



Compact ADI Method for Two-Dimensional Riesz Space Fractional Diffusion Equation

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Abstract. In this paper, a compact alternating direction implicit (ADI) method has been developed for solving two-dimensional Riesz space fractional diffusion equation. The precision of the discretization method used in spatial directions is twice the order of the corresponding fractional derivatives. It is proved that the proposed method is unconditionally stable via the matrix analysis method and the maximum error in achieving convergence is discussed. Numerical example is considered aiming to demonstrate the validity and applicability of the proposed technique.

1. Introduction

Fractional calculus is a natural extension of the integer order calculus [21, 25]. Recently, many problems in physics [17, 29], biology [14, 20], finance [6] and hydrology [2, 3, 10] have been formulated on fractional partial differential equations, containing derivatives of fractional order in space, time or both. Fractional derivatives play a key role in modelling particle transport in anomalous diffusion. The space fractional diffusion equation describes Lévy flights [3, 19]. The time fractional diffusion equation depicts traps, and the time-space fractional diffusion equation characterizes the competition between Lévy flights and traps [35]. The regularity criterion is important for the diffusion equations that are proposed in dynamic systems. Sadek et al. [27] established the Serrin-type regularity criteria for the 3D nematic liquid crystal flows in the terms of the multiplier space $X_r^{R^3}$. Sadek and Ragusa [28] studied the regularity criterion in terms of the homogeneous Besov space for the incompressible Boussinesq equations.

Numerical methods to different types of fractional diffusion models are increasingly appearing in the sciences. Some models can be mentioned, which are groundwater flow, the flow of heat transfer in furnaces, dissolving gases in liquids and fluid flow in a porous medium. In addition, the analytical solutions of such equations are usually difficult to obtain, so in order to gain their numerical solutions becomes more important and emergent. For one-dimensional problems, Cui [8] extended a compact finite difference method for the fractional diffusion equation with the Riemann-Liouville derivative via the Grünwald-Letnikov discretization. Alikhanov [1] constructed a widespread difference approximation

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of the Caputo fractional derivative for the time fractional diffusion equation with variable coefficients. Borhanifar and Valizadeh [4] considered Mittag-Leffler-Padé approximations for space and time fractional diffusion equations by using shifted Grünwald estimate in space, rational recurrence formula in time, and discussed their stabilities and truncation errors. Çelik and Duman [7] used the fractional centered difference that introduced by Ortigueira [23] to solve the Riesz fractional diffusion equation and also for this type equation, some of the authors employed the matrix stemming from the discretization of the Riesz space derivative by compact difference scheme and parameter spline function [38] and fractional centered difference formula [26]. For two-dimensional problems, Bu et al. [5] developed the Galerkin finite element method for the numerical study of the two-dimensional Riesz space fractional diffusion equations combined with a backward difference method. Tadjeran and Meerschaert [32] applied a mixed Crank-Nicolson-ADI method not only as a discretization, but also as a Richardson extrapolation to obtain a numerical solution of the two-dimensional space fractional diffusion equation and they examined of being unconditionally stable and second order accuracy of the method. H. Wang and K. Wang [33] investigated an $O(N \log^2 N)$ alternating-direction finite difference method for the two dimensional fractional diffusion equations and at the same time, Zhang and Sun [37] explored ADI schemes for the two-dimensional fractional sub-diffusion equation. Zeng et al. [36] derived approximate solution via Crank-Nicolson ADI spectral method for the two-dimensional Riesz space fractional nonlinear reaction diffusion equation. Gao and Sun [15] handled high order compact ADI schemes for the 2D time-fractional advection-diffusion equation. The numerical solutions of the two-dimensional Riesz space fractional diffusion equations have been challenging. The main purpose of this paper is to solve the two-dimensional Riesz space fractional diffusion equations using the compact difference scheme with the operator-splitting techniques, that is, using the compact ADI scheme.

Let Ω be a rectangular domain in R^2 with boundary $\Gamma = \partial\Omega$ and $J = (0, T]$ be the time interval, $T > 0$. In this paper, we consider the following two dimensional Riesz space fractional diffusion equation for a solute concentration u

$$\frac{\partial u(x, y, t)}{\partial t} = C_x \frac{\partial^\alpha u(x, y, t)}{\partial |x|^\alpha} + C_y \frac{\partial^\beta u(x, y, t)}{\partial |y|^\beta} + s(x, y, t), \quad (x, y, t) \in \Omega \times J, \quad (1)$$

$$u(x, y, 0) = f(x, y), \quad (x, y) \in \Omega, \quad (2)$$

$$u(x, y, t) = 0, \quad (x, y, t) \in \Gamma \times J, \quad (3)$$

where C_x and C_y are the average fluid velocities in the x- and y-directions. We restrict $1 < \alpha, \beta \leq 2$ and assume $C_x, C_y \geq 0$. The solution $u = u(x, y, t)$ is assumed to be sufficiently smooth and has the necessary continuous partial derivatives up to certain orders.

The outline of the paper is organized as follows. Preliminaries and basic definitions are presented in the next section. Section 3 is devoted to the construction and explanation of numerical algorithm that the Crank-Nicolson scheme and the alternating directions implicit method is combined together. In Section 4, the stability and the convergence order of the numerical scheme are theoretically analyzed. One example is given in Section 5 and some conclusions are drawn in Section 6.

2. Preliminaries and basic definitions

Definition 2.1. The Riesz fractional operator for $n - 1 < \gamma \leq n$ on a finite interval $a \leq x \leq b$ is defined as [16, 30]

$$\frac{\partial^\gamma v(x, t)}{\partial |x|^\gamma} = -\vartheta_\gamma ({}_a D_x^\gamma + {}_x D_b^\gamma) v(x, t), \quad (4)$$

where

$$\vartheta_\gamma = \frac{1}{2\cos(\frac{\pi\gamma}{2})}, \quad \gamma \neq 1,$$

$${}_a D_x^\gamma v(x, t) = \frac{1}{\Gamma(n - \gamma)} \frac{\partial^n}{\partial x^n} \int_a^x (x - \xi)^{n-\gamma-1} v(\xi, t) d\xi,$$

$${}_x D_b^\gamma v(x, t) = \frac{(-1)^n}{\Gamma(n - \gamma)} \frac{\partial^n}{\partial x^n} \int_x^b (\xi - x)^{n-\gamma-1} v(\xi, t) d\xi.$$

Lemma 2.2. For a function $h(x)$ defined on the infinite domain $-\infty < x < \infty$, the following equality holds:

$$-(-\Delta)^{\frac{\gamma}{2}} h(x) = -\frac{1}{2\cos(\frac{\pi\gamma}{2})} [{}_{-\infty} D_x^\gamma h(x) + {}_x D_\infty^\gamma h(x)] = \frac{\partial^\gamma}{\partial |x|^\gamma} h(x). \tag{5}$$

Proof. See Ref. [34] \square

Definition 2.3. [12] Let the Laplacian $(-\Delta)$ has a complete set of orthonormal eigenfunctions φ_n corresponding to eigenvalues λ_n^2 on a bounded region Ω with the homogeneous boundary conditions, then

$$(-\Delta)^{\frac{\gamma}{2}} f = \begin{cases} (-\Delta)^m f, & \gamma = 2m, \quad m = 0, 1, 2, \dots, \\ (-\Delta)^{\frac{\gamma}{2}-m} (-\Delta)^m f, & m - 1 < \frac{\gamma}{2} < m, \quad m = 1, 2, \dots, \\ \sum_{n=1}^\infty \lambda_n^\gamma \langle f, \varphi_n \rangle \varphi_n, & \gamma < 0. \end{cases}$$

Lemma 2.4. [22] The eigenvalues and eigenvectors of the following tridiagonal Toeplitz matrix

$$A = \begin{pmatrix} b & a & & & \\ c & b & a & & \\ & \ddots & \ddots & \ddots & \\ & & c & b & a \\ & & & c & b \end{pmatrix}_{n \times n}$$

are given by

$$\lambda_j = b + 2a \sqrt{c/a} \cos(j\pi/(n + 1)), \quad j = 1, 2, \dots, n, \tag{6}$$

while the corresponding eigenvectors are:

$$x_j = \begin{pmatrix} (c/a)^{1/2} \sin(1j\pi/(n + 1)) \\ (c/a)^{2/2} \sin(2j\pi/(n + 1)) \\ (c/a)^{3/2} \sin(3j\pi/(n + 1)) \\ \vdots \\ (c/a)^{n/2} \sin(nj\pi/(n + 1)) \end{pmatrix}, \quad j = 1, 2, \dots, n,$$

i.e., $Ax_j = \lambda_j x_j, j = 1, 2, \dots, n$. Moreover, the matrix A is diagonalizable and $P = (x_1 \ x_2 \ \dots \ x_n)$ diagonalizes A , i.e., $P^{-1}AP = D$, where $D = \text{diag}(\lambda_1 \ \lambda_2 \ \dots \ \lambda_n)$.

Definition 2.5. [18] Let $f, g : \mathbb{R} - \{0\} \rightarrow \mathbb{R}$ be real functions. We say $f = O(g)$ as $x \rightarrow 0$ if there are constants C and $r > 0$ such that

$$|f(x)| \leq C |g(x)| \quad \text{whenever} \quad 0 < |x| < r.$$

also following properties of asymptotic estimates are hold for "O" [13]:

$$O(f(x)) + O(f(x)) = O(f(x)),$$

$$O(f(x))O(g(x)) = O(f(x)g(x)).$$

Lemma 2.6. If $g(x)$ be a smooth function on \mathbb{R} that is discretized in a finite interval $[a, b]$ include n nodal points as $x_i = a + ih$ in which $h = \frac{b-a}{n}$ then $\frac{1}{h^2} \frac{\delta_x^2}{1 + \frac{\delta_x^2}{12}}$ operator approximates the second derivative of the $g(x)$ from the fourth order at inner nodal points of $[a, b]$.

Proof. According to being smooth the function $g(x)$, there is continuous function $f(x)$ that

$$g''(x) = \frac{d^2 g(x)}{dx^2} = f(x).$$

Alternatively, we can write, $g(x)$ is an exact solution of the above differential equation. To prove the lemma, needs to show the following relationship is confirmed in the internal nodes $x_i, i = 1, 2, \dots, n - 1$.

$$\frac{1}{h^2} \delta_x^2 g(x_i) - \left(1 + \frac{\delta_x^2}{12}\right) f(x_i) = O(h^4),$$

we apply the relevant operators on $g(x_i)$ and then $f(x_i)$

$$\frac{1}{h^2} \delta_x^2 g(x_i) = \frac{g(x_i + h) - 2g(x_i) + g(x_i - h)}{h^2}, \tag{7}$$

$$\left(1 + \frac{\delta_x^2}{12}\right) f(x_i) = \left(1 + \frac{\delta_x^2}{12}\right) g''(x_i) = g''(x_i) + \frac{g''(x_i + h) - 2g''(x_i) + g''(x_i - h)}{12}. \tag{8}$$

By substituting the Taylor series of the function $g(x_i + h)$ and $g(x_i - h)$ about $x = x_i$ in the formula (7) and using the average value theorem for derivatives, we have

$$\frac{1}{h^2} \delta_x^2 g(x_i) = g''(x_i) + \frac{g^{(4)}(x_i)}{12} h^2 + \frac{g^{(6)}(x_i)}{360} h^4 + \frac{g^{(8)}(\xi)}{20160} h^6, \quad x_i - h < \xi < x_i + h. \tag{9}$$

Similarly, $g''(x_i + h)$ and $g''(x_i - h)$ in the formula (8) replace with Taylor expansion theirs centered at $x = x_i$ and applying the average value theorem for derivatives, we have

$$\left(1 + \frac{\delta_x^2}{12}\right) f(x_i) = g''(x_i) + \frac{1}{12} [g^{(4)}(x_i) h^2 + \frac{g^{(6)}(x_i)}{12} h^4 + \frac{g^{(8)}(\zeta)}{360} h^6], \quad x_i - h < \zeta < x_i + h. \tag{10}$$

The result is following equation by subtracting formula (10) from formula (9), utilized the average value theorem

$$\frac{1}{h^2} \delta_x^2 g(x_i) - \left(1 + \frac{\delta_x^2}{12}\right) f(x_i) = -\frac{1}{240} g^{(6)}(x_i) h^4 - \frac{1}{60480} g^{(8)}(\eta) h^6, \quad x_i - h < \eta < x_i + h,$$

therefore

$$g''(x_i) = \frac{1}{h^2} \frac{\delta_x^2}{1 + \frac{\delta_x^2}{12}} g(x_i) + O(h^4).$$

□

3. Derivation of compact ADI scheme

In this section, we develop a compact ADI finite difference scheme for the problem (1)–(3). Let $h_x = \frac{R_1 - L_1}{M_1}$, $h_y = \frac{R_2 - L_2}{M_2}$, and $k_t = \frac{T}{N}$ be the spatial and temporal step sizes respectively, where M_1, M_2 and N are some given positive integers. Denote $x_i = L_1 + ih_x, y_j = L_2 + jh_y, t_n = nk_t$ for $i = 0, 1, \dots, M_1, j = 0, 1, \dots, M_2$ and

$n = 0, 1, \dots, N$. We let $u(x_i, y_j, t_n)$ be the exact solution of (1)–(3) at the mesh point (x_i, y_j, t_n) and $u_{i,j}^n$ represents the solution of an approximating difference scheme at the same mesh point.

Based on Lemma 2.2 the Riesz fractional derivative $\frac{\partial^\alpha}{\partial |x|^\alpha} h(x)$ and the fractional Laplacian operator $-(\Delta)^{\frac{\alpha}{2}} h(x)$ are equivalent. Thus the two-dimensional Riesz space fractional diffusion equation (1) is in the following form

$$\frac{\partial u(x, y, t)}{\partial t} = -[C_x(-\Delta_x)^{\frac{\alpha}{2}} + C_y(-\Delta_y)^{\frac{\beta}{2}}]u(x, y, t) + s(x, y, t). \tag{11}$$

The next stage is to translate each of fractional Riesz derivatives into their corresponding fractional operators at the point (x_i, y_j, t) . From (24) and (25) in Appendix A we have

$$((-\Delta_x)^{\frac{\alpha}{2}} u)_{i,j} \simeq \left(-\frac{1}{h_x^2} \frac{\delta_x^2}{1 + \frac{\delta_x^2}{12}}\right)^{\frac{\alpha}{2}} u_{i,j}, \tag{12}$$

and

$$((-\Delta_y)^{\frac{\beta}{2}} u)_{i,j} \simeq \left(-\frac{1}{h_y^2} \frac{\delta_y^2}{1 + \frac{\delta_y^2}{12}}\right)^{\frac{\beta}{2}} u_{i,j}. \tag{13}$$

Substituting (12)-(13) into (11) yields

$$\frac{\partial u_{i,j}^n}{\partial t} = -[C_x \mathcal{D}_{\alpha,x} + C_y \mathcal{D}_{\beta,y}]u_{i,j}^n + s_{i,j}^n, \tag{14}$$

in which

$$\left(-\frac{1}{h_x^2} \frac{\delta_x^2}{1 + \frac{\delta_x^2}{12}}\right)^{\frac{\alpha}{2}} = \mathcal{D}_{\alpha,x}, \quad \left(-\frac{1}{h_y^2} \frac{\delta_y^2}{1 + \frac{\delta_y^2}{12}}\right)^{\frac{\beta}{2}} = \mathcal{D}_{\beta,y}.$$

Finally, temporal discretization by Crank-Nicolson method for (14) results in

$$\frac{u_{i,j}^{n+1} - u_{i,j}^n}{k_t} = -[C_x \mathcal{D}_{\alpha,x} + C_y \mathcal{D}_{\beta,y}] \frac{u_{i,j}^n + u_{i,j}^{n+1}}{2} + \frac{s_{i,j}^n + s_{i,j}^{n+1}}{2}. \tag{15}$$

After rearrangement and multiplying (15) by k_t , we have

$$\left[1 + \frac{k_t}{2}(C_x \mathcal{D}_{\alpha,x} + C_y \mathcal{D}_{\beta,y})\right]u_{i,j}^{n+1} = \left[1 - \frac{k_t}{2}(C_x \mathcal{D}_{\alpha,x} + C_y \mathcal{D}_{\beta,y})\right]u_{i,j}^n + \frac{k_t}{2}(s_{i,j}^n + s_{i,j}^{n+1}). \tag{16}$$

We note that the compact finite difference method (16) can be rewritten as the following directional splitting factorization form [9]

$$\left[1 + \frac{k_t}{2}C_x \mathcal{D}_{\alpha,x}\right]\left[1 + \frac{k_t}{2}C_y \mathcal{D}_{\beta,y}\right]u_{i,j}^{n+1} = \left[1 - \frac{k_t}{2}C_x \mathcal{D}_{\alpha,x}\right]\left[1 - \frac{k_t}{2}C_y \mathcal{D}_{\beta,y}\right]u_{i,j}^n + \frac{k_t}{2}(s_{i,j}^n + s_{i,j}^{n+1}), \tag{17}$$

which introduces an additional perturbation error equal to $\frac{k_t}{4} \mathcal{D}_{\alpha,x} \mathcal{D}_{\beta,y} (u_{i,j}^{n+1} - u_{i,j}^n)$.

The additional term is of higher order and do not affect the accuracy of the scheme. In order to simplify the computation, we may re-write the scheme (17) in the Peaceman-Rachford ADI form [24] as

$$\left[1 + \frac{k_t}{2}C_x \mathcal{D}_{\alpha,x}\right]u_{i,j}^* = \left[1 - \frac{k_t}{2}C_y \mathcal{D}_{\beta,y}\right]u_{i,j}^n + \frac{k_t}{2}(s_{i,j}^n + s_{i,j}^{n+1}), \tag{18}$$

$$\left[1 + \frac{k_t}{2}C_y \mathcal{D}_{\beta,y}\right]u_{i,j}^{n+1} = \left[1 - \frac{k_t}{2}C_x \mathcal{D}_{\alpha,x}\right]u_{i,j}^* + \frac{k_t}{2}(s_{i,j}^n + s_{i,j}^{n+1}), \tag{19}$$

where $u_{i,j}^*$ is an intermediate value.

The corresponding algorithm is employed as follows:

(1) First solve on each fixed horizontal slice $y = y_k$ ($k = 1, 2, \dots, M_2 - 1$), a set of $M_1 - 1$ equations at the points x_i , $i = 1, 2, \dots, M_1 - 1$ defined by (18) to obtain the middle solution slice $u_{i,k}^*$.

(2) Next alternating the spatial direction, and for each $x = x_k$ ($k = 1, 2, \dots, M_1 - 1$) solving a set of $M_2 - 1$ equations defined by (19) at the points y_j , $j = 1, 2, \dots, M_2 - 1$, to get $u_{i,j}^{n+1}$.

4. Stability and convergence analysis

In this section, we prove consistency and stability for the compact difference scheme (17).

Theorem 4.1. *The compact difference scheme (17) is unconditionally stable.*

Proof. To prove the stability of the difference scheme (17), we examine the matrix $(I + S_x)^{-1}(I - S_x) \otimes (I + T_y)^{-1}(I - T_y)$ that stands as the tensor operator in formula (17).

Appendices A and B, show that the eigenvalues of matrices S_x and T_y are positive. Therefore all the eigenvalues of the matrices $(I + S_x)$ and $(I + T_y)$ are greater than one, and thus this matrices are invertible. Positivity of eigenvalues of the matrix S_x and T_y result that every eigenvalue of the matrices of $(I + S_x)^{-1}(I - S_x)$ and $(I + T_y)^{-1}(I - T_y)$ have the modulus less than one. Therefore, the spectral radius of the matrices $(I + S_x)^{-1}(I - S_x)$ and $(I + T_y)^{-1}(I - T_y)$ are less than one. $(I + S_x)^{-1}(I - S_x)$ and $(I + T_y)^{-1}(I - T_y)$ are real and symmetric due to symmetry of S_x and T_y (see Appendix B). So the norm of the matrices of $(I + S_x)^{-1}(I - S_x)$ and $(I + T_y)^{-1}(I - T_y)$ are less than one.

Hence, the difference scheme (17) is unconditionally stable. \square

Theorem 4.2. *The truncation error of the difference scheme (17) is $O(h_x^{2\alpha}) + O(h_y^{2\beta}) + O(k_t^2)$.*

Proof. Let $u(x_i, y_j, t_n)$ be the exact solution of (1)–(3) and $u_{i,j}^n$ be the solution of the numerically recurrence scheme (17). First, we derive the principal error term associated with discretization of the Riesz fractional derivative operators. we note that by considering the arbitrary order γ and variable z , based on the multiplication property of the order "O", (see Definition 2.5) we have the following relation

$$(O(h_z^4))^{\frac{\gamma}{2}} = O((h_z^4)^{\frac{\gamma}{2}}) = O(h_z^{2\gamma}).$$

By applying the Lemma 2.2 on the Eqs. (24), (25), (see Appendix A) and smoothness of the exact solution u , we have

$$\left(\frac{\partial^\alpha u}{\partial |x|^\alpha}\right)_{i,j} = -\mathcal{D}_{\alpha,x} u_{i,j} + O(h_x^{2\alpha}),$$

and

$$\left(\frac{\partial^\beta u}{\partial |y|^\beta}\right)_{i,j} = -\mathcal{D}_{\beta,y} u_{i,j} + O(h_y^{2\beta}).$$

Second, we discuss the local truncation error for scheme (17). We use the two-dimensional case of (12)–(13) and Crank-Nicolson scheme to do the discretization in space and time directions, respectively. Substitution into the expression for (1) yields

$$\left[1 + \frac{k_t}{2} C_x \mathcal{D}_{\alpha,x} + \frac{k_t}{2} C_y \mathcal{D}_{\beta,y}\right] u(x_i, y_j, t_{n+1}) = \left[1 - \frac{k_t}{2} C_x \mathcal{D}_{\alpha,x} - \frac{k_t}{2} C_y \mathcal{D}_{\beta,y}\right] u(x_i, y_j, t_n) + R_{i,j}^{n+1} \quad (20)$$

where

$$|R_{i,j}^{n+1}| \leq c k_t (O(h_x^{2\alpha}) + O(h_y^{2\beta}) + O(k_t^2)). \quad (21)$$

Finally, we give the global discretization error for numerically approximated scheme (17). Taking $e_{i,j}^n = u(x_i, y_j, t_n) - u_{i,j}^n$, and subtracting (17) from (20), yields

$$(I + S_x)(I + T_y)e^{n+1} = (I - S_x)(I - T_y)e^n + R^{n+1} \tag{22}$$

where S_x and T_y are defined in (26) and (27) of Appendix B, respectively, and

$$e^n = [e_{1,1}^n, e_{2,1}^n, \dots, e_{M_1-1,1}^n, e_{1,2}^n, e_{2,2}^n, \dots, e_{M_1-1,2}^n, \dots, e_{1,M_2-1}^n, e_{2,M_2-1}^n, \dots, e_{M_1-1,M_2-1}^n]^T,$$

$$R^n = [R_{1,1}^n, R_{2,1}^n, \dots, R_{M_1-1,1}^n, R_{1,2}^n, R_{2,2}^n, \dots, R_{M_1-1,2}^n, \dots, R_{1,M_2-1}^n, R_{2,M_2-1}^n, \dots, R_{M_1-1,M_2-1}^n]^T.$$

Now from (21) one can write

$$|R_{i,j}^{n+1}| \leq ck_t(\mathcal{O}(h_x^{2\alpha}) + \mathcal{O}(h_y^{2\beta}) + \mathcal{O}(k_t^2)). \tag{23}$$

Since S_x and T_y commute, then from (22)

$$e^{n+1} = (I + S_x)^{-1}(I - S_x)(I + T_y)^{-1}(I - T_y)e^n + (I + S_x)^{-1}(I + T_y)^{-1}R^{n+1}.$$

With taking the 2-norm on both sides of the above relation, we have

$$\|e^{n+1}\| \leq \|e^n\| + \|R^{n+1}\|.$$

Since from Theorem 4.1 one can write

$$\|(I + S_x)^{-1}(I - S_x)(I + T_y)^{-1}(I - T_y)\| \leq \|(I + S_x)^{-1}(I - S_x)\| \cdot \|(I + T_y)^{-1}(I - T_y)\| \leq 1$$

and

$$\|(I + S_x)^{-1}(I + T_y)^{-1}\| \leq \|(I + S_x)^{-1}\| \cdot \|(I + T_y)^{-1}\| \leq 1.$$

We use mathematical induction to create the relation between error in final step and errors created in earlier steps, i.e.,

$$\|e^{n+1}\| \leq \|e^n\| + \|R^{n+1}\| \leq \|e^{n-1}\| + \|R^n\| + \|R^{n+1}\|.$$

Since $\|e^0\| = \|u(x_i, y_j, t_0) - u_{i,j}^0\|$ from (23) we conclude that

$$\|e^n\| \leq \sum_{k=1}^n \|R^k\| \leq C(\mathcal{O}(h_x^{2\alpha}) + \mathcal{O}(h_y^{2\beta}) + \mathcal{O}(k_t^2)),$$

where $C = nck_t$.

It is shown that the solution to (1)–(3) can be approximated by numerical scheme (17) with the discretization error $\mathcal{O}(h_x^{2\alpha}) + \mathcal{O}(h_y^{2\beta}) + \mathcal{O}(k_t^2)$. \square

By Theorems 4.1 and 4.2 and Lax’s equivalence theorem [31], the scheme (17) is convergent.

5. Numerical experiments

In this section, we will present an example of two dimensional Riesz space fractional diffusion equations. We shall compare the numerical solutions with the exact solutions. To demonstrate the accuracy of preferred method, we have computed not only maximum errors, but also estimated convergence rates separately in

spatial and temporal directions.

The maximum absolute errors between the exact and the numerical solutions

$$E_{\infty}(h, k_t) = \max_{i,j} |u(x_i, y_j, t_N) - u_{i,j}^N|,$$

are measured in our examples. Furthermore, the spatial convergence order, denoted by

$$\text{Convergence Rate1} = \log_2(E_{\infty}(2h, k_t)/E_{\infty}(h, k_t)),$$

for sufficiently small k_t , and the temporal convergence order, denoted by

$$\text{Convergence Rate2} = \log_2(E_{\infty}(h, 2k_t)/E_{\infty}(h, k_t)),$$

when h is sufficiently small, are reporting. The numerical results given by these examples justify our theoretical results.

Example 5.1. We consider the following two dimensional Riesz space fractional diffusion equation with the initial and homogeneous Dirichlet boundary conditions:

$$\frac{\partial u(x, y, t)}{\partial t} = C_x \frac{\partial^{\alpha} u(x, y, t)}{\partial |x|^{\alpha}} + C_y \frac{\partial^{\beta} u(x, y, t)}{\partial |y|^{\beta}} + s(x, y, t), \quad 0 < t < 2, \quad 0 < x, y < \pi,$$

$$u(x, y, 0) = x^2 y^2 (\pi - x)(\pi - y), \quad 0 \leq x, y \leq \pi,$$

$$u(0, y, t) = u(\pi, y, t) = u(x, 0, t) = u(x, \pi, t) = 0, \quad 0 \leq t \leq 2, \quad 0 \leq x, y \leq \pi,$$

with source function

$$s(x, y, t) = \frac{C_x y^2 (\pi - y) e^{-t}}{2 \cos(\frac{\pi \alpha}{2})} \Theta(x, \alpha) + \frac{C_y x^2 (\pi - x) e^{-t}}{2 \cos(\frac{\pi \beta}{2})} \Theta(y, \beta) - x^2 y^2 (\pi - x)(\pi - y) e^{-t}$$

where $\Theta(z, \gamma) = \frac{2\pi z^{2-\gamma}}{\Gamma(3-\gamma)} - \frac{6z^{3-\gamma}}{\Gamma(4-\gamma)} + \frac{\pi^2(\pi-z)^{-\gamma}}{\Gamma(1-\gamma)} - \frac{2\pi(\pi-z)^{1-\gamma}}{\Gamma(2-\gamma)} + \frac{2(\pi-z)^{2-\gamma}}{\Gamma(3-\gamma)}$ and $C_x = C_y = 0.25$. The corresponding exact solution is $u(x, y, t) = x^2 y^2 (\pi - x)(\pi - y) e^{-t}$.

The table 1 shows maximum absolute errors and related estimated convergence rates with different values for $h_x = h_y$ as $0.1\pi, 0.05\pi, 0.025\pi, 0.0125\pi$ and 0.00625π , fixed value $k_t = 0.001$ whereas Table 2 presents them with different values for k_t as $0.1, 0.05, 0.025, 0.0125$ and 0.00625 and fixed value $h_x = h_y = 0.001\pi$. Whose fractional derivative orders $\alpha = 1.8, \beta = 1.6$ and $\alpha = 1.8, \beta = 1.8$ are considered separately in two tables. From Tables 1 and 2, we find the experimental convergence orders are approximately twice the smallest fractional derivative and two in spatial and temporal directions, respectively. The numerical Example results are provided to show that the proposed approximation method is computationally efficient.

6. Conclusions

In the present work, a high order compact ADI method for solving the two dimensional Riesz space fractional diffusion equation has been established. The method is spatially twice the smallest fractional derivative- and temporally second-order accuracy. It is shown through a matrix analysis that it is unconditionally stable. Numerical results are provided to verify the accuracy and efficiency of the preferred method.

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Table 1: The maximum errors and convergence rates for the compact ADI method for solving 2D Riesz space FDE with halved spatial step sizes and $k_t = 0.001$

$h_x = h_y$	Max Error $\alpha = 1.8, \beta = 1.6$	Convergence Rate	Max Error $\alpha = \beta = 1.8$	Convergence Rate
0.10000 π	7.36901e – 003		5.17481e – 003	
0.05000 π	9.30631e – 004	2.98519	4.92617e – 004	3.39297
0.02500 π	1.10146e – 004	3.07879	4.45651e – 005	3.46648
0.01250 π	1.24084e – 005	3.15003	3.85100e – 006	3.53261
0.00625 π	1.35755e – 006	3.19224	3.21297e – 007	3.58325

Table 2: The maximum errors and convergence rates for the compact ADI method for solving 2D Riesz space FDE with halved temporal step sizes and $h_x = h_y = 0.001\pi$

k_t	Max Error $\alpha = 1.8, \beta = 1.6$	Convergence Rate	Max Error $\alpha = \beta = 1.8$	Convergence Rate
0.10000	8.53972e – 003		6.78334e – 003	
0.05000	2.50889e – 003	1.76714	2.04295e – 003	1.73134
0.02500	6.85153e – 004	1.87255	5.74989e – 004	1.82905
0.01250	1.77936e – 004	1.94507	1.53015e – 004	1.90986
0.00625	4.34099e – 005	2.03526	3.89268e – 005	1.97484

Appendix A

We consider the fourth-order compact approximations for the second-order derivative operators based on Lemma 2.6 (also see [11])

$$\left(\frac{\partial^2 u}{\partial x^2}\right)_{i,j} = \frac{1}{h_x^2} \frac{\delta_x^2}{1 + \frac{\delta_x^2}{12}} u_{i,j} + \mathcal{O}(h_x^4), \quad \text{for } i = 0, 1, \dots, M_1 \text{ and fix } j \tag{24}$$

$$\left(\frac{\partial^2 u}{\partial y^2}\right)_{i,j} = \frac{1}{h_y^2} \frac{\delta_y^2}{1 + \frac{\delta_y^2}{12}} u_{i,j} + \mathcal{O}(h_y^4), \quad \text{for } j = 0, 1, \dots, M_2 \text{ and fix } i \tag{25}$$

where δ_x^2 and δ_y^2 are the standard second-order central difference operators in x- and y- directory respectively. If the boundary values at $i = 0$ and $i = M_1, j > 0$, are known, these $(M_1 - 1)$ equation for $i = 1, 2, \dots, M_1 - 1$ can be written in matrix form

$$\left(\left(\frac{1}{h_x^2} \frac{\delta_x^2}{1 + \frac{\delta_x^2}{12}}\right)_{i,j}\right) = A_x^{-1} B_x, \quad i, j = 1, 2, \dots, M_1 - 1,$$

where $A_x = \frac{h_x^2}{12} \text{diag}(1, 10, 1)$ and $B_x = \text{diag}(1, -2, 1)$ are tridiagonal matrices of $M_1 - 1$ order. And if the boundary values at $j = 0$ and $j = M_2, i > 0$, are known, these $(M_2 - 1)$ equation for $j = 1, 2, \dots, M_2 - 1$ can be written in matrix form

$$\left(\left(\frac{1}{h_y^2} \frac{\delta_y^2}{1 + \frac{\delta_y^2}{12}}\right)_{i,j}\right) = A_y^{-1} B_y, \quad i, j = 1, 2, \dots, M_2 - 1,$$

where $A_y = \frac{h_y^2}{12} \text{diag}(1, 10, 1)$ and $B_y = \text{diag}(1, -2, 1)$ are tridiagonal matrices of $M_2 - 1$ order.

Referring to the Lemma 2.4 our achievement on that the eigenvalues of the matrix of the $\frac{-1}{h_x^2} \frac{\delta_x^2}{1 + \frac{\delta_x^2}{12}}$ operator is as follows:

$$\begin{aligned} \lambda_j &= \left[\frac{h_x^2}{12} (10 + 2 \cos(j\pi/M_1)) \right]^{-1} \times (-1) \times [-2 + 2 \cos(j\pi/M_1)] \\ &= \frac{12}{h_x^2} \frac{2 - 2 \cos(j\pi/M_1)}{12 - 2 + 2 \cos(j\pi/M_1)} \\ &= \frac{12}{h_x^2} \frac{4 \sin^2(j\pi/2M_1)}{12 - 4 \sin^2(j\pi/2M_1)} \\ &= \frac{12}{h_x^2} \sin^2(j\pi/2M_1) (3 - \sin^2(j\pi/2M_1))^{-1} \end{aligned}$$

Since the eigenvalues of matrix $(\frac{-1}{h_x^2} \frac{\delta_x^2}{1 + \frac{\delta_x^2}{12}})$ are distinct positive. So there is a pair of matrices D_x and P that D_x is a diagonal matrix which members are eigenvalues of matrix $-A_x^{-1}B_x$ and the columns of the matrix P are eigenvectors corresponding to the these eigenvalues and we have

$$-A_x^{-1}B_x = PD_xP^{-1}$$

And similarly, in the direction of the second axis, there are pair matrices D_y and Q which have the following relation

$$-A_y^{-1}B_y = QD_yQ^{-1}.$$

As respects the eigenvalues of the matrices $-A_x^{-1}B_x$ and $-A_y^{-1}B_y$ are positive and distinct, and the matrices A_x, B_x, A_y and B_y are all symmetric, so the matrices $-A_x^{-1}B_x$ and $-A_y^{-1}B_y$ are symmetric positive definite.

Appendix B

In this section, the matrix form of the operators $\frac{k_t}{2}C_x\mathcal{D}_{\alpha,x}$ and $\frac{k_t}{2}C_y\mathcal{D}_{\beta,y}$, which is displayed by S_x and T_y respectively, is represented.

$$S_x = m \left\{ \frac{k_t}{2} C_x \mathcal{D}_{\alpha,x} \right\} = \frac{k_t}{2} C_x P D_x^{\frac{\alpha}{2}} P^{-1} \tag{26}$$

where $D_x = \text{diag}(\lambda_1 \ \lambda_2 \ \dots \ \lambda_{M_1-1})$ and $P = (x_1 \ x_2 \ \dots \ x_{M_1-1})$ in which

$$x_i = \begin{pmatrix} \sin(i\pi/M_1) \\ \sin(2i\pi/M_1) \\ \sin(3i\pi/M_1) \\ \vdots \\ \sin((M_1 - 1)i\pi/M_1) \end{pmatrix}, \quad \lambda_i = \frac{12 \sin^2(i\pi/2M_1)}{h_x^2 (3 - \sin^2(i\pi/2M_1))}, \quad i = 1, 2, \dots, M_1 - 1,$$

and similarly

$$T_y = m \left\{ \frac{k_t}{2} C_y \mathcal{D}_{\beta,y} \right\} = \frac{k_t}{2} C_y Q D_y^{\frac{\beta}{2}} Q^{-1} \tag{27}$$

where $D_y = \text{diag}(\gamma_1 \ \gamma_2 \ \dots \ \gamma_{M_2-1})$ and $Q = (y_1 \ y_2 \ \dots \ y_{M_2-1})$ in which

$$y_j = \begin{pmatrix} \sin(j\pi/M_2) \\ \sin(2j\pi/M_2) \\ \sin(3j\pi/M_2) \\ \vdots \\ \sin((M_2 - 1)j\pi/M_2) \end{pmatrix}, \quad \gamma_j = \frac{12 \sin^2(j\pi/2M_2)}{h_y^2 (3 - \sin^2(j\pi/2M_2))}, \quad j = 1, 2, \dots, M_2 - 1.$$

By attention to the positivity of the eigenvalues of the matrices $-A_x^{-1}B_x$ and $-A_y^{-1}B_y$ for every orders $M_1 - 1$ and $M_2 - 1$ respectively, eigenvalues of matrices of S_x and T_y are positive and we have from Appendix A that the two matrices $-A_x^{-1}B_x$ and $-A_y^{-1}B_y$ are real and symmetric therefore the two matrices S_x and T_y are real and symmetric. Moreover, note that the two matrices S_x and T_y commute, i.e.

$$S_x \otimes T_y = T_y \otimes S_x.$$

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