^*-\lambda E,\quad \eta\in\mathcal{Q}_-.$$

For any λ, equalities (63), (64) *take the form*

$$y(t)=(T^{-1}(\lambda)f)(t)=\int_a^b\mathbf{K}(t,s,\lambda)d\mathbf{m}(s)f(s),$$

where $\mathbf{K}(t,s,\lambda)=\widetilde{W}(t,\lambda)(M(\lambda)+2^{-1}\mathrm{sgn}(s-t)iJ)\widetilde{W}^*(s,\overline{\lambda})$.

Example 4.8. *Suppose that $\mathbf{m}=\mu$ is the usual Lebesque measure and the set $\mathcal{S}_{\mathbf{p}}$ of single-point atoms of the measure $\mathbf{p}$ can be arranged as an increasing sequence converging to b. In this case, the description of $T^{-1}(\lambda)$ is obtained in [9].*

Example 4.9. *Suppose that $\mathcal{S}_{\mathbf{m}} \neq \varnothing$ and $\mathbf{m} = \mu + \widehat{\mathbf{m}}$, where $\mu = \mathbf{m}_0$ is the usual Lebesque measure on $[a, b]$ and $\mu(\Delta) = \mathbf{m}(\Delta)$ for all Borel sets such that $\Delta \cap \mathcal{S}_{\mathbf{m}} = \varnothing$. So, $\mathcal{S}_{\mathbf{m}} = \mathcal{S}_{\widehat{\mathbf{m}}}$ and $\mathbf{m}(\{\beta\}) = \widehat{\mathbf{m}}(\{\beta\})$ for all $\beta \in \mathcal{S}_{\mathbf{m}}$. We arrange the elements of $\mathcal{S}_{\mathbf{m}}$ in the form of a finite or infinite sequence $\{\tau_k\}$. Let $\Bbbk_2$ be the number of elements in $\mathcal{S}_{\mathbf{m}}$. We denote $\widehat{Q}_{k,0} = \ker \mathbf{m}(\{\tau_k\})$, $\widehat{Q}_k = H \ominus \widehat{Q}_{k,0}$, where $\tau_k \in \mathcal{S}_{\mathbf{m}}$. Let $\mathbf{m}_k$ be the restriction of the operator $\mathbf{m}(\{\tau_k\})$ to $\widehat{Q}_k$. The operator $\mathbf{m}_k$ is self-adjoint and $\mathcal{R}(\mathbf{m}_k) \subset \widehat{Q}_k$. By $\widehat{Q}_k^-$ denote the completion of $\widehat{Q}_k$ with respect to norm $\|\xi\|_- = (\mathbf{m}_k\xi, \xi)^{1/2}$, where $\xi \in \widehat{Q}_k$. Let $\widehat{\mathcal{Q}}_-$ be linear space of sequences $\widetilde{\eta} = \{\eta_k\}$ such that $\eta_k \in \widehat{Q}_k^-$ ($k \in \mathbb{N}$ if $\Bbbk_2 = \infty$, and $1 \leqslant k \leqslant \Bbbk_2$ if $\Bbbk_2$ is finite) and the series $\sum_{k=1}^{\infty} \|\eta_k\|_-^2$ converges if $\Bbbk_2 = \infty$. Then $\mathfrak{H} = L_2(H; a, b) \oplus \widehat{\mathcal{Q}}_-$.*

Suppose $\mathbf{p} = 0$ and $a \notin \mathcal{S}_{\mathbf{m}}$, $b \notin \mathcal{S}_{\mathbf{m}}$. (The case of an arbitrary continuous measure $\mathbf{p}$ can be considered similarly.) Then $\mathfrak{H}_0 = \{0\}$, $\Bbbk_1 = 1$, $W(t, 0) = E$, and $\mathcal{Q}_- = H \oplus \widehat{\mathcal{Q}}_-$. It follows from Lemma 3.3 and (14) *that a pair $\{y, f\} \in L_0$ if and only if*

$$y(t) = -iJ \int_a^t f(s)ds, \quad y(b) = 0, \quad \mathbf{m}(\beta) f(\beta) = 0 \ \ (\beta \in \mathcal{S}_{\mathbf{m}}).$$

Using Theorem 3.25 for $\lambda = 0$, we obtain that a pair $\{y, f\} \in L_0^*$ if and only if

$$y(t) = \eta_0 + \sum_{\tau_k \leqslant t} \mathfrak{X}_{\{\tau_k\}}(t)\eta_k - iJ \int_a^t d\mathbf{m}(s) f(s), \tag{66}$$

where $\eta_0 \in H$, $\tau_k \in \mathcal{S}_{\mathbf{m}}$, $\eta_k \in \widehat{Q}_k^-$, and the sequence $\widetilde{\eta} = \{\eta_0, \eta_k\}$ belongs to $\mathcal{Q}_-$ (here $k \in \mathbb{N}$ if $\Bbbk_2 = \infty$, and $1 \leqslant k \leqslant \Bbbk_2$ if $\Bbbk_2$ is finite). It follows from Lemma 3.15 (for$\lambda = 0$) that the function $\mathfrak{X}_{\mathcal{S}_{\mathbf{m}}}(t) \int_a^t d\mathbf{m}(s) f(s) \in \ker L_0^*$. Therefore, equality (66) *can be written as*

$$y(t) = \xi_0 + \sum_{\tau_k \leqslant t} \mathfrak{X}_{\{\tau_k\}}\xi_k - \mathfrak{X}_{[a,b]\setminus\mathcal{S}_{\mathbf{m}}}(t) iJ \int_a^t d\mathbf{m}(s) f(s), \quad \xi_0 \in H, \quad \xi_k \in \widehat{Q}_k^-, \quad \widetilde{\xi} = \{\xi_0, \xi_k\} \in \mathcal{Q}_-.$$

By (6), *it follows that $W(t, \lambda) = \exp(-iJ\lambda t)$. Using* (31), *we get*

$$u_1(t, \lambda, \tau)x = -\mathfrak{X}_{[a,b]\setminus\mathcal{S}_{\mathbf{m}}} W(t, \lambda) iJ \int_a^t W^*(s, \overline{\lambda}) d\mathbf{m}(s) \lambda \mathfrak{X}_{\{\tau\}}(s)x, \quad x \in H, \quad \tau \in \mathcal{S}_{\mathbf{m}}.$$

Hence, $u_1(t, \lambda, \tau)x + \mathfrak{X}_{\{\tau\}}(t)x$ is equal to zero if $t < \tau$, and $\mathfrak{X}_{\{\tau\}}(t)x$ if $t = \tau$, and $-\lambda\mathfrak{X}_{[a.b]\setminus\mathcal{S}_{\mathbf{m}}} W(t, \lambda) iJW^*(\tau, \overline{\lambda})\mathbf{m}(\{\tau\})x$ if $t > \tau$. We denote $v_0(t, \lambda) = \mathfrak{X}_{[a,b]\setminus\mathcal{S}_{\mathbf{m}}} W(t, \lambda)$, $v_k(t, \lambda) = u_1(t, \lambda, \tau_k)W(\tau_k, \lambda)x + \mathfrak{X}_{\{\tau_k\}}(t)W(\tau_k, \lambda)x$ ($k \in \mathbb{N}$ if $\Bbbk_2 = \infty$, and $1 \leqslant k \leqslant \Bbbk_2$ if $\Bbbk_2$ is finite). By Lemma 3.18, it follows that the linear span of functions $v_0(\cdot, \lambda)\xi_0$, $v_k(\cdot, \lambda)\xi_k$ ($\xi_0, \xi_k \in H$) is dense in $\ker(L_{10}^* - \lambda E)$. The operator $V_N(t, \lambda)$ has the form $V_N(t, \lambda) = (v_0(t, \lambda), ..., v_{N-1}(t, \lambda))$. As above, by $\mathcal{V}(\lambda)$ we denote the operator $\mathcal{V}(\lambda)\colon \mathcal{Q}_- \to \mathfrak{H}$ such that $\mathcal{V}(\lambda)\widetilde{\eta} = \mathcal{V}_N(\lambda)\widetilde{\eta}_N$ for all $N \in \mathbb{N}$, where $\mathcal{V}_N(\lambda)$ is the operator $\widetilde{\xi}_N \to V_N(\cdot, \lambda)\widetilde{\xi}_N$, $\widetilde{\xi} = (\widetilde{\xi}_N, 0, ...)$, $\widetilde{\xi}_N \in \widetilde{Q}_N^-$.

Thus, in this example, equalities (61), (62) *will take form* (67), (68), *respectively,* (*see equalities below*)

$$y(t) = (T^{-1}(\lambda)f)(t) = \int_a^b \widetilde{V}(t, \lambda)M(\lambda)\widetilde{V}^*(s, \overline{\lambda}) d\mathbf{m}(s) f(s) + 2^{-1} \int_a^b W(t, \lambda) \operatorname{sgn}(s - t) iJW^*(s, \overline{\lambda}) f(s)ds -$$
$$- 2^{-1} \int_a^b \mathfrak{X}_{\mathcal{S}_{\mathbf{m}}}(t) W(t, \lambda) \operatorname{sgn}(s - t) iJW^*(s, \overline{\lambda}) f(s)ds - \lambda^{-1}\mathfrak{X}_{\mathcal{S}_{\mathbf{m}}}(t) f(t), \quad \lambda \neq 0, \quad f \in \mathfrak{H}, \tag{67}$$

$$y(t) = (T^{-1}(0)f)(t) = \int_a^b \widetilde{V}(t, 0)M(0)\widetilde{V}^*(s, 0) d\mathbf{m}(s) f(s) + 2^{-1} \int_a^b W(t, 0) \operatorname{sgn}(s - t) iJW^*(s, 0) f(s)ds +$$
$$+ 2^{-1} \int_a^b \mathfrak{X}_{[a,b]\setminus\mathcal{S}_{\mathbf{m}}}(t) W(t, 0) \operatorname{sgn}(s - t) iJW^*(s, 0) d\mathbf{m}(s) \mathfrak{X}_{\mathcal{S}_{\mathbf{m}}}(s) f(s), \quad f \in \mathfrak{H}. \tag{68}$$

We note that if $\mathcal{S}_{\mathbf{m}} = \varnothing$, then equalities (67), (68) *coincide with* (65) *for all λ.*

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