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# Global Optimal Approximate Solutions of Best Proximity Points

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**Abstract.** In this paper, we introduce the concept of contractive pair maps and give some necessary and sufficient conditions for existence and uniqueness of best proximity points for such pairs. In our approach, some conditions have been weakened. An application has been presented to demonstrate the usability of our results. Also, we introduce the concept of cyclic $\psi$-contraction and cyclic asymptotic $\psi$-contraction and give some existence and convergence theorems on best proximity point for cyclic $\psi$-contraction and cyclic asymptotic $\psi$-contraction mappings. The presented results extend, generalize and improve some known results from best proximity point theory and fixed-point theory.

## 1. Introduction and Preliminaries

Let $\Omega$ be a metric space and let $\Delta$ and $\Lambda$ be nonempty subsets of $\Omega$. Let

$$\begin{aligned}\Delta^\circ &= \{\delta \in \Delta : d(\delta, \lambda) = d(\Delta, \Lambda) \textit{ for some } \lambda \in \Lambda\},\\ \Lambda^\circ &= \{\delta \in \Lambda : d(\delta, \lambda) = d(\Delta, \Lambda) \textit{ for some } \lambda \in \Delta\}.\end{aligned}$$

If there is a pair $(\delta_0, \lambda_0) \in \Delta \times \Lambda$ for which

$$d(\delta_0, \lambda_0) = d(\Delta, \Lambda) = \sup_{\delta\in\Delta} \inf_{\lambda\in\Lambda} d(\delta, \lambda),$$

where $d(\Delta, \Lambda)$ is the distance between $\Delta$ and $\Lambda$, then the pair $(\delta_0, \lambda_0)$ is called a best proximity pair for $\Delta$ and $\Lambda$. Best proximity pair derives as an extension of the notion of best approximation.

We can find the best proximity points of the sets $\Delta$ and $\Lambda$, by considering a map $\Gamma : \Delta \cup \Lambda \to \Delta \cup \Lambda$. We say that the point $\delta \in \Delta \cup \Lambda$ is a best proximity point of the pair $(\Delta, \Lambda)$, if $d(\delta, \Gamma\delta) = d(\Delta, \Lambda)$ and we denote the set of all best proximity points of $(\Delta, \Lambda)$ by $P_\Gamma(\Delta, \Lambda)$, that is,

$$P_\Gamma(\Delta, \Lambda) = \{\delta \in \Delta \cup \Lambda : d(\delta, \Gamma\delta) = d(\Delta, \Lambda)\}.$$

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Best proximity point also derives as an extension of the notion of fixed point of mappings, because every best proximity point is a fixed point of $\Gamma$ whenever $\Delta \cap \Lambda \neq \emptyset$.

A best proximity point theorem for contractive mappings has been considered in [12]. Eldred et al. [4] have extracted a best proximity point theorem for relatively nonexpansive mappings, an alternative treatment to which has been focused in Sankar Raj and Veeramani [16]. A best proximity point theorem for contractions has been presented in [13]. Anuradha and Veeramani [1] have examined best proximity point theorems for proximal pointwise contractions. Best proximity point theorems for various contractions have been discussed by many authers( [4]-[16]).

This paper contains two sections. In the first section, we introduce the concept of contractive pair maps and give some necessary and sufficient conditions for existence and uniqueness of best proximity points for such pairs which causes the weakness of some conditions of [12]. An application has been presented to demonstrate our results.

In the second section, we introduce the concept of cyclic $\psi$-contraction and cyclic asymptotic $\psi$-contraction which are important generalizations of the cyclic contraction by substituting the constant $k$ by a real-valued control function $\psi : [0, \infty) \to [0, \infty)$ and so we give some existence and convergence theorems for best proximity point of cyclic $\psi$-contractive and cyclic asymptotic $\psi$-contractive mappings.

## 2. Best proximity points by contraction pair maps

First we give a simple and useful result in best proximity points. It is notable that some results in this section are extensions of the Eldred and Veeramani results in [5]. We start by a new definition.

**Definition 2.1.** *Let $\Delta$ and $\Lambda$ be nonempty subsets of a metric space $\Omega$ and let $\Gamma : \Delta \to \Lambda$ and $\Upsilon : \Lambda \to \Delta$. The pair $(\Gamma, \Upsilon)$ is said to be a contractive pair if,*

$$d(\Gamma\delta, \Upsilon\lambda) \leq kd(\delta, \lambda) + (1 - k)dist(\Delta, \Lambda) \tag{1}$$

*for some $k \in (0, 1)$ and for all $\delta \in \Delta$ and $\lambda \in \Lambda$.*

Note that (1) implies that $\Gamma$ satisfies $d(\Gamma\delta, \Upsilon\lambda) \leq d(\delta, \lambda)$, for all $\delta \in \Delta$ and $\lambda \in \Lambda$. For example, let $\Delta = \{(\delta, 0) : \delta \in [0, 1]\}$ and $\Lambda = \{(\delta, 1) : \delta \in [0, 1]\}$. Define the pair $\Gamma, \Upsilon$ by $\Gamma(\delta, 0) = (\frac{1}{2}, 1)$ and $\Upsilon(\delta, 1) = (\frac{1}{2}, 0)$. Then it is easy to see that $(\Gamma, \Upsilon)$ is a contractive pair.

**Proposition 2.2.** *Let $\Delta$ and $\Lambda$ be nonempty closed subsets of a complete metric space $\Omega$. Let $(\Gamma, \Upsilon)$ be a contraction pair, $(\delta_0, \lambda_0) \in \Delta \times \Lambda$ and $\delta_{n+1} := \Upsilon\lambda_n$ and $\lambda_{n+1} := \Gamma\delta_n$, for all $n \in \mathbb{N}$. Suppose that $\{\delta_n\}$ and $\{\lambda_n\}$ have convergent subsequences in $\Delta$ and $\Lambda$. Then there exists $(\delta, \lambda) \in \Delta \times \Lambda$ such that $d(\delta, \Gamma\delta) = d(\lambda, \Upsilon\lambda) = dist(\Delta, \Lambda)$.*

*Proof.* We know that

$$\begin{aligned}
d(\delta_{n+1}, \lambda_{n+1}) = d(\Gamma\delta_n, \Upsilon\lambda_n) &\leq kd(\delta_n, \lambda_n) + (1 - k)dist(\Delta, \Lambda) \\
&\leq k^2 d(\delta_{n-1}, \lambda_{n-1}) + (1 - k^2)dist(\Delta, \Lambda) \\
&\vdots \\
&\leq k^n d(\delta_0, \lambda_0) + (1 - k^n)dist(\Delta, \Lambda),
\end{aligned}$$

that is,

$$d(\delta_{n+1}, \lambda_{n+1}) \leq k^n d(\delta_0, \lambda_0) + (1 - k^n)dist(\Delta, \Lambda).$$

Therefore, $d(\delta_n, \lambda_n) \to dist(\Delta, \Lambda)$. Let $\{\delta_{n_k}\}$ be a subsequence of $\{\delta_n\}$ converging to some $\delta \in \Delta$. Now

$$dist(\Delta, \Lambda) \leq d(\delta, \lambda_{n_k}) \leq d(\delta, \delta_{n_k}) + d(\delta_{n_k}, \lambda_{n_k}).$$

Thus $d(\delta, \lambda_{n_k})$ converges to $dist(\Delta, \Lambda)$. Since

$$dist(\Delta, \Lambda) \leq d(\delta_{n_k}, \Gamma\delta) = d(\Upsilon\lambda_{n_k}, \Gamma\delta) \leq d(\lambda_{n_k}, \delta).$$

Thus, $d(\delta, \Gamma\delta) = dist(\Delta, \Lambda)$. ■ □

**Theorem 2.3.** *(Compare to [8, Theorem 2.1]) Let $\Delta$ and $\Lambda$ be nonempty closed subsets of a metric space $\Omega$ and $(\Gamma, \Upsilon)$ a contractive pair. If $\Gamma$ and $\Upsilon$ be contraction, then there exists $(\delta, \lambda) \in \Delta\times\Lambda$ such that $d(\delta, \Gamma\delta) = d(\lambda, \Upsilon\lambda) = dist(\Delta, \Lambda)$.*

*Proof.* By Proposition 2.2, it is sufficient to show that $\{\delta_{2n}\}$ and $\{\lambda_{2n}\}$ are convergence sequences in $\Delta$ and $\Lambda$.
Note that

$$d(\delta_{2n}, \delta_{2n+2}) = d(\Gamma\lambda_{2n-1}, \Gamma\lambda_{2n+1}) < d(\lambda_{2n-1}, \lambda_{2n+1})$$

and

$$d(\lambda_{2n-1}, \lambda_{2n+1}) = d(\Gamma\delta_{2n-2}, \Gamma\delta_{2n}) < d(\delta_{2n-2}, \delta_{2n+1}).$$

Hence, $\{d(\delta_{2n}, \delta_{2n+2})\}$ is monotonic decreasing and bounded below. So,

$$\lim_{n\to\infty} d(\delta_{2n}, \delta_{2n+2})$$

exists. Let $\lim_{n\to\infty} d(\delta_{2n}, \delta_{2n+2}) = \delta$. It is clear that $0 \leq \delta$. Assume that $\delta > 0$. Hence,

$$\delta = \lim_{n\to\infty} d(\delta_{2n}, \delta_{2n+2}) < \lim_{n\to\infty} d(\delta_{2n-2}, \delta_{2n}) = \delta.$$

So, $\delta = 0$.

Now, we show that $\{\delta_{2n}\}$ is a Cauchy sequence. Assume that $\{\delta_{2n}\}$ is not Cauchy. Then there exist $\varepsilon > 0$ and integers $2m_k, 2n_k \in LambdabbN$ such that $2m_k > 2n_k \geq k$ and $d(\delta_{2n_k}, \delta_{2m_k}) \geq \varepsilon$ for $k = 0, 1, 2, \cdots$. Also, choosing $m_k$ as small as possible, it may be assumed that

$$d(\delta_{2n_k}, \delta_{2m_k-2}) < \varepsilon.$$

Hence, for each $k \in LambdabbN$, we have

$$\begin{aligned} \varepsilon \leq d(\delta_{2n_k}, \delta_{2m_k}) &\leq d(\delta_{2n_k}, \delta_{2m_k-2}) + d(\delta_{2m_k-2}, \delta_{2m_k}) \\ &\leq \varepsilon + d(\delta_{2m_k-2}, \delta_{2m_k}) \end{aligned}$$

and since $d(\delta_{2m_k-2}, \delta_{2m_k}) \to 0$, hence $\lim_{k\to\infty} d(\delta_{2n_k}, \delta_{2m_k}) = \varepsilon$. Observe that

$$\begin{aligned} d(\delta_{2n_k}, \delta_{2m_k}) &\leq d(\delta_{2n_k}, \delta_{2n_k+2}) + d(\delta_{2n_k+2}, \delta_{2m_k+2}) + d(\delta_{2n_k+2}, \delta_{2m_k}) \\ &< d(\delta_{2n_k}, \delta_{2n_k+2}) + d(\lambda_{2n_k+1}, \lambda_{2m_k+1}) + d(\delta_{2n_k+2}, \delta_{2m_k}). \end{aligned}$$

Letting $k \to \infty$, we obtain that

$$\varepsilon < \lim_{k\to\infty} d(\lambda_{2n_k+1}, \lambda_{2m_k+1}).$$

On the other hand,

$$\lim_{k\to\infty} d(\lambda_{2n_k+1}, \lambda_{2m_k+1}) < \lim_{k\to\infty} d(\delta_{2n_k}, \delta_{2m_k}) = \varepsilon$$

which is a contradiction. Hence, $\{\delta_{2n}\}$ is a Cauchy sequence in $\Delta$ and so $\{\delta_{2n}\}$ converge to $\delta \in \Delta$. Similarly, $\{\lambda_{2n}\}$ converges to $\lambda \in \Lambda$. ■ □

**Remark 2.4.** *Under the contractive pair assumption in Theorem 2.3, compactness of $\Delta$ and $\Lambda$ had been omitted with respect to [8,Theorem 2.1].*

A metric space X is boundedly compact if all closed bounded subsets of X are compact. Every boundedly compact metric space is complete.

**Corollary 2.5.** *Let $\Delta$ and $\Lambda$ be nonempty closed subsets of a complete metric space $\Omega$. Suppose that the mappings $\Gamma : \Delta \to \Lambda$ and $\Upsilon : \Lambda \to \Delta$ are such that*

$$d(\Gamma\delta, \Upsilon\lambda) \leq \alpha d(\delta, \lambda) + \beta[d(\delta, \Gamma\delta) + d(\lambda, \Upsilon\lambda)] + \gamma dist(\Delta, \Lambda)$$

*for all $\delta \in \Delta$ and $\lambda \in \Lambda$, where $\alpha, \beta, \gamma \geq 0$ and $\alpha + 2\beta + \gamma < 1$. If either $\Delta$ or $\Lambda$ is boundedly compact, then there exist $\delta \in \Delta$ and $\lambda \in \Lambda$ such that $d(\delta, \Gamma\delta) = d(\lambda, \Upsilon\lambda) = dist(\Delta, \Lambda)$.*

*Proof.* Suppose that $(\delta_0, \lambda_0) \in \Delta \times \Lambda$ and define $\delta_{n+1} = \Upsilon\lambda_n$, $\lambda_{n+1} = \Gamma\delta_n$, for all $n \in \mathbb{N}$. Now, we have

$$\begin{aligned} d(\lambda_{n+1}, \delta_{n+2}) &= d(\Gamma\delta_n, \Upsilon\lambda_{n+1}) \\ &\leq \alpha d(\delta_n, \lambda_n) + \beta[d(\delta_n, \Gamma\delta_n) + d(\lambda_{n+1}, \Upsilon\lambda_{n+1})] \\ &+ \gamma\, dist(\Delta, \Lambda) \end{aligned}$$

which implies that

$$(1 - \beta)d(\lambda_{n+1}, \delta_{n+2}) \leq (\alpha + \beta)d(\delta_n, \lambda_{n+1}) + \gamma dist(\Delta, \Lambda)$$

and hence,

$$d(\lambda_{n+1}, \delta_{n+2}) \leq \frac{\alpha + \beta}{1 - \beta} d(\delta_n, \lambda_{n+1}) + \frac{\gamma}{1 - \beta} dist(\Delta, \Lambda).$$

Therefore,

$$d(\lambda_{n+1}, \delta_{n+2}) \leq kd(\lambda_n, \delta_{n+1}) + (1 - k)dist(\Delta, \Lambda),$$

where $k = \frac{\alpha+\beta}{1-\beta} < 1$. Therefore, by Theorem 2.3 there exist $\delta \in \Delta$ and $\lambda \in \Lambda$ such that $d(\delta, \Gamma\delta) = d(\lambda, \Upsilon\lambda) = dist(\Delta, \Lambda)$. ■ □

**Corollary 2.6.** *Let $\Delta$ and $\Lambda$ be nonempty closed subsets of a complete metric space $\Omega$. Suppose that the mappings $\Gamma : \Delta \to \Lambda$ and $\Upsilon : \Lambda \to \Delta$ are such that*

$$d(\Gamma\delta, \Upsilon\lambda) \leq a_1 d(\delta, \lambda) + a_2 d(\delta, \Gamma\delta) + a_3 d(\lambda, \Upsilon\lambda) + a_4 dist(\Delta, \Lambda) \tag{2}$$

*for all $\delta \in \Delta$ and $\lambda \in \Lambda$, where $a_i \geq 0$, $i = 1, 2, 3, 4$ and $\sum_{i=1}^{4} a_i < 1$. If either $\Delta$ or $\Lambda$ is boundedly compact, then there exist $\delta \in \Delta$ and $\lambda \in \Lambda$ such that $d(\delta, \Gamma\delta) = d(\lambda, \Upsilon\lambda) = dist(\Delta, \Lambda)$.*

*Proof.* In 2 it is sufficient to interchange the roles of $\delta$ and $\lambda$; and adding the new inequality to (3). ■ □

In the following, we give an important result from Theorem 2.3 in normed spaces for nonexpensive maps.

**Theorem 2.7.** *Let $\Omega$ be a normed space, $\Delta$, $\Lambda$ be subsets of $\Omega$ such that $\Delta^\circ$ and $\Lambda^\circ$ are nonempty and convex. Also, suppose that the mappings $\Gamma : \Delta \to \Lambda$ and $\Upsilon : \Lambda \to \Delta$ are such that $\|\Gamma\delta - \Upsilon\lambda\| \leq \|\delta - \lambda\|$ for all $(\delta, \lambda) \in \Delta \times \Lambda$, and $\Gamma$ and $\Upsilon$ be nonexpensive maps. Then there exists $(\delta, \lambda) \in \Delta \times \Lambda$ such that $\|\delta - \Gamma\delta\| = \|\lambda - \Upsilon\lambda\| = dist(\Delta, \Lambda)$.*

*Proof.* First, we show that $\Gamma : \Delta^\circ \to \Lambda^\circ$ and $\Upsilon : \Lambda^\circ \to \Delta^\circ$. Let $\delta \in \Delta^\circ$. Then there is $\lambda \in \Lambda^\circ$ such that $\|\delta - \lambda\| = dist(\Delta, \Lambda)$. Since $\|\Gamma\delta - \Upsilon\lambda\| \leq \|\delta - \lambda\|$, therefore, $\Gamma\delta \in \Lambda^\circ$.

Since $\Delta^\circ$ is nonempty, there are $\delta_0 \in \Delta$ and $\lambda_0 \in \Lambda$ such that $\|\delta_0 - \lambda_0\| = dist(\Delta, \Lambda)$. For every positive integer $n \in \mathbb{N}$, define

$$\Gamma_n(\delta) = \frac{1}{n}\lambda_0 + (1 - \frac{1}{n})\Gamma\delta \quad \delta \in \Delta$$

and

$$\Upsilon_n(\delta) = \frac{1}{n}\delta_0 + (1 - \frac{1}{n})\Upsilon\delta \quad \delta \in \Lambda.$$

Then, for every $\delta, \lambda \in \Delta \cup \Lambda$,

$$\begin{aligned} \|\Gamma_n\delta - \Upsilon_n y\| &\leq (1 - \frac{1}{n})\|\Gamma\delta - \Upsilon\lambda\| + \frac{1}{n}dist(\Delta, \Lambda) \\ &\leq (1 - \frac{1}{n})\|\delta - \lambda\| + (1 - \frac{1}{n}) + \frac{1}{n}dist(\Delta, \Lambda). \end{aligned}$$

Therefore, for every $n \in \mathbb{N}$ the pair $(\Gamma_n, \Upsilon_n)$ is a contractive pair. Hence, by Theorem 2.3, for every $n \in \mathbb{N}$, there exists $\delta_n \in \Delta^\circ$ such that

$$\|\delta_n - \Gamma_n\delta_n\| = dist(\Delta, \Lambda).$$

Since $\Delta^\circ$ is boundedly compact, there exists $\delta \in \Delta^\circ$ such that $\delta_n \rightharpoonup \delta$ (by passing to a subsequence, if necessary). Because $\|\delta_n - \Gamma\delta_n\| \to dist(\Delta, \Lambda)$, it follows that $\|\delta - \Gamma\delta\| = dist(\Delta, \Lambda)$. ■ □

**Corollary 2.8.** *Let $\Delta$ and $\Lambda$ be nonempty subsets of a normed space $\Omega$ such that $\Delta^\circ$ is a convex compact subset. Suppose that $\Gamma : \Delta \to \Lambda$ and $\Upsilon : \Lambda \to \Delta$ be continuous maps such that*

$$\|\Gamma\delta - \Upsilon\lambda\| \leq \|\delta - \lambda\| \quad \delta \in \Delta, \lambda \in \Lambda.$$

*Then the mapping $\Upsilon\Gamma$ has a fixed point.*

*Proof.* Similarly, as in Theorem 2.7, $\Gamma : \Delta^\circ \to \Lambda^\circ$ and $\Upsilon : \Lambda^\circ \to \Delta^\circ$ and so $\Upsilon\Gamma : \Delta^\circ \to \Delta^\circ$. Let $a \in \Delta^\circ$ and for every $n \in \mathbb{N}$ define $U_n : \Delta^\circ \to \Delta^\circ$ such that $U_n\delta = (1 - \frac{1}{n})\Upsilon\Gamma\delta + \frac{1}{n}a$. Hence,

$$\|U_n\delta - U_n y\| \leq (1 - \frac{1}{n})\|\delta - \lambda\|.$$

Therefore, for every $n \in \mathbb{N}$ there exists $\delta_n \in \Delta^\circ$ such that $U_n\delta_n = \delta_n$. Since $\Delta^\circ$ is compact, there exists a subsequence $\{\delta_{n_i}\}$ of $\{\delta_n\}$ such that $\delta_{n_i} \to \delta_0 \in \Delta^\circ$. Because $\Upsilon\Gamma$ is continuous, we have $\Upsilon\Gamma\delta = \delta$. ■.
□

In the following, we show that under some conditions $P_\Gamma(\Delta, \Lambda)$ is a nonempty compact set.

**Theorem 2.9.** *Let $\Delta$ and $\Lambda$ be nonempty subsets of a normed space $\Omega$ such that $\Delta$ be compact. Suppose that the mappings $\Gamma : \Delta \to \Lambda$ and $\Upsilon : \Lambda \to \Delta$ are such that*

$$\|\Gamma\delta - \Upsilon\lambda\| < \|\delta - \lambda\| \qquad (\delta, \lambda) \in \Delta \times \Lambda \setminus \Delta^\circ \times \Lambda^\circ.$$

*If $\Gamma$ be upper semicontinuous, then $P_\Gamma(\Delta, \Lambda)$ is a nonempty compact set.*

*Proof.* There exists $z_0 \in \Delta$ such that $\|z_0 - \Gamma z_0\| = \inf_{z \in \Delta} \|z - \Gamma z\|$. If $\|z_0 - \Gamma z_0\| > dist(\Delta, \Lambda)$, then we have $\|\Gamma z_0 - \Upsilon\Gamma z_0\| < \|z_0 - \Gamma z_0\|$ which is a contradiction with the fact that $z_0$ is minimum. Therefore, $\|z_0 - \Gamma z_0\| = dist(\Delta, \Lambda)$ and so, $P_\Gamma(\Delta, \Lambda)$ is nonempty.

Suppose that $z_n \in P_\Gamma(\Delta, \Lambda)$. Then $\|z_n - \Gamma z_n\| = dist(\Delta, \Lambda)$. There exist subsequence $z_{n_k}$ and $z_0 \in \Delta$ such that

$$\|z_0 - \Gamma z_0\| = \lim_{n\to\infty} \|z_n - \Gamma z_n\| = dist(\Delta, \Lambda)$$

and so $z_0 \in P_\Gamma(\Delta, \Lambda)$ i.e., $P_\Gamma(\Delta, \Lambda)$ is compact. ■ □

In the following, we give a new condition which guarantees that $P_\Gamma(\Delta, \Lambda)$ will be a singleton.

**Theorem 2.10.** *Let $\Delta$ and $\Lambda$ be nonempty subsets of a strictly convex Banach space $\Omega$ such that $\Delta$ be a convex compact subset. Suppose that the mappings $\Gamma : \Delta \to \Lambda$ and $\Upsilon : \Lambda \to \Delta$ are such that*

$$\|\Gamma\delta - \Upsilon\lambda\| < \|\delta - \lambda\| \qquad (\delta, \lambda) \in \Delta \times \Lambda \setminus \Delta^\circ \times \Lambda^\circ.$$

*If $\Gamma$ be upper semicontinuous and $(\Delta - \Delta) \cap (\Lambda - \Lambda) = \emptyset$, then $P_\Gamma(\Delta, \Lambda)$ is a singleton.*

*Proof.* By Theorem 2.9 $P_\Gamma(\Delta, \Lambda) \neq \emptyset$. Suppose that there exist two points $\delta, \lambda \in P_\Gamma(\Delta, \Lambda)$ such that $\delta \neq \lambda$. Also, $\delta - \Gamma\delta \neq \lambda - \Gamma\lambda$. By strict convexity of $\Omega$ we have $\|\frac{\delta+\lambda}{2} - \frac{\Gamma\delta+\Gamma\lambda}{2}\| < dist(\Delta, \Lambda)$. Since $\Delta$ is convex, $\frac{\delta+\lambda}{2} \in \Delta$ and $\frac{\Gamma\delta+\Gamma\lambda}{2} \in \Lambda$ which is a contradiction. Therefore, $\delta - \Gamma\delta = \lambda - \Gamma\lambda$ and so, $\delta - \lambda = \Gamma\delta - \Gamma\lambda \in (\Delta - \Delta) \cap (\Lambda - \Lambda) \neq \emptyset$, which is a contradiction. Therefore, $\delta = \lambda$. ■
□

As an application of Theorem 2.9, we present the following result. Recall that by a domain in the complex plane, we mean an open connected set.

**Corollary 2.11.** *(Compare to [8, Theorem 3.1] ) Let $\Delta$ and $\Lambda$ be nonempty subsets of a domain $D$ of a complex plane such that $\Delta$ is a compact set. Suppose that $f(z)$ and $g(z)$ be analytic functions in $D$ such that*

$(a)$ $\quad f(\Delta) \subseteq \Lambda,\ g(\Lambda) \subseteq \Delta,$

$(b)$ $\quad |f(z_1) - g(z_2)| < |z_1 - z_2| \quad (\delta, \lambda) \in \Delta \times \Lambda \setminus \Delta^\circ \times \Lambda^\circ.$

*Then there exists $(z^*, w^*) \in \Delta \times \Lambda$ such that*

$$|z^* - f(z^*)| = |w^* - g(w^*)| = |z^* - w^*| = dist(\Delta, \Lambda).$$

As an application of Theorem 2.10, we obtain the following result.

**Corollary 2.12.** *Let $\Delta$ and $\Lambda$ be nonempty subsets of a domain $D$ of the complex plane such that $\Delta$ be convex compact and $(\Delta - \Delta) \cap (\Lambda - \Lambda) = \emptyset$. Suppose that $f(z)$ and $g(z)$ be analytic function in $D$ such that*

$(a)$ $\quad f(\Delta) \subseteq \Lambda,\ g(\Lambda) \subseteq \Delta,$

$(b)$ $\quad |f(z_1) - g(z_2)| < |z_1 - z_2| \quad$ *for all* $(\delta, \lambda) \in \Delta \times \Lambda \setminus \Delta^\circ \times \Lambda^\circ.$

*Then there exists a unique $(z^*, w^*) \in \Delta \times \Lambda$ such that*

$$|z^* - f(z^*)| = |w^* - g(w^*)| = |z^* - w^*| = dist(\Delta, \Lambda).$$

## 3. Best proximity points for cyclic $\psi$-contractions

In this section, we consider some important generalizations of cyclic contractions in which the constant $k$ is replaced by some real-valued control function $\psi : [0, \infty) \to [0, \infty)$.

**Definition 3.1.** *Let $\Omega$ be a complete metric space and let $\Delta$ and $\Lambda$ be subsets of $\Omega$. $\Gamma : \Delta \cup \Lambda \to \Delta \cup \Lambda$ is a cyclic $\psi$-contraction if it satisfies:*

$(i)$ $\Gamma(\Delta) \subset \Lambda, \Gamma(\Lambda) \subset \Delta$
$(ii)$ $d(\Gamma\delta, \Gamma\lambda) \leq \psi(d(\delta, \lambda))$, *for all* $\delta, \lambda \in \Delta \cup \Lambda$,

*where $\psi : [0, \infty) \to [0, \infty)$ is an upper semicontinuous function on $\mathbb{R} - \{0, d\}$ from the right such that $\psi(t) < t$ for each $t \neq 0, d$, where $d := d(\Delta, \Lambda)$.*

In the following, we give a generalization of the Boyd and Wong's fixed point theorem [2].

**Theorem 3.2.** *Let $\Delta$ and $\Lambda$ be closed subsets of complete metric space $\Omega$ such that $diam(\Delta) < d(\Delta,\Lambda)$. Suppose that $\Gamma : \Delta\cup\Lambda \to \Delta\cup\Lambda$ is a cyclic $\psi$-contraction. Then $P_\Gamma(\Delta,\Lambda) \neq \emptyset$. Further, if $\delta_0 \in \Delta$ and $\delta_{n+1} = \Gamma\delta_n$, then $\{\delta_{2n}\}$ converges to the best proximity point.*

*Proof.* Fix $\delta \in \Delta\cup\Lambda$ and define a sequence $\{\delta_n\}$ in $\Delta\cup\Lambda$ by $\delta_n = \Gamma^n\delta$, $n \in \mathbb{N}_0$. We divide the proof into 4 steps:

**Step 1**. $\lim_{n\to\infty} d(\delta_n,\delta_{n+1}) = d(\Delta,\Lambda)$.

Note that

$$d(\delta_{n+1},\delta_{n+2}) = d(\Gamma\delta_n,\Gamma\delta_{n+1}) \leq \psi(d(\delta_n,\delta_{n+1})).$$

Hence, $\{d(\delta_n,\delta_{n+1})\}$ is monotonic decreasing and bounded below. Therefore, $\lim_{n\to\infty} d(\delta_n,\delta_{n+1})$ exists. Let $\lim_{n\to\infty} d(\delta_n,\delta_{n+1}) = d \geq d(\Delta,\Lambda)$. Assume that $d > d(\Delta,\Lambda)$. By the right upper semicontinuity of $\psi$,

$$d = \lim_{n\to\infty} d(\delta_{n+1},\delta_{n+2}) \leq \lim_{n\to\infty} \psi(d(\delta_n,\delta_{n+1})) \leq \psi(d) < d.$$

So, $d = d(\Delta,\Lambda)$.

**Step 2**. $\lim_{n\to\infty} d(\delta_{2n},\delta_{2n+2}) = 0$.

Note that

$$d(\delta_{2n},\delta_{2n+2}) = d(\Gamma\delta_{2n-1},\Gamma\delta_{2n+1}) \leq \psi(d(\delta_{2n-1},\delta_{2n+1})).$$

Hence, $\{d(\delta_{2n},\delta_{2n+2})\}$ is monotonic decreasing and bounded below. Hence, $\lim_{n\to\infty} d(\delta_{2n},\delta_{2n+2})$ exists. Let $\lim_{n\to\infty} d(\delta_{2n},\delta_{2n+2}) = \delta$. It is clear that $0 \leq \delta < d(\Delta,\Lambda)$. Assume that $\delta > 0$. By the right upper semicontinuity of $\psi$,

$$\begin{aligned}\delta &= \lim_{n\to\infty} d(\delta_{2n},\delta_{2n+2})\\ &\leq \lim_{n\to\infty} \psi(d(\delta_{2n-1},\delta_{2n+1}))\\ &\leq \lim_{n\to\infty} \psi(\psi(d(\delta_{2n-2},\delta_{2n})))\\ &\leq \psi(\psi(\delta)) < \psi(\delta) < \delta,\end{aligned}$$

so $\delta = 0$.

**Step 3**. $\{\delta_{2n}\}$ is a Cauchy sequence.

Assume that $\{\delta_{2n}\}$ is not Cauchy. Then there exist $\varepsilon > 0$ and integers $2m_k, 2n_k \in \mathbb{N}$ such that $2m_k > 2n_k \geq k$ and $d(\delta_{2n_k},\delta_{2m_k}) \geq \varepsilon$ for $k = 0,1,2,\cdots$. Also, choosing $m_k$ as small as possible, it may be assumed that

$$d(\delta_{2n_k},\delta_{2m_k-2}) < \varepsilon.$$

Hence, for each $k \in \mathbb{N}$, we have

$$\begin{aligned}\varepsilon \leq d(\delta_{2n_k},\delta_{2m_k}) &\leq d(\delta_{2n_k},\delta_{2m_k-2}) + d(\delta_{2m_k-2},\delta_{2m_k})\\ &\leq \varepsilon + d(\delta_{2m_k-2},\delta_{2m_k})\end{aligned}$$

and since $d(\delta_{2m_k-2},\delta_{2m_k}) \to 0$, hence $\lim_{k\to\infty} d(\delta_{2n_k},\delta_{2m_k}) = \varepsilon$. Observe that

$$\begin{aligned}d(\delta_{2n_k},\delta_{2m_k}) &\leq d(\delta_{2n_k},\delta_{2n_k+2}) + d(\delta_{2n_k+2},\delta_{2m_k+2}) + d(\delta_{2n_k+2},\delta_{2m_k})\\ &\leq d(\delta_{2n_k},\delta_{2n_k+2}) + \psi(d(\delta_{2n_k+1},\delta_{2m_k+1})) + d(\delta_{2n_k+2},\delta_{2m_k}).\end{aligned}$$

Letting $k \to \infty$, we obtain that

$$\varepsilon = \lim_{k\to\infty} d(\delta_{2n_k}, \delta_{2m_k}) \leq \lim_{k\to\infty} \psi(d(\delta_{2n_k+1}, \delta_{2m_k+1})).$$

On the other hand,

$$\lim_{k\to\infty} d(\delta_{2n_k+1}, \delta_{2m_k+1}) \leq \lim_{k\to\infty} \psi(d(\delta_{2n_k}, \delta_{2m_k})).$$

So, using the upper semicontinuity of $\psi$ from the right we have

$$\varepsilon \leq \psi(\psi(\varepsilon)) < \psi(\varepsilon) < \varepsilon$$

which is a contradiction. Hence, $\{\delta_{2n}\}$ is a Cauchy sequence in $\Delta$.

**Step 4**. Existence of best proximity pair.

Because $\{\delta_{2n}\}$ is Cauchy, $\Omega$ is complete and $\Delta$ is closed, $\lim_{n\to\infty} \delta_{2n} = \delta \in \Delta$. Now,

$$d(\Delta, \Lambda) \leq d(\delta, \delta_{2n-1}) \leq d(\delta, \delta_{2n}) + d(\delta_{2n}, \delta_{2n-1}).$$

Thus, by step 1 we infer that $d(\delta, \delta_{2n-1})$ converges to $d(\Delta, \Lambda)$. Since

$$d(\Delta, \Lambda) \leq d(\delta_{2n}, \Gamma\delta) \leq \psi(d(\delta_{2n-1}, \delta)),$$

by upper semicontinuity of $\psi$ we have

$$d(\Delta, \Lambda) \leq \lim_{n\to\infty} d(\delta_{2n}, \Gamma\delta) \leq \lim_{n\to\infty} \psi(d(\delta_{2n-1}, \delta)) \leq \psi(d(\Delta, \Lambda)) = d(\Delta, \Lambda).$$

So, $d(\delta, \Gamma\delta) = d(\Delta, \Lambda)$. ■
□

**Theorem 3.3.** *Let $\Delta$ and $\Lambda$ be two nonempty closed and convex subsets of a uniformly convex Banach space $\Omega$ such that diam$(\Delta) < d(\Delta, \Lambda)$. Suppose that $\Gamma : \Delta \cup \Lambda \to \Delta \cup \Lambda$ is a cyclic $\psi$-contraction. Then there exist a unique $\delta \in \Delta$ such that $\|\delta - \Gamma\delta\| = d(\Delta, \Lambda)$. Further, if $\delta_0 \in \Delta$ and $\delta_{n+1} = \Gamma\delta_n$, then $\{\delta_{2n}\}$ converges to the best proximity point.*

*Proof.* By Theorem 2.2 $P_\Gamma(\Delta, \Lambda) \neq \emptyset$. Suppose that $\delta, \lambda \in P_\Gamma(\Delta, \Lambda)$ such that $\delta \neq \lambda$. Since $\|\delta - \Gamma\delta\| = d(\Delta, \Lambda)$ and $\|\lambda - \Gamma\lambda\| = d(\Delta, \Lambda)$ where necessarily from uniformly convexity of $\Omega$, $\Gamma^2\delta = \delta$ and $\Gamma^2\lambda = \lambda$. Since $\delta \neq \lambda$, we have $d(\Delta, \Lambda) < \|\Gamma\delta - \lambda\|$ and so $\psi(\|\Gamma\delta - \lambda\|) < \|\Gamma\delta - \lambda\|$. Therefore, $\|\delta - \Gamma\lambda\| = \|\Gamma^2\delta - \Gamma\lambda\| \leq \psi(\|\Gamma\delta - \lambda\|) < \|\Gamma\delta - \lambda\|$. Similarly, $\|\Gamma\delta - \lambda\| < \|\delta - \Gamma\lambda\|$ which is a contradiction. Therefore, $\delta = \lambda$. Hence, the proof is completed. ■ □

**Exercise 3.4.** *Let $\Delta$ and $\Lambda$ be subsets of $\mathbb{R}^2$ defined by*

$$\Delta = \{(\delta, 0) : \delta \geq 1\}$$

*and*

$$\Lambda = \{(0, \lambda) : \lambda \geq 1\}.$$

*Suppose that*

$$\Gamma(\delta, \lambda) = (\sqrt{\lambda}, \sqrt{\delta})$$

*and*

$$\psi(t) = \begin{cases} \sqrt{t} & t < d(\Delta, \Lambda) \\ \sqrt{d(\Delta, \Lambda)t} & t > d(\Delta, \Lambda). \end{cases}$$

*Then $\Gamma$ is a cyclic $\psi$-contraction on $\Delta \cup \Lambda$ and $\|(0, 1) - \Gamma((1, 0))\| = d(\Delta, \Lambda)$.*

*Proof.* Here, $d(\Delta,\Lambda)=\sqrt{2}$. For $(\delta,0)\in\Delta$ and $(0,\lambda)\in\Lambda$ we have

$$\begin{aligned}
\|\Gamma(\delta,0)-\Gamma(0,\lambda)\| &= \|(0,\sqrt{\delta})-(\sqrt{\lambda},0)\| \\
&= \|(\sqrt{\lambda},\sqrt{\delta})\| \\
&= \sqrt{\delta+\lambda}\le\sqrt{\sqrt{2}\sqrt{\delta^2+\lambda^2}} \\
&\le \sqrt{d(\Delta,\Lambda)\|(\delta,0)-(0,\lambda)\|}=\psi(\|(\delta,0)-(0,\lambda)\|).
\end{aligned}$$

Then $\Gamma$ is a cyclic $\psi$-contraction on $\Delta\cup\Lambda$ and $\|(0,1)-\Gamma((1,0))\|=\sqrt{2}=d(\Delta,\Lambda)$. □

We now present a larger class of mappings called cyclic asymptotic contractions.

**Definition 3.5.** *Let $\Omega$ be a complete metric space and let $\Delta$ and $\Lambda$ be subsets of $\Omega$. A mapping $\Gamma:\Delta\cup\Lambda\to\Delta\cup\Lambda$ is a cyclic asymptotic $\psi$-contraction if:*

(*i*) $\Gamma(\Delta)\subset\Lambda, \Gamma(\Lambda)\subset\Delta$,
(*ii*) $d(\Gamma\delta,\Gamma\lambda)\le\psi(d(\delta,\lambda))$ *for all* $\delta,\lambda\in\Delta\cup\Lambda$. *where* $\psi:(0,d(\Delta,\Lambda))\cup(d(\Delta,\Lambda),\infty)\to(0,d(\Delta,\Lambda))\cup(d(\Delta,\Lambda),\infty)$

*is nondecreasing and*

$$\lim_{n\to\infty}\psi^n(t)=\begin{cases}0, & 0<t<d(\Delta,\Lambda),\\ d(\Delta,\Lambda), & d(\Delta,\Lambda)<t.\end{cases}\tag{3}$$

The following theorem demonstrates that asymptotic contractions possess unique fixed points. Also, in the following result the continuity constrain on cyclic $\psi$-contraction is substituted by 3.

**Theorem 3.6.** *Let $\Delta$ and $\Lambda$ be closed subsets of complete metric space $\Omega$ such that $diam(\Delta)<d(\Delta,\Lambda)$. Suppose that $\Gamma:\Delta\cup\Lambda\to\Delta\cup\Lambda$ be a cyclic asymptotic $\psi$-contraction. Then $P_\Gamma(\Delta,\Lambda)\neq\emptyset$. Further, if $\delta_0\in\Delta$ and $\delta_{n+1}=\Gamma\delta_n$, then $\{\delta_{2n}\}$ converges to the best proximity point.*

*Proof.* Fix $\delta_0\in\Omega$ and let $\delta_n=\Gamma_n\delta_0$ for all $n\in\mathbb{N}$. Note that $d(\Delta,\Lambda)\le\limsup_{n\to\infty}d(\delta_n,\delta_{n+1})\le\limsup_{n\to\infty}\psi^n(d(\delta_0,\delta_1))=d(\Delta,\Lambda)$. Hence,

$$\lim_{n\to\infty}d(\delta_n,\delta_{n+1})=d(\Delta,\Lambda).$$

On the other hand, $0\le\limsup_{n\to\infty}d(\delta_n,\delta_{n+2})\le\limsup_{n\to\infty}\psi^n(d(\delta_0,\delta_2))=0$. Hence,

$$\lim_{n\to\infty}d(\delta_n,\delta_{n+2})=0.$$

Because $\psi^n(t)\to0$ for $0<t<d(\Delta,\Lambda)$, $\psi(s)<s$ for any $s>0$. Since $\lim_{n\to\infty}d(\delta_n,\delta_{n+2})=0$, given $\varepsilon>0$, it is possible to choose $n$ such that

$$d(\delta_{2n},\delta_{2n+2})\le\varepsilon-\psi(\varepsilon).$$

Now, for $z\in\Lambda_\varepsilon[\delta_{2n}]=\{\delta\in\Delta:d(\delta,\delta_{2n})\le\varepsilon\}$, we have

$$\begin{aligned}
d(\Gamma z,\delta_{2n}) &\le d(\Gamma z,\Gamma\delta_{2n})+d(\Gamma\delta_{2n},\delta_{2n}) \\
&\le \psi(d(z,\delta_{2n-1}))+d(\delta_{2n+1},\delta_{2n}) \\
&\le \psi(\varepsilon)+(\varepsilon-\psi(\varepsilon))=\varepsilon.
\end{aligned}$$

Therefore, $\Gamma:\Lambda_\varepsilon[\delta_{2n}]\to\Lambda_\varepsilon[\delta_{2n}]$ and it follows that $d(\delta_{2m},\delta_{2n})\le\varepsilon$ for all $m\ge n$. Hence, $\{\delta_{2n}\}$ is a Cauchy sequence. The rest of the proof follows as in Theorem 3.2. ■ □

## References


[1] J. Anuradha and P. Veeramani, Proximal pointwise contraction. Topology Appl., 156(18)(2009) 2942-2948.
[2] D. W. Boyd and J. S. W. Wong, On nonlinear contractions, Proc. Amer. Math. Soc., 20(2) (1969), 458–464.
[3] J. Caballero, J Harjani and K Sadarangani, A best proximity point theorem for Geraghty-contractions, Fixed Point Theory Appl., 2012, 2012:231.
[4] A. A. Eldred, W. A. Kirk and P. Veeramani, Proximinal normal structure and relatively nonexpansive mappings, Studia Math., 171(3)(2005) 283-293.
[5] A. A. Eldred and P. Veeramani, Existence and convergence of best proximity points, J. Math. Anal. Appl., 323(2006), 1001-1006.
[6] M. R. Haddadi, *Best proximity point iteration for nonexpensive mapping in Banach spaces*, J. Nonlinear Sci. Appl. 6 (2013), 152-161
[7] M. R. Haddadi and S. M. Moshtaghioun, Some Results on the Best Proximity Pair, Abstract and Applied Analysis, 2011 (2011), ID 158430, 9 pages.
[8] H. Kumar Nashine, P. Kumam and C. Vetro, Best proximity point theorems for rational proximal contractions, Fixed Point Theory and Applications, (2013), 2013:95.
[9] A. P. Mihaela, Best proximity point theorem for weak cyclic Kannan contractions, Filomat, 25(1)(2011) 145–154.
[10] J. O. Olaleru, O. V. Olisama and M. Abbas, Coupled best proximity points for generalised Hardy–Rogers type cyclic ($\omega$)-contraction, Int. J. Math. Anal. Optim., 1, (2015) 33–54.
[11] V. O. Olisama, J. O. Olaleru and H. Olaoluwa, Quadruple best proximity points of Q-cyclic contraction pair in abstract metric spaces, Asian J. Math. Appl., 2015,(2015) Article ID 0206.
[12] S. Sadiq Basha, Best proximity points: global optimal approximate solution, J Global Optim., 49(1) (2011) 15-21.
[13] S. Sadiq Basha, Extensions of Banach contraction principle, Numer. Funct Anal. Optim., 31 (2010) 569- 576.
[14] S. Sadiq Basha, Best proximity point theorems in the frameworks of fairly and proximally complete spaces, J. Fixed Point Theory Appl., 19(3) (2017) 1939–1951.
[15] S. Sadiq Basha, Best Proximity Point Theorems for Some Special Proximal Contractions,Numerical Functional Analysis and Optimization, 40(10) (2019) 1–12.
[16] Raj V. Sankar and P. Veeramani, Best proximity pair theorems for relatively nonexpansive mappings, Appl Gen Topol., 10(1) (2009) 21–28.