Filomat 35:5 (2021), 1477–1487
https://doi.org/10.2298/FIL2105477N

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

On S-2-Absorbing Primary Submodules

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Abstract. This article introduces the concept of S-2-absorbing primary submodule as a generalization of 2-absorbing primary submodule. Let S be a multiplicatively closed subset of a ring R and M an R-module. A proper submodule N of M is said to be an S-2-absorbing primary submodule of M if $(N :_R M) \cap S = \phi$ and there exists a fixed element $s \in S$ such that whenever $abm \in N$ for some $a, b \in R$ and $m \in M$, then either $sam \in N$ or $sbm \in N$ or $sab \in \sqrt{(N :_R M)}$. We give several examples, properties and characterizations related to the concept. Moreover, we investigate the conditions that force a submodule to be S-2-absorbing primary.

1. Introduction

Throughout this article, all rings are commutative with nonzero identity and all modules are unital. Let R always represent such a ring and M represent such an R-module. Prime submodules play a crucial role in module theory, since they interfere with many classes of algebra and represent an important tool in helping to understand the structure of modules. This importance was an incentive for many researchers to work to generalize this term. Badawi started in [6] one of these endeavors, as he introduced the notion of 2-absorbing ideals as a generalization of prime ideals. Later, the concept of absorption has been studied intensively. See, for example [4, 5, 7, 8, 11, 13] and [12]. Recall from [7, 8] that a submodule P of M is said to be 2-absorbing (2-absorbing primary) if whenever $xym \in P$ for some $x, y \in R$ and $m \in M$, then either $xm \in P$ or $ym \in P$ or $xy \in (P :_R M)$ ($xy \in \sqrt{(P :_R M)}$), respectively. Recently, a new approach has been introduced to generalize prime submodules using a multiplicatively closed subset S of R (i.e. $1 \in S$ and $ss' \in S$ for each $s, s' \in S$). The authors in [14] defined a submodule N of M to be S-prime if $(N :_R M) \cap S = \phi$ and there exists an $s \in S$ such that $rm \in N$ for some $r \in R$ and $m \in M$ implies that $sr \in (N :_R M)$ or $sm \in N$. Expectedly, a proper ideal I of R is an S-prime (2-absorbing) ideal if and only if I is an S-prime (2-absorbing) submodule of R-module R, respectively. We aim in this article to introduce the concept of S-2-absorbing primary submodule and to investigate some properties related to it.

In the interest of completeness, we start with some definitions and notations that appear throughout this article. Let N be an R-submodule of M, L be a nonempty subset of M and A be an ideal of R. Then the residuals of N by L and N by A are defined as follows: $(N :_R L) = \{a \in R : aL \subseteq N\}$ and $(N :_M A) = \{x \in M : Ax \subseteq N\}$. In particular, let $ann_R(M)$ denote the ideal $(0 :_R M)$. If $ann_R(M) = 0$, then M is called a faithful module. Recall from [9] that, an R-module M is called a multiplication module if each submodule L of M has the

2010 *Mathematics Subject Classification*. Primary 16P40, 13A15, 16D60
Keywords. S-2-Absorbing Primary Submodules, 2-Absorbing Primary Submodules, S-Primary submodules
Received: 17 April 2020; Revised: 23 August 2020; Accepted: 24 August 2020
Communicated by Dijana Mosić
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form $L = AM$ for some ideal A of R, or equivalently, $L = (L :_R M)M$. A proper submodule L of M is called irreducible if L can not be written as an intersection of two submodules of M that properly contain it.

In this paper, we study the concept of S-2-absorbing primary submodules of a module which can be considered a generalization of many clases of submodules such as S-prime, primary, 2-absorbing, S-2-absorbing and 2-absorbing primary submodules. Also, this concept combines the two previous methods that we mentioned above in generalizing the prime submodules. A submodule N of M is said to be an S-2-absorbing primary (S-2-absorbing [15]) submodule if $(N :_R M) \cap S = \phi$ and there exists a fixed element $s \in S$ such that whenever $abm \in N$ for some $a, b \in R$ and $m \in M$ implies that either $sab \in \sqrt{(N :_R M)}$ ($sab \in (N :_R M)$) or $sam \in N$ or $sbm \in N$, respectively. Note that directly from the definition, if a submodule N of M is 2-absorbing primary or S-2-absorbing, then N is also S-2-absorbing primary provided that $(N :_R M) \cap S = \phi$. However, the converses are not true (See Example 2.3 and Example 2.4). Moreover, in a quick overview of the concept, we see that when $S \subseteq u(R)$, 2-absorbing and S-2-absorbing are identical concepts, where $u(R)$ is the set of all units in R. Among other results in this study, in Section 2, we study some basic properties of S-2-absorbing primary submodules and transfer some well known results to fit the new notion. Also, we give some relations between 2-absorbing primary and S-2-absorbing primary under some certain types of S (See Proposition 2.5). In Theorem 2.10, we show that N is an S-2-absorbing primary submodule of M if and only if there exists a fixed element $s \in S$ such that $IJK \subseteq N$ for some ideals I, J of R and some submodule K of M implies that either $sIK \subseteq N$ or $sJK \subseteq N$ or $sIJ \subseteq \sqrt{(N :_R M)}$. Furthermore, we investigate the conditions that force a submodule to be S-2-absorbing primary (Theorem 2.20, Proposition 2.21 and Proposition 2.22).

In Section 3, we study some additional traditional properties related to these types of concepts. We explore the behavior of S-2-absorbing primary under homomorphism, quotient module, in trivial extension and in cartesian product of modules (Proposition 3.2, Corollary 3.3, Proposition 3.1 and Proposition 3.8). Also, using the same sense in this paper, we generalize primary submodules to S-primary. In Theorem 3.6, we give a characterization of primary submodule in terms of primary ideal and S-primary submodules. Finally, we show that every irreducible submodule is S-primary provided that M is a Noetherian module (Proposition 3.7).

2. Characterizations of S-2-Absorbing Primary Submodules

Definition 2.1. *Let S be a multiplicatively closed subset of a ring R and M an R-module. A submodule N of M is said to be S-2-absorbing primary if $(N :_R M) \cap S = \phi$ and there exists a fixed element $s \in S$ such that whenever $a, b \in R$ and $m \in M$ with $abm \in N$, then $sam \in N$ or $sbm \in N$ or $sab \in \sqrt{(N :_R M)}$.*

Definition 2.2. *Let S be a multiplicatively closed subset of a ring R. An ideal I of R is said to be S-2-absorbing primary if $I \cap S = \phi$ and there exists an $s \in S$ such that for any $a, b, c \in R$ with $abc \in I$, then $sab \in I$ or $sac \in I$ or $sbc \in \sqrt{I}$.*

Example 2.3. *Suppose that $S \subseteq R$ is a multiplicatively closed subset and N is an R-submodule of M. If $S \subseteq u(R)$, then N is 2-absorbing primary if and only if N is S-2-absorbing primary.*

In other words, the above example shows that S-2-absorbing primary submodule need not to be S-2-absorbing. This can be seen by taking $S = u(R)$ and recalling that 2-absorbing primary does not imply 2-absorbing. The following example illustrate the idea in case S is not trivial.

Example 2.4. *Consider the $\mathbb{Z}$-submodule $N = q\mathbb{Z} \times p^3\mathbb{Z}$ of the module $M = \mathbb{Z} \times \mathbb{Z}$ and the multiplicatively closed subset $S = \{q^n : n \geq 0\}$ of $\mathbb{Z}$, where $p \neq q$ are prime integers. N is not S-2-absorbing. Since for any $n \geq 0$ and $s = q^n$, $p^2(q, p) \in N$, but $sp(q, p) \notin N$ and $sp^2 \notin (N :_R M) = qp^3\mathbb{Z}$. At the same time, N is S-2-absorbing primary. This can easily be shown by taking $s = q$ and whenever $ab(m_1, m_2) \in N$ then $sam \in N$ or $sbm \in N$ or $sab \in \sqrt{(N :_R M)} = qp\mathbb{Z}$ for any $a, b, m_1, m_2 \in \mathbb{Z}$.*

Let $S^{-1}M$ denote the module of fractions of M with respect to the multiplicatively closed subset $S \subseteq R$ over the quotient ring $S^{-1}R$. Each submodule of $S^{-1}M$ has the form

$$S^{-1}N = \left\{ \frac{n}{s} \in S^{-1}M : \text{for some } n \in N \text{ and } s \in S \right\}$$

where N is a submodule of M. Recall from [10] that a saturation set of S is defined by $S^{\star}=\left\{r\in R:\frac{r}{1}\text{ is a unit of } S^{-1}R\right\}$ and S is called a saturated set if $S=S^{\star}$. Indeed, $S^{\star}$ is a saturated set that contains S. Suppose that M is an R-module. The set $U_M(R)=\{a\in R: aM=M\}$ is a saturated multiplicatively closed subset of R that contains the set of all units $u(R)$ of R.

Proposition 2.5. *Let M be an R-module and $S\subseteq R$ a multiplicatively closed subset. Then the following statements hold:*

(i) Suppose that $S_1\subseteq S_2$ are two multiplicatively closed subsets of R. If N is an S_1-2-absorbing primary submodule such that $(N:_R M)\cap S_2=\phi$, then N is also an S_2-2-absorbing primary submodule. In particular, every 2-absoring primary submodule N with $(N:_R M)\cap S=\phi$ is S-2-absorbing primary.

(ii) Assume that $S^{\star}$ is the saturation of S. Then a submodule N of M is S-2-absorbing primary if and only if it is $S^{\star}$-2-absorbing primary.

(iii) Suppose that $M=Rm$ is a cyclic R-module and $S\subseteq U_M(R)$. Then N is a 2-absorbing primary submodule if and only if N is an S-2-absorbing primary submodule.

(iv) If N is an S-2-absorbing primary submodule, then $S^{-1}N$ is a 2-absorbing primary $S^{-1}R$-submodule.

Proof. (i) It is clear.
(ii) Suppose that N is S-2-absorbing primary. It is obvious that $(N:_R M)$ and $S^{\star}$ are disjoint. Since $S\subseteq S^{\star}$, then by (i), N is $S^{\star}$-2-absorbing primary. For the converse, let $abm\in N$ for some $a,b\in R$ and $m\in M$. Since N is an $S^{\star}$-2-absorbing primary submodule, there exists an $s\in S^{\star}$ such that $sam\in N$ or $sbm\in N$ or $sab\in\sqrt{(N:_R M)}$. As $\frac{s}{1}$ is a unit of $S^{-1}R$, there exist $x\in R$ and $t\in S$ with $\frac{s}{1}\frac{x}{t}=1$. Hence, $usx=ut$ for some $u\in S$. Now, put $s'=ut\in S$. This yields $s'am=ux(sam)\in N$ or $s'bm=ux(sbm)\in N$ or $s'ab=ux(sab)\in\sqrt{(N:_R M)}$. Therefore, N is an S-2-absorbing primary submodule.
(iii) ($\Rightarrow$) Take $S_1=\{1\}$ and $S_2=S$. Then by part (i), the result follows.
($\Leftarrow$) Suppose that N is an S-2-absorbing primary submodule. Let $abx\in N$ for some $x=rm\in M$ and $a,b,r\in R$. Then there exists an $s\in S$ such that $sax\in N$ or $sbx\in N$ or $sab\in\sqrt{(N:_R M)}$. If $sax\in N$, then $sarRm=arRm\subseteq N$ and hence $ax=arm\in N$. Similarly when $sbx\in N$, we get $bx\in N$. If $sab\in\sqrt{(N:_R M)}$, then $(sab)^k\in(N:_R M)$ for some $k\in\mathbb{N}$. Note that M is a multiplication module and so $(ab)^kM=(sab)^kM\subseteq(N:_R M)M=N$. Thus, N is a 2-absorbing primary submodule.
(iv) Assume that N is S-2-absorbing primary. Let $\frac{a}{s}\frac{b}{t}\frac{m}{u}\in S^{-1}N$, where $\frac{a}{s},\frac{b}{t}\in S^{-1}R$ and $\frac{m}{u}\in S^{-1}M$. Then $(va)bm\in N$ for some $v\in S$. Hence, there exists an $s'\in S$ such that $s'vam\in N$ or $s'bm\in N$ or $s'vab\in\sqrt{(N:_R M)}$, which implies that $\frac{a}{s}\frac{m}{u}=\frac{s'vam}{s'vsu}\in S^{-1}N$ or $\frac{b}{t}\frac{m}{u}=\frac{s'bm}{s'tu}\in S^{-1}N$ or $\frac{a}{s}\frac{b}{t}\in S^{-1}\sqrt{(N:_R M)}\subseteq\sqrt{(S^{-1}N:_{S^{-1}R}S^{-1}M)}$. Therefore, $S^{-1}N$ is a 2-absorbing primary submodule of $S^{-1}M$. □

The converses of Proposition 2.5 (i) and (iv) are not true in general. The following examples illustrate that.

Example 2.6. *(i) Consider the $\mathbb{Z}$-module $\mathbb{Z}_{72}$ and $S=reg(\mathbb{Z})=\mathbb{Z}-\{0\}$ to be a multiplicatively closed subset of $\mathbb{Z}$. Then the submodule $N=\{0,36\}$ is not 2-absorbing primary. Since $2.2.9\in N$ and neither $2.9\in N$ nor $2.2\in\sqrt{(N:_R M)}=6\mathbb{Z}$. On the other hand, it is clear that the submodule N is S-2-absorbing primary (For example, by taking $s=36$).*
(ii) Consider the $\mathbb{Z}$-module $\mathbb{Q}\times\mathbb{Z}$ and $S=\mathbb{Z}-\{0\}$. Take the submodule $N=\mathbb{Z}\times\{0\}$. N is not an S-2-absorbing primary submodule. Since for any element $s\in S$, choose a prime integer p such that $gcd(s,p)=1$ and then note that, $p^2(\frac{1}{p^2},0)\in N$, but $sp^2\notin\sqrt{(N:_{\mathbb{Z}}\mathbb{Q}\times\mathbb{Z})}=\{0\}$ and $sp(\frac{1}{p^2},0)\notin N$. Moreover, $S^{-1}\mathbb{Z}=\mathbb{Q}$ is a field. Hence, every submodule of $S^{-1}(\mathbb{Q}\times\mathbb{Z})$ is 2-absorbing primary.

Here we point out that some modules do not have any S-2-absorbing primary submodule. Let $p\neq q$ be two fixed prime integers and consider the $\mathbb{Z}$-module $\mathbb{Z}(p^{\infty})=\{\lambda\in\mathbb{Q}/\mathbb{Z}:\lambda=\frac{a}{p^n}+\mathbb{Z}\text{ for some } a\in\mathbb{Z}\text{ and } n\geq 0\}$ and the multiplicatively closed subset $S=\{q^n:n\geq 0\}$ of $\mathbb{Z}$. Every proper submodule of $\mathbb{Z}(p^{\infty})$ has the form $G_t=\{\lambda\in\mathbb{Q}/\mathbb{Z}:\lambda=\frac{a}{p^t}+\mathbb{Z}\text{ for some } a\in\mathbb{Z}\}$ for some $t\geq 0$. G_t is not an S-2-absorbing primary submodule of $\mathbb{Z}(p^{\infty})$. Since for any $n\geq 0$ and $s=q^n$, $p^2\left(\frac{1}{p^{t+2}}+\mathbb{Z}\right)\in G_t$, while $sp\left(\frac{1}{p^{t+2}}+\mathbb{Z}\right)\notin G_t$ and

$sp^2 \notin \sqrt{(G_t :_{\mathbb{Z}} \mathbb{Z}(p^\infty))} = 0$.

The following proposition shows that the converses of Proposition 2.5 (i) and (iv) can be true under some specific conditions. Recall from [2] that a multiplicatively closed set S is said to satisfy the maximal multiple condition if there exists $s \in S$ such that s' divides s for each $s' \in S$. Note that all finite multiplicatively closed sets and the set of units in R are examples of multiplicatively closed set satisfying the maximal multiple condition.

Proposition 2.7. *Let N be a submodule of an R-module M. Then the following statements hold:*

(i) Suppose that $S_1 \subseteq S_2$ are two multiplicatively closed subsets of R and for every $u \in S_2$, there exists $v \in R$ such that $uv \in S_1$. If N is S_2-2-absorbing primary, then N is also S_1-2-absorbing primary.

(ii) Let S be a multiplicatively closed subset of R satisfying the maximal multiple condition. If $S^{-1}N$ is a 2-absorbing primary submodule of $S^{-1}M$, then, N is an S-2-absorbing primary submodule of M.

Proof. (i) It is explicit.
(ii) Assume that S is a multiplicatively closed subset of R satisfying the maximal multiple condition. Then there exists $s \in S$ such that s' divides s for each $s' \in S$, that is, $Rs \subseteq Rs'$. Let $abm \in N$ for some $a, b \in R$ and $m \in M$. Since $S^{-1}N$ is 2-absorbing primary, then $\frac{a}{1}\frac{m}{1} \in S^{-1}N$ or $\frac{b}{1}\frac{m}{1} \in S^{-1}N$ or $\frac{a}{1}\frac{b}{1} \in \sqrt{(S^{-1}N :_{S^{-1}R} S^{-1}M)}$. If $\frac{a}{1}\frac{m}{1} \in S^{-1}N$ or $\frac{b}{1}\frac{m}{1} \in S^{-1}N$, we get $sam \in N$ or $sbm \in N$. If $\frac{a}{1}\frac{b}{1} \in \sqrt{(S^{-1}N :_{S^{-1}R} S^{-1}M)}$, then for every $x \in M$, $\frac{a^k}{1}\frac{b^k}{1}\frac{x}{1} \in S^{-1}N$ for some $k \in \mathbb{N}$. which implies that $(sab)^k x \in N$. Hence, $sab \in \sqrt{(N :_R M)}$. Therefore, N is an S-2-absorbing primary submodule. □

Lemma 2.8. *Let S be a multiplicatively closed subset of R. If N is a submodule of M with $(N :_R M) \cap S = \phi$, then the following statements are equivalent:*

(i) N is an S-2-absorbing primary submodule of M.

(ii) There exists an $s \in S$ such that whenever $Iam \subseteq N$ for some ideal I of R, $a \in R$ and $m \in M$ implies either $sam \in N$ or $sIm \subseteq N$ or $sIa \subseteq \sqrt{(N :_R M)}$.

Proof. (i)⇒(ii) Assume that N is an S-2-absorbing primary submodule of M. Then there exists an $s \in S$ such that $xyu \in N$ for some $x, y \in R$ and $u \in M$ implies $sxy \in \sqrt{(N :_R M)}$ or $sxu \in N$ or $syu \in N$. Let $Iam \subseteq N$ for some $a \in R$, $m \in M$ and an ideal I of R. Let $sam \notin N$ and $sIa \nsubseteq \sqrt{(N :_R M)}$. So there exists $b \in I$ such that $sab \notin \sqrt{(N :_R M)}$. Since $abm \in N$ and N is S-2-absorbing primary, we obtain $sbm \in N$. We show that $sIm \subseteq N$. Let $c \in I$, then $(b + c)am \in N$ and hence either $s(b + c)m \in N$ or $s(b + c)a \in \sqrt{(N :_R M)}$. If $s(b + c)m \in N$, then by $sbm \in N$ it follows that $scm \in N$. If $s(b + c)a \in \sqrt{(N :_R M)}$, then $sca \notin \sqrt{(N :_R M)}$. But we have $cam \in N$, so $scm \in N$. Thus, $sIm \subseteq N$.
(ii)⇒(i) It is clear. □

Lemma 2.9. *Let S be a multiplicatively closed subset of R. If N is a submodule of M with $(N :_R M) \cap S = \phi$, then the following statements are equivalent:*

(i) N is an S-2-absorbing primary submodule of M.

(ii) There exists an $s \in S$ such that $IJm \subseteq N$ for some ideals I, J of R and $m \in M$, then either $sIm \subseteq N$ or $sJm \subseteq N$ or $sIJ \subseteq \sqrt{(N :_R M)}$.

Proof. (i)⇒(ii) Suppose that N is an S-2-absorbing primary submodule of M. Then we keep in mind that there exists a fixed $s \in S$ that satisfies the S-2-absorbing primary condition. Let $IJm \subseteq N$ for some ideals I, J of R and $m \in M$ and assume that $sI \nsubseteq (N :_R m)$ and $sJ \nsubseteq (N :_R m)$. We are going to show that $sIJ \subseteq \sqrt{(N :_R M)}$. Let $c \in I$ and $d \in J$. There is an $a \in I - (N :_R m)$ such that $sam \notin N$. As $aJm \subseteq N$, then by Lemma 2.8, we get $saJ \subseteq \sqrt{(N :_R M)}$ and so $s[I - (N :_R sm)]J \subseteq \sqrt{(N :_R M)}$. Similarly, there exists $b \in J - (N :_R m)$ such that $sIb \subseteq \sqrt{(N :_R M)}$ and $sI[J - (N :_R sm)] \subseteq \sqrt{(N :_R M)}$. Hence we have $sab \in \sqrt{(N :_R M)}$, $sad \in \sqrt{(N :_R M)}$ and $scb \in \sqrt{(N :_R M)}$. Since $a+c \in I$ and $b+d \in J$, it gives that $(a+c)(b+d)m \in N$. Thus, $s(a+c)m \in N$ or $s(b+d)m \in N$ or $s(a + c)(b + d) \in \sqrt{(N :_R M)}$. If $s(a + c)m \in N$, then $scm \notin N$ which implies that $c \in I - (N :_R sm)$ and so

$scd \in \sqrt{(N :_R M)}$. Similarly, by $s(b+d)m \in N$, we conclude that $scd \in \sqrt{(N :_R M)}$. If $s(a+c)(b+d) \in \sqrt{(N :_R M)}$, then $sab + sad + scb + scd \in \sqrt{(N :_R M)}$ and this yields $scd \in \sqrt{(N :_R M)}$. Therefore, $sIJ \subseteq \sqrt{(N :_R M)}$.
(ii)⇒(i) It is clear. □

Theorem 2.10. *Let S be a multiplicatively closed subset of R and N a submodule of R-module M with $(N :_R M) \cap S = \phi$.Then, N is an S-2-absorbing primary submodule of M if and only if there exists a fixed $s \in S$ such that whenever $IJK \subseteq N$ for some ideals I, J of R and some submodule K of M, then either $sIK \subseteq N$ or $sJK \subseteq N$ or $sIJ \subseteq \sqrt{(N :_R M)}$.*

Proof. (⇒) Suppose that $s \in S$ satisfies S-2-absorbing primary condition. Assume that $IJK \subseteq N$ for some ideals I, J of R and a submodule K of M and $sIJ \nsubseteq \sqrt{(N :_R M)}$. Then by Lemma 2.9, for any $x \in K$ we obtain $sIx \subseteq N$ or $sJx \subseteq N$. If for every $x \in K$, $sIx \subseteq N$, then we are done. Similarly, if for all $x \in K$, $sJx \subseteq N$, we are done. Suppose that there exist $x_1, x_2 \in K$ such that $sIx_1 \nsubseteq N$ and $sJx_2 \nsubseteq N$. Hence, $sJx_1 \subseteq N$ and $sIx_2 \subseteq N$. Since $IJ(x_1 + x_2) \subseteq N$, then either $sI(x_1 + x_2) \subseteq N$ or $sJ(x_1 + x_2) \subseteq N$. If $sI(x_1 + x_2) \subseteq N$, it follows that $sIx_1 \subseteq N$ which is a contradiction. Similarly by $sJ(x_1 + x_2) \subseteq N$, we obtain a contradiction. Thus, either $sIK \subseteq N$ or $sJK \subseteq N$.
(⇐) It is clear. □

Corollary 2.11. *Let S be a multiplicatively closed subset of R and I an ideal of R with $I \cap S = \phi$. Then the following statements are equivalent:*
(i) I is an S-2-absorbing primary ideal of R.
(ii) There exists an $s \in S$ such that $I_1I_2I_3 \subseteq I$ for some ideals I_1, I_2 and I_3 of R implies $sI_1I_2 \subseteq I$ or $sI_1I_3 \subseteq I$ or $sI_2I_3 \subseteq \sqrt{I}$.

Proposition 2.12. *Let S be a multiplicatively closed subset of R. If N is an S-2-absorbing primary submodule of M, then $(N :_R M)$ is an S-2-absorbing primary ideal of R.*

Proof. Suppose that N is an S-2-absorbing primary submodule of M. Let $abc \in (N :_R M)$ for some $a, b, c \in R$. Then we get $RaRb(cM) \subseteq N$. Hence by Theorem 2.10, there is an $s \in S$ such that $sRaRb \subseteq \sqrt{(N :_R M)}$ or $sRa(cM) \subseteq N$ or $sRb(cM) \subseteq N$. Thus, either $sab \in \sqrt{(N :_R M)}$ or $sac \in (N :_R M)$ or $sbc \in (N :_R M)$. Therefore, $(N :_R M)$ is an S-2-absorbing primary ideal of R. □

The converse of Proposition 2.12 is not true in general. If $(N :_R M)$ is an S-2-absorbing primary ideal, then N may not be S-2-absorbing primary. Consider the $\mathbb{Z}$-module $M = \mathbb{Z} \times \mathbb{Z}_{12}$ and $S = \{3^n : n \in \mathbb{N} \cup \{0\}\}$. Let N be the zero submodule, then $(N :_R M) = \{0\}$ is an S-2-absorbing primary ideal. On the other hand, N is not S-2-absorbing primary. Since for any $n \in \mathbb{N} \cup \{0\}$ and $s = 3^n$, we have $2.2(0, 3) \in N$, but $s2^2 \notin \sqrt{(N :_R M)} = \{0\}$ and $s2(0, 3) \notin N$.

Proposition 2.13. *Let S be a multiplicatively closed subset of R. If A is an S-2-absorbing primary ideal of R, then $\sqrt{A}$ is an S-2-absorbing ideal of R.*

Proof. Suppose that A is S-2-absorbing primary and $s \in S$ satisfies S-2-absorbing primary condition. Clearly, $\sqrt{A} \cap S = \phi$. Now, let $a, b, c \in R$ such that $abc \in \sqrt{A}$, $sac \notin \sqrt{A}$ and $sbc \notin \sqrt{A}$. Then, there exists a positive integer k such that $(abc)^k = a^kb^kc^k \in A$. Hence by assumption, we conclude that $s^ka^kb^k = (sab)^k \in A$, and thus $sab \in \sqrt{A}$. Therefore, $\sqrt{A}$ is an S-2-absorbing ideal of R. □

Proposition 2.14. *Let S be a multiplicatively closed subset of R and N an S-2-absorbing primary submodule of R-module M. Then the following statements hold:*
(i) If $(N :_R m) \cap S = \phi$, for some $m \in M - N$, then $(N :_R m)$ is an S-2-absorbing primary ideal of R.
(ii) If $((N :_M r) :_R M) \cap S = \phi$, for some $r \in R - (N :_R M)$, then $(N :_M r)$ is an S-2-absorbing primary submodule of M.

Proof. (i) Assume that $m \in M - N$ and $abc \in (N :_R m)$ for some $a, b, c \in R$. Since N is S-2-absorbing primary and $(ab)cm \in N$, then there exists an $s \in S$ such that either $sabm \in N$ or $scm \in N$ or $sabc \in \sqrt{(N :_R M)}$. Assume that $(sa)bc \in \sqrt{(N :_R M)}$, Then by Proposition 2.13, $\sqrt{(N :_R M)}$ is an S-2-absorbing ideal of R. Let $t \in S$ be the element which satisfies the S-2-absorbing condition and put $s' = ts \in S$ and fix it. Hence, $s'ab = tsab \in \sqrt{(N :_R M)} \subseteq \sqrt{(N :_R m)}$ or $s'ac = tsac \in \sqrt{(N :_R M)} \subseteq \sqrt{(N :_R m)}$ or $s'bc = tsbc \in \sqrt{(N :_R M)} \subseteq \sqrt{(N :_R m)}$. If the first two terms $sabm \in N$ or $scm \in N$ are satisfied, then it follows that $s'ab \in (N :_R m)$ or $s'bc \in (N :_R m)$ and we are done. Thus, in each case, we have $(N :_R m)$ is an S-2-absorbing primary ideal of R.
(ii) Suppose that $r \in R - (N :_R M)$. Let $a, b \in R$ and $m \in M$ such that $abm \in (N :_M r)$. This implies that $abrm \in N$. Since N is an S-2-absorbing primary submodule of M, then there exists $s \in S$ such that either $sarm \in N$ or $sbrm \in N$ or $sab \in \sqrt{(N :_R M)}$. From the first two cases, we get that $sam \in (N :_M r)$ or $sbm \in (N :_M r)$. If $sab \in \sqrt{(N :_R M)}$, Then there exists a positive integer k such that $(sab)^k M \subseteq N \subseteq (N :_M r)$. Hence, $sab \in \sqrt{((N :_M r) :_R M)}$. Therefore, $(N :_M r)$ is an S-2-absorbing primary submodule of M. □

Next proposition shows that in a finitely generated faithful multiplication module, if a submodule N is S-2-absorbing primary, then M-$rad(N)$ is also S-2-absorbing primary, where M-$rad(N)$ is the M-radical of N. To see this, we recall the following lemma.

Lemma 2.15. *[11]Let M be a finitely generated multiplication R-module. Then for any submodule N of M, $\sqrt{(N :_R M)} = (M\text{-}rad(N) :_R M)$.*

Proposition 2.16. *Let S be a multiplicatively closed subset of R and M a finitely generated multiplication R-module. If N is an S-2-absorbing primary submodule of M, then M-$rad(N)$ is an S-2-absorbing primary submodule of M.*

Proof. First, it is obvious that $(M\text{-}rad(N) :_R M) \cap S = \phi$. Let $IJK \subseteq M\text{-}rad(N)$ for some ideals I, J of R and some submodule K of M. Then by Lemma 2.15, $IJ(K :_R M)M \subseteq (M\text{-}rad(N) :_R M)M = \sqrt{(N :_R M)}M$. This implies that $IJ(K :_R M) \subseteq \sqrt{(N :_R M)}$. Since N is S-2-absorbing primary, so by Proposition 2.12 and Proposition 2.13, $\sqrt{(N :_R M)}$ is an S-2-absorbing ideal. Then there exists a fixed $s \in S$ such that either $sIJ \subseteq \sqrt{(N :_R M)}$ or $sI(K :_R M) \subseteq \sqrt{(N :_R M)}$ or $sJ(K :_R M) \subseteq \sqrt{(N :_R M)}$. Thus, we obtain $sIJ \subseteq \sqrt{(M\text{-}rad(N) :_R M)}$ or $sIK = sI(K :_R M)M \subseteq \sqrt{(N :_R M)}M = M\text{-}rad(N)$ or $sJK = sJ(K :_R M)M \subseteq \sqrt{(N :_R M)}M = M\text{-}rad(N)$. □

Recall from [1] that the product of two submodules K and L of a multiplication R-module M is defined as $KL = IJM$, where $K = IM$ and $L = JM$ for some ideals I and J of R. Moreover, we point out that the product of two submodules is independent of the presentations of submodules of M [1, Theorem 3.4].

Proposition 2.17. *Let M be a finitely generated multiplication R-module, S be a multiplicatively closed subset of R and N a submodule of M with $(N :_R M) \cap S = \phi$. Then the following statements are equivalent:*
(i) N is an S-2-absorbing primary submodule of M.
(ii) There exists an $s \in S$ such that $KLP \subseteq N$ for some submodules K, L, and P of M implies $sKL \subseteq N$ or $sKP \subseteq N$ or $sLP \subseteq M\text{-}rad(N)$.

Proof. Follows directly by using Theorem 2.10 and Lemma 2.15. □

Lemma 2.18. *Let M be an R-module and S a multiplicatively closed subset of R. Suppose that N is an S-2-absorbing primary submodule of M. Then the following statements hold:*
(i) There exists $s \in S$ such that $(N :_M s^3) = (N :_M s^n)$ for all $n \geq 3$.
(ii) There exists $s \in S$ such that $(N :_R s^3 M) = (N :_R s^n M)$ for all $n \geq 3$.

Proof. (i): Assume that N is an S-2-absorbing primary submodule of M and $s \in S$ is the element which satisfies the condition of S-2-absorbing primary. Let $m \in (N :_M s^4)$. Then $s^4 m = s^2 s^2 m \in N$. We deduce that $s(s^2 m) = s^3 m \in N$ or $s^5 \in \sqrt{(N :_R M)}$. If $s^5 \in \sqrt{(N :_R M)}$, then $s^{5k} \in (N :_R M) \cap S$ for some positive integer k, which is a contradiction. Hence, $m \in (N :_M s^3)$. Since the other inclusion is always satisfied, so $(N :_M s^3) = (N :_M s^4)$. Now, assume that $(N :_M s^3) = (N :_M s^k)$ for all $k < n$. We will show that

$(N :_M s^3) = (N :_M s^n)$. Let $x \in (N :_M s^n)$. Then $s^n x = s^2(s^{n-2})x \in N$. As N is an S-2-absorbing primary submodule of M, we obtain $s^3 x \in N$ or $s^{n-1}x \in N$ or $s^{n+1} \in \sqrt{(N :_R M)}$. Since the last case gives a contradiction, we have $x \in (N :_M s^3) \cup (N :_M s^{n-1}) = (N :_M s^3)$. Thus, we have $(N :_M s^3) = (N :_M s^n)$.
(ii) : Follows directly from (i). □

In next theorem, we characterize S-2-absorbing primary submodules in terms of 2-absorbing primary residual submodules.

Theorem 2.19. *Let M be an R-module and $S \subseteq R$ a multiplicatively closed subset. Suppose that N is a submodule of M with $(N :_R M) \cap S = \phi$. Then the following statements are equivalent:*
(i) N is an S-2-absorbing primary submodule.
(ii) $(N :_M s)$ is a 2-absorbing primary submodule for some $s \in S$.

Proof. (ii)⇒(i) Suppose that $(N :_M s)$ is a 2-absorbing primary submodule for some $s \in S$. Let $abm \in N \subseteq (N :_M s)$ for some $a, b \in R$ and $m \in M$. Then, either $ab \in \sqrt{((N :_M s) :_R M)}$ or $am \in (N :_M s)$ or $bm \in (N :_M s)$. Hence, $sab \in \sqrt{(N :_R M)}$ or $sam \in N$ or $sbm \in N$. Therefore, N is an S-2-absorbing primary submodule.
(i)⇒(ii) Assume that N is an S-2-absorbing primary submodule and $s \in S$ satisfies the S-2-absorbing primary condition. Then by Lemma 2.18, we have $(N :_M s^3) = (N :_M s^n)$ and $(N :_R s^3 M) = (N :_R s^n M)$ for all $n \geq 3$. We show that $(N :_M s^6) = (N :_M s^3)$ is a 2-absorbing primary submodule of M. Take $abm \in (N :_M s^6)$ for some $a, b \in R$ and $m \in M$. Then we get $s^6(abm) = (s^2 a)(s^2 b)(s^2 m) \in N$. Since N is an S-2-absorbing primary submodule, we deduce that either $s(s^2 a)(s^2 b) = s^5 ab \in \sqrt{(N :_R M)}$ or $s(s^2 a)(s^2 m) = s^5 am \in N$ or $s(s^2 b)(s^2 m) = s^5 bm \in N$. If $s^5 ab \in \sqrt{(N :_R M)}$, then for some positive integer k, $(s^5 ab)^k \in (N :_R M)$, which implies that $(ab)^k \in (N :_R s^{5k} M) = (N :_R s^6 M) = ((N :_M s^6) :_R M)$ and so $ab \in \sqrt{((N :_M s^6) :_R M)}$. If $s^5 am \in N$ or $s^5 bm \in N$, it follows that $am \in (N :_M s^5) = (N :_M s^6)$ or $bm \in (N :_M s^5) = (N :_M s^6)$. Thus, $(N :_M s^6)$ is a 2-absorbing primary submodule of M. □

The following results examine the causes and conditions that make a submodule to be S-2-absorbing primary.

Theorem 2.20. *Let N be a submodule of R-module M and $S \subseteq R$ a multiplicatively closed subset. Suppose that $(N :_R M)$ is a prime ideal of R with $(N :_R M) \cap S = \phi$. Then the following statements are equivalent:*
(i) N is an S-2-absorbing primary submodule of M.
(ii) There exists an $s \in S$ such that for any $x_1, x_2 \in M$, if $(N :_R x_1) - (N :_R s^2 x_2) \cup \sqrt{(N :_R M)} \neq \phi$, then $N = (N + Rx_1) \cap (N + Rsx_2)$.

Proof. (i)⇒(ii) Assume that N is an S-2-absorbing primary submodule of M and $s \in S$ satisfies the S-2-absorbing primary condition. Let $ab \in (N :_R x_1) - (N :_R s^2 x_2) \cup \sqrt{(N :_R M)}$, where $a, b \in R$. Then $abx_1 \in N$ and $s^2 abx_2 \notin N$ and $s^k ab \notin \sqrt{(N :_R M)}$ for $k \in \mathbb{N}$. It is clear that $N \subseteq (N + Rx_1) \cap (N + Rsx_2)$. Let $n \in (N + Rx_1) \cap (N + Rsx_2)$. Then $n = n_1 + r_1 x_1 = n_2 + r_2 s x_2$, where $n_1, n_2 \in N$ and $r_1, r_2 \in R$. Hence, we have $abn = abn_1 + abr_1 x_1 = abn_2 + abr_2 s x_2$ and since $abr_1 x_1, abn_1, abn_2 \in N$, so $abr_2 s x_2 = (sab)(r_2)x_2 \in N$. Since N is an S-2-absorbing primary submodule of M and $s^2 abx_2 \notin N$, therefore either $r_2 s x_2 \in N$ or $s^2 abr_2 \in \sqrt{(N :_R M)}$. Take the case $s^2 abr_2 \in \sqrt{(N :_R M)}$. Then there exists a positive integer l such that $(s^2 ab)^l (r_2)^l \in (N :_R M)$. Since $(N :_R M)$ is prime and $s^k ab \notin \sqrt{(N :_R M)}$ implies $(s^2 ab)^l \notin (N :_R M)$. Therefore $(r_2)^l \in (N :_R M)$, which gives $r_2 \in (N :_R M)$ and so $r_2 s x_2 \in N$. Thus, in both cases, we have $n = n_2 + r_2 s x_2 \in N$. Therefore, $N = (N + Rx_1) \cap (N + Rsx_2)$.
(ii)⇒(i) Let $s \in S$ satisfy condition (ii) and fix $t = s^2 \in S$. Suppose that $abm \in N$ where $a, b \in R$, $m \in M$ and $s^2 am \notin N$ and $s^2 ab \notin \sqrt{(N :_R M)}$. We have to show that $s^2 bm \in N$. Now, we have $a \in (N :_R bm) - (N :_R s^2 m) \cup \sqrt{(N :_R M)}$. Put $x_1 = bm, x_2 = m$ in given assumption, then we have $N = (N + Rbm) \cap (N + Rsm)$. Thus, we obtain $s^2 bm \in N$. Therefore, N is an S-2-absorbing primary submodule of M. □

Proposition 2.21. *Let N be a submodule of R-module M and $S \subseteq R$ a multiplicatively closed subset. Suppose that $(N :_R M) \cap S = \phi$ and there exists an $s \in S$ such that $(N :_M sa) = (N :_M sa^2)$ for all $a \in R - \sqrt{(N :_R M)}$. Then, if N is an irreducible submodule of M, then N is an S-2-absorbing primary submodule of M.*

Proof. Assume that $s \in S$ satisfies $(N :_M sa) = (N :_M sa^2)$ for all $a \in R - \sqrt{(N :_R M)}$. Fix s and let $a, b \in R$ and $m \in M$ such that $abm \in N$ but $sab \notin \sqrt{(N :_R M)}$. Then, we have to show that $sam \in N$ or $sbm \in N$. On contrary, we assume that $sam \notin N$ and $sbm \notin N$. It is obvious that $a \notin \sqrt{(N :_R M)}$ and $b \notin \sqrt{(N :_R M)}$. Now, $(N + Rsam)$ and $(N + Rsbm)$ are two submodules of M that properly contain N, so $N \subseteq (N + Rsam) \cap (N + Rsbm)$. Let $n \in (N + Rsam) \cap (N + Rsbm)$. Then $n = n_1 + r_1sam = n_2 + r_2sbm$, where $n_1, n_2 \in N$ and $r_1, r_2 \in R$. Hence, $an = an_1 + r_1sa^2m = an_2 + r_2sabm$. Since $an_1, an_2, r_2sabm \in N$, we get $r_1sa^2m \in N$. Thus, $r_1m \in (N :_M sa^2) = (N :_M sa)$. This implies that $r_1sam \in N$ and so $n \in N$. Thus, $N = (N + Rsam) \cap (N + Rsbm)$ which is a contradiction, since N is an irreducible submodule. Therefore, N is an S-2-absorbing primary submodule of M. □

Let M be an R-module and S a multiplicatively closed subset of ring R. We say that M is an S-cancellative module if there exists an $s \in S$ such that whenever $rx = ry$ for elements $x, y \in M$ and $r \in R$, then $x = sy$ and $y = sx$. Recall that a submodule K of M is called pure if $aM \cap K = aK$ for every $a \in R$.

Proposition 2.22. *Let S be a multiplicatively closed subset of ring R and M an S-cancellative R-module M. If a proper submodule N is pure with $(N :_R M) \cap S = \phi$, then the following satements are satisfied:*
(i) N is an S-2-absorbing primary submodule of M with $\sqrt{(N :_R M)} = \{0\}$.
(ii) N is an S-2-absorbing submodule of M.

Proof. (i) Suppose that $s \in S$ satisfies the S-cancellative property. Let $abx \in N$ for some $a, b \in R$ and $x \in M$. Since N is pure, $abx \in abM \cap N = abN$. Hence, $abx = aby$ for some $y \in N$. By S-cancellative property we get, $bx = sby$ and $by = sbx$ and so $sbx \in N$. Thus, N is an S-2-absorbing primary submodule of M. Now, Let $r \in \sqrt{(N :_R M)}$ for some nonzero element r of R. Since N is proper, let $m \in M - N$. Then there exists a positive integer k such that $r^km \in r^kM \cap N = r^kN$. So $r^km = r^kn$ for some $n \in N$. This implies that $m = sn$ and $n = sm$ for some $s \in S$. Hence, $m \in N$ which is a contradiction. Therefore, $\sqrt{(N :_R M)} = \{0\}$.
(ii) From part (i) we have $\{0\} \subseteq (N :_R M) \subseteq \sqrt{(N :_R M)} = \{0\}$. Hence in this case, S-2-absorbing and S-2-absorbing primary are equivalent. □

3. Some More Properties of S-2-Absorbing Primary Submodules

Let M be an R-module. The idealization of M or trivial extension $R \propto M = R \oplus M$ is a commutative ring with componentwise addition and multiplication defined by $(r_1, x_1)(r_2, x_2) = (r_1r_2, r_1x_2 + r_2x_1)$ for each $r_1, r_2 \in R$ and $x_1, x_2 \in M$ [3]. Assume that A is an ideal of R and N is a submodule of M. Then $A \propto N$ is an ideal of $R \propto M$ if and only if $AM \subseteq N$ [3, Theorem 3.1]. Note that by [3, Theorem 3.2], $\sqrt{A \propto N} = \sqrt{A} \propto M$. Suppose that S is a multiplicatively closed subset of R and N is a submodule of M. Then $S \propto N$ is a multiplicatively closed subset of $R \propto M$ [3, Theorem 3.8]. Now, we characterize S-2-absorbing primary ideals of R in terms of $S \propto M$-2-absorbing primary ideals of $R \propto M$.

Proposition 3.1. *Suppose that $S \subseteq R$ is a multiplicatively closed subset and I is an ideal of R with $I \cap S = \phi$. Then the following statements are equivalent:*
(i) I is an S-2-absorbing primary ideal of R.
(ii) $I \propto M$ is an $S \propto 0$-2-absorbing primary ideal of $R \propto M$.
(iii) $I \propto M$ is an $S \propto M$-2-absorbing primary ideal of $R \propto M$.

Proof. (i)⇒(ii) Let $(a, m_1)(b, m_2)(c, m_3) = (abc, abm_3 + acm_2 + bcm_1) \in I \propto M$ for some $a, b, c \in R$ and $m_1, m_2, m_3 \in M$. Then we get $abc \in I$. By the assumption, there is an $s \in S$ such that $sab \in I$ or $sac \in I$ or $sbc \in \sqrt{I}$. Then we obtain $(s, 0)(a, m_1)(b, m_2) = (sab, sam_2 + sbm_1) \in I \propto M$ or $(s, 0)(a, m_1)(c, m_3) = (sac, sam_3 + scm_1) \in I \propto M$ or $(s, 0)(b, m_2)(c, m_3) = (sbc, sbm_3 + scm_2) \in \sqrt{I} \propto M = \sqrt{I \propto M}$, where $(s, 0) \in S \propto 0$. Thus, $I \propto M$ is an $S \propto 0$-2-absorbing primary ideal of $R \propto M$.
(ii)⇒(iii) : It is obvious from Proposition 2.5, since $S \propto 0 \subseteq S \propto M$.
(iii)⇒(i) Suppose that $abc \in I$ for some $a, b, c \in R$. Then $(a, 0)(b, 0)(c, 0) \in I \propto M$. Since $I \propto M$ is an $S \propto M$-2-absorbing primary ideal of $R \propto M$, there is an $(s, x) \in S \propto M$ such that $(s, x)(a, 0)(b, 0) = (sab, abx) \in I \propto M$ or $(s, x)(b, 0)(c, 0) = (sbc, bcx) \in I \propto M$ or $(s, x)(a, 0)(c, 0) = (sac, acx) \in \sqrt{I \propto M} = \sqrt{I} \propto M$ and hence we have $sab \in I$ or $sbc \in I$ or $sac \in \sqrt{I}$. Thus, I is an S-2-absorbing primary ideal of R. □

Proposition 3.2. *Suppose that $f : M \to M'$ is an R-homomorphism and S is a multiplicatively closed subset of R. The following statements hold:*
(i) If N' is an S-2-absorbing primary submodule of M' and $(f^{-1}(N') :_R M) \cap S = \phi$, then $f^{-1}(N')$ is an S-2-absorbing primary submodule of M.
(ii) If f is an epimorphism and N is an S-2-absorbing primary submodule of M containing $Ker(f)$, then $f(N)$ is an S-2-absorbing primary submodule of M'.

Proof. (i) Let $abm \in f^{-1}(N')$ for some $a, b \in R$ and $m \in M$. Then we have $f(abm) = abf(m) \in N'$. Since N' is an S-2-absorbing primary submodule, there exists $s \in S$ such that either $sab \in \sqrt{(N' :_R M')}$ or $saf(m) = f(sam) \in N'$ or $sbf(m) = f(sbm) \in N'$. If $sab \in \sqrt{(N' :_R M')}$ then we conclude that $sab \in \sqrt{(f^{-1}(N') :_R M)}$, since $(N' :_R M') \subseteq (f^{-1}(N') :_R M)$. On the other hand, if $f(sam) \in N'$ or $f(sbm) \in N'$, we obtain either $sam \in f^{-1}(N')$ or $sbm \in f^{-1}(N')$. Thus, $f^{-1}(N')$ is an S-2-absorbing primary submodule of M.
(ii) Assume that N is an S-2-absorbing primary submodule of M containing $Ker(f)$. If $(f(N) :_R M') \cap S \neq \phi$, there is an $s \in S$ such that $s \in (f(N) :_R M')$. This implies that $sM' \subseteq f(N)$ and so $f(sM) = sf(M) = sM' \subseteq f(N)$. Hence, we get $sM \subseteq sM + Ker(f) \subseteq N + Ker(f) = N$. Thus, $s \in (N :_R M) \cap S$, which is a contradiction. Now, assume that $aby \in f(N)$ for some $a, b \in R$ and $y \in M'$. Then there exist $n \in N$ and $x \in M$ such that $y = f(x)$ and $aby = abf(x) = f(abx) = f(n)$. This implies $f(abx - n) = 0$ which gives $abx - n \in Ker(f) \subseteq N$ and so $abx \in N$. Since N is an S-2-absorbing primary submodule of M, there exists an $s \in S$ such that $sab \in \sqrt{(N :_R M)}$ or $sax \in N$ or $sbx \in N$. As $(N :_R M) \subseteq (f(N) :_R M')$, consequently, we conclude that $sab \in \sqrt{(f(N) :_R M')}$ or $f(sax) = saf(x) = say \in f(N)$ or $f(sbx) = sbf(x) = sby \in f(N)$. Therefore, $f(N)$ is an S-2-absorbing primary submodule of M'. □

Corollary 3.3. *Let K be a submodule of an R-module M and $S \subseteq R$ be a multiplicatively closed subset. The following statements hold:*
(i) If N' is an S-2-absorbing primary submodule of M with $(N' :_R K) \cap S = \phi$, then $K \cap N'$ is an S-2-absorbing primary submodule of K.
(ii) Suppose that N is a submodule of M containing K. Then N is an S-2-absorbing primary submodule of M if and only if N/K is an S-2-absorbing primary submodule of M/K.

Proof. (i) Consider that the injection $i : K \to M$ defined by $i(x) = x$ for all $x \in K$. Then we have $i^{-1}(N') = K \cap N'$. Now, we show that $(i^{-1}(N') :_R K) \cap S = \phi$. Let $s \in (i^{-1}(N') :_R K) \cap S$, then we have $sK \subseteq i^{-1}(N') = K \cap N' \subseteq N'$ and so $s \in (N' :_R K) \cap S$, which gives a contradiction. Then, the result follows from Proposition 3.2.
(ii)($\Rightarrow$) Consider the canonical homomorphism $\pi : M \to M/K$ defined by $\pi(x) = x + K$ for all $x \in M$. Then, the result follows from Proposition 3.2.
($\Leftarrow$) Let $abm \in N$ for some $a, b \in R$ and $m \in M$. Then we have $ab(m + K) = abm + K \in N/K$. Thus, there exists an $s \in S$ such that $sab \in \sqrt{(N/K :_R M/K)} = \sqrt{(N :_R M)}$ or $sa(m + K) = sam + K \in N/K$ or $sb(m + K) = sbm + K \in N/K$ by the assumption. Hence, we get $sab \in \sqrt{(N :_R M)}$ or $sam \in N$ or $sbm \in N$. Consequently, N is an S-2-absorbing primary submodule of M. □

Definition 3.4. *Let S be a multiplicatively closed subset of a ring R and M an R-module. A submodule N of M is said to be S-primary if $(N :_R M) \cap S = \phi$ and there exists a fixed $s \in S$ such that whenever $r \in R$ and $m \in M$ with $rm \in N$, then either $sm \in N$ or $sr \in \sqrt{(N :_R M)}$.*

An ideal I is an S-primary ideal of R if I is an S-primary submodule of R-module R.

Remark 3.5. *It is clear from the definition of S-primary submodule that every primary is S-primary. Moreover, every S-primary is S-2-absorbing primary.*

Let N be an R-submodule of M. It is known that if N is primary , then $(N :_R M)$ is a primary ideal of R. However, the converse is not true in general. Now, we characterize certain primary submodules in terms of S -primary submodules.

Theorem 3.6. *Suppose that N is a submodule of M provided $(N :_R M) \subseteq J(R)$, where $J(R)$ is the Jacobson radical of R. The following statements are equivalent:*
(i) N is a primary submodule of M.
(ii) $(N :_R M)$ is a primary ideal of R and N is an $(R-\mathcal{M})$-primary submodule of M for each maximal ideal $\mathcal{M} \in Max(R)$.

Proof. (i)⇒(ii) Assume that N is a primary submodule of M. Then $(N :_R M)$ is a primary ideal of R. Since $(N :_R M) \subseteq J(R)$, $(N :_R M) \subseteq \mathcal{M}$ for each maximal ideal $\mathcal{M} \in Max(R)$, hence $(N :_R M) \cap (R - \mathcal{M}) = \phi$. The rest follows from Remark 3.5.
(ii)⇒(i) Suppose $(N :_R M)$ is a primary ideal and N is an $(R - \mathcal{M})$-primary submodule of M for each $\mathcal{M} \in Max(R)$. Let $rm \in N$ with $r \notin \sqrt{(N :_R M)}$ for some $r \in R$ and $m \in M$. Let $\mathcal{M} \in Max(R)$. As N is $(R - \mathcal{M})$-primary , then there exists for sure an $s_{\mathcal{M}} \in R - \mathcal{M}$ such that $s_{\mathcal{M}}r \in \sqrt{(N :_R M)}$ or $s_{\mathcal{M}}m \in N$. If $s_{\mathcal{M}}r \in \sqrt{(N :_R M)}$, then, since $\sqrt{(N :_R M)}$ is prime, we get $r \in \sqrt{(N :_R M)}$ or $(s_{\mathcal{M}})^k \in (N :_R M) \cap (R - \mathcal{M})$ for some positive integer k. In both cases we have a contradiction. So we have $s_{\mathcal{M}}m \in N$. Now consider the set $\mathcal{T} = \{s_{\mathcal{M}} : \exists \mathcal{M} \in Max(R), s_{\mathcal{M}} \notin \mathcal{M}$ and $s_{\mathcal{M}}m \in N\}$ Then, it is easy to see that $(\mathcal{T}) = R$. This yields $1 = a_1 s_{\mathcal{M}_1} + a_2 s_{\mathcal{M}_2} + \cdots + a_n s_{\mathcal{M}_n}$ for some $a_i \in R$ and $s_{\mathcal{M}_i} \notin \mathcal{M}_i$ with $s_{\mathcal{M}_i}m \in N$, where $\mathcal{M}_i \in Max(R)$ for each $i = 1, 2, \ldots, n$. This implies that $m = a_1 s_{\mathcal{M}_1} m + a_2 s_{\mathcal{M}_2} m + \cdots + a_n s_{\mathcal{M}_n} m \in N$. Therefore, N is a primary submodule. □

Proposition 3.7. *Let S be a multiplicatively closed subset of a ring R and M a Noetherian R-module. Then, every irreducible submodule is S-primary.*

Proof. Assume that N is not S-primary. Then for every $s \in S$, there exist $a \in R$ and $x \in M$ with $ax \in N$ such that $sx \notin N$ and $(sa)^n \notin (N :_R M)$ for every $n \in \mathbb{N}$. Consider the increasing chain of submodules $\{(N :_M (sa)^n)\}_{n\in\mathbb{N}}$ of M. Since M is Noetherian, there exists a positive integer k such that $(N :_M (sa)^k) = (N :_M (sa)^{k+1})$. We want to show that $N = (N + Rs^k a^k y) \cap (N + Rsx)$, where $y \in M$ such that $(sa)^k y \notin N$. It is clear that $N \subseteq (N + Rs^k a^k y) \cap (N + Rsx)$. Now, let $n = u + r_1 s^k a^k y = v + r_2 sx$, where $u, v \in N$ and $r_1, r_2 \in R$. Since $ax \in N$, we have $an = av + r_2 sax \in N$ and so $r_1 (sa)^{k+1} y \in N$. By $(N :_M (sa)^k) = (N :_M (sa)^{k+1})$, we obtain that $r_1 s^k a^k y \in N$ and hence $n \in N$. Since $sx \notin N$ and $(sa)^k y \notin N$, the submodules $(N + Rs^k a^k y)$ and $(N + Rsx)$ properly contain N. Thus, N is not irreducible. □

Let M_i be an R_i-module for each $i = 1, 2$. Suppose that $R = R_1 \times R_2$ and $M = M_1 \times M_2$. Then it is obvious that M is an R-module and all submodules of M have the form $N = N_1 \times N_2$, where N_i is a submodule of M_i for each $i = 1, 2$. Furthermore, if S_i is a multiplicatively closed subset of R_i, then $S = S_1 \times S_2$ is a multiplicatively closed subset of R. The following theorem studies the S-2-absorbing primary concept in cartesian product of modules.

Proposition 3.8. *Suppose that M_i is an R_i-module and S_i is a multiplicatively closed subset of R_i for each $i = 1, 2$. Let $M = M_1 \times M_2$, $R = R_1 \times R_2$ and $S = S_1 \times S_2$. Assume that $N = N_1 \times N_2$ is a submodule of M, where N_1 is a submodule of M_1 and N_2 is a submodule of M_2. Consider the following statements:*
(A) N is an S-2-absorbing primary submodule of M.
(B1) $(N_1 :_{R_1} M_1) \cap S_1 \neq \phi$ and N_2 is an S_2-2-absorbing primary submodule of M_2.
(B2) $(N_2 :_{R_2} M_2) \cap S_2 \neq \phi$ and N_1 is an S_1-2-absorbing primary submodule of M_1.
(B3) N_1 is an S_1-primary submodule of M_1 and N_2 is an S_2-primary submodule of M_2.
Then the following statements hold:
(i) (A) implies (B1) or (B2) or (B3).
(ii) (B1) or (B2) implies (A).

Proof. (i) Suppose that N is an S-2-absorbing primary submodule of M. First, by Proposition 2.13 note that $\sqrt{(N :_R M)} = \sqrt{(N_1 :_{R_1} M_1)} \times \sqrt{(N_2 :_{R_2} M_2)}$ is an S-2-absorbing ideal of R. So that $(N_1 :_{R_1} M_1) \cap S_1 = \phi$ or $(N_2 :_{R_2} M_2) \cap S_2 = \phi$. Suppose that $(N_1 :_{R_1} M_1) \cap S_1 \neq \phi$. We aim to show that N_2 is an S_2-2-absorbing primary submodule of M_2. Let $abm \in N_2$ for some $a, b \in R_2$ and $m \in M_2$. Then we get $(0_{R_1}, a)(0_{R_1}, b)(0_{M_1}, m) = (0_{M_1}, abm) \in N_1 \times N_2 = N$. As N is an S-2-absorbing primary submodule of M, there exists $s = (s_1, s_2) \in S$ such that $s(0_{R_1}, a)(0_{R_1}, b) = (0_{R_1}, s_2 ab) \in \sqrt{(N :_R M)}$ or $s(0_{R_1}, a)(0_{M_1}, m) = (0_{M_1}, s_2 am) \in N$

or $s(0_{R_1}, b)(0_{M_1}, m) = (0_{M_1}, s_2bm) \in N$. This implies that either $s_2ab \in \sqrt{(N_2 :_{R_2} M_2)}$ or $s_2am \in N_2$ or $s_2bm \in N_2$. Thus, N_2 is an S_2-2- absorbing primary submodule of M_2. If $(N_2 :_{R_2} M_2) \cap S_2 \neq \phi$, by using the same argument, we get N_1 is an S_1-2-absorbing primary submodule of M_1. Now suppose that $(N_1 :_{R_1} M_1) \cap S_1 = \phi$ and $(N_2 :_{R_2} M_2) \cap S_2 = \phi$. We will prove that N_1 is an S_1-primary submodule of M_1 and N_2 is an S_2-primary submodule of M_2. First, by assumption, there is a fixed $s = (s_1, s_2) \in S$ satisfying N to be an S-2-absorbing primary submodule of M. Assume that N_1 is not an S_1-primary submodule of M_1. Then there exists $r \in R_1$ and $x \in M_1$ such that $rx \in N_1$ but $s_1r \notin \sqrt{(N_1 :_{R_1} M_1)}$ and $s_1x \notin N_1$. Moreover, $(N_2 :_{R_2} M_2) \cap S_2 = \phi$ and $s_2 \notin (N_2 :_{R_2} M_2)$, so there exists $y \in M_2$ such that $s_2y \notin N_2$. Now, we have $(r, 1)(1, 0)(x, y) = (rx, 0_{M_2}) \in N_1 \times N_2 = N$. Since N is an S-2-absorbing primary submodule of M, we obtain either $(s_1, s_2)(r, 1)(1, 0) = (s_1r, 0) \in \sqrt{(N :_R M)}$ or $(s_1, s_2)(r, 1)(x, y) = (s_1rx, s_2y) \in N$ or $(s_1, s_2)(1, 0)(x, y) = (s_1x, 0) \in N$. Hence we conclude that either $s_1r \in \sqrt{(N_1 :_{R_1} M_1)}$ or $s_1x \in N_1$ or $s_2y \in N_2$ which all of them are contradictions. Hence, N_1 is an S_1-primary submodule of M_1. Similarly, N_2 is an S_2-primary submodule of M_2.

(ii) Suppose that $(N_1 :_{R_1} M_1) \cap S_1 \neq \phi$ and N_2 is an S_2-2-absorbing primary submodule of M_2. We will prove that N is an S-2-absorbing primary submodule of M. First, it is clear that $(N :_R M) \cap S = \phi$. Let $a_1, a_2 \in R_1$, $b_1, b_2 \in R_2$, $m_1 \in M_1$ and $m_2 \in M_2$ such that $(a_1, b_1)(a_2, b_2)(m_1, m_2) = (a_1a_2m_1, b_1b_2m_2) \in N$. Since $(N_1 :_{R_1} M_1) \cap S_1 \neq \phi$, there is $s_1 \in S_1$ such that $s_1x \in N_1$ for all $x \in M_1$. On the other hand, there exists a fixed $s_2 \in S_2$ satisfying N_2 to be an S_2-2- absorbing primary submodule of M_2. Now, let $s = (s_1, s_2) \in S$. Since $b_1b_2m_2 \in N_2$ and N_2 is an S_2-2-absorbing primary submodule of M_2, we conclude either $s_2b_1b_2 \in \sqrt{(N_2 :_{R_2} M_2)}$ or $s_2b_1m_2 \in N_2$ or $s_2b_2m_2 \in N_2$. This yields that $s(a_1, b_1)(a_2, b_2) = (s_1a_1a_2, s_2b_1b_2) \in \sqrt{(N_1 :_{R_1} M_1)} \times \sqrt{(N_2 :_{R2} M_2)} = \sqrt{(N :_R M)}$ or $s(a_1, b_1)(m_1, m_2) = (s_1a_1m_1, s_2b_1m_2) \in N_1 \times N_2 = N$ or $s(a_2, b_2)(m_1, m_2) = (s_1a_2m_1, s_2b_2m_2) \in N_1 \times N_2 = N$. Thus, we conclude that N is an S-2-absorbing primary submodule of M. Similarly, if N_1 is an S_1-2-absorbing primary submodule of M_1 and $(N_2 :_{R_2} M_2) \cap S_2 \neq \phi$, N is an S-2-absorbing primary submodule of M. □

In the above theorem, the condition $(N_i :_{R_i} M_i) \cap S_i \neq \phi$ is necessary. Generally, if N_1 is S_1-2-absorbing primary submodule of M_1 and N_2 is S_2-2-absorbing primary submodule of M_2, then $N_1 \times N_2$ may not be an $S_1 \times S_2$-2-absorbing primary of $M_1 \times M_2$. The following example illustrate this.

Example 3.9. *Consider the submodules $N_1 = 3\mathbb{Z}$ and $N_2 = 8\mathbb{Z}$ of $\mathbb{Z}$-module $\mathbb{Z}$. Let $S_1 = \{2^n : n \in \mathbb{N} \cup \{0\}\}$ and $S_2 = \{3^n : n \in \mathbb{N} \cup \{0\}\}$. Note that $(N_i :_{R_i} M_i) \cap S_i = \phi$ for $i = 1, 2$. Moreover, N_1 and N_2 are S_1 and S_2-2-absorbing primary submodules of $\mathbb{Z}$, respectively. On the other hand, $N = N_1 \times N_2$ is not an $S = S_1 \times S_2$-2-absorbing primary submodule of $M = \mathbb{Z} \times \mathbb{Z}$. Since $(1, 2)(1, 2)(3, 2) \in N$, but for each $s = (s_1, s_2) \in S$, $s(1, 2)(1, 2) \notin \sqrt{(N :_{\mathbb{Z}\times\mathbb{Z}} M)} = 3\mathbb{Z} \times 2\mathbb{Z}$ and $s(1, 2)(3, 2) \notin N$.*

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