



## Existence and Uniqueness of Weak Solution of $p(x)$ -Laplacian in Sobolev Spaces With Variable Exponents in Complete Manifolds

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**Abstract.** The paper deals with the existence and uniqueness of a non-trivial solution to non-homogeneous  $p(x)$ -laplacian equations, managed by non polynomial growth operator in the framework of variable exponent Sobolev spaces on Riemannian manifolds. The mountain pass Theorem is used.

### 1. Introduction

Let  $(M, g)$  be a complete non-compact Riemannian manifold, we consider the following equation

$$-\Delta_{p(x)}u(x) + h(x, u(x), \nabla u(x)) + |u(x)|^{p(x)-2}u(x) = f(x, u(x)), \quad (1)$$

where  $-\Delta_{p(x)}u(x) = -\operatorname{div}(|\nabla u(x)|^{p(x)-2} \cdot \nabla u(x))$  is the  $p(x)$ -laplacian in  $(M, g)$ . The conditions assumed on the functions  $f$  and  $h$  are:

$(f_1)$  :  $f(x, 0) = 0$  and  $f$  is measurable to the first variable and continuous to the second variable.

$(f_2)$  : There exists  $\beta > p^+$  and some  $A > 0$  such as for each  $|\alpha| > A$  we have

$$0 < \int_M F(x, \alpha) dv_g(x) \leq \int_M f(x, \alpha) \cdot \frac{\alpha}{\beta} dv_g(x) \text{ a.e } x \in M,$$

where  $F(x, \alpha) = \int_0^\alpha f(x, t) dt$  being the primitive of  $f(x, \alpha)$ .

$(f_3)$  :  $\lim_{|\alpha| \rightarrow \infty} \frac{f(x, \alpha)}{|\alpha|^{p(x)-1}} = 0$  uniformly a.e  $x \in M$ .

#### Example 1.1.

$$f(x, \alpha) = c |\alpha|^{K(x)-1} \quad \forall c > 0, \beta < K(x) < p(x)$$

is a function satisfying the above conditions.

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$(h_1) : h(x, s) : M \times \mathbb{R} \rightarrow \mathbb{R}$  be a Carathéodory function such as for a.e  $x \in M$  and for all  $s \in \mathbb{R}$ ,

$$|h(x, s)| \leq \gamma(x) + l(s) \cdot |s|^{p(x)},$$

where  $l : \mathbb{R} \rightarrow \mathbb{R}^+$  is a continuous increasing positive function that belongs to  $L^\infty(M)$  and  $\gamma(x) \in L^1(M)$ .

The study of variational problems with non-standard growth conditions is an interesting topic in recent years.  $p(x)$ –growth condition can be regarded as an important case of non-standard  $(p, q)$ –growth conditions. Recently, the study of these kind of problems has attracted more and more attention. For example Fan Xian-Ling and Zhang Qi-Hu, in [9], proved the existence of solutions to the Dirichlet problem of  $p(x)$ –laplacian

$$\begin{cases} -\operatorname{div}(|\nabla u|^{p(x)-2} \nabla u) = f(x, u) & x \in \Omega, \\ u = 0 & x \in \partial\Omega, \end{cases}$$

with several sufficient conditions, and a criterion of existence for an infinite number of pairs of solutions to this problem. For more result we refer the reader to [5, 18]. The typical applications of variable exponent equations include models for electrorheological fluids [3], image restoration processing [7], non-Newtonian fluid dynamics [14], Poisson equation [8], elasticity equations [15, 20], and thermistor model [21].

Moving on to another field undergoing great development; the Sobolev space on Riemannian manifolds. The theory of Sobolev space for non compact manifold arose in the 1970s with the work of Aubin, Cantor Hoffman, and Spruck, many of the results presented in their lecture notes have been collected between the 1980s and the 1990s. It has been studied very intensively for over fifty years see [4, 12, 16] and also e.g [10, 11, 13, 17]. This is also the case for the applications already mentioned to scalar curvature and generalized scalar curvature equation, we quote [6]. Additionally, Yamabe problem for conformal metrics with prescribed scalar curvature [19], and to obtain isoperimetric type inequalities [16].

Considering that some basic properties of the standard Lebesgue space are not valid in the variable exponent case. For example, Zhikov [21] observed that in general smooth functions are not dense in  $W^{k,p(\cdot)}(\Omega)$ . Besides, the challenges coming are due to the absence of topological properties like convergence and embedding.

In this paper, we will be applying the Sobolev spaces with their variable exponents on the non-compact Riemannian manifolds theory to our equation (1). As for the structure of the paper, we will be sectioning it into: Recalling some definitions and Lemmas. We will then move to prove the existence of non-trivial solution using the mountain pass Theorem, and we will finish it by a demonstration of the uniqueness of non-trivial solution with  $f$  as the Contraction Lipschitz Continuous function.

## 2. Framework Space: Notations and Basic Properties

First of all, we must recall the most important and pertinent properties and notations, by that, referring to [4, 12, 18] for more details.

**Definition 2.1.** Let  $\nabla$  be the Levi-Civita connection. If  $u$  is a smooth function on  $M$ , then  $\nabla^k u$  denotes the  $k$ –th covariant derivative of  $u$ , and  $|\nabla^k u|$  the norm of  $\nabla^k u$  defined in local coordinates by

$$|\nabla^k u|^2 = g^{i_1 j_1} \dots g^{i_k j_k} (\nabla^{k} u)_{i_1 \dots i_k} (\nabla^{k} u)_{j_1 \dots j_k}$$

where Einstein’s convention is used.

- Given a variable exponent  $p$  in  $\mathcal{P}(M)$  and a natural number  $k$ , introduce

$$C_k^{p(\cdot)}(M) = \{ u \in C^\infty(M) \text{ such that } \forall j \ 0 \leq j \leq k \ |\nabla^j u| \in L^{p(\cdot)} \}$$

on  $C_k^{p(\cdot)}(M)$  define the norm

$$\| u \|_{L_k^{p(\cdot)}} = \sum_{j=0}^k \| \nabla^j u \|_{L^{p(\cdot)}}$$

**Definition 2.2.** Given  $(M, g)$  a smooth Riemannian manifold, and  $\gamma : [a, b] \rightarrow M$  a curve of class  $C^1$ , the length of  $\gamma$  is

$$l(\gamma) = \int_a^b \sqrt{g\left(\frac{d\gamma}{dt}, \frac{d\gamma}{dt}\right)} dt,$$

and for a pair of points  $x, y \in M$ , we define the distance  $d_g(x, y)$  between  $x$  and  $y$  by

$$d_g(x, y) = \inf \{ l(\gamma) : \gamma : [a, b] \rightarrow M \text{ such that } \gamma(a) = x \text{ and } \gamma(b) = y \}$$

**Definition 2.3.** A function  $s : M \rightarrow \mathbb{R}$  is log-Hölder continuous if there exists a constant  $c$  such that for every pair of points  $\{x, y\}$  in  $M$  we have

$$|s(x) - s(y)| \leq \frac{c}{\log\left(e + \frac{1}{d_g(x,y)}\right)}.$$

We note by  $\mathcal{P}^{\log}(M)$  the set of log-Hölder continuous variable exponents.

**Proposition 2.4.** Let  $p \in \mathcal{P}^{\log}(M)$ , and let  $(\Omega, \phi)$  be a chart such that

$$\frac{1}{2} \delta_{ij} \leq g_{ij} \leq 2 \delta_{ij}$$

as bilinear forms, where  $\delta_{ij}$  is the delta Kronecker symbol. Then  $\rho\phi^{-1} \in \mathcal{P}^{\log}(\phi(\Omega))$ .

**Definition 2.5.** We say that the  $n$ -manifold  $(M, g)$  has property  $B_{\text{vol}}(\lambda, v)$  if its geometry is bounded in the following sense:

- $Rc(g) \geq \lambda(n - 1)g$  for some  $\lambda$
- There exists some  $v > 0$  such that  $|B_1(x)|_g \geq v \forall x \in M$ .

**Proposition 2.6.** Let  $(M, g)$  be a complete Riemannian  $n$ -manifold. Then, if the embedding  $L^1_1(M) \hookrightarrow L^{\frac{n}{n-1}}(M)$  holds, then whenever the real numbers  $p$  and  $q$  satisfy

$$1 \leq p < n,$$

and

$$p \leq q \leq p^* = \frac{np}{n-p},$$

the embedding  $L^p_1(M) \hookrightarrow L^q(M)$  also holds.

**Proposition 2.7.** Assume that the complete  $n$ -manifold  $(M, g)$  has property  $B_{\text{vol}}(\lambda, v)$  for some  $(\lambda, v)$ . Then there exist positive constants  $\delta_0 = \delta_0(n, \lambda, v)$  and  $A = A(n, \lambda, v)$ , we have, if  $R \leq \delta_0$ , if  $x \in M$  if  $1 \leq p \leq n$ , and if  $u \in L^p_{1,0}(B_R(x))$  the estimate

$$\|u\|_{L^q} \leq A_q \|\nabla u\|_{L^p},$$

where  $\frac{1}{q} = \frac{1}{p} - \frac{1}{n}$ .

**Proposition 2.8.** Assume that for some  $(\lambda, v)$  the complete  $n$ -manifold  $(M, g)$  has property  $B_{\text{vol}}(\lambda, v)$ . Let  $q \in \mathcal{P}(M)$  be uniformly continuous with  $p^+ < n$ . Then  $L^p_1(M) \hookrightarrow L^{q(\cdot)}(M) \forall p \in \mathcal{P}(M)$  such that  $p \ll q \ll p^* = \frac{np}{n-p}$ . In fact, for  $\|u\|_{L^{p(\cdot)}}$  sufficiently small we have the estimate

$$\varrho_{q(\cdot)}(u) \leq G(\varrho_{p(\cdot)}(u) + \varrho_{p(\cdot)}(|\nabla u|)),$$

where  $G$  is a positive constant depend on  $n, \lambda, v, p$  and  $q$ .

**3. Existence of Non-trivial Solution**

**Definition 3.1.**  $u \in L_1^{p(\cdot)}(M)$  is said to be a weak solution of the problem (1) if for every  $\phi \in D(M)$  we have

$$\int_M |\nabla u(x)|^{p(x)-2} g(\nabla u(x), \nabla \phi(x)) dv_g(x) + \int_M h(x, u(x)) \cdot \phi(x) dv_g(x) + \int_M |u(x)|^{p(x)-2} u(x) \phi(x) dv_g(x) = \int_M f(x, u(x)) \phi(x) dv_g(x) \tag{2}$$

**Theorem 3.2.** Suppose that  $(f_1) - (f_3)$  and  $(h_1)$  are true. Then the problem (1) possesses a non-trivial weak solution.

To access our main premises, the ones shown in the first Theorem, we have to initially demonstrate few lemmas related to the mountain pass Theorem and Palais-Smale condition. Considering the functional

$$A(u) = \int_M \frac{1}{p(x)} (|\nabla u(x)|^{p(x)} + |u(x)|^{p(x)}) dv_g(x) + \int_M H(x, u(x)) dv_g(x) - \int_M F(x, u(x)) dv_g(x),$$

with  $H(x, u(x)) = \int_0^u h(x, t) dt$  being the primitive of  $h(x, \alpha)$ . And  $H(x, 0) = 0$ .

It's follow from (2) and the hypothesis of  $(f_1) - (f_3)$ ,  $(h_1)$  and the above definition of  $H$ , that  $A$  is  $C^1$  functional. Let  $DA = DB - DC$  the differential of  $A = B - C$  with

$$DB = \int_M |\nabla u(x)|^{p(x)-2} g(\nabla u(x), \nabla \phi(x)) dv_g(x) + \int_M h(x, u(x)) \cdot \phi(x) dv_g(x) + \int_M |u(x)|^{p(x)-2} u(x) \phi(x) dv_g(x),$$

and

$$DC = \int_M f(x, u(x)) \phi(x) dv_g(x).$$

**Lemma 3.3.** The functional  $A$  satisfies mountain pass geometry in the sense that:

- i/  $A(0) = 0$ .
- ii/ There exists  $r, \eta > 0$  such that  $A(u) \geq \eta$  if  $\|u\| > r$ .
- iii/ There exists  $u, \|u\| > r$  such that  $A(u) \leq 0$ .

*Proof.* i/  $A(0) = 0$  is obvious.

ii/ we need the assumptions  $(f_2)$ ,  $(f_3)$  and  $(h_1)$ , then we obtain

$$\int_M F(x, u(x)) dv_g(x) \leq \frac{1}{\beta} \int_M |u(x)|^{p(x)} dv_g(x) \leq \frac{1}{\beta} \varrho_{p(\cdot)}(u),$$

and

$$\left| \int_M H(x, u(x)) dv_g(x) \right| \leq \int_M \left( \int_0^u |h(x, t)| dt \right) dv_g(x) \leq c_1 + c_2 \varrho_{p(\cdot)}(|u|), \quad \text{where } c_1, c_2 \text{ two positive constants.}$$

Choose  $\|u\| = r$  sufficiently small, so that  $\varrho_{p(\cdot)}(u) \leq \|u\|_{p(\cdot)}^p$  since  $\|u\| = r < 1$ . Now using the Poincaré inequality, we get

$$\begin{aligned} A(u) &= \int_M \frac{1}{p(x)} (|\nabla u(x)|^{p(x)} + |u(x)|^{p(x)}) dv_g(x) + \int_M H(x, u(x)) dv_g(x) \\ &\quad - \int_M F(x, u(x)) dv_g(x) \\ &\geq \frac{1}{p^+} (\varrho_{p(\cdot)}(|\nabla u|) + \varrho_{p(\cdot)}(u)) - c_2 \varrho_{p(\cdot)}(u) - \frac{1}{\beta} \varrho_{p(\cdot)}(u) \\ &\geq \frac{1}{G p^+} \rho_{q(\cdot)}(u) - \left(c_2 + \frac{1}{\beta}\right) \rho_{p(\cdot)}(u) \\ &\geq \frac{1}{G p^+} \|u\|^{q^-} - \left(c_2 + \frac{1}{\beta}\right) \|u\|^{p^+}, \end{aligned}$$

as in [11], since  $q^- \geq p^+$  and if  $\|u\|_{L^{p(\cdot)}(M)} = \gamma$  small enough, we have  $\|u\|_{L^{p(\cdot)}(M)}^{q^-} \ll \|u\|_{L^{p(\cdot)}(M)}^{p^+}$ . Hence

$$A(u) > 0.$$

We can prove *iii*/ by using  $(f_2)$  and  $(h_1)$ , for  $t > 0$  and  $u \neq 0$ . So,

$$\begin{aligned} A(tu) &= \int_M \frac{1}{p(x)} (|\nabla tu|^{p(x)} - |tu|^{p(x)}) dv_g(x) + \int_M H(x, tu) dv_g(x) \\ &\quad - \int_M F(x, tu) dv_g(x) \\ &= \int_M \frac{t^{p(x)}}{p(x)} (|\nabla tu|^{p(x)} - |tu|^{p(x)}) dv_g(x) + \int_M H(x, tu) dv_g(x) \\ &\quad - \int_M F(x, tu) dv_g(x) \\ &\leq \int_M \frac{t^{p(x)}}{p(x)} (|\nabla tu|^{p(x)} - |tu|^{p(x)}) dv_g(x) + \frac{1}{\beta} \int_M |tu|^{p(x)} dv_g(x) + c_1 \\ &\quad + c_2 \int_M |tu|^{p(x)} dv_g(x). \end{aligned}$$

This implies:

$$A(tu) \leq \frac{t^{p^+ - p^-}}{p^-} \left( \rho_{p(\cdot)}(|\nabla u|) - \rho_{p(\cdot)}(u) \right) + c_3 t^{p^-} \rho_{p(\cdot)}(u), \tag{3}$$

dividing (3) by  $t^{p^+}$  and passing the limit  $t \rightarrow \infty$  we get  $A(tu) \rightarrow -\infty$ , since  $p^+ > p^-$ . Hence,  $A(u)$  satisfies the hypothesis of mountain pass Theorem.  $\square$

**Lemma 3.4.** *The functional  $A$  satisfies Palais-Smale condition.*

*Proof.* Let  $\{u_n\}$  be a Palais-Smale sequence, such as the associated sequence of real numbers  $\{A(u_n)\}$  is bounded, and  $DA(u_n) \rightarrow 0$  in  $(L_1^{p(\cdot)}(M))'$ . We will first demonstrate that  $(u_n)$  is bounded in  $L_1^{p(\cdot)}(M)$ . And prove it throughout contradiction.

Let  $\|u_n\|_{L_1^{p(\cdot)}(M)} \rightarrow \infty$  as  $n \rightarrow \infty$ . Then, we have

$$\begin{aligned} A(u_n) - \frac{1}{\beta} \langle DA(u_n), u_n \rangle &= \left(\frac{1}{p^+} - \frac{1}{\beta}\right) \int_M (|\nabla u_n|^{p(x)} - |u_n|^{p(x)}) dv_g(x) \\ &\quad + \int_M \left(H(x, u_n) - h(x, u_n) \cdot \frac{u_n}{\beta}\right) dv_g(x) \\ &\quad + \int_M \left(f(x, u_n) \cdot \frac{u_n}{\beta} - F(x, u_n)\right) dv_g(x), \end{aligned}$$

by  $(f_2)$ , we obtain

$$\begin{aligned} A(u_n) - \frac{1}{\beta} \langle DA(u_n), u_n \rangle &\geq \left(\frac{1}{p^+} - \frac{1}{\beta}\right) \int_M (|\nabla u_n|^{p(x)} - |u_n|^{p(x)}) dv_g(x) \\ &\quad + \int_M \left(H(x, u_n) - h(x, u_n) \cdot \frac{u_n}{\beta}\right) dv_g(x) \\ &\geq J(u_n) + I(u_n), \end{aligned}$$

with,

$$J(u_n) = \left(\frac{1}{p^+} - \frac{1}{\beta}\right) \int_M (|\nabla u_n|^{p(x)} - |u_n|^{p(x)}) dv_g(x),$$

and

$$I(u_n) = \int_M \left(H(x, u_n) - h(x, u_n) \cdot \frac{u_n}{\beta}\right) dv_g(x).$$

$$J(u_n) \geq \left(\frac{1}{p^+} - \frac{1}{\beta}\right) (\varrho_{p(\cdot)}(u_n) + \varrho_{p(\cdot)}(|\nabla u_n|)),$$

or  $\varrho_{p(\cdot)}(|\nabla u_n|) + \varrho_{p(\cdot)}(u_n) \geq 2^{1-p^+} \|u_n\|_{L_1^{p(\cdot)}(M)}^{p^+}$ . So,

$$\begin{aligned} J(u_n) &\geq \left(\frac{1}{p^+} - \frac{1}{\beta}\right) 2^{1-p^+} \|u_n\|_{L_1^{p(\cdot)}(M)}^{p^+} \\ &\geq \left(\frac{1}{p^+} - \frac{1}{\beta}\right) \cdot \frac{2^{1-p^+}}{G^{p^-}} \cdot \|u_n\|_{L^q(\cdot)(M)}^{p^-}, \end{aligned}$$

with  $G$  being the positive constant of the embedding  $L_1^{p(\cdot)}(M) \hookrightarrow L^q(\cdot)(M)$ . And by  $(h_1)$ , we get

$$\int_M h(x, u_n) \cdot u_n dv_g(x) \leq c_1 + c_2 \|u_n\|_{L_1^{p(\cdot)}(M)}^{p(x)+1}.$$

And

$$\begin{aligned} \int_M H(x, u_n) dv_g(x) &= \int_M \left(\int_0^{u_n} f(x, t) dt\right) dv_g(x) \\ &\geq -c_1 - c_2 \varrho_{p(\cdot)}(|u_n|). \end{aligned}$$

Thus,

$$\begin{aligned} I(u_n) &\geq -c_1 - c_2 \cdot \varrho_{p(\cdot)}(|u_n|) - \frac{c_1}{\beta} - \frac{c_2}{\beta} \cdot \|u_n\|_{L_1^{p(\cdot)}(M)}^{p^++1} \\ &\geq -c_1 \left(1 + \frac{1}{\beta}\right) - c_2 \left(1 + \frac{1}{\beta} \cdot \|u_n\|_{L_1^{p(\cdot)}(M)}\right) \cdot \varrho_{p(\cdot)}(|u_n|). \end{aligned}$$

Hence,

$$A(u_n) - \frac{1}{\beta} < DA(u_n), u_n > \geq \left( \frac{1}{p^+} - \frac{1}{\beta} \right) \cdot 2^{1-p^+} \cdot \|u_n\|_{L_1^{p(\cdot)}(M)}^{p^+} - c_1 \left( 1 + \frac{1}{\beta} \right) - c_2 \left( 1 + \frac{1}{\beta} \cdot \|u_n\|_{L_1^{p(\cdot)}(M)} \right) \cdot \varrho_{p(\cdot)}(|u_n|). \tag{4}$$

Now, dividing both sides of (4) by  $\|u_n\|_{L_1^{p(\cdot)}(M)}$  and passing to the limit  $n \rightarrow \infty$ , we get  $0 \geq \infty$  as  $\beta \geq p^+ > 1$  which is absurd.

Hence,  $(u_n)$  is bounded in  $L_1^{p(\cdot)}(M)$ .

• Since  $\{u_n\}$  is bounded in  $L_1^{p(\cdot)}(M)$ , there exists a subsequence of  $\{u_n\}$ , noted again by  $\{u_n\}$ , that converges in  $L_1^{p(\cdot)}(M)$ . We will prove that  $\{u_n\}$  is Cauchy in  $L_1^{p(\cdot)}(M)$  i.e that

$$\lim_{m,n \rightarrow \infty} \varrho_{p(\cdot)}(u_m - u_n) = 0 = \lim_{m,n \rightarrow \infty} \varrho_{p(\cdot)}(|\nabla(u_m - u_n)|). \tag{5}$$

And to finish the demonstration of the Theorem 1, it suffice to show that the subsequence of  $\{u_n\}$  still noted by  $\{u_m\}$  is a Cauchy sequence in  $L_1^{p(\cdot)}(M)$ .

Consider the following functionals

$$B_1(u) = \int_{p(x)<2} (|\nabla u(x)|^{p(x)} + |u(x)|^{p(x)} + H(x, u)) dv_g(x), \tag{6}$$

and

$$B_2(u) = \int_{p(x)\geq 2} (|\nabla u(x)|^{p(x)} + |u(x)|^{p(x)} + H(x, u)) dv_g(x), \tag{7}$$

on  $L_1^{p(\cdot)}(M)$ , and note that

$$\varrho_{p(\cdot)}(u_m - u_n) + \varrho_{p(\cdot)}(|\nabla(u_m - u_n)|) = B_1(u_m - u_n) - B_2(u_m - u_n), \tag{8}$$

consider also the following inequalities

$$|\zeta - \mu|^p \leq \frac{2}{p-1} \left[ (|\zeta|^{p-2} \zeta - |\mu|^{p-2} \mu) \cdot (\zeta - \mu) \right]^{\frac{p}{2}} \cdot (|\zeta| - |\mu|)^{\frac{2p-q^2}{2}} \text{ for } 1 \leq p \leq 2, \tag{9}$$

and

$$|\zeta - \mu|^p \leq 2^p (|\zeta|^{p-2} \zeta - |\mu|^{p-2} \mu) \cdot (\zeta - \mu) \text{ for } p \geq 2, \tag{10}$$

where,  $\zeta, \mu \in \mathbb{R}^N$  and (2) we deduce that

$$\begin{aligned} B_2(u_m - u_n) &\leq \int_{p(x)\geq 2} (|\nabla(u_m - u_n)|^{p(x)} + |u_m - u_n|^{p(x)} + H(x, u_m) \\ &\quad - H(x, u_n)) dv_g(x) \\ &\leq 2^{p^+} \left[ \int_{p(x)\geq 2} (|\nabla u_m|^{p(x)-2} \nabla u_m - |\nabla u_n|^{p(x)-2} \nabla u_n) \cdot \nabla(u_m - u_n) dv_g(x) \right. \\ &\quad \left. + \int_{p(x)\geq 2} (|u_m|^{p(x)-2} u_m - |u_n|^{p(x)-2} u_n) \cdot (u_m - u_n) dv_g(x) \right] \\ &\quad + c_2 \int_{p(x)\geq 2} |u_m|^{p(x)} dv_g(x) - c_2 \int_{p(x)\geq 2} |u_n|^{p(x)} dv_g(x) + c_3. \end{aligned}$$

Thus,

$$2^{-p^+} B_2(u_m - u_n) \leq \langle u_m - u_n, DB(u_m) - DB(u_n) \rangle. \tag{11}$$

And using (6) and (9), we obtain

$$\begin{aligned} \frac{p^- - 1}{2} B_1(u_m - u_n) &\leq \int_{p(x)<2} g(|\nabla u_m|^{p(x)-2} \nabla u_m - |\nabla u_n|^{p(x)-2} \nabla u_n, \nabla(u_m - u_n)) \\ &\quad \times (|\nabla u_m| + |\nabla u_n|)^{\frac{2q(x)-q(x)^2}{2}} dv_g(x) \\ &\quad + \int_{p(x)<2} (|\nabla u_m|^{p(x)-2} u_m - |\nabla u_n|^{p(x)-2} u_n) \cdot (u_m - u_n)^{\frac{p(x)}{2}} \\ &\quad \times (|u_m| + |u_n|)^{\frac{2q(x)-q(x)^2}{2}} dv_g(x) \\ &\quad + \int_{p(x)<2} H(x, u_m) dv_g(x) - \int_{p(x)<2} H(x, u_n) dv_g(x). \end{aligned} \tag{12}$$

**Remark 3.5.** If  $a$  and  $b$  are two positive functions on  $M$ , then by Hölder’s inequality

$$\int_{p<2} a^{\frac{p}{2}} b^{\frac{2p-p^2}{2}} \leq 2 \| \mathbb{K}_{p<2} a^{\frac{p}{2}} \|_{L^{\frac{2}{p}}} \cdot \| \mathbb{K}_{p<2} b^{\frac{2p-p^2}{2}} \|_{L^{\frac{2}{2-p}}}. \tag{13}$$

where  $\mathbb{K}$  is the indicator function of  $M$ , moreover, since

$$\| \mathbb{K}_{p<2} a^{\frac{p}{2}} \|_{L^{\frac{2}{p}}} \leq \max\{ \varrho_1(a), \varrho_1(a)^{\frac{p^-}{2}} \}$$

and

$$\| \mathbb{K}_{p<2} b^{\frac{2p-p^2}{2}} \|_{L^{\frac{2}{2-p}}} \leq \max\{ \varrho_p(b)^{\frac{2-p^-}{2}}, 1 \},$$

we get,

$$\int_{p<2} a^{\frac{p}{2}} \cdot b^{\frac{2p-p^2}{2}} \leq 2 \max\{ \varrho_1(a), \varrho_1(a)^{\frac{p^-}{2}} \} \max\{ \varrho_p(b)^{\frac{2-p^-}{2}}, 1 \}. \tag{14}$$

Thus, using (14) twice in (12) we infer that

$$(P^- - 1) B_1(u_m - u_n) \leq P(\langle u_m - u_n, DB(u_m) - DB(u_n) \rangle) Q(\|u_m\|_{L_1^p}, \|u_n\|_{L_1^p}), \tag{15}$$

for some constructible continuous functions  $P(x)$  and  $Q(y, z)$  with  $P(0) = 0$ .

Since the sequence  $\{ \|u_n\|_{L_1^{p(\cdot)}} \}$  is bounded, from (15), (11) and (8) we conclude that if

$$\lim_{m,n \rightarrow \infty} \langle u_m - u_n, DB(u_m) - DB(u_n) \rangle = 0.$$

To obtain such that a limit, recall that  $DA = DB - DC$  hence

$$\begin{aligned} \langle u_m - u_n, DB(u_m) - DB(u_n) \rangle &= \langle u_m - u_n, DA(u_m) - DA(u_n) \rangle \\ &\quad + \langle u_m - u_n, DC(u_m) - DC(u_n) \rangle, \end{aligned}$$

we estimate

$$\begin{aligned} \langle u_m - u_n, DC(u_m) - DC(u_n) \rangle &\leq \frac{1}{G} \|N_f(u_m) - N_f(u_n)\|_{(L_1^{p(\cdot)}(M))'} \\ &\quad \times \|u_m - u_n\|_{L_1^{q(\cdot)}(M)}, \end{aligned}$$



where  $G$  being the positive constant of the embedding  $L_1^{p(\cdot)}(M) \hookrightarrow L^{q(\cdot)}(M)$  and  $N_f : L^{p(\cdot)}(M) \rightarrow (L^{p(\cdot)}(M))'$  is the Nemytskii operator induced by  $f$ , with  $N_f[u](x) = f(x, u(x))$ .

Hence, using the continuity of Nemytskii map, the convergence of  $\{u_n\}$  in  $L^{p(\cdot)}(M)$  the boundedness of  $\{\|u_n\|_{L^{q(\cdot)}(M)}\}$  and the hypothesis on  $DA(u_n) \rightarrow \infty$  as  $n \rightarrow \infty$ , we deduce that the functional  $A$  satisfies the Palais-Smale Condition.  $\square$

#### 4. Uniqueness of Non-trivial Solution

Now, let us show the uniqueness of solution.

**Theorem 4.1.** *Assume that condition  $(f_1) - (f_3)$  and  $(h_1)$  are true. With  $f \in L^{p'(x)}(M)$  is a contraction Lipschitz continuous function such that  $\exists \alpha$  with  $0 \leq \alpha < 1$*

$$|f(x, u_1) - f(x, u_2)| \leq \alpha |u_1 - u_2|. \tag{16}$$

Then, the problem (1) has a unique non-trivial solution.

*Proof.* Suppose that  $u_1$  and  $u_2$  satisfy (1). Then we have

$$\begin{aligned} & \int_M |\nabla u_1|^{p(x)-2} \cdot \nabla u_1 \cdot (\nabla u_1 - \nabla u_2) \, dv_g(x) + \int_M h(x, u_1) \cdot (u_1 - u_2) \, dv_g(x) \\ & + \int_M |u_1|^{p(x)-2} \cdot u_1 \cdot (u_1 - u_2) \, dv_g(x) = \int_M f(x, u_1) \cdot (u_1 - u_2) \, dv_g(x), \end{aligned}$$

and

$$\begin{aligned} & \int_M |\nabla u_2|^{p(x)-2} \cdot \nabla u_2 \cdot (\nabla u_2 - \nabla u_1) \, dv_g(x) + \int_M h(x, u_2) \cdot (u_2 - u_1) \, dv_g(x) \\ & + \int_M |u_2|^{p(x)-2} \cdot u_2 \cdot (u_2 - u_1) \, dv_g(x) = \int_M f(x, u_2) \cdot (u_2 - u_1) \, dv_g(x). \end{aligned}$$

Adding the above two equations yields.

$$\begin{aligned} & \int_M \left( |\nabla u_1|^{p(x)-2} \cdot \nabla u_1 - |\nabla u_2|^{p(x)-2} \cdot \nabla u_2 \right) \cdot (\nabla u_1 - \nabla u_2) \, dv_g(x) \\ & + \int_M \left( h(x, u_1) - h(x, u_2) \right) \cdot (u_1 - u_2) \, dv_g(x) \\ & + \int_M \left( |u_1|^{p(x)-2} \cdot u_1 - |u_2|^{p(x)-2} \cdot u_2 \right) \cdot (u_1 - u_2) \, dv_g(x) \\ & = \int_M (f(x, u_1) - f(x, u_2)) \cdot (u_1 - u_2) \, dv_g(x), \end{aligned}$$

by  $(f_2)$ ,  $(f_3)$  and (16) we obtain

$$\begin{aligned} & \int_M \left( |\nabla u_1|^{p(x)-2} \cdot \nabla u_1 - |\nabla u_2|^{p(x)-2} \cdot \nabla u_2 \right) \cdot (\nabla u_1 - \nabla u_2) \, dv_g(x) \\ & + \int_M \left( h(x, u_1) - h(x, u_2) \right) \cdot (u_1 - u_2) \, dv_g(x) \\ & + \int_M \left( |u_1|^{p(x)-2} \cdot u_1 - |u_2|^{p(x)-2} \cdot u_2 + \alpha (u_1 - u_2) \right) \cdot (u_1 - u_2) \, dv_g(x) \geq 0. \end{aligned}$$

Then,

$$\begin{aligned} & \int_M \left( |h(x, u_1)| - |h(x, u_2)| \right) \cdot (u_1 - u_2) \, dv_g(x) \geq \\ & \int_M \left( |\nabla u_1|^{p(x)-2} \cdot |\nabla u_1| - |\nabla u_2|^{p(x)-2} \cdot |\nabla u_2| \right) \cdot (|\nabla u_1| - |\nabla u_2|) \, dv_g(x) \\ & + \int_M \left( |u_1|^{p(x)-2} \cdot |u_1| - |u_2|^{p(x)-2} \cdot |u_2| \right) \cdot (|u_1| - |u_2|) \, dv_g(x) \\ & \geq \int_M \left( |\nabla u_1|^{p(x)-1} - |\nabla u_2|^{p(x)-1} \right) \cdot (|\nabla u_1| - |\nabla u_2|) \, dv_g(x) \\ & + \int_M \left( |u_1|^{p(x)-1} - |u_2|^{p(x)-1} \right) \cdot (|u_1| - |u_2|) \, dv_g(x) \\ & + \alpha \int_M (|u_1| - |u_2|)^2 \, dv_g(x). \end{aligned}$$

So, by  $(h_1)$  we have

$$\begin{aligned} & \int_M \left( |h(x, u_1)| - |h(x, u_2)| \right) \cdot (u_1 - u_2) \, dv_g(x) \\ & \leq \int_M (l(u_1) \cdot |u_1|^{p(x)} - l(u_2) \cdot |u_2|^{p(x)}) \cdot (|u_1| - |u_2|) \, dv_g(x). \end{aligned}$$

Hence,

$$\begin{aligned} & \int_M (l(u_1) \cdot |u_1|^{p(x)} - l(u_2) \cdot |u_2|^{p(x)}) \cdot (|u_1| - |u_2|) \, dv_g(x) \\ & \geq \int_M \left( |\nabla u_1|^{p(x)-1} - |\nabla u_2|^{p(x)-1} \right) \cdot (|\nabla u_1| - |\nabla u_2|) \, dv_g(x) \\ & + \int_M \left( |u_1|^{p(x)-1} - |u_2|^{p(x)-1} \right) \cdot (|u_1| - |u_2|) \, dv_g(x) \\ & + \alpha \int_M (|u_1| - |u_2|)^2 \, dv_g(x). \end{aligned}$$

Which implies that

$$\begin{aligned} 0 & \geq \int_M \left( |\nabla u_1|^{p(x)-1} - |\nabla u_2|^{p(x)-1} \right) \cdot (|\nabla u_1| - |\nabla u_2|) \, dv_g(x) \\ & + \int_M \left( |u_1|^{p(x)-1} - l(u_1) \cdot |u_1|^{p(x)} - |u_2|^{p(x)-1} - l(u_2) \cdot |u_2|^{p(x)} \right) \\ & \quad \times (|u_1| - |u_2|) \, dv_g(x) \\ & + \alpha \int_M (|u_1| - |u_2|)^2 \, dv_g(x) \\ & \geq \int_M \left( |\nabla u_1|^{p(x)-1} - |\nabla u_2|^{p(x)-1} \right) \cdot (|\nabla u_1| - |\nabla u_2|) \, dv_g(x). \end{aligned}$$

Hence, we obtain that  $u_1 = u_2$  almost everywhere. This complete the proof of uniqueness.  $\square$

### 5. Conclusion

In sum, the first and second Theorems, are valid and substantiated. We can then say that the solution of the problem (1) is a unique and non-trivial one.

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