Filomat 35:5 (2021), 1489–1494
https://doi.org/10.2298/FIL2105489D

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Regular Methods of Summability and the Banach-Saks Property for Double Sequences

Raluca Dumitru[a], Jose A. Franco[a], Richard F. Patterson[a]

[a]*Department of Mathematics and Statistics*
University of North Florida
Jacksonville, FL 32224

Abstract. A Banach space B is said to satisfy the Banach-Saks property with respect to a regular summability method if every bounded subsequence has a summable subsequence. We show that if a Banach space satisfies the Banach-Saks property with respect to a Robison-Hamilton regular summability method, for every bounded double sequence there exists a β-subsequence whose subsequences are all summable to the same limit.

1. Introduction

A Banach space B is said to have the Banach-Saks property with respect to a regular summabiliy method $\langle a_{i,j}\rangle_{i,j}$ if for every bounded sequence, there exists a summable subsequence. Erdös and Magidor showed that if the Banach space B has the Banach-Saks property with respect to a summabiliy method $\langle a_{i,j}\rangle$ then every bounded sequence has a summable subsequence such that every subsequence of the subsequence is also $\langle a_{i,j}\rangle$-summable [2]. In this short note, we take advantage of a new type of subsequence of a double sequence recently introduced by Dumitru and Franco [1] to generalize the result of Erdös and Magidor to double sequences and Robison-Hamilton regular summability methods.

1.1. *Definitions and Notation*

In [1], a new type of double subsequence of a double sequence was introduced. Let $\psi : \mathbb{N}\times\mathbb{N} \to \mathbb{N}$ be defined recursively in the following way

$$\psi(1, n) = (n-1)^2 + 1,$$
$$\psi(m, 1) = m^2,$$
$$\psi(m, n) = \begin{cases} \psi(m-1, n) + 1 & \text{if } 1 < m \le n, \\ \psi(m, n-1) - 1 & \text{if } 1 < n < m. \end{cases}$$

2010 *Mathematics Subject Classification*. Primary 40B05; Secondary 40A05
Keywords. Banach-Saks; Double Sequences; β-subsequences.
Received: 17 April 2020; Revised: 13 July 2020; Accepted: 15 July 2020
Communicated by Eberhard Malkowsky
Email addresses: raluca.dumitru@unf.edu (Raluca Dumitru), jose.franco@unf.edu (Jose A. Franco), rpatters@unf.edu (Richard F. Patterson)

In matrix form, this looks like the following,

$$\begin{pmatrix} \psi(1,1) & \psi(1,2) & \psi(1,3) & \psi(1,4) & \cdots \\ \psi(2,1) & \psi(2,2) & \psi(2,3) & \psi(2,4) & \cdots \\ \psi(3,1) & \psi(3,2) & \psi(3,3) & \psi(3,4) & \cdots \\ \psi(4,1) & \psi(4,2) & \psi(4,3) & \psi(4,4) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} = \begin{pmatrix} 1 & 2 & 5 & 10 & \cdots \\ 4 & 3 & 6 & 11 & \cdots \\ 9 & 8 & 7 & 12 & \cdots \\ 16 & 15 & 14 & 13 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Then, define a *β-section* $S_\beta \subseteq \mathbb{N} \times \mathbb{N}$ by

$$S_\beta := \left\{ (m,n) \in \mathbb{N} \times \mathbb{N} \;\middle|\; \frac{1}{\beta} \le \frac{m}{n} \le \beta \right\}.$$

Definition 1.1 (β-subsequence [1]). *Let $x = [x_{k,l}]$ be a double sequence and let $\beta > 1$ be an extended real. The double sequence $y^{(\pi,\beta)}$ is called a* β-subsequence *of the double sequence x if and only if there exists a strictly increasing function $\pi : \psi(S_\beta) \to \psi(S_\beta)$ such that*

$$y^{(\pi,\beta)}_{p,q} = \begin{cases} z_{\psi(p,q)}, & \text{if } \frac{1}{\beta} > \frac{p}{q} \text{ or } \frac{p}{q} > \beta \\ z_{\pi(\psi(p,q))}, & \text{if } \frac{1}{\beta} \le \frac{p}{q} \le \beta \end{cases}$$

where $z_i = x_{\psi^{-1}(i)}$. If $\beta = +\infty$, the inequalities are understood in the limit sense.

Definition 1.2 (Summability Method [6]). *Let A be a four dimensional summability method that maps the complex double sequences x into the double sequence Ax where the m,n-th term of Ax is given by*

$$(Ax)_{m,n} = \sum_{k,l=1}^{\infty} a_{m,n,k,l} x_{k,l}.$$

Definition 1.3 (P-convergence [5]). *A double sequence $x = [x_{k,l}]$ has a* Pringsheim limit *L if and only if for every $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that*

$$|x_{k,l} - L| < \epsilon,$$

whenever $k,l > N$. In this case, we say x is P-convergent *and we denote it by*

$$L = \lim_{k,l\to\infty} x_{k,l}.$$

Unless otherwise specified, the notation $\lim_{k,l\to\infty}$ is reserved in this article to limits in the Pringsheim sense.

Definition 1.4 (RH-regular [6]). *Let A be a four dimensional matrix. A is said to be* RH-regular *if it maps every bounded P-convergent sequence into a P-convergent sequence with the same P-limit.*

Hamilton and Robison provide a characterization of RH-regularity that will be useful for the rest of the article.

Theorem 1.5 (Hamilton [4], Robison [6]). *A 4-dimensional matrix A is RH-regular if and only if*

(RH1) $\lim_{m,n\to\infty} a_{m,n,k,l} = 0$ *for each* $(k,l) \in \mathbb{N}^2$;

(RH2) $\lim_{m,n\to\infty} \sum_{k,l=0}^{\infty,\infty} a_{m,n,k,l} = 1$;

(RH3) $\lim_{m,n\to\infty} \sum_{k=0}^{\infty} |a_{m,n,k,l}| = 0$, *for each* $l \in \mathbb{N}$;

(RH4) $\lim_{m,n\to\infty} \sum_{l=0}^{\infty} |a_{m,n,k,l}| = 0$, *for each* $k \in \mathbb{N}$;

(RH5) $\lim_{m,n\to\infty} \sum_{k,l=0}^{\infty,\infty} |a_{m,n,k,l}|$ *is P-convergent;*

(RH6) *there exist finite positive integers A and B such that*

$$\sum_{\substack{k>B \\ l>B}} |a_{m,n,k,l}| < A$$

for each $(m,n) \in \mathbb{N}^2$.

In order to keep our notation consistent to [3] and [2], we introduce the following definitions.

Definition 1.6. *Let S be a set and* κ *a cardinal. Then,*

1. $2^S := \{X \mid X \subseteq S\}$ *and*
2. $[S]^\kappa = \{X \subseteq S \mid |X| = \kappa\}$.

Let ω denote the set of natural numbers and let $P(\omega)$ denote the set of all infinite subsets of ω.

Definition 1.7. *A subset S of* 2^ω *is* Ramsey *if and only if there exists* $M \in [\omega]^{|\omega|}$ *such that either* $[M]^{|\omega|} \subseteq S$ *or* $[M]^{|\omega|} \subseteq 2^\omega \setminus S$.

In other words, an infinite subset S of 2^ω is *Ramsey* if and only if there exists an infinite subset of the natural numbers M such that every infinite subset of M belongs to S or every infinite subset of M does not belong to S. Lastly, in the proof of the following theorem we use the concept of a Borel set. Therefore, we remind the reader of this definition.

Definition 1.8 (Borel Sets)**.** *Let X be a topological space. The* Borel σ-algebra *of X is the smallest* σ*-algebra that contains all open sets of X. Elements of the Borel* σ*-algebra are called* Borel sets.

We remark that all Borel sets in $P(\omega)$ are Ramsey sets [3].

2. Main Theorem

Theorem 2.1. *Let* $\langle e_{i,j}\rangle_{i,j\in\mathbb{N}}$ *be a bounded double sequence of elements in a Banach space B and* $\langle a_{i,j,k,l}\rangle_{i,j\in\mathbb{N}}$ *a RH-regular summability method. Then, there exists a* β*-subsequence* $\langle e_{i_\gamma,j_\delta}\rangle_{\gamma,\delta\in\mathbb{N}}$ *such that:*

1. *every* β*-subsequence of* $\langle e_{i_\gamma,j_\delta}\rangle_{\gamma,\delta\in\mathbb{N}}$ *is summable with respect to* $\langle a_{i,j,k,l}\rangle_{i,j,k,l\in\mathbb{N}}$*, where they all are summed to the same limit; or*
2. *no* β*-subsequence of* $\langle e_{i_\gamma,j_\delta}\rangle_{\gamma,\delta\in\mathbb{N}}$ *is summable with respect to* $\langle a_{i,j,k,l}\rangle_{i,j,k,l\in\mathbb{N}}$.

Proof. The proof is adapted from [2]. As in [2], we consider the topology on $P(\omega)$ generated by the subbasis $\{A_n\}_{n\in\omega} \cup \{B_n\}_{n\in\omega}$, where

$$A_n = \{X \in P(\omega) \mid n \notin X\}, \quad B_n = \{X \in P(\omega) \mid n \in X\}.$$

There exists a unique bijective and increasing map $\tau : \psi(S_\beta) \to \mathbb{N}$ (see Figure 1). We impose the topology on $P(\psi(S_\beta))$ induced by this map and the topology on $P(\omega)$.

Consider a set $X \in P(\psi(S_\beta))$. It is clear that there exists a unique bijective and monotonically increasing function from $\psi(S_\beta)$ to X. Denote this function by $\pi_X : \psi(S_\beta) \to X$. Now, we consider β-subsequence of $\langle e_{i,j}\rangle_{i,j\in\mathbb{N}}$ corresponding to X to be the β-subsequence $\langle e_{i,j}^{(\pi_X,\beta)}\rangle_{i,j\in\mathbb{N}}$ as defined in Definition 1.1.

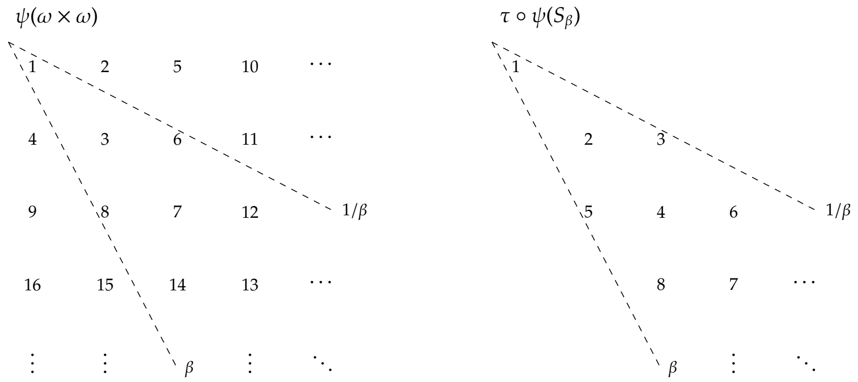

Figure 1: Pictorial representation of the map τ.

Partition $P(\omega)$ into two sets,

$$A = \{X \in P(\omega) \mid \langle e_{i,j}^{(\pi_{\tau^{-1}(X)},\beta)}\rangle_{i,j\in\mathbb{N}} \text{ is } \langle a_{i,j,k,l}\rangle_{i,j,k,l\in\mathbb{N}} \text{ summable}\},$$
$$B = P(\omega)\setminus A.$$

We will show next that A is a Ramsey set. If this is the case, then there exists an $M \in P(\omega)$ such that either all infinite subsets of M are in A, or else they all are not in A. Since each of those M's corresponds to a β-subsequence of $\langle e_{i,j}^{(\pi_{\tau^{-1}(X)},\beta)}\rangle_{i,j\in\mathbb{N}}$, then they would all be either $\langle a_{i,j,k,l}\rangle_{i,j,k,l\in\mathbb{N}}$-summable, or else they would all be not $\langle a_{i,j,k,l}\rangle_{i,j,k,l\in\mathbb{N}}$-summable.

It suffices to show that A is a Borel set in $P(\omega)$.

To simplify the notation, define

$$\langle d_{r,s}^X\rangle_{r,s\in\mathbb{N}} := \langle e_{i,j}^{(\pi_{\tau^{-1}(X)},\beta)}\rangle_{i,j\in\mathbb{N}}$$

and consider

$$B_{\epsilon,m,n,p,q} = \left\{X \in P(\omega) \mid \left\| \sum_{i,j=1,1}^{\infty,\infty} a_{m,n,k,l} d_{k,l}^X - \sum_{i,j=1,1}^{\infty,\infty} a_{p,q,k,l} d_{k,l}^X \right\| < \epsilon \right\}.$$

With respect to this definition,

$$A = \bigcap_{k=1}^{\infty} \bigcup_{N=1}^{\infty} \bigcap_{m,n,p,q\geq N} B_{1/k,m,n,p,q}.$$

As a result, to show that A is a Borel set, it suffices to show that $B_{\epsilon,m,n,p,q}$ is open. Let $\epsilon' > 0$ be such that

$$\left\| \sum_{i,j=1,1}^{\infty,\infty} a_{m,n,k,l} d_{k,l}^X - \sum_{i,j=1,1}^{\infty,\infty} a_{p,q,k,l} d_{k,l}^X \right\| < \epsilon' < \epsilon.$$

Let $T > 0$ be an upper bound of $\langle e_{i,j}\rangle_{i,j\in\mathbb{N}}$ and by (RH2) pick $J > 0$ large enough so that following inequalities are simultaneously satisfied

$$T\left(\sum_{i,j=1,J}^{J-1,\infty}|a_{m,n,k,l}|+\sum_{i,j=1,J}^{J-1,\infty}|a_{p,q,k,l}|\right)<\frac{\epsilon-\epsilon'}{4},\quad \text{by (RH3)},$$
$$T\left(\sum_{i,j=J,1}^{\infty,J-1}|a_{m,n,k,l}|+\sum_{i,j=J,1}^{\infty,J-1}|a_{p,q,k,l}|\right)<\frac{\epsilon-\epsilon'}{4},\quad \text{by (RH4)},$$
$$T\left(\sum_{i,j=J,J}^{\infty,\infty}|a_{m,n,k,l}|+\sum_{i,j=J,J}^{\infty,\infty}|a_{p,q,k,l}|\right)<\frac{\epsilon-\epsilon'}{4},\quad \text{by (RH5)}.$$

Let $X\in B_{\epsilon,m,n,p,q}$. We construct next an open neighborhood C of X such that $C\subseteq B_{\epsilon,m,n,p,q}$. We start by defining the set

$$S_K=\{c\in\omega\mid \pi_{\tau^{-1}(X)}\circ\tau^{-1}(c)<\psi(K,K)\},$$

where $K=\max\{p\in\mathbb{N}\mid 1/\beta\le p/J\le\beta\}$.

Finally, we define

$$C=\{Y\in P(\omega)\mid Y\cap S_K=X\cap S_K\}.$$

It can be verified that if $Y\in C$, then $d^X_{k,l}=d^Y_{k,l}$. In particular,

$$\left\|\sum_{i,j=1,1}^{J-1,J-1}a_{m,n,k,l}d^Y_{k,l}-\sum_{i,j=1,1}^{J-1,J-1}a_{p,q,k,l}d^Y_{k,l}\right\|=\left\|\sum_{i,j=1,1}^{J-1,J-1}a_{m,n,k,l}d^X_{k,l}-\sum_{i,j=1,1}^{J-1,J-1}a_{p,q,k,l}d^X_{k,l}\right\|.$$

The set C is open in the topology on $P(\omega)$ and clearly $X\in C$. We now show that $C\subseteq B_{\epsilon,m,n,p,q}$.

$$\begin{aligned}\left\|\sum_{i,j=1,1}^{\infty,\infty}a_{m,n,k,l}d^Y_{k,l}-\sum_{i,j=1,1}^{\infty,\infty}a_{p,q,k,l}d^Y_{k,l}\right\| &\le \left\|\sum_{i,j=1,1}^{J-1,J-1}a_{m,n,k,l}d^Y_{k,l}-\sum_{i,j=1,1}^{J-1,J-1}a_{p,q,k,l}d^Y_{k,l}\right\|\\
&\quad+\left\|\sum_{i,j=J,J}^{\infty,\infty}a_{m,n,k,l}d^Y_{k,l}-\sum_{i,j=J,J}^{\infty,\infty}a_{p,q,k,l}d^Y_{k,l}\right\|\\
&\quad+\left\|\sum_{i,j=1,J}^{J-1,\infty}a_{m,n,k,l}d^Y_{k,l}-\sum_{i,j=1,J}^{J-1,\infty}a_{p,q,k,l}d^Y_{k,l}\right\|\\
&\quad+\left\|\sum_{i,j=J,1}^{\infty,J-1}a_{m,n,k,l}d^Y_{k,l}-\sum_{i,j=J,1}^{\infty,J-1}a_{p,q,k,l}d^Y_{k,l}\right\|\\
&\le \left\|\sum_{i,j=1,1}^{J-1,J-1}a_{m,n,k,l}d^X_{k,l}-\sum_{i,j=1,1}^{J-1,J-1}a_{p,q,k,l}d^X_{k,l}\right\|\\
&\quad+T\left(\sum_{i,j=J,J}^{\infty,\infty}|a_{m,n,k,l}|+\sum_{i,j=J,J}^{\infty,\infty}|a_{p,q,k,l}|\right)\\
&\quad+T\left(\sum_{i,j=1,J}^{J-1,\infty}|a_{m,n,k,l}|+\sum_{i,j=1,J}^{J-1,\infty}|a_{p,q,k,l}|\right)\\
&\quad+T\left(\sum_{i,j=J,1}^{\infty,J-1}|a_{m,n,k,l}|+\sum_{i,j=J,1}^{\infty,J-1}|a_{p,q,k,l}|\right)\end{aligned}$$

and thus

$$\left\|\sum_{i,j=1,1}^{\infty,\infty} a_{m,n,k,l}d_{k,l}^{Y} - \sum_{i,j=1,1}^{\infty,\infty} a_{p,q,k,l}d_{k,l}^{Y}\right\| < \left\|\sum_{i,j=1,1}^{\infty,\infty} a_{m,n,k,l}d_{k,l}^{X} - \sum_{i,j=1,1}^{\infty,\infty} a_{p,q,k,l}d_{k,l}^{X}\right\| + \left\|\sum_{i,j=J,J}^{\infty,\infty} a_{m,n,k,l}d_{k,l}^{X} - \sum_{i,j=J,J}^{\infty,\infty} a_{p,q,k,l}d_{k,l}^{X}\right\| + \frac{3(\epsilon-\epsilon')}{4} < \epsilon' + \epsilon - \epsilon' = \epsilon.$$

Therefore, $C \subseteq B_{\epsilon,m,n,p,q}$. Hence every element of $B_{\epsilon,m,n,p,q}$ has an open neighborhood C included in $B_{\epsilon,m,n,p,q}$, therefore $B_{\epsilon,m,n,p,q}$ is open.

As noted above, this implies that A is a Ramsey set. Hence there exists an infinite subset of the natural numbers M such that every infinite subset of M belongs to A or every infinite subset of M does not belong to A. If $M \notin A$, then for any infinite $X \subset M$ the subsequence $\langle d_{r,s}^{X}\rangle_{r,s\in\mathbb{N}}$ is not $\langle a_{i,j,k,l}\rangle_{i,j,k,l\in\mathbb{N}}$-summable. In this case, conclusion (2) is obtained.

Otherwise, if $M \in A$ it is clear that for all infinite $X \subset M$ the subsequence $\langle d_{r,s}^{X}\rangle_{r,s\in\mathbb{N}}$ is $\langle a_{i,j,k,l}\rangle_{i,j,k,l\in\mathbb{N}}$-summable. Moreover, one can argue in the same way as in [2] to show that for all infinite $X \subset M$ the subsequences $\langle d_{r,s}^{X}\rangle_{r,s\in\mathbb{N}}$ sum to the same limit. □

Corollary 2.2. *Assume that* $\langle e_{i,j}\rangle_{i,j\in\mathbb{N}}$ *and* $\langle a_{i,j,k,l}\rangle_{i,j\in\mathbb{N}}$ *are as in Theorem 2.1. Assume further, that B satisfies the Banach-Saks property with respect to the summability method* $\langle a_{i,j,k,l}\rangle_{i,j\in\mathbb{N}}$*. Then, there exists a* β*-subsequence* $\langle e_{i_\gamma,j_\delta}\rangle_{\gamma,\delta\in\mathbb{N}}$ *such that every* β*-subsequence of* $\langle e_{i_\gamma,j_\delta}\rangle_{\gamma,\delta\in\mathbb{N}}$ *is summable with respect to* $\langle a_{i,j,k,l}\rangle_{i,j,k,l\in\mathbb{N}}$*, where they all are summed to the same limit.*

References

[1] R. Dumitru and J. A. Franco. Fundamental Theorems of Summability Theory for a New Type of Subsequences of Double Sequences. *J. Class. Anal.*, 15(1):23–33, 2019.
[2] P. Erdős and M. Magidor. A note on regular methods of summability and the Banach-Saks property. *Proc. Amer. Math. Soc.*, 59(2):232–234, 1976.
[3] Fred Galvin and Karel Prikry. Borel sets and Ramsey's theorem. *J. Symbolic Logic*, 38:193–198, 1973.
[4] H. J. Hamilton. Transformations of multiple sequences. *Duke Math. J.*, 2(1):29–60, 1936.
[5] A. Pringsheim. Zur Theorie der zweifach unendlichen Zahlenfolgen. *Math. Ann.*, 53(3):289–321, 1900.
[6] G. M. Robison. Divergent double sequences and series. *Trans. Amer. Math. Soc.*, 28(1):50–73, 1926.