Filomat 35:5 (2021), 1565–1587
https://doi.org/10.2298/FIL2105565C

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

The Perturbation Bound for the T-Drazin Inverse of Tensor and its Application

Ying-nan Cui[a], Hai-feng Ma[a]

[a]*School of Mathematical Science, Harbin Normal University, Harbin 150025, P. R. China.*

Abstract. In this paper, let $\mathcal{A}$ and $\mathcal{B}$ be $n \times n \times p$ complex tensors and $\mathcal{B} = \mathcal{A} + \mathcal{E}$. Denote the T-Drazin inverse of $\mathcal{A}$ by $\mathcal{A}^D$. We give a perturbation bound for $\|\mathcal{B}^D - \mathcal{A}^D\|/\|\mathcal{A}^D\|$ under condition $(\mathcal{W})$. Considering the solution of singular tensor equation $\mathcal{A} * x = b$, $(b \in \mathcal{R}(\mathcal{A}^D))$ at the same time. The optimal perturbation of T-Drazin inverse of tensors and the solution of a system of tensor equations have been given.

1. Introduction

The Drazin inverse plays an important role in many applications [1, 7, 20, 21, 25, 35]. There have been some papers on Drazin inverse of the perturbation bounds of matrix [27–31, 33, 34, 37]. Furthermore, we consider the perturbation of the Drazin inverse under the T-product of tensor. There are three monographs on the tensor [5, 19, 32]. Tensors are hyper dimensional matrices, which are the extensions of matrices. We study the generalized inverses of tensor based on Einstein product, in order to overcome high-dimension of tensor [10, 15, 22, 24]. In addition, the T-product of tensor [9, 11, 12, 14, 26] is another product which has been proven to be a useful tool in many applications[2, 9, 11, 12, 14, 16, 23, 38]. Recently, Ji and Wei [10] presented the Drazin inverse of an even-order tensor with the Einstein product. Che and Wei [3, 4, 32, 36] present the randomized algorithms for the tensor decomposition and the tensor equations.

The T-Jordan canonical form of the T-Drazin of third-order tensor inverse and the generalized tensor function are given by Miao, Qi and Wei in [17, 18], but its perturbation has not been developed yet. The perturbation of T-Drazin inverse and its application are introduced in this paper.

In this paper, let $\mathbb{C}^{n\times n\times p}$ and $\mathbb{R}^{n\times n\times p}$ be two sets of the $n \times n \times p$ tensors over the complex field $\mathbb{C}$ and the real field $\mathbb{R}$, respectively. Let $\mathcal{A} \in \mathbb{C}^{n\times n\times p}$, and $\rho_T(\mathcal{A})$ denote the T-spectral radius of $\mathcal{A}$. For positive integers k and n, $[k] = [1, \cdots, n]$. We call $\mathcal{O}$ as a zero tensor in case of all the entries of the tensor are zero.

Now, a concept is proposed for multiplying third order tensors [9, 11, 12], based on viewing a tensor as a stake of frontal slices. Suppose $\mathcal{A} \in \mathbb{R}^{m\times n\times p}$ and $\mathcal{B} \in \mathbb{R}^{n\times s\times p}$ are third order tensors, denote their frontal

2010 *Mathematics Subject Classification*. Primary 15A09; Secondary 65F20.
Keywords. perturbation, T-Drazin inverse, T-product, tensor.
Received: 21 April 2020; Accepted: 08 May 2020
Communicated by Yimin Wei
Corresponding author: Hai-feng Ma
Research supported by the National Natural Science Foundation of China under grant 11971136, the bilateral project between China and Poland (no.37-18), and Innovative Research Project for Postgraduate Students of Harbin Normal University (no. HSDSSCX2019-34).
Email addresses: 994998246@qq.com (Ying-nan Cui), haifengma@aliyun.com (Hai-feng Ma)

faces as $A^{(k)} \in \mathbb{R}^{m\times n}$ and $B^{(k)} \in \mathbb{R}^{n\times s}$, respectively ($k = 1, 2, \cdots, p$). $\mathcal{A} \in \mathbb{C}^{n\times n\times p}$ is called as F-square tensor , if every frontal face of $\mathcal{A}$ is square. The operation of "bcirc" was introduced in [9, 11, 12],

$$bcirc(\mathcal{A}) := \begin{pmatrix} A^{(1)} & A^{(p)} & A^{(p-1)} & \cdots & A^{(2)} \\ A^{(2)} & A^{(1)} & A^{(p)} & \cdots & A^{(3)} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ A^{(p)} & A^{(p-1)} & \cdots & A^{(2)} & A^{(1)} \end{pmatrix}, unfold(\mathcal{A}) := \begin{pmatrix} A^{(1)} \\ A^{(2)} \\ \vdots \\ A^{(p)} \end{pmatrix},$$

and $fold\,(unfold(\mathcal{A})) := \mathcal{A}$. We define the corresponding inverse operation $bcirc^{-1} : \mathbb{R}^{mp\times np} \longrightarrow \mathbb{R}^{m\times n\times p}$ such that $bcirc^{-1}\,(bcirc(\mathcal{A})) = \mathcal{A}$.

Definition 1.1. [9, 11, 12](T-product) *Let $\mathcal{A} \in \mathbb{R}^{m\times n\times p}$ and $\mathcal{B} \in \mathbb{R}^{n\times s\times p}$ be two real tensors. Then the T-product $\mathcal{A} * \mathcal{B}$ is an $m \times s \times p$ real tensor defined by*

$$\mathcal{A} * \mathcal{B} := fold\,(bcirc(\mathcal{A})unfold(\mathcal{B}))\,.$$

Definition 1.2. [9, 11, 12](Transpose and conjugate transpose) *If $\mathcal{A}$ is a third order tensor of size $m \times n \times p$, then the transpose $\mathcal{A}^T$ is obtained by transposing each of the frontal slices and then reversing the order of transposed frontal slices 2 through n. The conjugate transpose $\mathcal{A}^H$ is obtained by conjugate transposing each of the frontal slices and then reversing the order of transposed frontal slices 2 through n.*

Definition 1.3. [9, 11, 12](Identity tensor) *The $n \times n \times p$ identity tensor $\mathcal{I}_{nnp}$ is the tensor whose first frontal slice is the $n \times n$ identity matrix, and whose other frontal slices are all zeros. It is easy to check that*

$$\mathcal{A} * \mathcal{I}_{nnp} = \mathcal{I}_{mmp} * \mathcal{A} = \mathcal{A} \; for \; \mathcal{A} \in \mathbb{R}^{m\times n\times p}.$$

For a frontal square $\mathcal{A}$ of size $n \times n \times p$, it has inverse tensor $\mathcal{B} \in \mathbb{R}^{n\times n\times p} (= \mathcal{A}^{-1})$, provided that

$$\mathcal{A} * \mathcal{B} = \mathcal{I}_{nnp} \; and \; \mathcal{B} * \mathcal{A} = \mathcal{I}_{nnp}.$$

Definition 1.4. [17, 18] *Let $\mathcal{A} \in \mathbb{R}^{m\times n\times p}$, then*
(1) *The T-range space of $\mathcal{A}$, $\mathcal{R}(\mathcal{A}) := Ran\left((F_p \otimes I_m)bcirc(\mathcal{A})(F_p^H \otimes I_n)\right)$,"Ran" means the range space,*
(2) *The T-null space of $\mathcal{A}$, $\mathcal{N}(\mathcal{A}) := Null\left((F_p \otimes I_m)bcirc(\mathcal{A})(F_p^H \otimes I_n)\right)$, "Null" represents the null space,*
(3) *The tensor norm $\|\mathcal{A}\| := \|bcirc(\mathcal{A})\|$,*
where F_n is the discrete Fourier matrix of size $n \times n$, which is defined as [2].

$$F_{n\times n} = \frac{1}{\sqrt{n}} \begin{pmatrix} 1 & 1 & 1 & 1 & \cdots & 1 \\ 1 & w & w^2 & w^3 & \cdots & w^{n-1} \\ 1 & w^2 & w^4 & w^6 & \cdots & w^{2(n-1)} \\ 1 & w^3 & w^6 & w^9 & \cdots & w^{3(n-1)} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & w^{n-1} & w^{2(n-1)} & w^{3(n-1)} & \cdots & w^{(n-1)(n-1)} \end{pmatrix},$$

where $w = e^{-2\pi \mathbf{i}/n}$ is the primitive n-th root of unity in which $\mathbf{i} = \sqrt{-1}$. F_p^H is the conjugate transpose of F_p.

Lemma 1.5. [12] *Suppose $\mathcal{A} \in \mathbb{C}^{n\times n\times p}$ and $\mathcal{B} \in \mathbb{C}^{n\times s\times p}$, then*

$$bcirc(\mathcal{A} * \mathcal{B}) = bcirc(\mathcal{A})bcirc(\mathcal{B}).$$

Remark 1.6. *Let $\mathcal{A}, \mathcal{B}, \mathcal{C} \in \mathbb{C}^{n\times n\times p}$ be F-square tensors. Then $\|\mathcal{A} * \mathcal{B} * \mathcal{C}\| \leq \|\mathcal{A}\|\|\mathcal{B}\|\|\mathcal{C}\|$.*

Proof. Since Lemma 1.5, we obtain

$$bcirc(\mathcal{A} * \mathcal{B} * \mathcal{C}) = bcirc(\mathcal{A})bcirc(\mathcal{B})bcirc(\mathcal{C}). \tag{1}$$

Take norm on both sides of (1) at the same time, then

$$\begin{aligned}\|bcirc(\mathcal{A} * \mathcal{B} * \mathcal{C})\| &= \|bcirc(\mathcal{A})bcirc(\mathcal{B})bcirc(\mathcal{C})\| \\ &\le \|bcirc(\mathcal{A})\|\|bcirc(\mathcal{B})\|\|bcirc(\mathcal{C})\|.\end{aligned}$$

According to (3) of Definition 1.4, we have

$$\|\mathcal{A} * \mathcal{B} * \mathcal{C}\| \le \|\mathcal{A}\|\|\mathcal{B}\|\|\mathcal{C}\|.$$

□

Definition 1.7. [17](T-index) *Let $\mathcal{A} \in \mathbb{C}^{n\times n\times p}$ be a complex tensor. The T-index of $\mathcal{A}$ is defined as*

$$Ind_T(\mathcal{A}) = Ind\left(bcirc(\mathcal{A})\right).$$

Definition 1.8. [17](T-Drazin inverse) *Let $\mathcal{A}, \mathcal{X} \in \mathbb{C}^{n\times n\times p}$, satisfying the following three equations*

$$\mathcal{A} * \mathcal{X} = \mathcal{X} * \mathcal{A}, \tag{2}$$

$$\mathcal{X} * \mathcal{A} * \mathcal{X} = \mathcal{X}, \tag{3}$$

$$\mathcal{A}^k * \mathcal{X} * \mathcal{A} = \mathcal{A}^k, \tag{4}$$

where $Ind_T(\mathcal{A}) = k$, then $\mathcal{X}$ is called by T-Drazin inverse of $\mathcal{A}$, which is denoted as $\mathcal{A}^D$.

Definition 1.9. [17](Nilpotent tensor) *Let $\mathcal{A} \in \mathbb{C}^{n\times n\times p}$ be nilpotent, if there exists a positive integer $s \in \mathbb{Z}$ such that $\mathcal{A}^s = 0$. If $s \in \mathbb{Z}$ is the smallest positive integer satisfying the equation $\mathcal{A}^s = 0$, then s is called the nilpotent index of $\mathcal{A}$.*

Definition 1.10. [17](T-core-nilpotent decomposition) *Let $\mathcal{A} \in \mathbb{C}^{n\times n\times p}$ be a complex tensor, $\mathcal{N}_A$ is T-nilpotent-part of $\mathcal{A}$, and $\mathcal{C}_A$ is T-core-part of $\mathcal{A}$, satisfying*

$$\mathcal{N}_A = \mathcal{A} - \mathcal{C}_A = (\mathcal{I} - \mathcal{A} * \mathcal{A}^D) * \mathcal{A},$$

then $\mathcal{A} = \mathcal{C}_A + \mathcal{N}_A$ is called T-core-nilpotent decomposition of $\mathcal{A}$.

The construction of T-core-nilpotent decomposition of a tensor is introduced in [17]. Suppose $\mathcal{A} \in \mathbb{C}^{n\times n\times p}$, $\mathcal{P}$ is an invertible tensor, $\mathcal{J} \in \mathbb{C}^{n\times n\times p}$ is an F-bidiagonal tensor, and $Ind_T(\mathcal{A}) = k$, then the T-Jordan decomposition of $\mathcal{A}$ is $\mathcal{A} = \mathcal{P}^{-1} * \mathcal{J} * \mathcal{P}$, and

$$bcirc(\mathcal{J}) = (F_p \otimes I_n)\begin{pmatrix} J_1 & & & \\ & J_2 & & \\ & & \ddots & \\ & & & J_p \end{pmatrix}(F_p^H \otimes I_n),$$

where J_i can be block partitioned as

$$J_i = \begin{pmatrix} C_i & O \\ O & N_i \end{pmatrix} = \begin{pmatrix} C_i & O \\ O & O \end{pmatrix} + \begin{pmatrix} O & O \\ O & N_i \end{pmatrix} = J_i^C + J_i^N,\ (i = 1, 2, \cdots, p)$$

and C_i is a nonsingular matrix, N_i is nilpotent with $\max\limits_{1<i<p} Ind(N_i) = k$, then

$$bcirc(\mathcal{J}) = bcirc(\mathcal{J}^C) + bcirc(\mathcal{J}^N),$$

that is

$$\mathcal{A} = \mathcal{P}^{-1} * \mathcal{J} * \mathcal{P} = \mathcal{P}^{-1} * (\mathcal{J}^C + \mathcal{J}^N) * \mathcal{P} = \mathcal{C}_A + \mathcal{N}_A,$$

which is the construction of T-core-nilpotent decomposition of $\mathcal{A}$.

Theorem 1.11. [17] *Let* $\mathcal{A} \in \mathbb{C}^{n\times n\times p}$*, then there is an invertible tensor* $\mathcal{P} \in \mathbb{C}^{n\times n\times p}$ *and F-bidiagonal tensor* $\mathcal{J} \in \mathbb{C}^{n\times n\times p}$*, and the T-Jordan canonical form is,*

$$\mathcal{A} = \mathcal{P}^{-1} * \mathcal{J} * \mathcal{P},$$

where the diagonal elements of $\mathcal{J}_i (i = 1, 2, \cdots, p)$ *are the T-eigenvalues of* $\mathcal{A}$*. The decomposition of matrix bcirc(*$\mathcal{J}$*) is given, as follows*

$$bcirc(\mathcal{J}) = (F_p \otimes I_n) \begin{pmatrix} J_1 & & & \\ & J_2 & & \\ & & \ddots & \\ & & & J_p \end{pmatrix} (F_p^H \otimes I_n),$$

where J_i *can be partitioned as* $J_i = \begin{pmatrix} J_i^1 & O \\ O & J_i^0 \end{pmatrix}$*,* J_i^1 *is the core of the matrix* J_i*, and* J_i^0 *is nilpotent,* $(i = 1, 2, \cdots, p)$.

Further, the T-Drazin inverse is denoted as

$$\mathcal{A}^D = \mathcal{P}^{-1} * \mathcal{J}^D * \mathcal{P}.$$

The decomposition of bcirc($\mathcal{J}^D$*) is*

$$bcirc(\mathcal{J}^D) = (F_p \otimes I_n) \begin{pmatrix} J_1^D & & & \\ & J_2^D & & \\ & & \ddots & \\ & & & J_p^D \end{pmatrix} (F_p^H \otimes I_n),$$

where $J_i^D = \begin{pmatrix} (J_i^1)^{-1} & O \\ O & O \end{pmatrix}$ *is the Drazin inverse of the matrix* J_i. $(i = 1, 2, \cdots, p)$

Remark 1.12. *From the T-Jordan canonical form, we know that for any complex tensor* $\mathcal{A} \in \mathbb{C}^{n\times n\times p}$ *with* $Ind_T(\mathcal{A}) = k$ *and* $rank_T(\mathcal{A}^k) = r$*, there exists nonsingular tensor* $\mathcal{P} \in \mathbb{C}^{n\times n\times p}$ *such that*

$$\mathcal{A} = \mathcal{P}^{-1} * \mathcal{J} * \mathcal{P} = \mathcal{P}^{-1} * \begin{pmatrix} \mathcal{J}_1 & \mathcal{O} \\ \mathcal{O} & \mathcal{J}_4^0 \end{pmatrix} * \mathcal{P},$$

and

$$\mathcal{A}^D = \mathcal{P}^{-1} * \mathcal{J}^D * \mathcal{P} = \mathcal{P}^{-1} * \begin{pmatrix} \mathcal{J}_1^{-1} & \mathcal{O} \\ \mathcal{O} & \mathcal{O} \end{pmatrix} * \mathcal{P},$$

where $\mathcal{J}_1$ *is the core part of tensor* $\mathcal{J}$*, and* $\mathcal{J}_4^0$ *is nilpotent.*

Theorem 1.13. [10, 17, 18](T-linear system) *Let* $\mathcal{A} \in \mathbb{C}^{n\times n\times p}$ *be an F-square invertible tensor with* $Ind_T(\mathcal{A}) = k$*. If the T-linear tensor system*

$$\mathcal{A} * x = b, \ x \in \mathcal{R}(\mathcal{A}^k),$$

where $x, b \in \mathbb{C}^{n\times 1\times p}$*, has an unique solution, then it is given by*

$$x = \mathcal{A}^D * b. \tag{5}$$

Theorem 1.14. *If* $\mathcal{N} = \begin{pmatrix} \mathcal{A} & \mathcal{B} \\ O & C \end{pmatrix} \in \mathbb{C}^{2n\times 2n\times p}$, *where* $\mathcal{A}$ *and* C *are F-square tensors,* $Ind_T(\mathcal{A}) = k$, $Ind_T(C) = l$, *then*

$$\mathcal{N}^D = \begin{pmatrix} \mathcal{A}^D & \mathcal{X} \\ O & C^D \end{pmatrix} \in \mathbb{C}^{2n\times 2n\times p},$$

where

$$\mathcal{X} = \sum_{s=0}^{l-1} (\mathcal{A}^D)^{s+2} * \mathcal{B} * C^s * (\mathcal{I} - C * C^D) + (\mathcal{I} - \mathcal{A} * \mathcal{A}^D) * \sum_{s=0}^{k-1} \mathcal{A}^s * \mathcal{B} * (C^D)^{s+2} - \mathcal{A}^D * \mathcal{B} * C^D.$$

Proof. There are some decompositions of matrixes bcirc($\mathcal{A}$), bcirc($\mathcal{X}$), bcirc(C), bcirc($\mathcal{B}$), such that

$$bcirc(\mathcal{A}) = (F_p \otimes I_n) \begin{pmatrix} A_1 & & & \\ & A_2 & & \\ & & \ddots & \\ & & & A_p \end{pmatrix} (F_p^H \otimes I_n),\ bcirc(\mathcal{A}^D) = (F_p \otimes I_n) \begin{pmatrix} A_1^D & & & \\ & A_2^D & & \\ & & \ddots & \\ & & & A_p^D \end{pmatrix} (F_p^H \otimes I_n),$$

$$bcirc(\mathcal{B}) = (F_p \otimes I_n) \begin{pmatrix} B_1 & & & \\ & B_2 & & \\ & & \ddots & \\ & & & B_p \end{pmatrix} (F_p^H \otimes I_n),\ bcirc(\mathcal{B}^D) = (F_p \otimes I_n) \begin{pmatrix} B_1^D & & & \\ & B_2^D & & \\ & & \ddots & \\ & & & B_p^D \end{pmatrix} (F_p^H \otimes I_n),$$

$$bcirc(C) = (F_p \otimes I_n) \begin{pmatrix} C_1 & & & \\ & C_2 & & \\ & & \ddots & \\ & & & C_p \end{pmatrix} (F_p^H \otimes I_n),\ bcirc(C^D) = (F_p \otimes I_n) \begin{pmatrix} C_1^D & & & \\ & C_2^D & & \\ & & \ddots & \\ & & & C_p^D \end{pmatrix} (F_p^H \otimes I_n),$$

and

$$bcirc(\mathcal{X}) = (F_p \otimes I_n) \begin{pmatrix} T_1 & & & \\ & T_2 & & \\ & & \ddots & \\ & & & T_p \end{pmatrix} (F_p^H \otimes I_n),$$

where

$$\begin{aligned} T_i &= (A_i^D)^2 \left(\sum_{s=0}^{n} (A_i^D)^s B_i C_i^s \right) (I - CC^D) + (I - AA^D) \left(\sum_{s=0}^{n} A_i^s B_i (C_i^D)^s \right) (C_i^D)^2 - A_i^D B_i C_i^D \\ &= (A_i^D)^2 \left(\sum_{s=0}^{l-1} (A_i^D)^s B_i C_i^s \right) (I - CC^D) + (I - AA^D) \left(\sum_{s=0}^{k-1} A_i^s B_i (C_i^D)^s \right) (C_i^D)^2 - A_i^D B_i C_i^D, \end{aligned}$$

$i = 1, 2, \cdots, p$.
Expand the term $\mathcal{A} * \mathcal{X}$ as follows. Since Lemma 1.5, we obtain

$$\begin{aligned} bcirc(\mathcal{A} * \mathcal{X}) &= bcirc(\mathcal{A}) bcirc(\mathcal{X}) \\ &= (F_p \otimes I_n) \begin{pmatrix} A_1 T_1 & & & \\ & A_2 T_2 & & \\ & & \ddots & \\ & & & A_p T_p \end{pmatrix} (F_p^H \otimes I_n), \end{aligned}$$

where

$$\begin{aligned}
A_iT_i &= \sum_{s=0}^{l-1}(A_i^D)^{s+1}B_iC_i^s - \sum_{s=0}^{l-1}(A_i^D)^{s+1}B_iC_i^{s+1}C_i^D \\
&- \sum_{s=0}^{k-1}A_i^{s+1}B_i(C_i^D)^{s+2} - \sum_{s=0}^{k-1}A_i^DA_i^{s+2}B_i(C_i^D)^{s+2} - A_iA_i^DB_iC_i \\
&= \left(A_i^DB_i + \sum_{s=0}^{l-2}(A_i^D)^{s+2}B_iC_i^{s+1}\right) - \left(A_i^DB_iC_iC_i^D + \sum_{s=0}^{l-2}(A_i^D)^{s+2}B_iC_i^{s+2}C_i^D\right) \\
&+ \left(\sum_{s=1}^{k-1}(A_i)^sB_i(C_i^D)^{s+1} + A_i^kB_i(C_i^D)^{k+1}\right) - \left(\sum_{s=1}^{k-1}(A_i)^DA_i^{s+1}B_i(C_i^D)^{s+1} + A_i^kB_i(C_i^D)^{k-1}\right) \\
&- A_iA_i^DB_iC_i \\
&= A_i^DB_i + \sum_{s=0}^{l-2}(A_i^D)^{s+2}B_iC_i^{s+1} - A_i^DB_iC_iC_i^D - \sum_{s=0}^{l-2}(A_i^D)^{s+2}B_iC_i^{s+2}C_i^D \\
&+ \sum_{s=1}^{k-1}A_i^sB_i(C_i^D)^{s+1} - \sum_{s=1}^{k-1}A_i^DA_i^{s+1}B_i(C_i^D)^{s+1} - A_iA_i^DB_iC_i. \ (i = 1, 2\cdots p)
\end{aligned}$$

Now we expand the term $\mathcal{X} * \mathcal{C}$ as follows.
By Lemma 1.5, then

$$\begin{aligned}
bcirc(\mathcal{X} * \mathcal{C}) &= bcirc(\mathcal{X})bcirc(\mathcal{C}) \\
&= (F_p \otimes I_n)\begin{pmatrix} T_1C_1 & & & \\ & T_2C_2 & & \\ & & \ddots & \\ & & & T_pC_p \end{pmatrix}(F_p^H \otimes I_n),
\end{aligned}$$

where

$$\begin{aligned}
T_iC_i &= \sum_{s=0}^{l-1}(A_i^D)^{s+2}B_iC_i^{s+1} - \sum_{s=0}^{l-1}(A_i^D)^{s+2}B_iC_i^{s+2}C_i^D \\
&+ \sum_{s=0}^{k-1}A_i^sB_i(C_i^D)^{s+1} - \sum_{s=0}^{k-1}A_i^DA_i^{s+1}B_i(C_i^D)^{s+1} - A_i^DB_iC_i^DC_i \\
&= \left(\sum_{s=0}^{l-2}(A_i^D)^{s+2}B_iC_i^{s+1} + (A_i^D)^{l+1}B_iC_i^l\right) - \left(\sum_{s=0}^{l-2}(A_i^D)^{s+2}B_iC_i^{s+2}C^D + (A_i^D)^{l+1}B_iC_i^l\right) \\
&+ \left(B_iC_i^D + \sum_{s=1}^{k-1}A_i^sB_i(C_i^D)^{s+1}\right) - \left(A_i^DA_iB_iC_i^D + \sum_{s=1}^{k-1}A_i^DA_i^{s+1}B_i(C_i^D)^{s+1}\right) \\
&- A_i^DB_iC_i^DC_i. \ (i = 1, 2\cdots p)
\end{aligned}$$

According to $bcirc(\mathcal{A}), bcirc(\mathcal{B}), bcirc(\mathcal{C}), bcirc(\mathcal{A}^D)$ and $bcirc(\mathcal{C}^D)$, we obtain

$$\begin{aligned}
bcirc(\mathcal{A}^D * \mathcal{B}) &= bcirc(\mathcal{A}^D)bcirc(\mathcal{B}) \\
&= (F_p \otimes I_n)\begin{pmatrix} A_1^DB_1 & & & \\ & A_2^DB_2 & & \\ & & \ddots & \\ & & & A_p^DB_p \end{pmatrix}(F_p^H \otimes I_n),
\end{aligned}$$

$$bcirc(\mathcal{B}*\mathcal{C}^D) = bcirc(\mathcal{B})bcirc(\mathcal{C}^D) = (F_p \otimes I_n)\begin{pmatrix} B_1C_1^D & & & \\ & B_2C_2^D & & \\ & & \ddots & \\ & & & B_pC_p^D \end{pmatrix}(F_p^H \otimes I_n),$$

then

$$\mathcal{A}^D*\mathcal{B}-\mathcal{B}*\mathcal{C}^D = (F_p \otimes I_n)\begin{pmatrix} A_1^DB_1 - B_1C_1^D & & & \\ & A_2^DB_2 - B_2C_2^D & & \\ & & \ddots & \\ & & & A_p^DB_p - B_pC_p^D \end{pmatrix}(F_p^H \otimes I_n),$$

and

$$\mathcal{A}*\mathcal{X}-\mathcal{X}*\mathcal{C} = (F_p \otimes I_n)\begin{pmatrix} A_1T_1 - T_1C_1 & & & \\ & A_2T_2 - T_2C_2 & & \\ & & \ddots & \\ & & & A_pT_p - T_pC_p \end{pmatrix}(F_p^H \otimes I_n).$$

It is easy to see that $\mathcal{A}*\mathcal{X}-\mathcal{X}*\mathcal{C} = \mathcal{A}^D*\mathcal{B}-\mathcal{B}*\mathcal{C}^D$, or $\mathcal{A}*\mathcal{X}+\mathcal{B}*\mathcal{C}^D = \mathcal{A}^D*\mathcal{B}+\mathcal{X}*\mathcal{C}$. From this it follows that

$$\begin{pmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{O} & \mathcal{C} \end{pmatrix}*\begin{pmatrix} \mathcal{A}^D & \mathcal{X} \\ \mathcal{O} & \mathcal{C}^D \end{pmatrix} = \begin{pmatrix} \mathcal{A}^D & \mathcal{X} \\ \mathcal{O} & \mathcal{C}^D \end{pmatrix}*\begin{pmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{O} & \mathcal{C} \end{pmatrix},$$

so that (2) of Definition 1.8 is satisfied. To show that (3) of Definition 1.8 holds, note that

$$\begin{pmatrix} \mathcal{A}^D & \mathcal{X} \\ \mathcal{O} & \mathcal{C}^D \end{pmatrix}*\begin{pmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{O} & \mathcal{C} \end{pmatrix}*\begin{pmatrix} \mathcal{A}^D & \mathcal{X} \\ \mathcal{O} & \mathcal{C}^D \end{pmatrix} = \begin{pmatrix} \mathcal{A}^D & \mathcal{A}^D*\mathcal{A}*\mathcal{X}+\mathcal{X}*\mathcal{C}*\mathcal{C}^D+\mathcal{A}^D*\mathcal{B}*\mathcal{C}^D \\ \mathcal{O} & \mathcal{C}^D \end{pmatrix}.$$

Thus, it is only necessary to show that $\mathcal{A}^D*\mathcal{A}*\mathcal{X}+\mathcal{X}*\mathcal{C}*\mathcal{C}^D+\mathcal{A}^D*\mathcal{B}*\mathcal{C}^D = \mathcal{X}$.

Finally, we will show that

$$\begin{pmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{O} & \mathcal{C} \end{pmatrix}^{n+2}*\begin{pmatrix} \mathcal{A}^D & \mathcal{X} \\ \mathcal{O} & \mathcal{C}^D \end{pmatrix} = \begin{pmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{O} & \mathcal{C} \end{pmatrix}^{n+1}.$$

First notice that for any $m > 0$,

$$\begin{pmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{O} & \mathcal{C} \end{pmatrix}^{m} = \begin{pmatrix} \mathcal{A}^m & \mathcal{S}_{(m)} \\ \mathcal{O} & \mathcal{C}^m \end{pmatrix},$$

where

$$\mathcal{S}_{(m)} = \sum_{s=0}^{m-1} \mathcal{A}^{m-1-s}*\mathcal{B}*\mathcal{C}^s, \tag{6}$$

it is seen that the decompose of matrix $bcirc(\mathcal{S}_{(m)})$ is

$$bcirc(\mathcal{S}_{(m)}) = (F_p \otimes I_n)\begin{pmatrix} S_1 & & & \\ & S_2 & & \\ & & \ddots & \\ & & & S_p \end{pmatrix}(F_p^H \otimes I_n),$$

and

$$S_i = \sum_{s=0}^{m-1} A_i^{m-1-s} B_i C_i^s, \ (i = 1, 2, \cdots, p)$$

Since $n+2 > k$ and $n+2 > l$, then

$$\begin{pmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{O} & \mathcal{C} \end{pmatrix}^{n+2} * \begin{pmatrix} \mathcal{A}^D & \mathcal{X} \\ \mathcal{O} & \mathcal{C}^D \end{pmatrix} = \begin{pmatrix} \mathcal{A}^{n+1} & \mathcal{A}^{n+2} * \mathcal{X} + \mathcal{S}_{(n+2)} * \mathcal{C}^D \\ \mathcal{O} & \mathcal{C}^{n+1} \end{pmatrix}.$$

Therefore, it is necessary to show that $\mathcal{A}^{n+2} * \mathcal{X} + \mathcal{S}_{(n+2)} * \mathcal{C}^D = \mathcal{S}_{(n+1)}$. Observe first since $l + k < n + 1$, by Definition 1.8, it is the case that

$$\mathcal{A}^n * (\mathcal{A}^D)^i = \mathcal{A}^{n-1} \ for \ i = 1, 2, \cdots, l-1.$$

Thus

$$\begin{aligned} \mathcal{A}^{n+2} * \mathcal{X} &= \mathcal{A}^n * \left(\sum_{s=0}^{l-1} (\mathcal{A}^D)^s * \mathcal{B} * \mathcal{C}^s \right) * (I - \mathcal{C} * \mathcal{C}^D) - \mathcal{A}^{n+1} * \mathcal{B} * \mathcal{C}^D \\ &= \left(\sum_{s=0}^{l-1} \mathcal{A}^{n-s} * \mathcal{B} * \mathcal{C}^s \right) * (I - \mathcal{C} * \mathcal{C}^D) - \mathcal{A}^{n+1} * \mathcal{B} * \mathcal{C}^D \\ &= \sum_{s=0}^{l-1} \mathcal{A}^{n-s} * \mathcal{B} * \mathcal{C}^s - \sum_{s=0}^{l-1} \mathcal{A}^{n-s} * \mathcal{B} * \mathcal{C}^{s+1} * \mathcal{C}^D - \mathcal{A}^{n+1} * \mathcal{B} * \mathcal{C}^D, \end{aligned}$$

that is

$$\mathcal{A}^{n+2} * \mathcal{X} = \sum_{s=0}^{l-1} \mathcal{A}^{n-s} * \mathcal{B} * \mathcal{C}^s - \sum_{s=0}^{l-1} \mathcal{A}^{n-s} * \mathcal{B} * \mathcal{C}^{s+1} * \mathcal{C}^D - \mathcal{A}^{n+1} * \mathcal{B} * \mathcal{C}^D, \tag{7}$$

the decomposition of matrix $bcirc(\mathcal{A}^{n+2} * \mathcal{X})$ is

$$\begin{aligned} bcirc(\mathcal{A}^{n+2} * \mathcal{X}) &= (F_p \otimes I_n) \begin{pmatrix} A_1^{n+2} T_1 & & & \\ & A_2^{n+2} T_2 & & \\ & & \ddots & \\ & & & A_p^{n+2} T_p \end{pmatrix} (F_p^H \otimes I_n) \\ &= (F_p \otimes I_n) \begin{pmatrix} U_1 & & & \\ & U_2 & & \\ & & \ddots & \\ & & & U_p \end{pmatrix} (F_p^H \otimes I_n), \end{aligned}$$

and

$$U_i = \sum_{s=0}^{l-1} A_i^{n-s} B_i C_i^s - \sum_{s=0}^{l-1} A_i^{n-s} B_i C_i^{s+1} C_i^D - A_i^{n+1} B_i C_i^D, \ (i = 1, 2, \cdots, p)$$

Since (6), then

$$\mathcal{S}_{(n+2)} * \mathcal{C}^D = \sum_{s=0}^{n+1} \mathcal{A}^{n+1-s} * \mathcal{B} * \mathcal{C}^s * \mathcal{C}^D = \sum_{s=0}^{l} \mathcal{A}^{n+1-s} * \mathcal{B} * \mathcal{C}^s * \mathcal{C}^D + \sum_{s=l+1}^{n+1} \mathcal{A}^{n+1-s} * \mathcal{B} * \mathcal{C}^{s-1}.$$

By writing

$$\sum_{s=0}^{l} \mathcal{A}^{n+1-s} * \mathcal{B} * C^{s} * C^{D} = \mathcal{A}^{n+1} * \mathcal{B} * C^{D} + \sum_{s=1}^{l} \mathcal{A}^{n+1-s} * \mathcal{B} * C^{s} * C^{D}$$
$$= \mathcal{A}^{n+1} * \mathcal{B} * C^{D} + \sum_{s=0}^{l-1} \mathcal{A}^{n-s} * \mathcal{B} * C^{s+1} * C^{D},$$

we obtain

$$\mathcal{S}_{(n+2)} * C^{D} = \mathcal{A}^{n+1} * \mathcal{B} * C^{D} + \sum_{s=0}^{l-1} \mathcal{A}^{n-s} * \mathcal{B} * C^{s+1} * C^{D} + \sum_{s=l+1}^{n+1} \mathcal{A}^{n+1-s} * \mathcal{B} * C^{s-1}, \tag{8}$$

the decomposition of matrix $bcirc(\mathcal{S}_{(n+2)} * C^{D})$ as follows

$$\begin{aligned} bcirc(\mathcal{S}_{(n+2)} * C^{D}) &= (F_p \otimes I_n) \begin{pmatrix} Q_1 & & & \\ & Q_2 & & \\ & & \ddots & \\ & & & Q_p \end{pmatrix} (F_p^H \otimes I_n) \\ &= (F_p \otimes I_n) \begin{pmatrix} A_1 B_1 C_1^D & & & \\ & A_2 B_2 C_2^D & & \\ & & \ddots & \\ & & & A_p B_p C_p^D \end{pmatrix} (F_p^H \otimes I_n) \\ &+ (F_p \otimes I_n) \begin{pmatrix} R_1 & & & \\ & R_2 & & \\ & & \ddots & \\ & & & R_p \end{pmatrix} (F_p^H \otimes I_n) \\ &+ (F_p \otimes I_n) \begin{pmatrix} V_1 & & & \\ & V_2 & & \\ & & \ddots & \\ & & & V_p \end{pmatrix} (F_p^H \otimes I_n), \end{aligned}$$

and

$$R_i = \sum_{s=0}^{l-1} A_i^{n-s} B_i C_i^{s+1} C_i^D, \; V_i = \sum_{s=l+1}^{n+1} A_i^{n+1-s} B_i C_i^{s-1},$$

then

$$Q_i = A_i^{n+1} B_i C_i^D + R_i + V_i = A_i^{n+1} B_i C_i^D + \sum_{s=0}^{l-1} A_i^{n-s} B_i C_i^{s+1} C_i^D + \sum_{s=l+1}^{n+1} A_i^{n+1-s} B_i C_i^{s-1}. \; (i = 1, 2, \cdots, p)$$

It is seen from (7) and (8) that

$$\begin{aligned}\mathcal{A}^{n+2} * \mathcal{X} + \mathcal{S}_{(n+2)} * \mathcal{C}^D &= \sum_{s=0}^{l-1} \mathcal{A}^{n-s} * \mathcal{B} * \mathcal{C}^s + \sum_{s=l+1}^{n+1} \mathcal{A}^{n+1-s} * \mathcal{B} * \mathcal{C}^{s-1} \\ &= \sum_{s=0}^{l-1} \mathcal{A}^{n-s} * \mathcal{B} * \mathcal{C}^s + \sum_{s=l}^{n} \mathcal{A}^{n-s} * \mathcal{B} * \mathcal{C}^s \\ &= \sum_{s=0}^{n} \mathcal{A}^{n-s} * \mathcal{B} * \mathcal{C}^s \\ &= \mathcal{S}_{(n+1)}.\end{aligned}$$

The proof is completed. □

Definition 1.15. (T-spectral radius) *Let $\mathcal{A} \in \mathbb{C}^{n\times n\times p}$ be an F-square tensor, then denote the spectral radius of $\mathcal{A}$ as*

$$\rho_T(\mathcal{A}) = \rho(bcirc(\mathcal{A})) = \rho\left((F_p \otimes I_n)bcirc(\mathcal{A})(F_p^H \otimes I_n)\right),$$

where $\rho_T(\mathcal{A})$ is called by T-spectral radius of $\mathcal{A}$.

Definition 1.16. [17](T-eigenvalue) *Let $\mathcal{A} \in \mathbb{C}^{n\times n\times p}$ be an F-square tensor, then denote the eigenvalue of $\mathcal{A}$ as*

$$\lambda_T(\mathcal{A}) = \lambda\left(bcirc(\mathcal{A})\right) = \lambda\left((F_p \otimes I_n)bcirc(\mathcal{A})(F_p^H \otimes I_n)\right),$$

where $\lambda_T(\mathcal{A})$ is called by T-eigenvalue of $\mathcal{A}$.

2. Perturbation bounds

Theorem 2.1. *Let $\mathcal{F} \in \mathbb{C}^{n\times n\times p}$ be an F-square tensor, suppose $\|\mathcal{F}\| < 1$, then $\mathcal{I} + \mathcal{F}$ is nonsingular, and*

$$\|(\mathcal{I} + \mathcal{F})^{-1}\| \leq \frac{1}{1 - \|\mathcal{F}\|}.$$

Proof. Assume $\mathcal{I} + \mathcal{F}$ is singular, then there is a nonzero $\mathcal{X} \in \mathbb{C}^{n\times n\times p}$, such that

$$(\mathcal{I} + \mathcal{F}) * \mathcal{X} = \mathcal{O},$$

furthermore

$$\mathcal{I} * \mathcal{X} = -\mathcal{F} * \mathcal{X}. \tag{9}$$

Take norm on both sides of (9) at the same time, we have

$$\|\mathcal{X}\| = \|\mathcal{I} * \mathcal{X}\| = \|\mathcal{F} * \mathcal{X}\| \leq \|\mathcal{F}\|\|\mathcal{X}\|.$$

According to $\|\mathcal{X}\| \leq \|\mathcal{F}\|\|\mathcal{X}\|$, which implies $\|\mathcal{F}\| \geq 1$, and it is contradictory to $\|\mathcal{F}\| < 1$. Therefore, $\mathcal{I} + \mathcal{F}$ is nonsingular.

Since $\mathcal{I} + \mathcal{F}$ is invertible, we have $(\mathcal{I} + \mathcal{F}) * (\mathcal{I} + \mathcal{F})^{-1} = \mathcal{I}$, then

$$(\mathcal{I} + \mathcal{F})^{-1} = \mathcal{I} - \mathcal{F} * (\mathcal{I} + \mathcal{F})^{-1}. \tag{10}$$

Take norm on both sides of (10) at the same time, we obtain

$$\begin{aligned}\|(\mathcal{I} + \mathcal{F})^{-1}\| &= \|\mathcal{I} - \mathcal{F} * (\mathcal{I} + \mathcal{F})^{-1}\| \\ &\leq \|\mathcal{I}\| + \|\mathcal{F} * (\mathcal{I} + \mathcal{F})^{-1}\| \\ &\leq 1 + \|\mathcal{F}\|\|(\mathcal{I} + \mathcal{F})^{-1}\|.\end{aligned}$$

And then

$$1 \geq (1 - \|\mathcal{F}\|)\|(\mathcal{I} + \mathcal{F})^{-1}\|,$$

therefore

$$\|(\mathcal{I} + \mathcal{F})^{-1}\| \leq \frac{1}{1 - \|\mathcal{F}\|}.$$

The proof is completed. □

Let $\mathcal{A}, \mathcal{B}, \mathcal{E} \in \mathbb{C}^{n\times n\times p}$ be F-square tensors, a condition $(\mathcal{W})$ [28] is given,

$$(\mathcal{W}),\ \mathcal{B} = \mathcal{A} + \mathcal{E} \text{ with } Ind_T(\mathcal{A}) = k,\ \mathcal{E} = \mathcal{A} * \mathcal{A}^D * \mathcal{E} * \mathcal{A} * \mathcal{A}^D, \text{ and } \|\mathcal{A}^D\|\|\mathcal{E}\| < 1.$$

Now, we consider the perturbation of the T-Drazin inverse. First, let us give two lemmas of the perturbation bounds of $\mathcal{B}^D - \mathcal{A}^D$.

Lemma 2.2. *Suppose condition $(\mathcal{W})$ holds, let $\mathcal{A} \in \mathbb{C}^{n\times n\times p}$ be a complex tensor, then there is an invertible tensor $\mathcal{P} \in \mathbb{C}^{n\times n\times p}$ and F-bidiagonal tensor $\mathcal{N} \in \mathbb{C}^{n\times n\times p}$. Further, the decomposition form of $\mathcal{E}$ is*

$$\mathcal{E} = \mathcal{P}^{-1} * \mathcal{N} * \mathcal{P} = \mathcal{P}^{-1} * \begin{pmatrix} \mathcal{N}_1 & \mathcal{O} \\ \mathcal{O} & \mathcal{O} \end{pmatrix} * \mathcal{P},$$

where $\mathcal{N}_1$ is the first block element of the tensor $\mathcal{N}$, and the matrix $bcirc(\mathcal{N})$ has the following decomposition

$$bcirc(\mathcal{N}) = (F_p \otimes I_n) \begin{pmatrix} N_1 & & & \\ & N_2 & & \\ & & \ddots & \\ & & & N_p \end{pmatrix} (F_p^H \otimes I_n),$$

where $N_i = \begin{pmatrix} N_i^1 & O \\ O & O \end{pmatrix}$, N_i^1 is the first block element of the matrix of N_i. $(i = 1, 2, \cdots, p)$

Proof. According to the Theorem 1.11, we have

$$\mathcal{A} = \mathcal{P}^{-1} * \mathcal{J} * \mathcal{P} = \mathcal{P}^{-1} * \begin{pmatrix} \mathcal{J}_1 & \mathcal{O} \\ \mathcal{O} & \mathcal{J}_4^0 \end{pmatrix} * \mathcal{P},$$

where $\mathcal{J}_1$ is the first block inverse element of tensor $\mathcal{J}$, and $\mathcal{J}_4^0$ is nilpotent.
Further, we obtain

$$\mathcal{A}^D = \mathcal{P}^{-1} * \mathcal{J}^D * \mathcal{P} = \mathcal{P}^{-1} * \begin{pmatrix} \mathcal{J}_1^{-1} & \mathcal{O} \\ \mathcal{O} & \mathcal{O} \end{pmatrix} * \mathcal{P},$$

where $\mathcal{J}_1^{-1}$ is the first block element of the tensor $\mathcal{J}^D$.

Next, the decomposition of $\mathcal{E}$ will be given.

Suppose that $\mathcal{E} = \mathcal{P}^{-1} * \begin{pmatrix} \mathcal{N}_1 & \mathcal{N}_2 \\ \mathcal{N}_3 & \mathcal{N}_4 \end{pmatrix} * \mathcal{P}$, then

$$\mathcal{A} * \mathcal{A}^D * \mathcal{E} = \mathcal{P}^{-1} * \begin{pmatrix} \mathcal{J}_1 & \mathcal{O} \\ \mathcal{O} & \mathcal{J}_4^0 \end{pmatrix} * \mathcal{P} * \mathcal{P}^{-1} * \begin{pmatrix} \mathcal{J}_1^{-1} & \mathcal{O} \\ \mathcal{O} & \mathcal{O} \end{pmatrix} * \mathcal{P} * \mathcal{P}^{-1} * \begin{pmatrix} \mathcal{N}_1 & \mathcal{N}_2 \\ \mathcal{N}_3 & \mathcal{N}_4 \end{pmatrix} * \mathcal{P} = \mathcal{P}^{-1} * \begin{pmatrix} \mathcal{N}_1 & \mathcal{N}_2 \\ \mathcal{O} & \mathcal{O} \end{pmatrix} * \mathcal{P}, \quad (11)$$

and

$$\mathcal{E} * \mathcal{A} * \mathcal{A}^D = \mathcal{P}^{-1} * \begin{pmatrix} \mathcal{N}_1 & \mathcal{N}_2 \\ \mathcal{N}_3 & \mathcal{N}_4 \end{pmatrix} * \mathcal{P} * \mathcal{P}^{-1} * \begin{pmatrix} \mathcal{J}_1 & \mathcal{O} \\ \mathcal{O} & \mathcal{J}_4^0 \end{pmatrix} * \mathcal{P} * \mathcal{P}^{-1} * \begin{pmatrix} \mathcal{J}_1^{-1} & \mathcal{O} \\ \mathcal{O} & \mathcal{O} \end{pmatrix} * \mathcal{P} = \mathcal{P}^{-1} * \begin{pmatrix} \mathcal{N}_1 & \mathcal{O} \\ \mathcal{N}_3 & \mathcal{O} \end{pmatrix} * \mathcal{P}, \quad (12)$$

According to $\mathcal{E} = \mathcal{A} * \mathcal{A}^D * \mathcal{E} = \mathcal{E} * \mathcal{A} * \mathcal{A}^D$, (11) and (12), we obtain

$$\mathcal{E} = \mathcal{P}^{-1} * \begin{pmatrix} \mathcal{N}_1 & \mathcal{N}_2 \\ \mathcal{N}_3 & \mathcal{N}_4 \end{pmatrix} * \mathcal{P} = \mathcal{P}^{-1} * \begin{pmatrix} \mathcal{N}_1 & \mathcal{N}_2 \\ O & O \end{pmatrix} * \mathcal{P} = \mathcal{P}^{-1} * \begin{pmatrix} \mathcal{N}_1 & O \\ \mathcal{N}_3 & O \end{pmatrix} * \mathcal{P} \tag{13}$$

Hence $\mathcal{E} = \mathcal{P}^{-1} * \begin{pmatrix} \mathcal{N}_1 & O \\ O & O \end{pmatrix} * \mathcal{P}$. The proof is completed. □

Lemma 2.3. *Suppose condition ($\mathcal{W}$) holds, let $\mathcal{A} \in \mathbb{C}^{n\times n\times p}$ be a complex tensor, $\mathcal{B} = \mathcal{A} + \mathcal{E}$, then there is an invertible tensor $\mathcal{P} \in \mathbb{C}^{n\times n\times p}$ and F-bidiagonal tensor $\mathcal{M} \in \mathbb{C}^{n\times n\times p}$, such that*
(1) $\mathcal{B}^D = \mathcal{P}^{-1} * \mathcal{M}^D * \mathcal{P}$, and the decomposition of the matrix $bcirc(\mathcal{M}^D)$ is

$$bcirc(\mathcal{M}^D) = (F_p \otimes I_n) \begin{pmatrix} M_1^D & & & \\ & M_2^D & & \\ & & \ddots & \\ & & & M_p^D \end{pmatrix} (F_p^H \otimes I_n),$$

where $M_i^D = \begin{pmatrix} (M_i^1)^{-1} & O \\ O & O \end{pmatrix}$, $(i = 1, 2, \cdots, p)$
(2) $\mathcal{A} * \mathcal{A}^D = \mathcal{B} * \mathcal{B}^D$.

Proof. (1) According to the Theorem 1.11, there is $\mathcal{N} \in \mathbb{C}^{n\times n\times p}$ *and* $\mathcal{J} \in \mathbb{C}^{n\times n\times p}$, then $\mathcal{A} = \mathcal{P}^{-1} * \mathcal{J} * \mathcal{P}$, $\mathcal{E} = \mathcal{P}^{-1} * \mathcal{N} * \mathcal{P}$, suppose $\mathcal{B} = \mathcal{A} + \mathcal{E} = \mathcal{P}^{-1} * \mathcal{M} * \mathcal{P}$, where

$$\begin{aligned} bcirc(\mathcal{M}) &= bcirc(\mathcal{J} + \mathcal{N}) \\ &= (F_p \otimes I_n) \begin{pmatrix} (N_1 + J_1) & & & \\ & (N_2 + J_2) & & \\ & & \ddots & \\ & & & (N_p + J_p) \end{pmatrix} (F_p^H \otimes I_n), \end{aligned}$$

and $J_i = \begin{pmatrix} J_i^1 & O \\ O & J_i^0 \end{pmatrix}$, $N_i = \begin{pmatrix} N_i^1 & O \\ O & O \end{pmatrix}$, J_i^1 is the first block element of the matrix of J_i, N_i^1 is the first block element of the matrix of N_i, and J_i^0 is nilpotent, $(i = 1, 2, \cdots, p)$
Therefore

$$bcirc(\mathcal{M}^D) = (F_p \otimes I_n) \begin{pmatrix} (N_1 + J_1)^D & & & \\ & (N_2 + J_2)^D & & \\ & & \ddots & \\ & & & (N_p + J_p)^D \end{pmatrix} (F_p^H \otimes I_n).$$

Moreover, it proves that $N_i^1 + J_i^1$ is invertible, where $N_i + J_i = \begin{pmatrix} N_i^1 + J_i^1 & O \\ O & O \end{pmatrix}$. $(i = 1, 2, \cdots, p)$

Now, by Theorem 1.11 and Lemma 2.2, we have

$$\begin{aligned} \mathcal{A}^D * \mathcal{E} &= \mathcal{P}^{-1} * \mathcal{J}^D * \mathcal{P} * \mathcal{P}^{-1} * \mathcal{N} * \mathcal{P} \\ &= \mathcal{P}^{-1} * \mathcal{J}^D * \mathcal{N} * \mathcal{P}, \end{aligned}$$

and the decomposition of $bcirc(\mathcal{J}^D * \mathcal{N})$ is

$$bcirc(\mathcal{J}^D * \mathcal{N}) = bcirc(\mathcal{J}^D) bcirc(\mathcal{N}) = (F_p \otimes I_n) \begin{pmatrix} J_1^D N_1 & & & \\ & J_2^D N_2 & & \\ & & \ddots & \\ & & & J_p^D N_P \end{pmatrix} (F_p^H \otimes I_n),$$

where $J_i^D N_i = \begin{pmatrix} (J_i^1)^{-1}N_i^1 & O \\ O & O \end{pmatrix}$, $(i = 1, 2, \cdots, p)$
By Definition 1.15, we have

$$\begin{aligned} \rho_T(\mathcal{J}^D * \mathcal{N}) &= \rho(bcirc(\mathcal{J}^D * \mathcal{N})) \\ &= \rho\left((F_p \otimes I_n)bcirc(\mathcal{J}^D * \mathcal{N})(F_p^H \otimes I_n)\right) \\ &= \max_i \rho\left((J_i^1)^{-1}N_i^1\right), \end{aligned}$$

that is

$$\rho_T(\mathcal{A}^D * \mathcal{E}) = \rho_T(\mathcal{J}^D * \mathcal{N}) = \max_i \rho\left((J_i^1)^{-1}N_i^1\right), \tag{14}$$

thus

$$\rho_T(\mathcal{A}^D * \mathcal{E}) \leq \|\mathcal{A}^D\|\|\mathcal{E}\| < 1. \tag{15}$$

On the other hand, it will prove that $J_i^1 + N_i^1 = J_i^1\left(I + (J_i^1)^{-1}N_i^1\right)$ is invertible. According to the inverse of J_i^1, we will only prove that $I + (J_i^1)^{-1}N_i^1$ is nonsingular. Now, we prove it by reduction to absurdity. Assume $I + (J_i^1)^{-1}N_i^1$ is singular, then there is a nonzero vector $x \in \mathbb{C}^{n\times 1}$, such that

$$\left(I + (J_i^1)^{-1}N_i^1\right)x = 0,$$

then

$$x = -\left((J_i^1)^{-1}N_i^1\right)x.$$

Therefore, -1 is the eigenvalue of matrix $(J_i^1)^{-1}N_i^1$, denoted $\lambda\left((J_i^1)^{-1}N_i^1\right) = -1$,
it implies $\rho\left((J_i^1)^{-1}N_i^1\right) \geq 1$.
According to (14), we obtain

$$\rho_T(\mathcal{A}^D * \mathcal{E}) = \max_i \rho\left((J_i^1)^{-1}N_i^1\right) \geq 1,$$

which is contradictory to (15). Hence $I + (J_1^1)^{-1}N_1^1$ is nonsingular.

(2) By Theorem 1.11, we have $\mathcal{A} = \mathcal{P}^{-1} * \mathcal{J} * \mathcal{P}$ and $\mathcal{A}^D = \mathcal{P}^{-1} * \mathcal{J}^D * \mathcal{P}$.
Similary, $\mathcal{B} = \mathcal{P}^{-1} * \mathcal{M} * \mathcal{P}$ and $\mathcal{B}^D = \mathcal{P}^{-1} * \mathcal{M}^D * \mathcal{P}$, then

$$\mathcal{A} * \mathcal{A}^D = \mathcal{P}^{-1} * \mathcal{J} * \mathcal{P} * \mathcal{P}^{-1} * \mathcal{J}^D * \mathcal{P} = \mathcal{P}^{-1} * \mathcal{J} * \mathcal{J}^D * \mathcal{P},$$

and

$$\mathcal{B} * \mathcal{B}^D = \mathcal{P}^{-1} * \mathcal{M} * \mathcal{P} * \mathcal{P}^{-1} * \mathcal{M}^D * \mathcal{P} = \mathcal{P}^{-1} * \mathcal{M} * \mathcal{M}^D * \mathcal{P},$$

By Lemma 1.5, we have

$$\begin{aligned} bcirc(\mathcal{J} * \mathcal{J}^D) &= bcirc(\mathcal{J})bcirc(\mathcal{J}^D) \\ &= (F_p \otimes I_n)\begin{pmatrix} J_1J_1^D & & & \\ & J_2J_2^D & & \\ & & \ddots & \\ & & & J_pJ_p^D \end{pmatrix}(F_p^H \otimes I_n), \end{aligned}$$

and

$$\begin{aligned} bcirc(\mathcal{M} * \mathcal{M}^D) &= bcirc(\mathcal{M})bcirc(\mathcal{M}^D) \\ &= (F_p \otimes I_n)\begin{pmatrix} M_1M_1^D & & & \\ & M_2M_2^D & & \\ & & \ddots & \\ & & & M_pM_p^D \end{pmatrix}(F_p^H \otimes I_n), \end{aligned}$$

where $J_iJ_i^D = \begin{pmatrix} J_i^1 & O \\ O & J_i^0 \end{pmatrix}\begin{pmatrix} (J_i^1)^{-1} & O \\ O & O \end{pmatrix} = \begin{pmatrix} I & O \\ O & O \end{pmatrix}$, J_i^1 is the first block element of the matrix of J_i, and J_i^0 is nilpotent, and $M_iM_i^D = \begin{pmatrix} M_i^1 & O \\ O & O \end{pmatrix}\begin{pmatrix} (M_i^1)^{-1} & O \\ O & O \end{pmatrix} = \begin{pmatrix} I & O \\ O & O \end{pmatrix}$, M_i^1 is the first block element of the matrix of M_i. $(i = 1, 2, \cdots, p)$

Hence, $\mathcal{A} * \mathcal{A}^D = \mathcal{B} * \mathcal{B}^D$. The proof is completed. □

Theorem 2.4. *Let $\mathcal{A}, \mathcal{B}, \mathcal{E} \in \mathbb{C}^{n\times n\times p}$ be F-square tensors, $\mathcal{A}^D$ is T-Drazin inverse of $\mathcal{A}$, if $\mathcal{E} = \mathcal{A}*\mathcal{A}^D*\mathcal{E} = \mathcal{E}*\mathcal{A}*\mathcal{A}^D$, $Ind_T(\mathcal{A}) = k$, $\mathcal{B} = \mathcal{A} + \mathcal{E}$ and $\|\mathcal{A}^D * \mathcal{E}\| < 1$, then*
(1) $\mathcal{B}^D - \mathcal{A}^D = -\mathcal{B}^D * \mathcal{E} * \mathcal{A}^D = -\mathcal{A}^D * \mathcal{E} * \mathcal{B}^D$,
(2) $\mathcal{B}^D = (\mathcal{I} + \mathcal{A}^D * \mathcal{E})^{-1} * \mathcal{A}^D = \mathcal{A}^D * (\mathcal{I} + \mathcal{E} * \mathcal{A}^D)^{-1}$,
(3) $\frac{\|\mathcal{B}^D-\mathcal{A}^D\|}{\|\mathcal{A}^D\|} \leq \frac{\|\mathcal{A}^D*\mathcal{E}\|}{1-\|\mathcal{A}^D*\mathcal{E}\|}$.

Proof. (1) According to Lemma 2.3, we have $\mathcal{A} * \mathcal{A}^D = \mathcal{B} * \mathcal{B}^D$, then

$$\begin{aligned} \mathcal{B}^D - \mathcal{A}^D &= -\mathcal{B}^D * \mathcal{E} * \mathcal{A}^D + \mathcal{B}^D - \mathcal{A}^D + \mathcal{B}^D * (\mathcal{B} - \mathcal{A}) * \mathcal{A}^D \\ &= -\mathcal{B}^D * \mathcal{E} * \mathcal{A}^D + \mathcal{B}^D - \mathcal{B}^D * \mathcal{A} * \mathcal{A}^D - \mathcal{A}^D + \mathcal{B}^D * \mathcal{B} * \mathcal{A}^D \\ &= -\mathcal{B}^D * \mathcal{E} * \mathcal{A}^D + \mathcal{B}^D - \mathcal{B}^D * \mathcal{B} * \mathcal{B}^D - \mathcal{A}^D + \mathcal{A}^D * \mathcal{A} * \mathcal{A}^D \\ &= -\mathcal{B}^D * \mathcal{E} * \mathcal{A}^D, \end{aligned}$$

that is

$$\mathcal{B}^D - \mathcal{A}^D = -\mathcal{B}^D * \mathcal{E} * \mathcal{A}^D. \tag{16}$$

Similarly,

$$\begin{aligned} \mathcal{B}^D - \mathcal{A}^D &= -\mathcal{A}^D * \mathcal{E} * \mathcal{B}^D + \mathcal{B}^D - \mathcal{A}^D + \mathcal{A}^D * (\mathcal{B} - \mathcal{A}) * \mathcal{B}^D \\ &= -\mathcal{A}^D * \mathcal{E} * \mathcal{B}^D + \mathcal{B}^D - \mathcal{A}^D * \mathcal{A} * \mathcal{B}^D - \mathcal{A}^D + \mathcal{A}^D * \mathcal{B} * \mathcal{B}^D \\ &= -\mathcal{A}^D * \mathcal{E} * \mathcal{B}^D + \mathcal{B}^D - \mathcal{B}^D * \mathcal{B} * \mathcal{B}^D - \mathcal{A}^D + \mathcal{A}^D * \mathcal{A} * \mathcal{A}^D \\ &= -\mathcal{A}^D * \mathcal{E} * \mathcal{B}^D, \end{aligned}$$

that is

$$\mathcal{B}^D - \mathcal{A}^D = -\mathcal{A}^D * \mathcal{E} * \mathcal{B}^D. \tag{17}$$

(2) By (16), we have

$$\mathcal{B}^D * (\mathcal{I} + \mathcal{E} * \mathcal{A}^D) = \mathcal{A}^D.$$

Since $\rho_T(\mathcal{E} * \mathcal{A}^D) = \rho_T(\mathcal{A}^D * \mathcal{E})$, then $\rho_T(\mathcal{E} * \mathcal{A}^D) = \rho_T(\mathcal{A}^D * \mathcal{E}) \leq \|\mathcal{A}^D * \mathcal{E}\| < 1$, therefore $\mathcal{I} + \mathcal{E} * \mathcal{A}^D$ is nonsingular, then

$$\mathcal{B}^D = \mathcal{A}^D * (\mathcal{I} + \mathcal{E} * \mathcal{A}^D)^{-1}. \tag{18}$$

By (17), we obtain

$$(\mathcal{I} + \mathcal{A}^D * \mathcal{E}) * \mathcal{B}^D = \mathcal{A}^D.$$

Since $\|\mathcal{A}^D * \mathcal{E}\| < 1$, therefore $\mathcal{I} + \mathcal{A}^D * \mathcal{E}$ is nonsingular, then

$$\mathcal{B}^D = (\mathcal{I} + \mathcal{A}^D * \mathcal{E})^{-1} * \mathcal{A}^D. \tag{19}$$

(3) By Theorem 2.1, and take norm on both sides of (19) at the same time, then

$$\begin{aligned}\|\mathcal{B}^D\| &= \|(\mathcal{I} + \mathcal{A}^D * \mathcal{E})^{-1} * \mathcal{A}^D\| \\ &\leq \|(\mathcal{I} + \mathcal{A}^D * \mathcal{E})^{-1}\|\|\mathcal{A}^D\| \\ &\leq \frac{\|\mathcal{A}^D\|}{1 - \|\mathcal{A}^D * \mathcal{E}\|}.\end{aligned}$$

Therefore

$$\|\mathcal{B}^D\| \leq \frac{\|\mathcal{A}^D\|}{1 - \|\mathcal{A}^D * \mathcal{E}\|}. \tag{20}$$

Take norm on both sides of (17) at the same time, then

$$\begin{aligned}\|\mathcal{B}^D - \mathcal{A}^D\| &= \| - \mathcal{A}^D * \mathcal{E} * \mathcal{B}^D\| \\ &\leq \|\mathcal{A}^D * \mathcal{E}\|\|\mathcal{B}^D\|.\end{aligned}$$

Divide $\|\mathcal{A}^D\|$ on both sides at the same time, we obtain

$$\frac{\|\mathcal{B}^D - \mathcal{A}^D\|}{\|\mathcal{A}^D\|} \leq \frac{\|\mathcal{A}^D * \mathcal{E}\|\|\mathcal{B}^D\|}{\|\mathcal{A}^D\|},$$

Since (20), then

$$\frac{\|\mathcal{B}^D - \mathcal{A}^D\|}{\|\mathcal{A}^D\|} \leq \frac{\|\mathcal{A}^D * \mathcal{E}\|\|\mathcal{B}^D\|}{\|\mathcal{A}^D\|} \leq \frac{\|\mathcal{A}^D * \mathcal{E}\|}{1 - \|\mathcal{A}^D * \mathcal{E}\|}.$$

Therefore

$$\frac{\|\mathcal{B}^D - \mathcal{A}^D\|}{\|\mathcal{A}^D\|} \leq \frac{\|\mathcal{A}^D * \mathcal{E}\|}{1 - \|\mathcal{A}^D * \mathcal{E}\|}. \tag{21}$$

The proof is completed. □

Corollary 2.5. *Suppose condition ($\mathcal{W}$) holds, let $\mathcal{A}, \mathcal{B}, \mathcal{E} \in \mathbb{C}^{n\times n\times p}$ be F-square tensors, then*

$$\frac{\|\mathcal{A}^D\|}{1 + \|\mathcal{A}^D\|\|\mathcal{E}\|} \leq \|\mathcal{B}^D\| \leq \frac{\|\mathcal{A}^D\|}{1 - \|\mathcal{A}^D\|\|\mathcal{E}\|}.$$

Proof. According to Theorem 2.4, we have $\mathcal{B}^D = \mathcal{A}^D * (\mathcal{I} + \mathcal{E} * \mathcal{A}^D)^{-1}$, then

$$\mathcal{A}^D = \mathcal{B}^D * (\mathcal{I} + \mathcal{E} * \mathcal{A}^D). \tag{22}$$

Taking norm on both sides of (22) at the same time, we obtain

$$\|\mathcal{A}^D\| = \|\mathcal{B}^D * (\mathcal{I} + \mathcal{E} * \mathcal{A}^D)\| \leq \|\mathcal{B}^D\|\|\mathcal{I} + \mathcal{E} * \mathcal{A}^D\|.$$

Hence

$$\|\mathcal{B}^D\| \geq \frac{\|\mathcal{A}^D\|}{\|\mathcal{I} + \mathcal{E} * \mathcal{A}^D\|}. \tag{23}$$

According to $\|(\mathcal{I} + \mathcal{E} * \mathcal{A}^D)\| \leq \|\mathcal{I}\| + \|\mathcal{E} * \mathcal{A}^D\| \leq 1 + \|\mathcal{E}\|\|\mathcal{A}^D\|$, then

$$\frac{1}{1 + \|\mathcal{E}\|\|\mathcal{A}^D\|} \leq \frac{1}{\|\mathcal{I} + \mathcal{E} * \mathcal{A}^D\|}.$$

Multiply $\|\mathcal{A}^D\|$ on both sides at the same time, we obtain

$$\frac{\|\mathcal{A}^D\|}{1 + \|\mathcal{E}\|\|\mathcal{A}^D\|} \leq \frac{\|\mathcal{A}^D\|}{\|\mathcal{I} + \mathcal{E} * \mathcal{A}^D\|}.$$

By (23), then

$$\frac{\|\mathcal{A}^D\|}{1 + \|\mathcal{E}\|\|\mathcal{A}^D\|} \leq \frac{\|\mathcal{A}^D\|}{\|\mathcal{I} + \mathcal{E} * \mathcal{A}^D\|} \leq \|\mathcal{B}^D\|.$$

On the other hand, by (20), it shows that

$$\|\mathcal{B}^D\| \leq \|\mathcal{A}^D\|\|(\mathcal{I} + \mathcal{A}^D * \mathcal{E})^{-1}\| \leq \frac{\|\mathcal{A}^D\|}{1 - \|\mathcal{A}^D\|\|\mathcal{E}\|}.$$

Therefore

$$\frac{\|\mathcal{A}^D\|}{1 + \|\mathcal{A}^D\|\|\mathcal{E}\|} \leq \|\mathcal{B}^D\| \leq \frac{\|\mathcal{A}^D\|}{1 - \|\mathcal{A}^D\|\|\mathcal{E}\|}.$$

The proof is completed. □

Theorem 2.6. *Let $\mathcal{A}, \mathcal{B} \in \mathbb{C}^{n\times n\times p}$ be F-square tensors, if $\|\mathcal{E}\|\|\mathcal{A}^D\| < 1$, and $\mathcal{K}_D(\mathcal{A}) = \|\mathcal{A}\|\|\mathcal{A}^D\|$, then*

$$\frac{\|\mathcal{B}^D - \mathcal{A}^D\|}{\|\mathcal{A}^D\|} \leq \frac{\mathcal{K}_D(\mathcal{A})\|\mathcal{E}\|/\|\mathcal{A}\|}{1 - \mathcal{K}_D(\mathcal{A})\|\mathcal{E}\|/\|\mathcal{A}\|}.$$

Proof. From (21), we have

$$\begin{aligned}\frac{\|\mathcal{B}^D - \mathcal{A}^D\|}{\|\mathcal{A}^D\|} &\leq \frac{\|\mathcal{A}^D * \mathcal{E}\|}{1 - \|\mathcal{A}^D * \mathcal{E}\|} \\ &\leq \frac{\|\mathcal{A}^D\|\|\mathcal{E}\|}{1 - \|\mathcal{A}^D\|\|\mathcal{E}\|} \\ &= \frac{\|\mathcal{A}\|\|\mathcal{A}^D\|\|\mathcal{E}\|/\|\mathcal{A}\|}{1 - \|\mathcal{A}\|\|\mathcal{A}^D\|\|\mathcal{E}\|/\|\mathcal{A}\|} \\ &= \frac{\mathcal{K}_D(\mathcal{A})\|\mathcal{E}\|/\|\mathcal{A}\|}{1 - \mathcal{K}_D(\mathcal{A})\|\mathcal{E}\|/\|\mathcal{A}\|},\end{aligned}$$

where $\mathcal{K}_D(\mathcal{A}) = \|\mathcal{A}\|\|\mathcal{A}^D\|$.
The proof is completed. □

Remark 2.7. *If $Ind_T(\mathcal{A}) = 1$, then condition $(\mathcal{W})$ is reduced to $\mathcal{B} = \mathcal{A} + \mathcal{E}$, $\mathcal{E} = \mathcal{A} * \mathcal{A}_g * \mathcal{E} * \mathcal{A} * \mathcal{A}_g$, and $\|\mathcal{A}_g\|\|\mathcal{E}\| < 1$. Thus under these assumes, we can get a perturbation bound for the group inverse of the tensor.*

Remark 2.8. *If $Ind_T(\mathcal{A}) = 0$,i.e., $\mathcal{A}$ is nonsingular, then condition $(\mathcal{W})$ is reduced to $\mathcal{B} = \mathcal{A}+\mathcal{E}$, and $\|\mathcal{A}^{-1}\|\|\mathcal{E}\| < 1$. We also obtain a perturbation bound on the common tensor inverse.*

3. Applications

In this section, we consider the T-linear system. Let $\mathcal{B} \in \mathbb{C}^{n\times n\times p}$ be an F-square tensor, and $y, b, c, f \in \mathbb{C}^{n\times 1\times p}$ are tensors.

$$\mathcal{B} * y = c, \ y \in \mathcal{R}(\mathcal{B}^D),$$

where $\mathcal{B} = \mathcal{A} + \mathcal{E}$, $c = b + f \in \mathcal{R}(\mathcal{B}^D)$.

Theorem 3.1. *Suppose condition* $(\mathcal{W})$ *holds, let* $y, x, b, c, f \in \mathbb{C}^{n\times 1\times p}$ *and* $\|\mathcal{A}^D\|\|\mathcal{E}\| < 1$, *then*

$$\frac{\|y - x\|}{\|x\|} \leq \frac{\mathcal{K}_D(\mathcal{A})}{1 - \mathcal{K}_D(\mathcal{A})\|\mathcal{E}\|/\|\mathcal{A}\|}\left(\frac{\|\mathcal{E}\|}{\|\mathcal{A}\|} + \frac{\|f\|}{\|b\|}\right).$$

Proof. According to Theorem 1.13, we obtain $x = \mathcal{A}^D * b$,
and by (5), one can obtain

$$x = \mathcal{A}^D * b.$$

Similarly

$$\begin{aligned} y &= \mathcal{B}^D * c \\ &= (\mathcal{A} + \mathcal{E})^D * (b + f). \end{aligned}$$

Since $\mathcal{B}^D - \mathcal{A}^D = -\mathcal{B}^D * \mathcal{E} * \mathcal{A}^D$, then

$$\begin{aligned} y - x &= (\mathcal{A} + \mathcal{E})^D * (b + f) - \mathcal{A}^D * b \\ &= (\mathcal{A} + \mathcal{E})^D * b + (\mathcal{A} + \mathcal{E})^D * f - \mathcal{A}^D * b \\ &= \left((\mathcal{A} + \mathcal{E})^D - \mathcal{A}^D\right) * b + (\mathcal{A} + \mathcal{E})^D * f \\ &= -\mathcal{B}^D * \mathcal{E} * \mathcal{A}^D * b + (\mathcal{A} + \mathcal{E})^D * f \\ &= -\left((\mathcal{A} + \mathcal{E})^D\right) * \mathcal{E} * x + (\mathcal{A} + \mathcal{E})^D * f. \end{aligned}$$

Hence

$$y - x = -\left((\mathcal{A} + \mathcal{E})^D\right) * \mathcal{E} * x + (\mathcal{A} + \mathcal{E})^D * f. \tag{24}$$

Due to Corollary 2.5, and take norm on both sides of (24) at the same time, then

$$\begin{aligned} \|y - x\| &= \| - (\mathcal{A} + \mathcal{E})^D * \mathcal{E} * x + (\mathcal{A} + \mathcal{E})^D * f\| \\ &\leq \|(\mathcal{A} + \mathcal{E})^D\|\|\mathcal{E}\|\|x\| + \|(\mathcal{A} + \mathcal{E})^D\|\|f\| \\ &= \|(\mathcal{A} + \mathcal{E})^D\| \left(\|\mathcal{E}\|\|x\| + \|f\|\right) \\ &= \|\mathcal{B}^D\|\left(\|\mathcal{E}\|\|x\| + \frac{\|f\|\|b\|}{\|b\|}\right) \\ &\leq \frac{\|\mathcal{A}\|\|\mathcal{A}^D\|\|x\|}{1 - \|\mathcal{A}^D * \mathcal{E}\|}\left(\|\mathcal{E}\| + \frac{\|f\|\|\mathcal{A}\|}{\|b\|}\right) \\ &\leq \frac{\|\mathcal{A}\|\|\mathcal{A}^D\|\|x\|}{1 - \|\mathcal{A}^D\|\|\mathcal{E}\|}\left(\|\mathcal{E}\| + \frac{\|f\|\|\mathcal{A}\|}{\|b\|}\right) \\ &\leq \frac{\mathcal{K}_D(\mathcal{A})\|x\|}{1 - \mathcal{K}_D(\mathcal{A})\|\mathcal{E}\|/\|\mathcal{A}\|}\left(\frac{\|\mathcal{E}\|}{\|\mathcal{A}\|} + \frac{\|f\|}{\|b\|}\right). \end{aligned}$$

The proof is completed. □

4. One-sided Perturbation of T-Drazin Inverse

Lemma 4.1. *Let $\mathcal{A} \in \mathbb{C}^{n\times n\times p}, \mathcal{E} \in \mathbb{C}^{n\times n\times p}$ be complex tensors, and $\mathcal{E} = \mathcal{A} * \mathcal{A}^D * \mathcal{E}$, then there is an invertible tensor $\mathcal{P} \in \mathbb{C}^{n\times n\times p}$ and F-bidiagonal tensor $\mathcal{N} \in \mathbb{C}^{n\times n\times p}$. Further, the decomposition form of $\mathcal{E}$ is*

$$\mathcal{E} = \mathcal{P}^{-1} * \mathcal{N} * \mathcal{P} = \mathcal{P}^{-1} * \begin{pmatrix} \mathcal{N}_1 & \mathcal{N}_2 \\ \mathcal{O} & \mathcal{O} \end{pmatrix} * \mathcal{P},$$

where $\mathcal{N}_1$ and $\mathcal{N}_2$ are block elements of tensor $\mathcal{N}$.
And the matrix bcirc($\mathcal{N}$) has the following decomposition

$$bcirc(\mathcal{N}) = (F_p \otimes I_n) \begin{pmatrix} N_1 & & & \\ & N_2 & & \\ & & \ddots & \\ & & & N_p \end{pmatrix} (F_p^H \otimes I_n),$$

where $N_i = \begin{pmatrix} N_i^1 & N_i^2 \\ O & O \end{pmatrix}$, N_i^1 and N_i^2 are block elements of the matrix of N_i. $(i = 1, 2, \cdots, p)$

Proof. According to the Theorem 1.11, we have

$$\mathcal{A} = \mathcal{P}^{-1} * \mathcal{J} * \mathcal{P} = \mathcal{P}^{-1} * \begin{pmatrix} \mathcal{J}_1 & \mathcal{O} \\ \mathcal{O} & \mathcal{J}_4^0 \end{pmatrix} * \mathcal{P}, \tag{25}$$

where the first block element $\mathcal{J}_1$ is inverse in tensor $\mathcal{J}$, and $\mathcal{J}_4^0$ is nilpotent.
Further, we obtain

$$\mathcal{A}^D = \mathcal{P}^{-1} * \mathcal{J}^D * \mathcal{P} = \mathcal{P}^{-1} * \begin{pmatrix} \mathcal{J}_1^{-1} & \mathcal{O} \\ \mathcal{O} & \mathcal{O} \end{pmatrix} * \mathcal{P}, \tag{26}$$

where the first block element $\mathcal{J}_1^{-1}$ of the tensor $\mathcal{J}^D$.

Next, the decomposition of $\mathcal{E}$ will be given. Suppose $\mathcal{E} = \mathcal{P}^{-1} * \mathcal{N} * \mathcal{P} = \mathcal{P}^{-1} * \begin{pmatrix} \mathcal{N}_1 & \mathcal{N}_2 \\ \mathcal{N}_3 & \mathcal{N}_4 \end{pmatrix} * \mathcal{P}$, then

$$\mathcal{A} * \mathcal{A}^D * \mathcal{E} = \mathcal{P}^{-1} * \begin{pmatrix} \mathcal{J}_1 & \mathcal{O} \\ \mathcal{O} & \mathcal{J}_4^0 \end{pmatrix} * \mathcal{P} * \mathcal{P}^{-1} * \begin{pmatrix} \mathcal{J}_1^{-1} & \mathcal{O} \\ \mathcal{O} & \mathcal{O} \end{pmatrix} * \mathcal{P} * \mathcal{P}^{-1} * \begin{pmatrix} \mathcal{N}_1 & \mathcal{N}_2 \\ \mathcal{N}_3 & \mathcal{N}_4 \end{pmatrix} * \mathcal{P} = \mathcal{P}^{-1} * \begin{pmatrix} \mathcal{N}_1 & \mathcal{N}_2 \\ \mathcal{O} & \mathcal{O} \end{pmatrix} * \mathcal{P}. \tag{27}$$

By $\mathcal{E} = \mathcal{A} * \mathcal{A}^D * \mathcal{E}$ and (27), we obtain

$$\mathcal{E} = \mathcal{P}^{-1} * \begin{pmatrix} \mathcal{N}_1 & \mathcal{N}_2 \\ \mathcal{N}_3 & \mathcal{N}_4 \end{pmatrix} * \mathcal{P} = \mathcal{P}^{-1} * \begin{pmatrix} \mathcal{N}_1 & \mathcal{N}_2 \\ \mathcal{O} & \mathcal{O} \end{pmatrix} * \mathcal{P} \tag{28}$$

Hence $\mathcal{E} = \mathcal{P}^{-1} * \mathcal{N} * \mathcal{P} = \mathcal{P}^{-1} * \begin{pmatrix} \mathcal{N}_1 & \mathcal{N}_2 \\ \mathcal{O} & \mathcal{O} \end{pmatrix} * \mathcal{P}$, and

$$bcirc(\mathcal{N}) = (F_p \otimes I_n) \begin{pmatrix} N_1 & & & \\ & N_2 & & \\ & & \ddots & \\ & & & N_p \end{pmatrix} (F_p^H \otimes I_n).$$

The proof is completed. □

Lemma 4.2. *Let $\mathcal{A} \in \mathbb{C}^{n\times n\times p}$, $\mathcal{E} \in \mathbb{C}^{n\times n\times p}$ be complex tensors, and $\mathcal{E} = \mathcal{A} * \mathcal{A}^D * \mathcal{E}$, $\|\mathcal{A}^D * \mathcal{E}\| < 1$, $\mathcal{B} = \mathcal{A} + \mathcal{E}$, such that*

$$\mathcal{B}^D = \mathcal{A}^D - \mathcal{A}^D * \mathcal{E} * (\mathcal{I} + \mathcal{A}^D * \mathcal{E})^{-1} * \mathcal{A}^D + \sum_{s=0}^{k-1} \left(\mathcal{A}^D - \mathcal{A}^D * \mathcal{E} * (\mathcal{I} + \mathcal{A}^D * \mathcal{E})^{-1} * \mathcal{A}^D\right)^{s+2} * \mathcal{E} * (\mathcal{I} - \mathcal{A} * \mathcal{A}^D) * \mathcal{A}^s.$$

Proof. According to the Theorem 1.11, then there is an invertible tensor $\mathcal{P} \in \mathbb{C}^{n\times n\times p}$ such that

$$\begin{aligned}\mathcal{B} &= \mathcal{A} + \mathcal{E}\\ &= \mathcal{P}^{-1} * \mathcal{J} * \mathcal{P} + \mathcal{P}^{-1} * \mathcal{N} * \mathcal{P}\\ &= \mathcal{P}^{-1} * \begin{pmatrix} \mathcal{J}_1 & \mathcal{O} \\ \mathcal{O} & \mathcal{J}_4^0 \end{pmatrix} * \mathcal{P} + \mathcal{P}^{-1} * \begin{pmatrix} \mathcal{N}_1 & \mathcal{N}_2 \\ \mathcal{O} & \mathcal{O} \end{pmatrix} * \mathcal{P}\\ &= \mathcal{P}^{-1} * \begin{pmatrix} \mathcal{J}_1 + \mathcal{N}_1 & \mathcal{N}_2 \\ \mathcal{O} & \mathcal{J}_4^0 \end{pmatrix} * \mathcal{P}.\end{aligned}$$

By Theorem 1.14, we have

$$\begin{aligned}\mathcal{B}^D &= \mathcal{P}^{-1} * \begin{pmatrix} \mathcal{J}_1 + \mathcal{N}_1 & \mathcal{N}_2 \\ \mathcal{O} & \mathcal{J}_4^0 \end{pmatrix}^D * \mathcal{P}\\ &= \mathcal{P}^{-1} * \begin{pmatrix} (\mathcal{J}_1 + \mathcal{N}_1)^D & \mathcal{X} \\ \mathcal{O} & (\mathcal{J}_4^0)^D \end{pmatrix} * \mathcal{P}\\ &= \mathcal{P}^{-1} * \begin{pmatrix} (\mathcal{J}_1 + \mathcal{N}_1)^{-1} & \mathcal{X} \\ \mathcal{O} & \mathcal{O} \end{pmatrix} * \mathcal{P},\end{aligned}$$

where $\mathcal{J}_4^0$ is nilpotent and

$$\begin{aligned}\mathcal{X} &= \sum_{s=0}^{k-1} \left((\mathcal{J}_1 + \mathcal{N}_1)^{-1}\right)^{s+2} * \mathcal{N}_2 * (\mathcal{J}_4^0)^s * \left(\mathcal{I} - \mathcal{J}_4^0 * (\mathcal{J}_4^0)^D\right)\\ &+ \left(\mathcal{I} - (\mathcal{J}_1 + \mathcal{N}_1) * (\mathcal{J}_1 + \mathcal{N}_1)^{-1}\right) * \sum_{s=0}^{l-1} (\mathcal{J}_1 + \mathcal{N}_1)^s * N_2 * (\mathcal{J}_4^0)^{s+2}\\ &- (\mathcal{J}_1 + \mathcal{N}_1)^D * \mathcal{N}_2 * (\mathcal{J}_4^0)^D\\ &= \sum_{s=0}^{k-1} \left((\mathcal{J}_1 + \mathcal{N}_1)^{-1}\right)^{s+2} * \mathcal{N}_2 * (\mathcal{J}_4^0)^s.\end{aligned}$$

Therefore

$$\begin{aligned}\mathcal{B}^D &= \mathcal{P}^{-1} * \begin{pmatrix} (\mathcal{J}_1 + \mathcal{N}_1)^{-1} & \sum_{s=0}^{k-1} ((\mathcal{J}_1 + \mathcal{N}_1)^{-1})^{s+2} * \mathcal{N}_2 * (\mathcal{J}_4^0)^s \\ \mathcal{O} & \mathcal{O} \end{pmatrix} * \mathcal{P}\\ &= \mathcal{A}^D - \mathcal{A}^D * \mathcal{E} * (\mathcal{I} + \mathcal{A}^D * \mathcal{E})^{-1} * \mathcal{A}^D\\ &+ \sum_{s=0}^{k-1} \left(\mathcal{A}^D - \mathcal{A}^D * \mathcal{E} * (\mathcal{I} + \mathcal{A}^D * \mathcal{E})^{-1} * \mathcal{A}^D\right)^{s+2} * \mathcal{E} * (\mathcal{I} - \mathcal{A} * \mathcal{A}^D) * \mathcal{A}^s.\end{aligned}$$

Moreover, it proves that $\mathcal{N}_1 + \mathcal{J}_1$ is invertible. Let consider spectral radius of $\mathcal{A}^D * \mathcal{E}$.
Since (26) and (28), then

$$\begin{aligned}\mathcal{A}^D * \mathcal{E} &= \mathcal{P}^{-1} * \mathcal{J}^D * \mathcal{P} * \mathcal{P}^{-1} * \mathcal{N} * \mathcal{P}\\ &= \mathcal{P}^{-1} * \mathcal{J}^D * \mathcal{N} * \mathcal{P},\end{aligned}$$

and the decomposition of the matrix $bcirc(\mathcal{J}^D * \mathcal{N})$ is

$$
\begin{aligned}
bcirc(\mathcal{J}^D * \mathcal{N}) &= bcirc(\mathcal{J}^D)bcirc(\mathcal{N}) \\
&= (F_p \otimes I_n)\begin{pmatrix} J_1^D N_1 & & & \\ & J_2^D N_2 & & \\ & & \ddots & \\ & & & J_p^D N_P \end{pmatrix}(F_p^H \otimes I_n),
\end{aligned}
$$

where $J_i^D N_i = \begin{pmatrix} (J_i^1)^{-1} N_i^1 & (J_i^1)^{-1} N_i^2 \\ O & O \end{pmatrix}$. $(i = 1, 2, \cdots, p)$

Similarly, we obtain

$$
\begin{aligned}
\mathcal{E} * \mathcal{A}^D &= \mathcal{P}^{-1} * \mathcal{N} * \mathcal{P} * \mathcal{P}^{-1} * \mathcal{J}^D * \mathcal{P} \\
&= \mathcal{P}^{-1} * \mathcal{N} * \mathcal{J}^D * \mathcal{P},
\end{aligned}
$$

and the decomposition of the matrix $bcirc(\mathcal{N} * \mathcal{J}^D)$ is

$$
\begin{aligned}
bcirc(\mathcal{N} * \mathcal{J}^D) &= bcirc(\mathcal{N})bcirc(\mathcal{J}^D) \\
&= (F_p \otimes I_n)\begin{pmatrix} N_1 J_1^D & & & \\ & N_2 J_2^D & & \\ & & \ddots & \\ & & & N_P J_p^D \end{pmatrix}(F_p^H \otimes I_n),
\end{aligned}
$$

where $N_i J_i^D = \begin{pmatrix} N_i^1 (J_i^1)^{-1} & N_i^2 (J_i^1)^{-1} \\ O & O \end{pmatrix}$. $(i = 1, 2, \cdots, p)$

By Definition 1.15, we have

$$
\begin{aligned}
\rho_T(\mathcal{J}^D * \mathcal{N}) &= \rho(bcirc(\mathcal{J}^D * \mathcal{N})) \\
&= \rho\left((F_p \otimes I_n)bcirc(\mathcal{J}^D * \mathcal{N})(F_p^H \otimes I_n)\right) \\
&= \max_i \rho\left((J_i^1)^{-1} N_i^1\right) \\
&= \max_i \rho\left(N_i^1 (J_i^1)^{-1}\right) \\
&= \rho\left((F_p \otimes I_n)bcirc(\mathcal{N} * \mathcal{J}^D))(F_p^H \otimes I_n)\right) \\
&= \rho(bcirc(\mathcal{N} * \mathcal{J}^D)) \\
&= \rho_T(\mathcal{N} * \mathcal{J}^D),
\end{aligned}
$$

that is

$$
\rho_T(\mathcal{A}^D * \mathcal{E}) = \max_i \rho\left((J_i^1)^{-1} N_i^1\right) = \max_i \rho\left(N_i^1 (J_i^1)^{-1}\right) = \rho_T(\mathcal{E} * \mathcal{A}^D), \tag{29}
$$

further

$$
\rho_T(\mathcal{E} * \mathcal{A}^D) = \rho_T(\mathcal{A}^D * \mathcal{E}) \le \|\mathcal{A}^D * \mathcal{E}\| < 1. \tag{30}
$$

On the other hand, it will prove that $\mathcal{J}_1 + \mathcal{N}_1 = \mathcal{J}_1 * \left(\mathcal{I} + (\mathcal{J}_1)^{-1} * \mathcal{N}_1\right)$ is invertible. According to the inverse of $\mathcal{J}_1$, we will only prove that $\mathcal{I} + (\mathcal{J}_1)^{-1} * \mathcal{N}_1$ is nonsingular. Now, we prove it by reduction to absurdity. Assume $\mathcal{I} + (\mathcal{J}_1)^{-1} * \mathcal{N}_1$ is singular, then there is a nonzero tensor $y \in \mathbb{C}^{n \times n \times p}$, such that

$$
\left(\mathcal{I} + (\mathcal{J}_1)^{-1} * \mathcal{N}_1\right) * y = \mathcal{O},
$$

then

$$y = -\left((\mathcal{J}_1)^{-1} * \mathcal{N}_1\right) * y,$$

and the decomposition of bcirc$\left((\mathcal{J}_1)^{-1} * \mathcal{N}_1\right)$ is

$$\begin{aligned} bcirc\left((\mathcal{J}_1)^{-1} * \mathcal{N}_1\right) &= bcirc\left((\mathcal{J}_1)^{-1}\right) bcirc(\mathcal{N}_1) \\ &= (F_p \otimes I_n) \begin{pmatrix} (J_1^1)^{-1}N_1^1 & & & \\ & (J_2^1)^{-1}N_2^1 & & \\ & & \ddots & \\ & & & (J_p^1)^{-1}N_p^1 \end{pmatrix} (F_p^H \otimes I_n). \end{aligned}$$

Therefore, by Definition 1.16, then -1 is the eigenvalue of tensor $\left((\mathcal{J}_1)^{-1} * \mathcal{N}_1\right)$, denoted

$$\begin{aligned} \lambda_T\left((\mathcal{J}_1)^{-1} * \mathcal{N}_1\right) &= \lambda\left(bcirc((\mathcal{J}_1)^{-1} * \mathcal{N}_1)\right) \\ &= \lambda\left((F_p \otimes I_n) bcirc((\mathcal{J}_1)^{-1} * \mathcal{N}_1)(F_p^H \otimes I_n)\right) \\ &= \lambda\left((J_i^1)^{-1}N_i^1\right) \\ &= -1, \end{aligned}$$

it implies $\max_i \rho\left((J_i^1)^{-1}N_i^1\right) \geq 1$.
According to (29), we obtain

$$\rho_T(\mathcal{E} * \mathcal{A}^D) = \rho_T(\mathcal{A}^D * \mathcal{E}) = \max_i \rho\left((J_i^1)^{-1}N_i^1\right) \geq 1,$$

which is contradictory to (30).
Hence $\mathcal{I} + (\mathcal{J}_1)^{-1} * \mathcal{N}_1$ is nonsingular. The proof is completed. □

Theorem 4.3. *Let $\mathcal{A} \in \mathbb{C}^{n\times n\times p}$, $\mathcal{E} \in \mathbb{C}^{n\times n\times p}$ be complex tensors, $\|\mathcal{A}^D * \mathcal{E}\| < 1$, and $\mathcal{B} = \mathcal{A} + \mathcal{E}$ with $Ind_T(\mathcal{A}) = k$. Suppose that $\mathcal{E} = \mathcal{A} * \mathcal{A}^D * \mathcal{E}$, then*

$$\frac{\|\mathcal{B}^D - \mathcal{A}^D\|}{\|\mathcal{A}^D\|} \leq \frac{\|\mathcal{A}^D * \mathcal{E}\|}{1 - \|\mathcal{A}^D * \mathcal{E}\|} + \sum_{s=0}^{k-1} \frac{\mathcal{K}_D(\mathcal{A})^{s+1}}{(1 - \|\mathcal{A}^D * \mathcal{E}\|)^{s+2}} \frac{\|\mathcal{E}\|}{\|\mathcal{A}\|} \|\mathcal{A} * \mathcal{A}^D\|,$$

where $\mathcal{K}_D(\mathcal{A}) = \|\mathcal{A}\|\|\mathcal{A}^D\|$.

Proof. Since Lemma 4.2, we have

$$\begin{aligned} \mathcal{B}^D - \mathcal{A}^D &= -\mathcal{A}^D * \mathcal{E} * (\mathcal{I} + \mathcal{A}^D * \mathcal{E})^{-1} * \mathcal{A}^D \\ &\quad + \sum_{s=0}^{k-1} \left(\mathcal{A}^D - \mathcal{A}^D * \mathcal{E} * (\mathcal{I} + \mathcal{A}^D * \mathcal{E})^{-1} * \mathcal{A}^D\right)^{s+2} * \mathcal{E} * (\mathcal{I} - \mathcal{A} * \mathcal{A}^D) * \mathcal{A}^s, \end{aligned} \tag{31}$$

taking norm on both sides of (31) at the same time, then

$$\begin{aligned} \|\mathcal{B}^D - \mathcal{A}^D\| &\leq \| - \mathcal{A}^D * \mathcal{E} * (\mathcal{I} + \mathcal{A}^D * \mathcal{E})^{-1} * \mathcal{A}^D\| \\ &\quad + \sum_{s=0}^{k-1} \|\left(\mathcal{A}^D - \mathcal{A}^D * \mathcal{E} * (\mathcal{I} + \mathcal{A}^D * \mathcal{E})^{-1} * \mathcal{A}^D\right)^{s+2} * \mathcal{E} * (\mathcal{I} - \mathcal{A} * \mathcal{A}^D) * \mathcal{A}^s\| \\ &\leq \|\mathcal{A}^D * \mathcal{E}\|\|(\mathcal{I} + \mathcal{A}^D * \mathcal{E})^{-1}\|\|\mathcal{A}^D\| \\ &\quad + \sum_{s=0}^{k-1} \left(\|\mathcal{A}^D\| + \|\mathcal{A}^D * \mathcal{E}\|\|(\mathcal{I} + \mathcal{A}^D * \mathcal{E})^{-1}\|\|\mathcal{A}^D\|\right)^{s+2} \|\mathcal{E}\|\|(\mathcal{I} - \mathcal{A} * \mathcal{A}^D)\|\|\mathcal{A}\|^s, \end{aligned}$$

by Theorem 2.1, we have

$$\begin{aligned}
\|\mathcal{B}^D - \mathcal{A}^D\| &\leq \|\mathcal{A}^D * \mathcal{E}\| \frac{1}{1 - \|\mathcal{A}^D * \mathcal{E}\|} \|\mathcal{A}^D\| \\
&+ \sum_{s=0}^{k-1} \left(\|\mathcal{A}^D\| + \|\mathcal{A}^D * \mathcal{E}\| \frac{1}{1 - \|\mathcal{A}^D * \mathcal{E}\|} \|\mathcal{A}^D\| \right)^{s+2} \|\mathcal{E}\| \|\mathcal{A} * \mathcal{A}^D\| \|\mathcal{A}\|^s \\
&= \|\mathcal{A}^D * \mathcal{E}\| \frac{1}{1 - \|\mathcal{A}^D * \mathcal{E}\|} \|\mathcal{A}^D\| \\
&+ \sum_{s=0}^{k-1} (\|\mathcal{A}^D\|)^{s+2} \left(1 + \|\mathcal{A}^D * \mathcal{E}\| \frac{1}{1 - \|\mathcal{A}^D * \mathcal{E}\|} \right)^{s+2} \|\mathcal{E}\| \|\mathcal{A} * \mathcal{A}^D\| \|\mathcal{A}\|^s,
\end{aligned}$$

that is

$$\begin{aligned}
\|\mathcal{B}^D - \mathcal{A}^D\| &\leq \|\mathcal{A}^D * \mathcal{E}\| \frac{1}{1 - \|\mathcal{A}^D * \mathcal{E}\|} \|\mathcal{A}^D\| \\
&+ \sum_{s=0}^{k-1} (\|\mathcal{A}^D\|)^{s+2} \left(1 + \|\mathcal{A}^D * \mathcal{E}\| \frac{1}{1 - \|\mathcal{A}^D * \mathcal{E}\|} \right)^{s+2} \|\mathcal{E}\| \|\mathcal{A} * \mathcal{A}^D\| \|\mathcal{A}\|^s, \qquad (32)
\end{aligned}$$

divide $\|\mathcal{A}^D\|$ on both sides of (32) at the same time, we obtain

$$\begin{aligned}
\frac{\|\mathcal{B}^D - \mathcal{A}^D\|}{\|\mathcal{A}^D\|} &\leq \|\mathcal{A}^D * \mathcal{E}\| \frac{1}{1 - \|\mathcal{A}^D * \mathcal{E}\|} \\
&+ \sum_{s=0}^{k-1} (\|\mathcal{A}^D\|)^{s+1} \left(1 + \|\mathcal{A}^D * \mathcal{E}\| \frac{1}{1 - \|\mathcal{A}^D * \mathcal{E}\|} \right)^{s+2} \|\mathcal{E}\| \|\mathcal{A} * \mathcal{A}^D\| \|\mathcal{A}\|^s \\
&= \|\mathcal{A}^D * \mathcal{E}\| \frac{1}{1 - \|\mathcal{A}^D * \mathcal{E}\|} \\
&+ \sum_{s=0}^{k-1} (\|\mathcal{A}^D\|)^{s+1} (\frac{1}{1 - \|\mathcal{A}^D * \mathcal{E}\|})^{s+2} \frac{\|\mathcal{E}\|}{\|\mathcal{A}\|} \|\mathcal{A} * \mathcal{A}^D\| \|\mathcal{A}\|^s \|\mathcal{A}\| \\
&= \|\mathcal{A}^D * \mathcal{E}\| \frac{1}{1 - \|\mathcal{A}^D * \mathcal{E}\|} \\
&+ \sum_{s=0}^{k-1} (\|\mathcal{A}^D\|)^{s+1} (\|\mathcal{A}\|)^{s+1} (\frac{1}{1 - \|\mathcal{A}^D * \mathcal{E}\|})^{s+2} \frac{\|\mathcal{E}\|}{\|\mathcal{A}\|} \|\mathcal{A} * \mathcal{A}^D\| \\
&= \frac{\|\mathcal{A}^D * \mathcal{E}\|}{1 - \|\mathcal{A}^D * \mathcal{E}\|} + \sum_{s=0}^{k-1} \frac{\mathcal{K}_D(\mathcal{A})^{s+1}}{(1 - \|\mathcal{A}^D * \mathcal{E}\|)^{s+2}} \frac{\|\mathcal{E}\|}{\|\mathcal{A}\|} \|\mathcal{A} * \mathcal{A}^D\|,
\end{aligned}$$

where $\mathcal{K}_D(\mathcal{A})^{s+1} = (\|\mathcal{A}\| \|\mathcal{A}^D\|)^{s+1}$. The proof is completed. □

References

[1] S. Campbell and C. Meyer. Generalized Inverse of Linear Transformations. London, Pitman 1979.
[2] R. Chan and X. Jin. An Introduction to Iterative Toeplitz Solvers. SIAM, Philadelphia, 2007.
[3] M. Che and Y. Wei. Randomized algorithms for the approximations of Tucker and the tensors train decompositions, Adv. Comput. Math., 45(2019), 395-428.
[4] M. Che, Y. Wei, and H. Yan. The computation of low multilinear rank approximations of tensors via power scheme and random projection, SIAM.J Matrix Anal. Appl., 41(2020), 605-636.
[5] M. Che and Y. Wei. Theory and Computation of Complex Tensors and its Applications, Springer, Singaproe. 2020.
[6] R. E. Cline and T. N. E. Greville. A Drazin inverse for rectangular matrices, Linear algebra Appl. 29 (1980), 53-62.

[7] D. S. Cvetkovic and Y. Wei, Algebraic Properties of Generalized Inverses. Developments in Mathematics, 52. Springer, Singapore, 2017.
[8] M. P. Drazin. Pseudoinvese in associative rings and semigroups, Amer. Math. Monthly 65 (1958), 506-514.
[9] N. Hao, M. E. Kilmer, K. Braman, and R. C. Hoover. Facial recognition using tensor-tensor decompositions. SIAM J. Imaging Sci. 6 (2013), 437-463.
[10] J. Ji and Y. Wei. The Drazin inverse of an even-order tensor and its application to singular tensor equations. Comput. Math. Appl. 75 (2018), 3402-3413.
[11] M. E. Kilmer, K. Braman, N. Hao, and R. C. Hoover. Third-order tensors as operators on matrices: a theoretical and computational framework with applications in imaging. SIAM J. Matrix Anal. Appl. 34 (2013), 148-172.
[12] M. E Kilmer and C. D. Martin. Factorization strategies for third-order tensors. Linear Algebra Appl. 435 (2011), 641-658.
[13] C.F. King. A note on Drazin inverse, Pacific J. Math. 70 (1977), 383-390.
[14] K. Lund, The tensor t-function: a definition for functions of third-order tensors. Numer. Linear Algebra Appl. 2020, e2288.
[15] H. Ma, N. Li, P. S. Stanimirovic, and V. Katsikis. Perturbation theory for Moore-Penrose inverse of tensor via Einstein product, Comp. Appl. Math. 38 (2019), Art 111.
[16] C. D. Matin, R. Shafer, and B. Latue. An order-p tensor factorization with applications in imaging, SIAM J. Sci. Comput. 35 (2013), A474-A490.
[17] Y. Miao, L. Qi, and Y. Wei. T-Jordan canonical form and T-Drazin inverse based on the T-Product. arXiv: 1902.07024v1, 2019. Commun. Appl. Math. Comput. 3(2021), 201-220.
[18] Y. Miao, L. Qi, Y. Wei. Generalized tensor function via the tensor singular value decomposition based on the T-product. Linear Algebra Appl. 590 (2020), 258-303.
[19] L. Qi and, Z. Luo. Tensor Analysis: Spectral Theory and Special Tensors, Society of Industrial and Applied Mathematics, Philadelphia. 2017.
[20] S. Qiao, X. Wang and Y. Wei, Two finite-time convergent Zhang neural network models for time-varying complex matrix Drazin inverse, Linear Algebra Appl. 542 (2018), 101-117.
[21] S. Qiao and Y.Wei, Acute perturbation of Drazin inverse and oblique projectors, Front. Math. China, 13 (2018), 1427-1445.
[22] J. K. Sahoo, R. Behera, P. S. Stanimirovic, V. N. Katsikis and H. Ma. Core and core-EP inverses of tensors, Comput. Appl. Math. 39 (2020), no.1, Art 9, 28 pp.
[23] S. Soltani, M. E. Kilmer, and P. C. Hansen. A tensor-based dictionary learning approach to tomo-graphic image reconstruction, BIT Numerical Mathematics. 56 (2016), 1425-1454.
[24] L. Sun, B. Zheng, Y. Wei and C. Bu, Generalized inverses of tensors via a general product of tensors, Front. Math. China. 13 (2018), 893-911.
[25] G. Wang, Y. Wei, and S. Qiao, *Generalized Inverses: Theory and Computations*, Developments in Mathematics 53. Singapore: Springer; Beijing: Science Press, 2018.
[26] X. Wang, M. Che and Y. Wei, Tensor neural network models for tensor singular value decompositions, Comput. Optim. Appl. 75 (2020), 753-777.
[27] Y. Wei, X. Li, and F. Bu, A pertubation bound of the Drazin inverse of a matrix by separation of simple invariant subspaces, SIAM J. Matrix Anal. Appl. 27 (2005), 72-81.
[28] Y. Wei, G. Wang. The Perturbation Theory for the Drazin Inverse and its Applications, Linear Algebra Appl. 258 (1997), 179-186.
[29] Y. Wei. A characterization and representation of the Drazin inverse, SIAM J. Matrix Anal. Appl. 17 (1996), 744-747.
[30] Y. Wei. Index splitting for the Drazin inverse and the sigular linear system, Appl. Math. Comput. 108 (2000), 115-124.
[31] Y. Wei. The Drazin inverse of updating of a square matrix with application to perturbation formula, Appl. Math. Comput. 108 (2000), 77-83.
[32] Y. Wei and W. Ding, Theory and Computation of Tensors: Multi-Dimensional Arrays, Academic Press, New York, 2016.
[33] Y. Wei and X. Li, An improvement on perturbation bounds for the Drazin inverse, Numer. Linear Algebra Appl., 10 (2003), 563-575.
[34] Y. Wei and H. Wu, Challenging problems on the perturbation of Drazin inverse, Annals of Operations Research, 103 (2001), 371-378.
[35] Y. Wei, P. S. Stanimirović and M. D. Petković, Numerical and Symbolic Computations of Generalized Inverses, World Scientific Publishing Co. Pte. Ltd., Singapore, 2018.
[36] Z. Xie, X. Jin and Y. Wei. Tensor methods for solving symmetric M-tensor systems. J. Sci. Comput., 74(2018), 412-425.
[37] Q. Xu, C. Song and Y. Wei, The stable perturbation of the Drazin inverse of the square matrices, SIAM J. Matrix Anal. Appl., 31 (2009), 1507-1520.
[38] P. Zhou, C. Lu, Z. Lin, and C. Zhang. Tensor factorization for low-rank tensor completion, IEEE Trans. Image Process. 27 (2018), 1152-1163.