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Some Integral Inequalities via (p, q) −Calculus On Finite Intervals

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Abstract. The aim of this paper is to construct (p, q)-calculus on finite intervals. The (p_k, q_k) −derivative and (p_k, q_k) −integral are defined and some basic properties are given. Also, (p_k, q_k) −analogue of Hölder, Minkowski integral inequalities are proved.

1. Introduction

All of the scientific works deal with the ambition for giving the meaning of the universe in which we live. Every new discovery we made come up looking, feeling, living and transmitting in a different perspective. For understanding and transmitting these happenings, we all need different type of methods. As mathematicians, the main purpose of our studies is to *analyze* the nature and express in mathematical ways. In this sense, calculus which is the main well-known way become our alphabet while we are translating the universe into some notions.

Quantum calculus is a field that searches mathematical formulas which turn the original version when q tends to 1. The history of quantum analysis goes back to eighteenth century to when Euler introduced q in 'Introductio' in the tracks of Newton's infinite series. In nineteenth century, Jackson defined an integral which is called q-Jackson integral in 1910 and q-analysis has gone through a period of rapidly development. For more details, see [7, 8, 10, 13] and the refrences therein.

In recent years, as being one of the most desirable area, many authors are interested in quantum calculus. One can easily see new contributions to the field almost every day. This is due to the fact that quantum calculus has not also important applications in mathematics but also in particle physics, theoretical physics, analytic number theory, and computer science. In mathematics, q- analysis is closely linked with theory of ordinary fractional calculus, optimal control problems, q-difference and q-integral equations. In [27] and [28] Tariboon et al. define quantum calculus on finite intervals namely q_k-calculus, prove some of its properties and extend some of the important integral inequalities to quantum calculus.

While the attention on q-calculus has been increasing, Post-quantum or (p, q)-calculus has appear as a generalization of q-calculus and the next step ahead of the q-calculus. (p, q) −integers which was first taken in [6] for generalizing q-oscillator algebras which is well known in the earlier pyhsics, was studied independently and at the same time by Chakrabarti and Jagannathan in [6], Bromidas et al. in [3], Wachs and White in [29], Arık et al. in [2]. Until then today, (p, q)-calculus has become an appropriate workspace for both mathematicians and pyhsicist. There are many researches about (p, q)-calculus on operator theory, special functions, integral inequlities and integral transforms, see [1, 4–6, 9, 11, 12] and [14]-[25].

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In this paper, we give the definition of (p,q)-derivative, (p,q)-integral on finite interval, (p,q)-integration by parts and some basic properties. We also prove (p,q)-anologue of frequently used some integral inequalities. With these definitions, the way is opened for proving (p,q)-anologues of many different and useful inequalities in finite intervals and so enlarging the results the more generalized form of them. Under the convience circumstances, some results become a direct consequances of their $q-$forms or ordinary forms but for some of them you need to reorganize the interval you study on. Trivally you work on more restricted area corresponding to $q-$form. That's why post quantum calculus have edge to quantum calculus is $0 < p < 1$.

The $(p,q)-$integers $[n]_{p,q}$ are defined by

$$[n]_{p,q} = \frac{p^n - q^n}{p - q} \tag{1}$$

where $0 < q < p \leq 1$. For $p = 1$

$$[n]_q = \frac{1 - q^n}{1 - q}$$

See [13]. One can easily see the relation between post quantum and quantum intergers as:

$$[n]_{p,q} = p^{n-1} [n]_{p/q}$$

where $0 < q < p \leq 1$. For each $k, n \in \mathbb{N}$, $n \geq k \geq 0$, the $(p,q)-$factorial and $(p,q)-$binomial are defined by

$$\begin{aligned} [n]_{p,q}! &= \prod_{k=1}^{n} [k]_{p,q}, \; n \geq 1, \; [0]_{p,q}! = 1 \\ \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} &= \frac{[n]_{p,q}!}{[n-k]_{p,q}! \, [k]_{p,q}!}. \end{aligned}$$

Let $f : \mathbb{R} \to \mathbb{R}$. The $(p,q)-$derivative of the function f is defined as

$$D_{p,q} f(x) = \frac{f(px) - f(qx)}{(p-q)x}, \; x \neq 0 \tag{2}$$

provided that $D_{p,q} f(0) = f'(0)$.

Let $f : C[0,a] \to \mathbb{R}$ $(a > 0)$ then the (p,q)-integration of f defined by

$$\begin{aligned} \int_0^a f(t) \, d_{p,q}t &= (q-p) a \sum_{n=0}^{\infty} \frac{p^n}{q^{n+1}} f\left(\frac{p^n}{q^{n+1}} a\right) \text{ if } \left|\frac{p}{q}\right| < 1 \\ \int_0^a f(t) \, d_{p,q}t &= (p-q) a \sum_{n=0}^{\infty} \frac{q^n}{p^{n+1}} f\left(\frac{q^n}{p^{n+1}} a\right) \text{ if } \left|\frac{p}{q}\right| > 1. \end{aligned} \tag{3}$$

The formula of $(p,q)-$integration by parts is given by

$$\int_a^b f(px) D_{p,q} g(x) \, d_{p,q}t = f(x) g(x)\Big|_a^b - \int_a^b g(qx) D_{p,q} f(x) \, d_{p,q}t. \tag{4}$$

All notions written above reduce to the $q-$analogs when $p = 1$. For more details,see the refrences mentioned in above.

2. (p,q)-Calculus on Finite Intervals

In this section, we define (p_k,q_k)-derivative and (p_k,q_k)-integral on finite intervals. Let $I_k := [u_k, u_{k+1}]$ be an interval and $0 < q_k < p_k \leq 1$ be constants.

Definition 2.1. *Let $f : I_k \to \mathbb{R}$ be a continuous function and assume that $u \in I_k$. Then the following equality*

$$\begin{aligned} D_{p_k,q_k} f(u) &= \frac{f(p_k u + (1-p_k) u_k) - f(q_k u + (1-q_k) u_k)}{(p_k - q_k)(u - u_k)}, u \neq u_k \\ D_{p_k,q_k} f(u_k) &= \lim_{u \to u_k} D_{p_k,q_k} f(u) \end{aligned} \tag{5}$$

is called the (p_k,q_k)-derivative of a function f at u.

Obviously, f is (p_k,q_k)-differentiable on I_k provided $D_{p_k,q_k} f(u)$ exists for all $u \in I_k$. In (5), if $p_k = 1$, then $D_{p_k,q_k} f = D_{q_k} f$ which is the q_k-derivative of the function f and also if $q_k \to 1, u_k = 0$, (5) reduces to q-derivative of the function f, see [13, 28].

Example 2.2. *For $u \in I_k$, if $f(u) = (u - u_k)^n$, then*

$$\begin{aligned} D_{p_k,q_k} f(u) &= \frac{(p_k u + (1-p_k) u_k - u_k)^n - (q_k u + (1-q_k) u_k - u_k)^n}{(p_k - q_k)(u - u_k)} \\ &= \frac{p_k^n (u-u_k)^n - q_k^n (u-u_k)^n}{(p_k - q_k)(u - u_k)} \\ &= [n]_{p,k} (u - u_k)^{n-1} \end{aligned} \tag{6}$$

where $[n]_{p_k,q_k} = \frac{p_k^n - q_k^n}{p_k - q_k}$. If $p_k = 1$ in (6), then (6) reduces

$$D_{q_k} f(u) = [n]_{q_k} (u - u_k)^{n-1}$$

which is given in [28]. Also if $q_k \to 1, u_k = 0$, it reduces q-derivative of the given function, see [13].

Theorem 2.3. *Suppose that $f, g : I_k \to \mathbb{R}$ is (p_k,q_k)-differentiable on I_k. Then:*
(a) If $f + g : I_k \to \mathbb{R}$ is (p_k,q_k)-differentiable on I_k, then

$$D_{p_k,q_k} (f(u) + g(u)) = D_{p_k,q_k} f(u) + D_{p_k,q_k} g(u). \tag{7}$$

(b) If $\lambda f : I_k \to \mathbb{R}$ is (p_k,q_k)-differentiable on I_k for any constant λ, then

$$D_{p_k,q_k} f(u) = \lambda D_{p_k,q_k} f(u). \tag{8}$$

(c) If $fg : I_k \to \mathbb{R}$ is (p_k,q_k)-differentiable on I_k, then

$$\begin{aligned} & D_{p_k,q_k} (fg)(u) \\ = \; & g(p_k u + (1-p_k) u_k) D_{p_k,q_k} f(u) + f(q_k u + (1-q_k) u_k) D_{p_k,q_k} g(u) \\ = \; & f(p_k u + (1-p_k) u_k) D_{p_k,q_k} g(u) + g(q_k u + (1-q_k) u_k) D_{p_k,q_k} f(u) \end{aligned} \tag{9}$$

(d) If $g(p_k u)\, g\left(q_k u + \left(1 - \frac{q_k}{p_k}\right) u_k\right) \neq 0$, then $\frac{f}{g}$ is (p_k,q_k)-differentiable on I_k with

$$D_{p_k,q_k} \left(\frac{f}{g}\right)(u) = \frac{g(p_k u + (1-p_k) u_k) D_{p_k,q_k} f(u) - f(p_k u + (1-p_k) u_k) D_{p_k,q_k} g(u)}{g(p_k u + (1-p_k) u_k)\, g(q_k u + (1-q_k) u_k)}. \tag{10}$$

Proof. The proofs of (a) and (b) are obvious.

(c) From Definition 2.1, we have

$$
\begin{aligned}
& D_{p_k,q_k}(fg)(u) \\
= & \frac{f(p_k u+(1-p_k)u_k)\,g(p_k u+(1-p_k)u_k)-f(q_k u+(1-q_k)u_k)\,g(p_k u+(1-p_k)u_k)}{(p_k-q_k)(u-u_k)} \\
& +\frac{f(q_k u+(1-q_k)u_k)\,g(p_k u+(1-p_k)u_k)-f(q_k u+(1-q_k)u_k)\,g(q_k u+(1-q_k)u_k)}{(p_k-q_k)(u-u_k)} \\
= & g(p_k u+(1-p_k)u_k)\,D_{p_k,q_k}f(u)+f(q_k u+(1-q_k)u_k)\,D_{p_k,q_k}g(u)
\end{aligned}
$$

The second equation can be proved in similar way by interchanging the functions f and g.

(d) From Definition 2.1, we have

$$
\begin{aligned}
& D_{p_k,q_k}\left(\frac{f}{g}\right)(u) \\
= & \frac{\left(\frac{f}{g}\right)(p_k u+(1-p_k)u_k)-\left(\frac{f}{g}\right)(q_k u+(1-q_k)u_k)}{(p_k-q_k)(u-u_k)} \\
= & \frac{f(p_k u+(1-p_k)u_k)\,g(q_k u+(1-q_k)u_k)-g(p_k u+(1-p_k)u_k)\,f(q_k u+(1-q_k)u_k)}{g(p_k u+(1-p_k)u_k)\,g(q_k u+(1-q_k)u_k)(p_k-q_k)(u-u_k)} \\
= & \frac{g(p_k u+(1-p_k)u_k)\,D_{p_k,q_k}f(u)-f(p_k u+(1-p_k)u_k)\,D_{p_k,q_k}g(u)}{g(p_k u+(1-p_k)u_k)\,g(q_k u+(1-q_k)u_k)}
\end{aligned}
$$

□

Definition 2.4. *Let $f: I_k \to \mathbb{R}$ be a continuous function. If $D_{p_k,q_k}f$ is (p_k,q_k)-differentiable on I_k, the second-order derivative is defines as $D^2_{p_k,q_k}f$ with $D_{p_k,q_k}\left(D_{p_k,q_k}f\right): I_k \to \mathbb{R}$. By this way, we obtain n-th order (p_k,q_k)-derivative $D^n_{p_k,q_k}f: I_k \to \mathbb{R}$.*

For instance, if $f: I_k \to \mathbb{R}$, then we have

$$
\begin{aligned}
D^2_{p_k,q_k}f(u) &= D_{p_k,q_k}\left(D_{p_k,q_k}f\right)(u) \\
&= \frac{D_{p_k,q_k}f(p_k u+(1-p_k)u_k)-D_{p_k,q_k}f(q_k u+(1-q_k)u_k)}{(p_k-q_k)(u-u_k)} \\
&= \frac{\frac{f(p_k(p_k u+(1-p_k)u_k)+(1-p_k)u_k)-f(q_k(p_k u+(1-p_k)u_k)+(1-q_k)u_k)}{(p_k-q_k)(p_k u+(1-p_k)u_k-u_k)}}{(p_k-q_k)(u-u_k)} \\
&\quad -\frac{\frac{f(p_k(q_k u+(1-q_k)u_k)+(1-p_k)u_k)-f(q_k(q_k u+(1-q_k)u_k)+(1-q_k)u_k)}{(p_k-q_k)(q_k u+(1-q_k)u_k-u_k)}}{(p_k-q_k)(u-u_k)} \\
&= \frac{f(p_k(p_k u+(1-p_k)u_k)+(1-p_k)u_k)-f(q_k(p_k u+(1-p_k)u_k)+(1-q_k)u_k)}{p_k(p_k-q_k)^2(u-u_k)^2} \\
&\quad -\frac{f(p_k(q_k u+(1-q_k)u_k)+(1-p_k)u_k)-f(q_k(q_k u+(1-q_k)u_k)+(1-q_k)u_k)}{q_k(p_k-q_k)^2(u-u_k)^2}
\end{aligned}
$$

$$\begin{aligned}
&= \frac{f\left(p_k^2 u + \left(1-p_k^2\right)u_k\right) - f\left(q_k p_k u + \left(1-q_k p_k\right)u_k\right)}{p_k\left(p_k-q_k\right)^2\left(u-u_k\right)^2} \\
&\quad - \frac{f\left(p_k q_k u + \left(1-p_k q_k\right)u_k\right) - f\left(q_k^2 u + \left(1-q_k^2\right)u_k\right)}{q_k\left(p_k-q_k\right)^2\left(u-u_k\right)^2} \\
&= \frac{q_k f\left(p_k^2 u + \left(1-p_k^2\right)u_k\right) - \left(p_k+q_k\right) f\left(p_k q_k u + \left(1-p_k q_k\right)u_k\right) + p_k f\left(q_k^2 u + \left(1-q_k^2\right)u_k\right)}{p_k q_k\left(p_k-q_k\right)^2\left(u-u_k\right)^2}
\end{aligned}$$

and $D^2_{p_k,q_k} f(u_k) = \lim_{u\to u_k} D^2_{p_k,q_k} f(u)$.

We define the (p_k, q_k)-integration as the inverse (p_k, q_k)-differention. Assume that T_{p_k,q_k} is a shifting operator defined by

$$T_{p_k,q_k} F(u) = F\left(\frac{q_k}{p_k}u + \left(1-\frac{q_k}{p_k}\right)u_k\right) \tag{11}$$

where $F(u)$ is the (p_k, q_k)-antiderivative of f. Applying mathematical induction to (11), we see

$$T^n_{p_k,q_k} F(u) = F\left(\frac{q_k^n}{p_k^n}u + \left(1-\frac{q_k^n}{p_k^n}\right)u_k\right) \tag{12}$$

where $n = 1, 2, \ldots$ and $T^0_{p_k,q_k} F(u) = F(u)$. From Definition 2.1, we have

$$f(u) = \frac{F\left(p_k u + \left(1-p_k\right)u_k\right) - F\left(q_k u + \left(1-q_k\right)u_k\right)}{\left(p_k-q_k\right)\left(u-u_k\right)}.$$

Making a change of variable, $\left(p_k u + \left(1-p_k\right)u_k\right) = t$, we have

$$\begin{aligned}
f\left(\frac{t-\left(1-p_k\right)u_k}{p_k}\right) &= \frac{F(t) - F\left(\frac{q_k}{p_k}t + \left(1-\frac{q_k}{p_k}\right)u_k\right)}{\left(\frac{p_k-q_k}{p_k}\right)\left(t-u_k\right)} \\
&= \frac{1-T_{p_k,q_k}}{\left(\frac{p_k-q_k}{p_k}\right)\left(u-u_k\right)} F(t).
\end{aligned}$$

Thus, we obtain

$$F(t) = \frac{1}{1-T_{p_k,q_k}}\left(1-\frac{q_k}{p_k}\right)\left(t-u_k\right) f\left(\frac{t-\left(1-p_k\right)u_k}{p_k}\right).$$

Therefore, applying the formula of expansion of geometric series to (12), we have the following formula

$$\begin{aligned}
F(t) &= \left(1-\frac{q_k}{p_k}\right)\sum_{n=0}^{\infty} T^n_{p_k,q_k}\left(t-u_k\right) f\left(\frac{t-\left(1-p_k\right)u_k}{p_k}\right) \\
&= \left(1-\frac{q_k}{p_k}\right)\sum_{n=0}^{\infty}\left(\frac{q_k^n}{p_k^n}t + \left(1-\frac{q_k^n}{p_k^n}\right)u_k - u_k\right) f\left(\frac{1}{p_k}\left(\frac{q_k^n}{p_k^n}t + \left(1-\frac{q_k^n}{p_k^n}\right)u_k\right) + \left(1-\frac{1}{p_k}\right)u_k\right) \\
&= \left(p_k-q_k\right)\left(t-u_k\right)\sum_{n=0}^{\infty}\frac{q_k^n}{p_k^{n+1}} f\left(\frac{q_k^n}{p_k^{n+1}}t + \left(1-\frac{q_k^n}{p_k^{n+1}}\right)u_k\right).
\end{aligned}$$

Thus, we get

$$F(u) = \left(p_k-q_k\right)\left(u-u_k\right)\sum_{n=0}^{\infty}\frac{q_k^n}{p_k^n+1} f\left(\frac{q_k^n}{p_k^{n+1}}u + \left(1-\frac{q_k^n}{p_k^{n+1}}\right)u_k\right)$$

Now, we define the (p_k, q_k)-integral of f on a finite interval as follows:

Definition 2.5. *Let $f : I_k \to \mathbb{R}$ is a continuous function. Then for $0 < q_k < p_k \leq 1$,*

$$\int_{u_k}^{u} f(s)\, d_{p_k,q_k}s = (p_k - q_k)(u - u_k) \sum_{n=0}^{\infty} \frac{q_k^n}{p_k^{n+1}} f\left(\frac{q_k^n}{p_k^{n+1}} u + \left(1 - \frac{q_k^n}{p_k^{n+1}}\right) u_k \right) \tag{13}$$

is called (p_k, q_k)-integral of f for $u \in I_k$.

Moreover, if $a \in (u_k, u)$, then (p_k, q_k)-integral is defined by

$$\begin{aligned} & \int_{a}^{u} f(s)\, d_{p_k,q_k}s \\ = & \int_{u_k}^{u} f(s)\, d_{p_k,q_k}s \int_{u_k}^{a} f(s)\, d_{p_k,q_k}s \\ = & (p_k - q_k)(u - u_k) \sum_{n=0}^{\infty} \frac{q_k^n}{p_k^{n+1}} f\left(\frac{q_k^n}{p_k^{n+1}} u + \left(1 - \frac{q_k^n}{p_k^{n+1}}\right) u_k \right) \\ & - (p_k - q_k)(a - u_k) \sum_{n=0}^{\infty} \frac{q_k^n}{p_k^{n+1}} f\left(\frac{q_k^n}{p_k^{n+1}} a + \left(1 - \frac{q_k^n}{p_k^{n+1}}\right) u_k \right). \end{aligned} \tag{14}$$

Note that if $u_k = 0$ and $p = 1$, then (14) reduces to q_k-integral of the function. See, [28].

Remark 2.6. *We assume $0 < q_k < p_k \leq 1$ for all of the above results. We shall mention that $0 < q_k < 1$, $0 < p_k \leq 1$ for interchanging p_k and q_k in the formulas. So, we have*

$$\begin{aligned} \int_{u_k}^{u} f(s)\, d_{p_k,q_k}s &= (p_k - q_k)(u - u_k) \sum_{n=0}^{\infty} \frac{q_k^n}{p_k^{n+1}} f\left(\frac{q_k^n}{p_k^{n+1}} u + \left(1 - \frac{q_k^n}{p_k^{n+1}}\right) u_k \right), \left|\frac{p}{q}\right| > 1 \\ \int_{u_k}^{u} f(s)\, d_{p_k,q_k}s &= (q_k - p_k)(u - u_k) \sum_{n=0}^{\infty} \frac{p_k^n}{q_k^{n+1}} f\left(\frac{p_k^n}{q_k^{n+1}} u + \left(1 - \frac{p_k^n}{q_k^{n+1}}\right) u_k \right), \left|\frac{p}{q}\right| < 1. \end{aligned} \tag{15}$$

where $0 < q_k < 1$, $0 < p_k \leq 1$.

Remark 2.7. *Note that, if we take $u_k = 0$ in (15), then (15) reduces to (3), [22, Definition 5.] . Also, if $p_k = 1$ in (13), then (13) reduces to q_k-integral of a function f defined by*

$$\int_{u_k}^{u} f(s)\, d_{q_k}s = (1 - q_k)(u - u_k) \sum_{n=0}^{\infty} q_k^n f\left(q_k^n u + \left(1 - q_k^n\right) u_k \right).$$

For more details, see [28].

Theorem 2.8. *The following formulas hold for $u \in I_k$:*

(a) $D_{p_k,q_k} \int_{u_k}^{u} f(s)\, d_{p_k,q_k}s = f(u)$

(b) $\int_{u_k}^{u} D_{p_k,q_k} f(s)\, d_{p_k,q_k}s = f(u)$

(c) $\int_{a}^{u} D_{p_k,q_k} f(s)\, d_{p_k,q_k}s = f(u) - f(a)$ *, for $a \in (u_k, u)$.*

Proof. (a) From Definition 2.1 and Definition 2.5, we obtain

$$\begin{aligned}
& D_{p_k,q_k} \int_{u_k}^{u} f(s)\, d_{p_k,q_k} s \\
= \; & D_{p_k,q_k} \left[(p_k - q_k)(u - u_k) \sum_{n=0}^{\infty} \frac{q_k^n}{p_k^{n+1}} f\left(\frac{q_k^n}{p_k^{n+1}} u + \left(1 - \frac{q_k^n}{p_k^{n+1}}\right) u_k \right) \right] \\
= \; & (p_k - q_k) \left\{ \frac{(p_k u + (1-p_k) u_k - u_k) \sum_{n=0}^{\infty} \frac{q_k^n}{p_k^{n+1}} f\left(\frac{q_k^n}{p_k^{n+1}} (p_k u + (1-p_k) u_k) + \left(1 - \frac{q_k^n}{p_k^{n+1}}\right) u_k \right)}{(p_k - q_k)(u - u_k)} \right. \\
& \left. - \frac{(q_k u + (1-q_k) u_k - u_k) \sum_{n=0}^{\infty} \frac{q_k^n}{p_k^{n+1}} f\left(\frac{q_k^n}{p_k^{n+1}} (q_k u + (1-q_k) u_k) + \left(1 - \frac{q_k^n}{p_k^{n}}\right) u_k \right)}{(p_k - q_k)(u - u_k)} \right\} \\
= \; & \left[\sum_{n=0}^{\infty} \frac{q_k^n}{p_k^n} f\left(\frac{q_k^n}{p_k^n} u + \left(1 - \frac{q_k^n}{p_k^n}\right) u_k \right) - \sum_{n=0}^{\infty} \frac{q_k^{n+1}}{p_k^{n+1}} f\left(\frac{q_k^{n+1}}{p_k^{n+1}} u + \left(1 - \frac{q_k^{n+1}}{p_k^{n+1}}\right) u_k \right) \right] \\
= \; & f(u).
\end{aligned}$$

(b)From Definition 2.1 and Definition 2.5, we get

$$\begin{aligned}
& \int_{u_k}^{u} D_{p_k,q_k} f(s)\, d_{p_k,q_k} s \\
= \; & \int_{u_k}^{u} \frac{f(p_k s + (1-p_k) u_k) - f\left(q_k s + \left(1 - \frac{q_k}{p_k}\right) u_k \right)}{(p_k - q_k)(s - u_k)} d_{p_k,q_k} s \\
= \; & (p_k - q_k)(u - u_k) \sum_{n=0}^{\infty} \frac{q_k^n}{p_k^{n+1}} \left[\frac{f\left(\frac{q_k^n}{p_k^{n+1}} (p_k u + (1-p_k) u_k) + \left(1 - \frac{q_k^n}{p_k^{n+1}}\right) u_k \right)}{(p_k - q_k)(u - u_k)} \right. \\
& \left. - \frac{f\left(\frac{q_k^n}{p_k^{n+1}} (q_k u + (1-q_k) u_k) + \left(1 - \frac{q_k^n}{p_k^{n+1}}\right) u_k \right)}{(p_k - q_k)(u - u_k)} \right] \\
= \; & (u - u_k) \sum_{n=0}^{\infty} \frac{q_k^n}{p_k^{n+1}} \left[\frac{f\left(\frac{q_k^n}{p_k^n} u + \left(1 - \frac{q_k^n}{p_k^n}\right) u_k \right)}{(u - u_k)} \right. \\
& \left. - \frac{f\left(\frac{q_k^{n+1}}{p_k^{n+1}} u + \left(1 - \frac{q_k^{n+1}}{p_k^{n+1}}\right) u_k \right)}{(u - u_k)} \right] \\
= \; & \sum_{n=0}^{\infty} f\left(\frac{q_k^n}{p_k^n} p_k u + \left(1 - \frac{q_k^n}{p_k^n}\right) u_k \right) - f\left(\frac{q_k^{n+1}}{p_k^{n+1}} p_k u + \left(1 - \frac{q_k^{n+1}}{p_k^{n+1}}\right) u_k \right) \\
& f(u).
\end{aligned}$$

(c) The proof is carried on from the part of (b). □

Theorem 2.9. *Let $f, g : I_k \to \mathbb{R}$ are continuous functions. The following formulas hold:*

(a) $\int_{u_k}^{u} [f(s) + g(s)]\, d_{p_k,q_k} s = \int_{u_k}^{u} f(s)\, d_{p_k,q_k} s + \int_{u_k}^{u} g(s)\, d_{p_k,q_k}$;

(b) $\int_{u_k}^{u} \lambda f(s)\, d_{p_k,q_k} s = \lambda \int_{u_k}^{u} f(s)\, d_{p_k,q_k} s$;

(c) $\int_{u_k}^{u} f(q_k s + (1-q_k) u_k) D_{p_k,q_k} g(p_k s) d_{p_k,q_k} s = (fg)(s)\big|_{u_k}^{u} - \int_{u_k}^{u} g(p_k s + (1-p_k) u_k) D_{p_k,q_k} f(s) d_{p_k,q_k} s$
or
$\int_{u_k}^{u} f(p_k s + (1-p_k) u_k) D_{p_k,q_k} g(q_k s) d_{p_k,q_k} s = (fg)(s)\big|_{u_k}^{u} - \int_{u_k}^{u} g(q_k s + (1-q_k) u_k) D_{p_k,q_k} f(s) d_{p_k,q_k} s$

where $u \in I_k$, $\lambda \in \mathbb{R}$.

Proof. The proofs of (a)-(b) are derived from Definition 2.5.
(c) From (9), we write

$$f(q_k u + (1-q_k) u_k) D_{p_k,q_k} g(u) = D_{p_k,q_k}(fg)(u) - g(p_k u + (1-p_k) u_k) D_{p_k,q_k} f(u).$$

By integrating over $[u_k, u]$ and using Theorem 2.9 part (b), we get

$$\begin{aligned} & \int_{u_k}^{u} f(q_k u + (1-q_k) u_k) D_{p_k,q_k} g(p_k s) d_{p_k,q_k} s \\ = \ & (fg)(u) - \int_{u_k}^{u} g(p_k u + (1-p_k) u_k) D_{p_k,q_k} f(s) d_{p_k,q_k} s. \end{aligned}$$

□

3. Integral Inequalities On Finite Intervals

Lets start with (p,q)-Hölder integral ineqality on $I = [a,b]$:

Theorem 3.1. *Let f and g be two functions defined on I, $0 < q < p \leq 1$ and $s_1, s_2 > 1$ with $\frac{1}{s_1} + \frac{1}{s_2} = 1$. Then*

$$\int_a^b |f(t) g(t)| \, {}_a d_{p,q} t \leq \left(\int_a^b |f(t)|^{s_1} \, {}_a d_{p,q} t \right)^{\frac{1}{s_1}} \left(\int_a^b |g(t)|^{s_2} \, {}_a d_{p,q} t \right)^{\frac{1}{s_2}}. \tag{16}$$

Proof. From Definition 2.5 and discrete Hölder inequality, we get

$$\begin{aligned} \int_a^b |f(t) g(t)| \, {}_a d_{p,q} t &= (p-q)(b-a) \sum_{n=0}^{\infty} \frac{q^n}{p^{n+1}} \left| f\left(\frac{q^n}{p^{n+1}} b + \left(1 - \frac{q^n}{p^{n+1}}\right) a \right) g \frac{q^n}{p^{n+1}} b + \left(1 - \frac{q^n}{p^{n+1}}\right) a \right| \\ &= (p-q)(b-a) \sum_{n=0}^{\infty} \left| f\left(\frac{q^n}{p^{n+1}} b + \left(1 - \frac{q^n}{p^{n+1}}\right) a \right) \left(\frac{q^n}{p^{n+1}} \right)^{\frac{1}{s_1}} \right| \\ &\quad \times \left| g\left(\frac{q^n}{p^{n+1}} b + \left(1 - \frac{q^n}{p^{n+1}}\right) a \right) \left(\frac{q^n}{p^{n+1}} \right)^{\frac{1}{s_2}} \right| \\ &\leq \left((p-q)(b-a) \sum_{n=0}^{\infty} \left| f\left(\frac{q^n}{p^{n+1}} b + \left(1 - \frac{q^n}{p^{n+1}}\right) a \right) \right|^{s_1} \left(\frac{q^n}{p^{n+1}} \right) \right)^{\frac{1}{s_1}} \\ &\quad \times \left((p-q)(b-a) \sum_{n=0}^{\infty} \left| g\left(\frac{q^n}{p^{n+1}} b + \left(1 - \frac{q^n}{p^{n+1}}\right) a \right) \right|^{s_2} \left(\frac{q^n}{p^{n+1}} \right) \right)^{\frac{1}{s_2}} \\ &= \left(\int_a^b |f(t)|^{s_1} \, {}_a d_{p,q} t \right)^{\frac{1}{s_1}} \left(\int_a^b |g(t)|^{s_2} \, {}_a d_{p,q} t \right)^{\frac{1}{s_2}}. \end{aligned}$$

Thus, the proof is complete. □

It easy to show that we obtain the same result in the statement $p < q$.

Corollary 3.2. *Under the assumptions of Theorem 3.1, if we take $s_1 = s_2 = 2$, then we have the following formula,*

$$\int_a^b \left|f(t)\,g(t)\right| \,{}_a d_{p,q}t \le \left(\int_a^b \left|f(t)\right|^2 \,{}_a d_{p,q}t\right)^{\frac{1}{2}} \left(\int_a^b \left|g(t)\right|^2 \,{}_a d_{p,q}t\right)^{\frac{1}{2}} \tag{17}$$

which we call (p,q)-Cauchy-Schwarz integral inequality.

Remark 3.3. *If $p = 1$, (16) and (17) reduces to q-Hölder integral inequality and q-Cauchy-Schwarz integral inequality respectively.*

Theorem 3.4. *Let f and g real-valued functions on $[a,b]$ such that $\left|f\right|^{s_1}$, $\left|g\right|^{s_1}$ and $\left|f+g\right|^{s_1}$ are (p,q)-integrable functions on $[a,b]$, $0 < q < p \le 1$ and $s_1 > 1$. Then*

$$\left(\int_a^b \left|f(t)+g(t)\right|^{s_1} \,{}_a d_{p,q}t\right)^{\frac{1}{s_1}} \le \left(\int_a^b \left|f(t)\right|^{s_1} \,{}_a d_{p,q}t\right)^{\frac{1}{s_1}} + \left(\int_a^b \left|g(t)\right|^{s_1} \,{}_a d_{p,q}t\right)^{\frac{1}{s_1}}. \tag{18}$$

Equality holds if and only if $f(t) = 0$ almost everwhere or $g(t) = \mu f(t)$ almost everywhere with a constant $\mu \ge 0$.

Proof. Since $\left|f\right|^{s_1}$, $\left|g\right|^{s_1}$ and $\left|f+g\right|^{s_1}$ are (p,q)-integrable on $[a,b]$, by using the triangle inequality, we can write

$$\begin{aligned}\int_a^b \left|f(t)+g(t)\right|^{s_1} \,{}_a d_{p,q}t &= \int_a^b \left|f(t)+g(t)\right| \left|f(t)+g(t)\right| \,{}_a d_{p,q}t \\ &\le \int_a^b \left|f(t)\right| \left|f(t)+g(t)\right|^{s_1-1} \,{}_a d_{p,q}t + \int_a^b \left|g(t)\right| \left|f(t)+g(t)\right|^{s_1-1} \,{}_a d_{p,q}t\,.\end{aligned}$$

Taking $s_1, s_2 > 1$ with $\dfrac{1}{s_1} + \dfrac{1}{s_2} = 1$ and using (p,q) –Hölder integral inequality, we have

$$\int_a^b \left|f(t)\right| \left|f(t)+g(t)\right|^{s_1-1} \,{}_a d_{p,q}t \le \left(\int_a^b \left|f(t)\right|^{s_1} \,{}_a d_{p,q}t\right)^{\frac{1}{s_1}} \left(\int_a^b \left|f(t)+g(t)\right|^{(s_1-1)s_2} \,{}_a d_{p,q}t\right)^{\frac{1}{s_2}} \tag{19}$$

and

$$\int_I \left|g(t)\right| \left|f(t)+g(t)\right|^{s_1-1} \,{}_a d_{p,q}t \le \left(\int_I \left|g(t)\right|^{s_1} \,{}_a d_{p,q}t\right)^{\frac{1}{s_1}} \left(\int_I \left|f(t)+g(t)\right|^{(s_1-1)s_2} \,{}_a d_{p,q}t\right)^{\frac{1}{s_2}}. \tag{20}$$

Since $(s_1 - 1)\,s_2 = s_1$, from (19) and (20), it easy to see that

$$\left(\int_a^b \left|f(t)+g(t)\right|^{s_1} \,{}_a d_{p,q}t\right)^{1-\frac{1}{s_2}} \le \left(\int_a^b \left|f(t)\right|^{s_1} \,{}_a d_{p,q}t\right)^{\frac{1}{s_1}} + \left(\int_a^b \left|g(t)\right|^{s_1} \,{}_a d_{p,q}t\right)^{\frac{1}{s_1}}$$

from which we obtain the required inequality. □

Remark 3.5. *If $p = 1$, (18) reduces to*

$$\left(\int_a^b \left|f(t)+g(t)\right|^{s_1} d_q t\right)^{\frac{1}{s_1}} \le \left(\int_a^b \left|f(t)\right|^{s_1} d_q t\right)^{\frac{1}{s_1}} + \left(\int_a^b \left|g(t)\right|^{s_1} d_q t\right)^{\frac{1}{s_1}}$$

which can be called q-Minkowski integral inequality.

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