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Better Numerical Approximation by λ-Durrmeyer-Bernstein Type Operators

Voichiţa Adriana Radu[a], Purshottam Narain Agrawal[b], Jitendra Kumar Singh[b]

[a]*Babes-Bolyai University, FSEGA, Department of Statistics-Forecasts-Mathematics, Cluj-Napoca, Romania*
[b]*Department of Mathematics, Indian Institute of Technology Roorkee, Roorkee-247667, India*

Abstract. The main object of this paper is to construct a new Durrmeyer variant of the λ-Bernstein type operators which have better features than the classical one. Some results concerning the rate of convergence in terms of the first and second moduli of continuity and asymptotic formulas of these operators are given. Moreover, we define a bivariate case of these operators and investigate the approximation degree by means of the total and partial modulus of continuity and the Peetre's K-functional. A Voronovskaja type asymptotic and Grüss-Voronovskaja theorem for the bivariate operators is also proven. Further, we introduce the associated GBS (Generalized Boolean Sum) operators and determine the order of convergence with the aid of the mixed modulus of smoothness for the Bögel continuous and Bögel differentiable functions. Finally the theoretical results are analyzed by numerical examples.

1. Introduction

In 1912, Bernstein defined his polynomials in order to prove Weierstrass's fundamental theorem. Bernstein polynomials attracted the most interest because of their remarkable and notable approximation properties. For more details we refer the readers to the excellent recent monographs e.g. [10], [19] and [23]. For Bernstein operators, the order of approximation has been studied in great detail for a long time, starting with the pioneer works of Popoviciu, Lorentz and Sikkema. In order to study the order of approximation, a new technique was introduced by Esser [16]. He gave the convergence estimates using the second order modulus of continuity. Different types of Bernstein operators, their combinations and generalizations were studied over the time, underlining the importance and usefulness of these famous operators.

For $f \in C(I)$, the space of continuous functions on $I = [0,1]$ endowed with the sup-norm, Cai et al. [12] introduced a new Bernstein type operator depending on the parameter $\lambda \in [-1,1]$

$$B_{n,\lambda}(f;x) = \sum_{k=0}^{n} \tilde{b}_{n,k}(\lambda;x) f\left(\frac{k}{n}\right), \tag{1}$$

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Email addresses: voichita.radu@econ.ubbcluj.ro (Voichiţa Adriana Radu), pnappfma@iitr.ac.in (Purshottam Narain Agrawal), jitendrasingh00004@gmail.com (Jitendra Kumar Singh)

where $\tilde{b}_{n,k}$ are defined as follows:

$$\begin{cases} \tilde{b}_{n,0}(\lambda;x) &= b_{n,0}(x) - \frac{\lambda}{n+1} b_{n+1,1}(x), \\ \tilde{b}_{n,i}(\lambda;x) &= b_{n,i}(x) + \lambda\left(\frac{n-2i+1}{n^2-1} b_{n+1,i}(x) - \frac{n-2i-1}{n^2-1} b_{n+1,i+1}(x)\right), \quad 1 \le i \le n-1, \\ \tilde{b}_{n,n}(\lambda;x) &= b_{n,n}(x) - \frac{\lambda}{n+1} b_{n+1,n}(x), \end{cases} \tag{2}$$

and $0 \le x \le 1$. In [27], Ye et al. introduced these new Bernstein-Bézier bases $\left\{\tilde{b}_{n,k}\right\}$, $k = 0, 1, \dots, n$, in order to obtain more flexibility by adding the shape parameter λ. Note that for $\lambda = 0$, we retrieve the Bernstein basis polynomials.

In this article, for $f \in C[0,1]$, we treat a Durrmeyer variant of the operators (1):

$$D_{n,\lambda}(f;x) = (n+1)\sum_{k=0}^{n} \tilde{b}_{n,k}(\lambda;x)\int_0^1 b_{n,k}(t)f(t)dt, \quad 0 \le x \le 1. \tag{3}$$

In particular when $\lambda = 0$, these operators include the classical Durrmeyer operators defined by

$$D_n(f;x) = (n+1)\sum_{k=0}^{n} b_{n,k}(x)\int_0^1 b_{n,k}(t)f(t)dt.$$

This new basis was used in order to construct a generalization of the λ-Bernstein operators namely U_n^ρ operators in [2]. For some other significant papers dealing with Durrmeyer operaotrs, we refer to [18, 20].

The goal of the present paper is to study the local and global approximation properties of the λ-Durrmeyer-Bernstein type operators for functions of one and two variables. We start with the values of the moments and central moments of the operators. Then, we present some Voronovskaja type asymptotic theorems and estimates of the rate of convergence in terms of the first and second moduli of continuity. We present some graphs and numerical examples to show the convergence of the operators to the initial function. Next, we define a bivariate case of these operators and investigate the degree of approximation by means of the total and partial modulii of continuity, the Peetre's K-functional and the Voronovskaja type theorem. Further, we introduce the associated GBS operators and determine the order of convergence of these operators with the aid of mixed modulus of smoothness. Finally, we show that the GBS operators yield a better rate of convergence than the bivariate operators for a certain function by illustrative graphics and a table.

2. Basic approximation properties

The following formulas for the initial moments and the central moments are easily derived by direct computations.

Lemma 2.1. *The λ-Durrmeyer operators (3) verify*

i) $D_{n,\lambda}(e_0;x) = 1$;

ii) $D_{n,\lambda}(e_1;x) = x + \dfrac{1-2x}{n+2} + \dfrac{-2x+1+x^{n+1}-(1-x)^{n+1}}{(n+2)(n-1)}\lambda$;

iii) $D_{n,\lambda}(e_2;x) = x^2 - \dfrac{2(3nx^2-2nx+3x^2-1)}{(n+3)(n+2)} + \dfrac{2\lambda(-2nx^2+x^{n+1}n+xn+2x^{n+1}-(1-x)^{n+1}-3x+1)}{(n+3)(n+2)(n-1)}$;

iv) $D_{n,\lambda}(e_3;x) = \dfrac{1}{(n+4)(n+3)(n+2)}\left\{n^3x^3 - 3n^2x^3 + 9n^2x^2 + 2nx^3 - 9nx^2 + 18nx + 6\right\}$
$+\dfrac{3\lambda}{(n+4)(n+3)(n+2)(n-1)}\left\{-2n^2x^3 + n^2x^2 + 2nx^3 + x^{n+1}n^2 - 11nx^2 + 5x^{n+1}n\right.$
$\left.+4xn + 6x^{n+1} - 2(1-x)^{n+1} - 8x + 2\right\}$;

v) $D_{n,\lambda}(e_4;x) = \frac{1}{(n+5)(n+4)(n+3)(n+2)}\Big\{n^4x^4 - 6n^3x^4 + 16n^3x^3 + 11n^2x^4 - 48n^2x^3$
$-6nx^4 + 72n^2x^2 + 32nx^3 - 72nx^2 + 96nx + 24\Big\} + \frac{4\lambda}{(n+5)(n+4)(n+3)(n+2)(n-1)}$
$\Big\{-2n^3x^4 + n^3x^3 + 6n^2x^4 - 24n^2x^3 - 4x^4n + x^{n+1}n^3 + 9n^2x^2 + 23nx^3 + 9x^{n+1}n^2$
$-63nx^2 + 26x^{n+1}n + 18nx + 24x^{n+1} - 6(1-x)^{n+1} - 30x + 6\Big\}.$

We denote by $\Omega_{n,k}(x,\lambda) = D_{n,\lambda}((t-x)^k;x)$, $k \geq 1$, the k-th order central moment.

Lemma 2.2. *The central moments for the λ-Durrmeyer operators (3) are:*

i) $\Omega_{n,1}(x,\lambda) = \frac{1-2x}{n+2} + \frac{1-2x+x^{n+1}-(1-x)^{n+1}}{(n+2)(n-1)}\lambda;$

ii) $\Omega_{n,2}(x,\lambda) = \frac{-2(nx^2-nx-3x^2+3x-1)}{(n+3)(n+2)} + \frac{\lambda}{(n+3)(n+2)(n-1)}\Big\{2(1-x)^{n+1}nx$
$-2x^{n+2}n + 6(1-x)^{n+1}x + 2x^{n+1}n + 12x^2 - 6x^{n+2} - 2(1-x)^{n+1} + 4x^{n+1} - 12x + 2\Big\}.$

Lemma 2.3. *The following statements hold:*

i) $\lim_{n\to\infty} n\Omega_{n,1}(x,\lambda) = 1-2x;$

ii) $\lim_{n\to\infty} n\Omega_{n,2}(x,\lambda) = 2x(1-x);$

iii) $\lim_{n\to\infty} n^2\Omega_{n,4}(x,\lambda) = 12x^2(1-x)^2;$

iv) $\lim_{n\to\infty} n^3\Omega_{n,6}(x,\lambda) = 120x^3(1-x)^3.$

For a detailed study on the moments of various linear positive operators and their approximation properties one can see [21].

Theorem 2.4. *The sequence $\{D_{n,\lambda}\}_{n\geq1}$ converges to f, uniformly on I, for any $f \in C(I)$ and $\lambda \in [-1,1]$.*

Proof. The proof of this theorem is based on the previous lemmas and the well known Bohman-Korovkin theorem. □

The expressions of central moments lead us to the fact that, for $\lambda \in [-1,1]$ and $n \geq 2$, we have the following upper bounds:

Lemma 2.5. *The following inequalities yield:*

i) $|\Omega_{n,1}(x,\lambda)| \leq \mu_{n,\lambda}$

ii) $\Omega_{n,2}(x,\lambda) \leq \delta_{n,\lambda}$,

where

$$\mu_{n,\lambda} = \frac{1}{n+2} + \frac{|\lambda|}{(n-1)(n+2)},\ \delta_{n,\lambda} = \frac{n+5}{2(n+2)(n+3)} + |\lambda|\frac{(n+24)}{2(n+2)(n+3)(n-1)}. \tag{4}$$

We estimate the rate of convergence by using the usual moduli of continuities $\omega_1(f;\delta)$ and $\omega_2(f;\delta)$ and also the general modulus of second order $\omega_2^*(f;\delta)$ introduced by Păltănea in [25]:

$$\omega_2^*(f;\delta) := sup\{|\Delta(f;u,y,v)|, u,v \in [0,1], u \neq v, u \leq y \leq v, y-u \leq \delta, v-y \leq \delta\},$$

where

$$\Delta(f;u,y,v) := \frac{v-y}{v-u}f(u) + \frac{y-u}{v-u}f(v) - f(y)$$

and f is any real valued functions. The following results are obtained using the general arguments from [3] and [25].

Theorem 2.6. *If $f \in C(I)$, $\lambda \in [-1,1]$ and $\delta > 0$, then for all $x \in I$ it follows that*

i) $\|D_{n,\lambda}(f) - f\| \le \left(1 + \delta^{-1}\sqrt{\delta_{n,\lambda}}\right)\omega_1(f;\delta)$,

On the other hand, if $0 < \delta \le \frac{1}{2}$, $x \in I$, *we have*

ii) $\|D_{n,\lambda}(f) - f\| \le \mu_{n,\lambda}\delta^{-1}\omega_1(f;\delta) + \left(1 + \frac{1}{2}\delta^{-2}\delta_{n,\lambda}\right)\omega_2(f;\delta)$,

iii) $\|D_{n,\lambda}(f) - f\| \le \mu_{n,\lambda}\delta^{-1}\omega_1(f;\delta) + \left(1 + \delta^{-2}\delta_{n,\lambda}\right)\omega_2^*(f;\delta)$.

Proof. Using ([3], Theorem 5.1.2), ([25], Corollary 2.2.1) for $s = 2$ and ([25], Theorem 2.2.3) for $s = 2$, we obtain the desired results. □

Theorem 2.7. *If f is differentiable on I with f' bounded on I, $\lambda \in [-1,1]$ and $\delta > 0$, then for all $x \in I$*

$$\|D_{n,\lambda}(f) - f\| \le \mu_{n,\lambda}\|f'\| + \left(\frac{\delta}{4} + \delta^{-1}\delta_{n,\lambda}\right)\omega_1(f';\delta).$$

Proof. Using ([25], Theorem 2.3.8) for $r = 2$ and Lemma 2.5, we get the desired result. □

Corollary 2.8. *For $f \in C(I)$, $\lambda \in [-1,1]$ and $x \in I$, it follows that*

i) $\|D_{n,\lambda}(f) - f\| \le 2\omega_1\left(f;\sqrt{\delta_{n,\lambda}}\right)$.

Moreover, if f is differentiable on I with f' bounded on I, then we have

ii) $\|D_{n,\lambda}(f) - f\| \le \mu_{n,\lambda}\|f'\| + \frac{5}{4}\sqrt{\delta_{n,\lambda}} \cdot \omega_1\left(f';\sqrt{\delta_{n,\lambda}}\right)$.

Proof. In Theorem 2.6 and Theorem 2.7, we consider $\delta = \sqrt{\delta_{n,\lambda}}$. □

3. Voronovskaja Type Theorems

In the following, using Ditzian-Totik modulus of smoothness, we prove a quantitative Voronovskaja type theorem for the operators $D_{n,\lambda}$.
For a function $h \in C(I)$, the first order Ditzian-Totik moduls of smoothness is defined by

$$\omega_1^{\varphi}(h;\delta) = \sup_{h\in(0,\delta]} \sup_{x\pm\frac{h}{2}\phi(x)\in I} |f(x + \frac{h}{2}\phi(x)) - f(x - \frac{h}{2}\phi(x))|,$$

where $\phi(x)$ is an admissible weight function. The associated K-functional is given by

$$K_{\phi}(h;\delta) = \inf_{g\in W_{\phi}(I)}\{\|h - g\| + \delta\|\phi g'\|\}, \quad \delta > 0,$$

where $W_{\phi}(I) = \{g : g \in AC_{loc}(I), \|\phi g'\| < \infty\}$ and $AC_{loc}(I)$ denotes the space of absolutely continuous functions on every interval $[a,b] \subset (0,1)$. It is well known [14] that there holds the following relation

$$K_{\phi}(h;\delta) \le C\omega_1^{\varphi}(h;\delta),$$

where C is some positive constant.

Theorem 3.1. *For any $f \in C^2(I)$ and n sufficiently large the following inequality holds*

$$\left|D_{n,\lambda}(f;x) - f(x) - A_n(x;\lambda)f'(x) - B_n(x;\lambda)f''(x)\right| \leq \frac{1}{n}C\varphi^2(x)\omega_1^{\varphi}\left(f'', n^{-1/2}\right),$$

where

$$A_n(x;\lambda) = \frac{1-2x}{n+2} + \frac{1-2x+x^{n+1}-(1-x)^{n+1}}{(n+2)(n-1)}\lambda;$$

$$B_n(x;\lambda) = \frac{-(nx^2-nx-3x^2+3x-1)}{(n+3)(n+2)} + \frac{\lambda}{(n+3)(n+2)(n-1)}\Big\{(1-x)^{n+1}nx - x^{n+2}n$$
$$+3(1-x)^{n+1}x + x^{n+1}n + 6x^2 - 3x^{n+2} - (1-x)^{n+1} + 2x^{n+1} - 6x + 1\Big\},$$

$\phi(x) = \sqrt{x(1-x)}$ *and C is a positive constant.*

Proof. For $f \in C^2(I)$, $t, x \in I$, by Taylor's expansion, we have

$$f(t) - f(x) = (t-x)f'(x) + \int_x^t (t-y)f''(y)dy.$$

Therefore,

$$f(t) - f(x) - (t-x)f'(x) - \frac{1}{2}(t-x)^2 f''(x) = \int_x^t (t-y)f''(y)dy - \int_x^t (t-y)f''(x)dy$$
$$= \int_x^t (t-y)[f''(y) - f''(x)]dy.$$

Applying $D_{n,\lambda}(\cdot;x)$ to both sides of the above relation and using the estimate of the quantity $\left|\int_x^t \left|f''(y) - f''(x)\right| |t-y| du\right|$ as in ([17], p. 337)

$$\left|\int_x^t |f''(y) - f''(x)||t-y|dy\right| \leq 2\|f''-g\|(t-x)^2 + 2\|\varphi g'\|\varphi^{-1}(x)|t-x|^3,$$

where $g \in W_\varphi(I)$ and using Lemma 2.3 it follows that there exists a constant $C > 0$ such that for n sufficiently large

$$\Omega_{n,2}(x,\lambda) \leq \frac{C}{2n}\varphi^2(x) \text{ and } \Omega_{n,4}(x,\lambda) \leq \frac{C}{12n^2}\varphi^4(x). \tag{5}$$

Applying the Cauchy-Schwarz inequality, we get

$$\left|D_{n,\lambda}(f;x) - f(x) - A_n(x;\lambda)f'(x) - B_n(x;\lambda)f''(x)\right|$$
$$\leq 2\|f''-g\|\Omega_{n,2}(x,\lambda) + 2\|\varphi g'\|\varphi^{-1}(x)D_{n,\lambda}(|t-x|^3;x)$$
$$\leq \frac{C}{n}\varphi^2(x)\|f''-g\| + 2\|\varphi g'\|\varphi^{-1}(x)\left\{\Omega_{n,2}(x,\lambda)\right\}^{1/2}\left\{\Omega_{n,4}(x,\lambda)\right\}^{1/2}$$
$$\leq \frac{C}{n}\varphi^2(x)\|f''-g\| + \varphi^2(x)\frac{C}{n\sqrt{n}}\|\varphi g'\| \leq \frac{C}{n}\varphi^2(x)\left\{\|f''-g\| + n^{-1/2}\|\varphi g'\|\right\}.$$

The theorem is proved by taking the infimum on the right hand side of the above relation over all $g \in W_\varphi[0,1]$. □

Corollary 3.2. *If $f \in C^2(I)$ and $A_n(x;\lambda)$ and $B_n(x;\lambda)$ are defined as in Theorem 3.1, then*

$$\lim_{n\to\infty} n\left\{D_{n,\lambda}(f;x) - f(x) - A_n(x;\lambda)f'(x) - B_n(x;\lambda)f''(x)\right\} = 0.$$

4. Numerical Results

In the following we explain the convergence of λ-Durrmeyer operators by graphical examples. Let us denote by $E_{n,\lambda}(f;x) = \left|f(x) - D_{n,\lambda}(f;x)\right|$, the error function of λ-Durrmeyer operators.

Let $f(x) = \left(x - \frac{3}{8}\right)\sin(2\pi x)$. The graphs of $D_{n,1}(f;x)$ with different values of n are given in Figure 1. The errors of the approximation of $D_{n,1}(f;x)$ to $f(x)$ are shown in Figure 2.

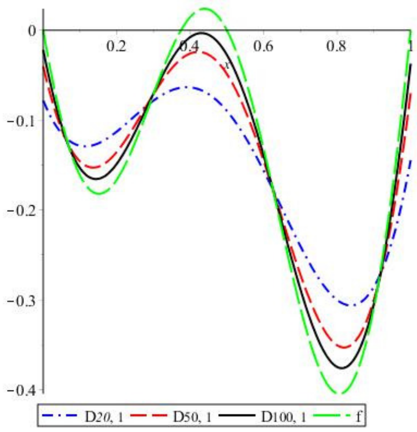

Figure 1:

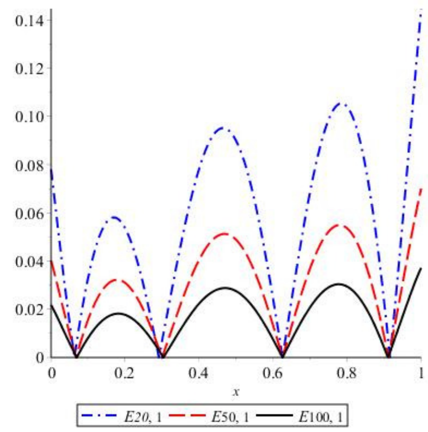

Figure 2:

For $\lambda = -1$, the convergence of λ-Durrmeyer operators to $f(x) = \cos(2\pi x) + \sin\left(\frac{1}{2}\pi x\right)$ is illustrated in Figure 3 and the error functions $E_{n,\lambda}$ are given in Figure 4.

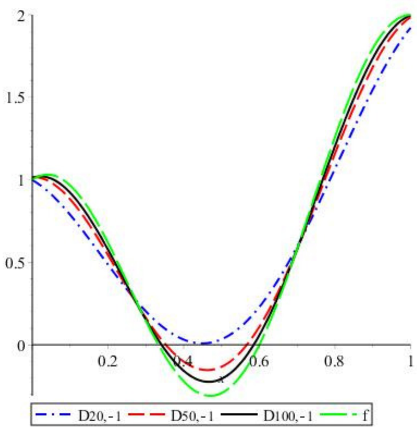

Figure 3:

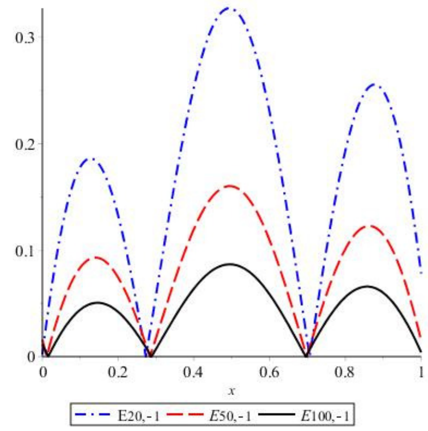

Figure 4:

Let $\lambda = 0.5$, $f(x) = \cos(2\pi x) + 2\cos\left(\frac{1}{2}\pi x\right)$. The convergence of λ-Durrmeyer operators is illustrated in Figure 5 and note that for increasing values of n, the graphs of λ-Durrmeyer operators tend to the graph of function f. Also, the error of approximation are given in Figure 6.

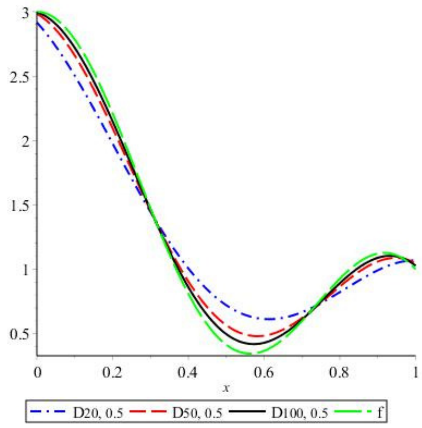

Figure 5:

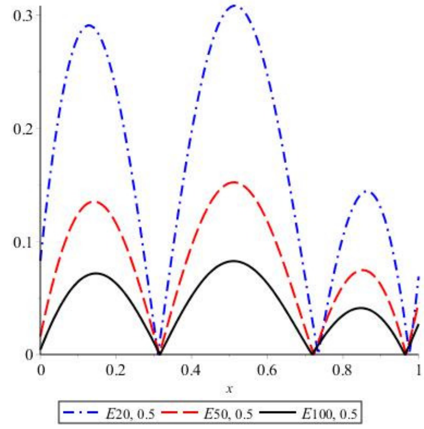

Figure 6:

In Figure 7 is illustrated the error of approximation of $f(x) = \sin(2\pi x) + 2\sin\left(\frac{1}{2}\pi x\right)$ by $D_{n,\lambda}$ for $n = 7$ and $\lambda = -1, 0, 1$. We note that for this special case, the approximation by λ-Durrmeyer operator is better than the classical operators obtained for $\lambda = 0$.

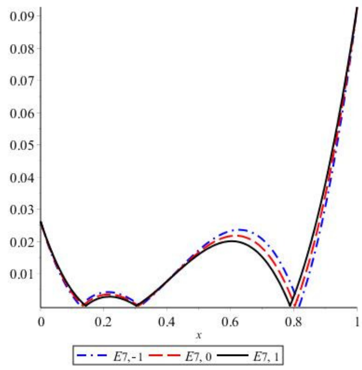

Figure 7: Error $\left|f(x) - D_{n,\lambda}(f;x)\right|$

Now, we introduce a bivariate case of the operators given by (3) and investigate the order of convergence of these operators. Furthermore, we define the associated GBS operator to approximate the Bögel continuous and Bögel differentiable functions introduced by Bögel [4]. In the last section of the paper, we give some numerical results to validate the results obtained in the paper and illustrate that the GBS operators yield better approximation than the bivariate operators for a certain function.

5. Construction of the bivariate operator

For $f \in C(I^2)$, $I^2 = I \times I$, endowed with the norm $\|f\| = \sup_{(x,y)\in I^2} |f(x,y)|$, we define the bivariate case of the operators given by (3) as

$$D_{n_1,n_2;\lambda_1,\lambda_2}(f(s,t);x,y) =$$

$$(n_1+1)(n_2+1)\sum_{k_1=0}^{n_1}\sum_{k_2=0}^{n_2}\tilde{b}_{n_1,n_2,k_1,k_2}(\lambda_1,\lambda_2;x,y)\int_0^1\int_0^1 p_{n_1,n_2,k_1,k_2}(s,t)f(s,t)dtds, \tag{6}$$

where

$$\tilde{b}_{n_1,n_2,k_1,k_2}(\lambda_1,\lambda_2;x,y) = \tilde{b}_{n_1,k_1}(\lambda_1,x)\tilde{b}_{n_2,k_2}(\lambda_2,y),\ (x,y)\in I^2,$$

and

$$p_{n_1,n_2,k_1,k_2}(s,t) = p_{n_1,k_1}(s)p_{n_2,k_2}(t) = \binom{n_1}{k_1} s^{k_1}(1-s)^{n_1-k_1}\binom{n_2}{k_2} t^{k_2}(1-t)^{n_2-k_2}$$

and $\tilde{b}_{n_1,k_1}(\lambda_1, x)$ is defined by replacing n, k, λ by $n_1,\ k_1,\ \lambda_1$ respectively in the definition of $\tilde{b}_{n,k}(\lambda, x)$ in (1) and $\tilde{b}_{n_2,k_2}(\lambda_2, y)$ is also defined similarly.
From the definition (6), it follows that

$$D_{n_1,n_2;\lambda_1,\lambda_2}((s-x)^i(t-y)^j; x, y) = D_{n_1;\lambda_1}((s-x)^i; x)D_{n_2;\lambda_2}((t-y)^j; y),$$

for all $i, j \in \mathbb{N} \cup \{0\}$ and $(x, y) \in I^2$.

Let $f : C(I^2) \to \mathbb{R}$, then for $\delta > 0$, the total modulus of continuity $\bar{\omega}(f;\delta) : [0,\infty) \times [0,\infty) \to \mathbb{R}$ is defined by

$$\bar{\omega}(f;\delta) = \sup\Big\{|f(s,t) - f(x,y)| : \sqrt{(s-x)^2 + (t-y)^2} < \delta, (s,t), (x,y) \in I^2\Big\},$$

and for $\delta_1, \delta_2 > 0$, the partial modulus of continuity $\omega_1(f;\delta_1)$ and $\omega_2(f;\delta_2)$ are defined as

$$\omega_1(f;\delta_1) = \sup\Big\{|f(s,t) - f(x,t)| : t \in I, |s-x| \leq \delta_1\Big\}$$

and

$$\omega_2(f;\delta_2) = \sup\Big\{|f(s,t) - f(s,y)| : s \in I, |t-y| \leq \delta_2\Big\}.$$

For $f \in C(I^2)$ and $0 < \varepsilon, \eta \leq 1$, the Lipschitz class $Lip_M(\varepsilon, \eta)$ for the bivariate case is defined as

$$|f(s,t) - f(x,y)| \leq M|s-x|^{\varepsilon}|t-y|^{\eta}$$

and the Peetre's K-functional is given by

$$K(f;\delta) = \inf_{g \in C^2(I^2)}\Big\{\|f-g\| + \delta\|g\|_{C^2(I^2)}\Big\},\ \delta > 0,$$

where $C^2(I^2) = \Big\{h \in C(I^2) : h''_{xx}, h''_{yy}, h''_{xy}, h''_{yx} \in C(I^2)\Big\}$ with the norm

$$\|h\|_{C^2(I^2)} = \|h\| + \sum_{i=1}^{2}\left(\left\|\frac{\partial^i h}{\partial x^i}\right\| + \left\|\frac{\partial^i h}{\partial y^i}\right\|\right).$$

The second order modulus of continuity is defined as

$$\bar{\omega}_2(f;\sqrt{\delta}) = \sup_{|h|\leq\delta, |k|\leq\delta}\Big\{|f(x,y) - 2f(x+h, y+k) + f(x+2h, y+2k)| : (x,y), (x+2h, y+2k) \in I^2\Big\}.$$

From ([11], page 192), it is known that

$$K(f;\delta) \leq M\Big\{\bar{\omega}_2(f;\sqrt{\delta}) + \min(1,\delta)\|f\|\Big\}, \tag{7}$$

holds for all $\delta > 0$ and some positive constant M which is independent of δ and f.

6. Approximation properties for the bivariate case

Theorem 6.1. *For $f \in C(I^2)$, the sequence $D_{n_1,n_2,\lambda_1,\lambda_2}(f)$ converges to f uniformly as $n_1, n_2 \to \infty$.*

Proof. As a consequence of the theorem given by Volkov[26] and Lemma 2.1, the result follows. □

The following result is an immediate consequence of definition of partial moduli of continuity, Lemma 2.5 and the Cauch-Schwarz inequality:

Theorem 6.2. *For $f \in C(I^2)$, we have*

$$\|D_{n_1,n_2,\lambda_1,\lambda_2}(f) - f\| \leq 2\omega_1(f;\delta_{n_1,\lambda_1}) + 2\omega_2(f;\delta_{n_2,\lambda_2}).$$

The next result provides the degree of approximation of f by $D_{n_1,n_2,\lambda_1,\lambda_2}(f)$ in terms of the total modulus of continuity of f.

Theorem 6.3. *Let $f \in C(I^2)$. Then there holds the following inequality*

$$\|D_{n_1,n_2,\lambda_1,\lambda_2}(f) - f\| \leq 2\bar{\omega}(f;\delta).$$

Proof. Using the definition of total modulus of continuity, Lemma 2.5 and the Cauchy-Schwarz inequality, the result easily follows. Hence the details are omitted. □

For the Lipschitz class functions we can formulate the next asertion:

Theorem 6.4. *For $f \in Lip_M(\varepsilon, \eta)$, we have*

$$\|D_{n_1,n_2,\lambda_1,\lambda_2}(f) - f\| \leq M\delta^{\varepsilon}_{n_1,\lambda_1}\delta^{\eta}_{n_2,\lambda_2}.$$

Proof. Since $f \in Lip_M(\varepsilon, \eta)$, for any $(s,t), (x,y) \in I^2$, we may write

$$|D_{n_1,n_2,\lambda_1,\lambda_2}(f;x,y) - f(x,y)| \quad \leq \quad MD_{n_1,\lambda_1}\Big(|s-x|^{\varepsilon};x\Big)D_{n_2,\lambda_2}\Big(|t-y|^{\eta};y\Big).$$

Now using the Hölder's inequality, with $p_1 = \dfrac{2}{\varepsilon}$, $q_1 = \frac{2}{2-\varepsilon}$ and $p_2 = \frac{2}{\eta}$, $q_2 = \frac{2}{2-\eta}$ and applying Lemma 2.5, we have the desired result. □

The following result provides the rate of approximation of f by $D_{n_1,n_2,\lambda_1,\lambda_2}(f)$ when f is continuously differentiable in I^2.

Theorem 6.5. *For $f \in C^1(I^2)$, the operator $D_{n_1,n_2,\lambda_1,\lambda_2}$ verifies the following inequality*

$$\|D_{n_1,n_2,\lambda_1,\lambda_2}(f) - f\| \leq \|f'_x\|\delta_{n_1,\lambda_1} + \|f'_y\|\delta_{n_2,\lambda_2},$$

where $C^1(I^2) = \{f \in C(I^2) : f'_x, f'_y$ *exist and are continuous in* $I^2\}$.

Proof. Let $(x,y) \in I^2$, be a fixed point. Then we may write

$$f(s,t) - f(x,y) = \int_x^s f'_u(u,t)du + \int_y^t f'_v(x,v)dv.$$

Now, applying the operator $D_{n_1,n_2,\lambda_1,\lambda_2}(;x,y)$ on the above equation, the Cauchy-Schwarz inequality and Lemma 2.5, we are led to the required result. □

In the following theorem we obtain the degree of approximation of f by $D_{n_1,n_2,\lambda_1,\lambda_2}(f)$ in terms of the first and second order moduli of continuity of f via the approach of Peetre's K-functional.

Theorem 6.6. *If $f \in C(I^2)$ then we have*

$$\|D_{n_1,n_2,\lambda_1,\lambda_2}(f) - f\| \le M\Big\{\bar{\omega}_2\Big(f; \frac{\sqrt{\rho_{n_1,n_2,\lambda_1,\lambda_2}}}{2}\Big) + min\{1, \frac{\rho_{n_1,n_2,\lambda_1,\lambda_2}}{4}\}\|f\|\Big\} + \bar{\omega}\Big(f; \sqrt{(\mu^2_{n_1,\lambda_1} + \mu^2_{n_2,\lambda_2})}\Big),$$

where $\rho_{n_1,n_2,\lambda_1,\lambda_2} = \frac{1}{2}\Big[(\sqrt{\delta_{n_1,\lambda_1}} + \sqrt{\delta_{n_2,\lambda_2}})^2 + (\mu_{n_1,\lambda_1} + \mu_{n_2,\lambda_2})^2\Big]$ *and $M > 0$ is a constant independent of f and $\rho_{n_1,n_2,\lambda_1,\lambda_2}$.*

Proof. Let us define the auxiliary operator $D^*_{n_1,n_2,\lambda_1,\lambda_2}$ associated with $D_{n_1,n_2,\lambda_1,\lambda_2}$ as

$$D^*_{n_1,n_2,\lambda_1,\lambda_2}(f; x, y) = D_{n_1,n_2,\lambda_1,\lambda_2}(f; x, y) - f\Big(D_{n_1,\lambda_1}(s; x), D_{n_2,\lambda_2}(t; y)\Big) + f(x, y). \tag{8}$$

Then using Lemma 2.1, we have

$$D^*_{n_1,n_2,\lambda_1,\lambda_2}(1; x, y) = 1,\ D^*_{n_1,n_2,\lambda_1,\lambda_2}(s; x, y) = x \text{ and } D^*_{n_1,n_2,\lambda_1,\lambda_2}(t; x, y) = y. \tag{9}$$

Let $g \in C^2(I^2)$ and $(s, t) \in I^2$. Using the Taylor's theorem for a function of two variables, we have

$$\begin{aligned} g(s, t) - g(x, y) &= g(s, y) - g(x, y) + g(s, t) - g(s, y) \\ &= \frac{\partial g(x, y)}{\partial x}(s - x) + \int_x^s (s - u)\frac{\partial^2 g(u, y)}{\partial u^2}du + \frac{\partial g(x, y)}{\partial y}(t - y) \\ &+ \int_y^t (t - v)\frac{\partial^2 g(x, v)}{\partial v^2}dv + \int_x^s\int_y^t \frac{\partial^2 g(u, v)}{\partial u \partial v}dv du. \end{aligned}$$

Applying the operator $D^*_{n_1,n_2,\lambda_1,\lambda_2}$ on both sides of of the above equation, for $f \in C(I^2)$ and any $g \in C^2(I^2)$, using the properties (9), we have

$$|D_{n_1,n_2,\lambda_1,\lambda_2}(f; x, y) - f(x, y)| \le 4\Big(\|f - g\| + \frac{\rho_{n_1,n_2,\lambda_1,\lambda_2}}{4}\|g\|_{C^2(I^2)}\Big) + \bar{\omega}\Big(f; \sqrt{(\mu^2_{n_1,\lambda_1} + \mu^2_{n_2,\lambda_2})}\Big).$$

Now, taking the infimum on the right side of the above equation over all $g \in C^2(I^2)$ and using relation (7), we reach to the required result. □

In our next result, we obtain a Voronovskaja type asymptotic theorem for the bivariate operators $D_{n_1,n_2,\lambda_1,\lambda_2}$.

Theorem 6.7. *Let $f \in C^2(I^2)$. Then*

$$\lim_{n\to\infty} n(D_{n,n,\lambda_1,\lambda_2}(f; x, y) - f(x, y)) = (1 - 2x)f'_x(x, y) + (1 - 2y)f'_y(x, y) + x(1 - x)f''_{xx}(x, y) + y(1 - y)f''_{yy}(x, y),$$

uniformly in $(x, y) \in I^2$.

Proof. Let $(x, y) \in I^2$ be arbitrary. By the Taylor's theorem we have

$$\begin{aligned} D_{n,n,\lambda_1,\lambda_2}(f(s, t); x, y) &= f(x, y) + f'_x(x, y)D_{n,n,\lambda_1,\lambda_2}((s - x); x, y) + f'_y(x, y)D_{n_1,n_2,\lambda_1,\lambda_2}((t - y); x, y) \\ &+ \frac{1}{2}f''_{xx}(x, y)D_{n,n,\lambda_1,\lambda_2}((s - x)^2; x, y) + f''_{xy}(x, y)D_{n,n,\lambda_1,\lambda_2}((s - x)(t - y); x, y) \\ &+ \frac{1}{2}f''_{yy}(x, y)D_{n,n,\lambda_1,\lambda_2}((t - y)^2; x, y) + D_{n,n,\lambda_1,\lambda_2}\Big(\chi(s, t; x, y)\sqrt{(s - x)^4 + (t - y)^4}; x, y\Big), \end{aligned}$$

where $\chi(s, t; x, y) \in C(I^2)$ *and* $\chi(s, t; x, y) \to 0,$ *as* $(s, t) \to (x, y)$.
Applying Lemma 2.3, we get

$$\begin{aligned} \lim_{n\to\infty} n(D_{n,n,\lambda_1,\lambda_2}(f; x, y) - f(x, y)) &= (1 - 2x)f'_x(x, y) + (1 - 2y)f'_y(x, y) + x(1 - x)f''_{xx}(x, y) + y(1 - y)f''_{yy}(x, y) \\ &+ \lim_{n\to\infty} nD_{n,n,\lambda_1,\lambda_2}\Big(\chi(s, t; x, y)\sqrt{(s - x)^4 + (t - y)^4}; x, y\Big), \end{aligned}$$

uniformly in $(x, y) \in I^2$. Now, we evaluate $\lim_{n\to\infty} nD_{n,n,\lambda_1,\lambda_2}\Big(\chi(s,t;x,y)\sqrt{(s-x)^4+(t-y)^4};x,y\Big)$. By using Hölder's inequality, Theorem 6.1 and Lemma 2.3, we get

$$\lim_{n\to\infty} nD_{n,n,\lambda_1,\lambda_2}\Big(\chi(s,t;x,y)\sqrt{(s-x)^4+(t-y)^4};x,y\Big) = 0,$$

as $n \to \infty$, uniformly in $(x, y) \in I^2$. Thus, the proof is completed. □

In the forthcoming result, we present a Grüss-Voronovskaja type theorem.

Theorem 6.8. *Let $f, g \in C^2\left(I^2\right)$ then the following equality holds true:*

$$\begin{aligned}\lim_{n\to\infty} n\{D_{n,n,\lambda_1,\lambda_2}(fg;x,y) - D_{n,n,\lambda_1,\lambda_2}(f;x,y)D_{n,n,\lambda_1,\lambda_2}(g;x,y)\} &= 2x(1-x)f'_x(x,y)g'_x(x,y)\\ &+ 2y(1-y)f'_y(x,y)g'_y(x,y),\end{aligned}$$

uniformly in $(x, y) \in I^2$.

Proof. Proceeding in a manner similar to the proof of ([13], Theorem. 2), and applying Theorem 6.1, in view of Lemma 2.3 and Theorem 6.7, we reach the desired result. □

7. Construction of GBS operator of λ-Durrmeyer-Bernstein type

Bögel [8] gave some new concepts in analysis known as Bögel continuity and Bögel differentiablity for a function of two variables. Using these concepts, Dobrescu and Matei [15] proved that the bivariate Bernstein polynomials can be uniformly approximated by the associated GBS (Generalized boolean sum) operators. Badea et al. [6] established a Korovkin type theorem known as "Test Function Theorem " to approximate Bögel continuous functions. Badea and Cottin [7] gave Korovkin type theorems for GBS operators. For further related research in this direction we refer the readers to [22], [1] and the references therein.

For any $(s, t), (x, y) \in I^2$, the mixed difference is denoted by $\Delta f[(s, t); (x, y)]$ and is defined as

$$\Delta f[(s,t);(x,y)] = f(x,y) - f(x,t) - f(s,y) + f(s,t).$$

A function $f : I^2 \to \mathbb{R}$ is said to be Bögel bounded or B-bounded on I^2 if there exists a constant K such that

$$|\Delta f[(s,t);(x,y)]| \le K,$$

for all $(s, t), (x, y) \in I^2$. Let $B_B(I^2)$ denote the space of all Bögel bounded functions on I^2.

A function $f : I^2 \to \mathbb{R}$ is said to be Bögel continuous on I^2 if for every $(x, y) \in I^2$, we have

$$\lim_{(s,t)\to(x,y)} \Delta f[(s,t);(x,y)] = 0.$$

We denote the space of all Bögel continuous functions on I^2 by $C_B(I^2)$.

A function $f : I^2 \to \mathbb{R}$ is called Bögel differentiable function at $(x, y) \in I^2$, if the limit

$$\lim_{(s,t)\to(x,y)} \frac{\Delta f[(s,t);(x,y)]}{(s-x)(t-y)},$$

exists and is finite. The space of all Bögel differentiable function on I^2 is denoted by $D_B(I^2)$.

For any $f \in C_B(I^2)$, we define the GBS operators associated with the operators given by (6) as follows:

$$\begin{aligned} S_{n_1,n_2,\lambda_1,\lambda_2}(f;x,y) &= D_{n_1,n_2;\lambda_1,\lambda_2}\Big(f(s,y)+f(x,t)-f(s,t);x,y\Big) \\ &= (n_1+1)(n_2+1)\sum_{k_1=0}^{n_1}\sum_{k_2=0}^{n_2}\tilde{b}_{n_1,n_2,k_1,k_2}(\lambda_1,\lambda_2;x,y) \\ & \int_0^1\int_0^1 p_{n_1,n_2,k_1,k_2}(s,t)\Big(f(s,y)+f(x,t)-f(s,t)\Big)dtds, \end{aligned} \tag{10}$$

for every $(x,y) \in I^2$. Clearly, (10) is a linear operator. Using the Korovkin type theorem given by Badea et. al [6], and Lemma 2.1, the sequence $S_{n_1,n_2,\lambda_1,\lambda_2}(f)$, converges to f uniformly on I^2 for all $f \in C_B(I^2)$.

For $f \in C_B(I^2)$, the mixed modulus of smoothness is defined as

$$\omega_{mixed}(f;\delta_1,\delta_2) = \sup\{|\Delta f[s,t;x,y]| : |s-x| < \delta_1, |t-y| < \delta_2\},$$

for all $(s,t),(x,y) \in I^2$ and for any $\delta_1,\delta_2 > 0$. For the basic properties of ω_{mixed} we refer to [5] and [7].

Applying the Shisha-Mond type theorem given by Badea et. al [5] to obtain the degree of approximation for Bögel continuous functions by GBS operators and Lemma 2.5, we have

Theorem 7.1. *For any $f \in C_B(I^2)$, the operator (10) satisfies the following inequality*

$$\|S_{n_1,n_2,\lambda_1,\lambda_2}(f) - f\| \leq 4\omega_{mixed}\Big(f;\delta_{n_1,\lambda_1},\delta_{n_2,\lambda_2}\Big).$$

The Lipschitz class of Bögel continuous functions is denoted by $Lip_M(\varepsilon,\eta)$, $0 < \varepsilon,\eta \leq 1, M > 0$ and is defined as

$$Lip_M(\varepsilon,\eta) = \Big\{f \in C_B(I^2) : |\Delta f[(s,t);(x,y)]| \leq M|s-x|^{\varepsilon}|t-y|^{\eta},\ for\ (s,t),(x,y) \in I^2\Big\}.$$

The following theorem provides the degree of approximation by the operators $S_{n_1,n_2,\lambda_1,\lambda_2}$ for Lipschitz class of Bögel continuous functions.

Theorem 7.2. *Let $f \in Lip_M(\varepsilon,\eta)$ then we have*

$$\|S_{n_1,n_2,\lambda_1,\lambda_2}(f) - f\| \leq M\delta_{n_1,\lambda_1}^{\varepsilon/2}\delta_{n_2,\lambda_2}^{\eta/2}.$$

Proof. From the definition of the operators $S_{n_1,n_2,\lambda_1,\lambda_2}(f;x,y)$, the linearity of the operator (6) and Lemma 2.1, for any $(x,y) \in I^2$ we have

$$S_{n_1,n_2,\lambda_1,\lambda_2}(f;x,y) = f(x,y) - D_{n_1,n_2,\lambda_1,\lambda_2}\Big(\Delta f[(s,t);(x,y)];x,y\Big). \tag{11}$$

Since $f \in Lip_M(\varepsilon,\eta)$, we get

$$\begin{aligned} |S_{n_1,n_2,\lambda_1,\lambda_2}(f;x,y) - f(x,y)| &\leq MD_{n_1,n_2,\lambda_1,\lambda_2}\Big(|s-x|^{\varepsilon}|t-y|^{\eta};x,y\Big) \\ &= MD_{n_1,n_2,\lambda_1,\lambda_2}\Big(|s-x|^{\varepsilon};x,y\Big)D_{n_1,n_2,\lambda_1,\lambda_2}\Big(|t-y|^{\eta};x,y\Big). \end{aligned}$$

Now, using Hölder's inequality with $p_1 = \frac{2}{\varepsilon}$, $q_1 = \frac{2}{2-\varepsilon}$ and $p_2 = \frac{2}{\eta}$, $q_2 = \frac{2}{2-\eta}$ and Lemma 2.5, the required result is immediate. □

The following theorem provides the order of approximation for Bögel differentiable functions by the operators defined by (10) in terms of the mixed modulus of smoothness.

Theorem 7.3. *Let the function $f \in D_B(I^2)$ with $D_B f$ be bounded on I^2. Then, we have*

$$\|S_{n_1,n_2,\lambda_1,\lambda_2}(f) - f\| \le \frac{C}{\sqrt{n_1 n_2}}\Big[\|D_B f\|_\infty + \omega_{mixed}\Big(D_B f; \frac{1}{\sqrt{n_1}}, \frac{1}{\sqrt{n_2}}\Big)\Big].$$

Proof. For $f \in D_B(I^2)$, we have

$$\Delta f[(s,t);(x,y)] = (s-x)(t-y)D_B f(u,v) \tag{12}$$

with $x < u < s$ and $y < v < t$ (cf [9],page 62). It is clear that

$$D_B f(u,v) = \Delta D_B f(u,v) + D_B f(u,y) + D_B f(x,v) - D_B f(x,y).$$

Since $D_B f \in B(I^2)$, from equation (12), we can write

$$\begin{aligned}
\Big|D_{n_1,n_2,\lambda_1,\lambda_2}\Big(\Delta f[(s,t);(x,y)];x,y\Big)\Big| &\le D_{n_1,n_2,\lambda_1,\lambda_2}\Big(|s-x||t-y||\Delta D_B f(u,v)|;x,y\Big)\\
&+ D_{n_1,n_2,\lambda_1,\lambda_2}\Big(|s-x||t-y|\Big(|D_B f(u,y)| + |D_B f(x,v)| + |D_B f(x,y)|\Big);x,y\Big)\\
&\le D_{n_1,n_2,\lambda_1,\lambda_2}\Big(|s-x||t-y|\omega_{mixed}\Big(D_B f; |u-x|, |v-y|\Big);x,y\Big)\\
&+ 3\cdot \|D_B f\|_\infty D_{n_1,n_2,\lambda_1,\lambda_2}\Big(|s-x||t-y|;x,y\Big).
\end{aligned}$$

Hence, considering (11) and the following property of ω_{mixed}

$$\omega_{mixed}\Big(f; |s-x|, |t-y|\Big) \le \Big(1 + \frac{|s-x|}{\delta_1}\Big)\Big(1 + \frac{|t-y|}{\delta_2}\Big)\omega_{mixed}(f;\delta_1,\delta_2),$$

for any $\delta_1, \delta_2 > 0$, we have

$$\begin{aligned}
|S_{n_1,n_2,\lambda_1,\lambda_2}(f;x,y) - f(x,y)| &= |D_{n_1,n_2,\lambda_1,\lambda_2}(\Delta f[(s,t);(x,y)];x,y)|\\
&\le \Big(D_{n_1,n_2,\lambda_1,\lambda_2}(|s-x||t-y|;x,y) + \delta_1^{-1} D_{n_1,n_2,\lambda_1,\lambda_2}((s-x)^2|t-y|;x,y)\\
&+ \delta_2^{-1} D_{n_1,n_2,\lambda_1,\lambda_2}(|s-x|(t-y)^2;x,y)\\
&+ \delta_1^{-1}\delta_2^{-1} D_{n_1,n_2,\lambda_1,\lambda_2}((s-x)^2(t-y)^2;x,y)\Big)\omega_{mixed}(D_B f;\delta_1,\delta_2)\\
&+ 3\cdot\|D_B f\|_\infty D_{n_1,n_2,\lambda_1,\lambda_2}\Big(|s-x||t-y|;x,y\Big).
\end{aligned}$$

Applying Cauchy-Schwarz inequality, in view of Lemma 2.5, we have

$$|S_{n_1,n_2,\lambda_1,\lambda_2}(f;x,y) - f(x,y)| \le \frac{C}{\sqrt{n_1 n_2}}\Big[\|D_B f\|_\infty + \omega_{mixed}\Big(D_B f; \frac{1}{\sqrt{n_1}}, \frac{1}{\sqrt{n_2}}\Big)\Big].$$

□

8. Numerical results

In this section we present some numerical results obtained by using Mathematica. In Figure 8, we plot the operators $D_{n_1,n_2;\lambda_1,\lambda_2}(f;x,y)$ (Green) for $n_1 = n_2 = 10$, $D_{n_1,n_2;\lambda_1,\lambda_2}(f;x,y)$ (Blue) for $n_1 = n_2 = 20$, and $f(x,y) = 3xy^2e^{x-y}$ (Yellow) on I^2 for $\lambda_1 = \lambda_2 = 1$.

In Figure 9, we compare the rate of convergence of the bivariate operators $D_{n_1,n_2;\lambda_1,\lambda_2}(f;x,y)$ (Green) and its GBS modification $S_{n_1,n_2;\lambda_1,\lambda_2}(f;x,y)$ (Blue) to the function

$f(x, y) = 5x^2(1 - y + y^2)$ (Yellow) on I^2 for $n_1 = n_2 = 50$ and $\lambda_1 = \lambda_2 = 1$. It is clearly seen that the GBS operators yield a better approximation than the bivariate operators.

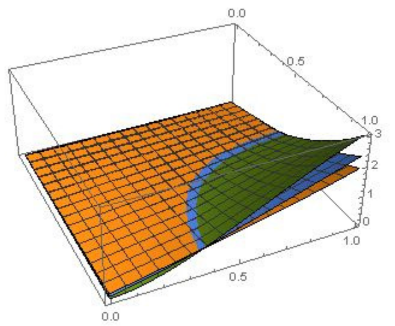

Figure 8:

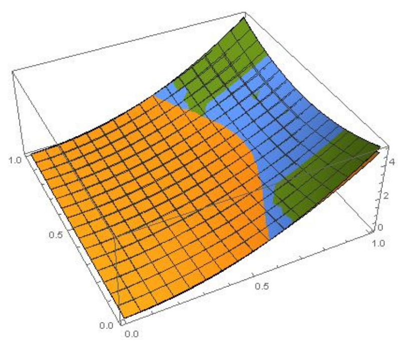

Figure 9:

In the table below, we compare the error in the approximation by the bivariate operators to the function $f(x, y) = 3xy^2e^{x-y}$ for $n_1 = n_2 = 10$ and $n_1 = n_2 = 20$ at certain points $(x, y) \in I^2$ for $\lambda_1 = \lambda_2 = 1$. It is clear that as n_1 and n_2 increase, the error in the approximation decreases.

x	y	$\lvert D_{n_1,n_2;\lambda_1,\lambda_2}(f; x, y) - f(x, y)\rvert$	$\lvert D_{n_1,n_2;\lambda_1,\lambda_2}(f; x, y) - f(x, y)\rvert$
0.0	0.0	0.0143905	0.0030483
0.1	0.4	0.0911713	0.0464795
0.2	0.4	0.1029830	0.0543372
0.3	0.8	0.0966132	0.0615227
0.4	0.9	0.0416243	0.0355476
0.5	1.0	0.0460035	0.0085817
0.6	0.6	0.0366915	0.0366915
0.7	0.9	0.3195692	0.1739941
0.8	0.4	0.0186596	0.0215596
0.9	0.5	0.1748071	0.1011293
1.0	1.0	1.0770612	0.6748435

In our final table, we compare an estimate of the error in the approximation of $f(x, y) = 5x^2(1 - y + y^2)$ at certain points in I^2 by the bivariate operators $D_{n_1,n_2;\lambda_1,\lambda_2}$ and its GBS case $S_{n_1,n_2;\lambda_1,\lambda_2}$ for $n_1 = n_2 = 50$ and $\lambda_1 = \lambda_2 = 1$. It is evident that the error in the approximation of f by $S_{n_1,n_2;\lambda_1,\lambda_2}(f)$ is much less than the error by $D_{n_1,n_2;\lambda_1,\lambda_2}(f)$.

x	y	$\lvert D_{n_1,n_2;\lambda_1,\lambda_2}(f; x, y) - f(x, y)\rvert$	$\lvert S_{n_1,n_2;\lambda_1,\lambda_2}(f; x, y) - f(x, y)\rvert$
0.1	0.1	0.0386415	0.0004711
0.2	0.4	0.0540595	0.0006730
0.3	0.5	0.0643545	0.0008823
0.4	0.4	0.0650762	0.0007382
0.5	0.6	0.0560842	0.0005641
0.6	0.2	0.0192291	0.0000285
0.7	0.6	0.0090968	0.0001957
0.8	0.2	0.0697866	0.0000888
0.9	0.6	0.0764991	0.0015097
1.0	1.0	0.3407891	0.0052795

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References

[1] A.M. Acu, T. Acar, C.V. Muraru, V.A. Radu, Some approximation properties by a class of bivariate operators, Mathematical Method in the Applied Sciences 42 (2019) 5551–5565.
[2] A.M. Acu, T. Acar, V.A. Radu, Approximation by modified U_n^ρ operators, Revista de la Real Academia de Ciencias Exactas Físicas y Naturales. Serie A. Matemáticas (RACSAM) 113 (2019) 2715–2729.
[3] F. Altomare, M. Campiti, Korovkin-type Approximation Theory and its Applications, de Gruyter, Berlin, New York, 1994.
[4] K. Bögel, Über mehrdimenssionale differentiation, integration and beschränkte variation, Journal für die reine und angewandte Mathematik 173 (1935) 5–30.
[5] C. Badea, I Badea, and H.H. Gonska, Notes on degree of approximation of B-continuous and B-differentiable functions, Approximation Theory and Its Applications 4(3) (1988), 95–108.
[6] C. Badea, I Badea, H.H. Gonska, A test function theorem and approximation by pseudopolynomials, Bulletin of the Australian Mathematical Society 34(1) (1986) 53–64.
[7] C. Badea, C. Cottin, Korovkin type theorems for generalised Boolean sum operators, In Approximation theory Colloquia Mathematica Societatis Janos Bolyai 58 (1991) 51–68.
[8] K. Bögel, Mehrdimensionaie differentiation von funktionen mehrerer reeller veränderlichen, Journal für die reine und angewandte Mathematik 170 (1934) 197–217.
[9] K. Bögel, Über Mehrdimensionaie differentiation, Jahresbericht der Deutschen Mathematiker-Vereinigung 65 (1963) 45–71.
[10] J. Bustamante, Bernstein Operators and Their Properties, Birkhäuser, Basel, 2017.
[11] P.L. Butzer, H. Berens, Semi groups of Operators and Approximation, Springer-Verlag, New York, 1967.
[12] Q.B. Cai, B.Y. Lian, G. Zhou, Approximation properties of λ-Bernstein operators, Journal of Inequalities and Applications 12 (2018), 1–11.
[13] R. Chauhan, B. Baxhaku and P.N. Agrawal, GBS operators of bivariate Bernstein-Durrmeyer type on a triangle, Mathematical Methods in the Applied Sciences 41(7) (2018) 2673–2683.
[14] Z. Ditzian, V. Totik, Moduli of Smoothness, Springer Science & Business Media, 1987.
[15] E. Dobrescu, I. Matei, The approximatio by Bernstein type polynomials of bidimentionally continuous functions, Analele Universitaţii din Timişoara. Seria Ştiinţe Fizice-Chimice 4 (1966) 85–90.
[16] H. Esser, On pointwise convergence estimates for positive linear operators on $C[a,b]$, Indagationes Mathematicae 38 (1976) 189–194.
[17] Z. Finta, Remark on Voronovskaja theorem for q-Bernstein operators, Studia Universitatis Babe¸s-Bolyai / Mathematica 56 (2011) 335–339.
[18] S. G. Gal, V. Gupta, Quantitative estimates for a new complex Durrmeyer operator in compact disks, Applied Mathematics and Computations 218(6) (2011) 2944–2951.
[19] V. Gupta, R.P. Agarwal, Convergence Estimates in Approximation Theory, Springer, Cham, 2014.
[20] V. Gupta, A. Aral, Bernstein Durrmeyer operators based on two parameters, Facta Universitatis Series: Mathematics and Informatics 31(1) (2016) 79–95.
[21] V. Gupta, M. T. Rassias, Moments of Linear Positive Operators and Approximation, Series: Springer Briefs in Mathematics, Springer Nature, Switzerland, 2019.
[22] V. Gupta, T.M. Rassias, P.N. Agrawal, A.M. Acu, Recent Advances in Constructive Approximation Theory, Springer, Cham, 2018.
[23] V. Gupta, G. Tachev, Approximation with Positive Linear Operators and Linear Combinations, Springer, Cham, 2017.
[24] B. Lenze, On Lipschitz-type maximal functions and their smoothness spaces, Nederl Akad Wetensch Indag Math 50(1) (1988) 53–63.
[25] R. Păltănea, Approximation Theory Using Positive Linear Operators, Birkhauser, Boston, 2004.
[26] V.I. Volkov, On the convergence of sequences of linear positive operators in the space of continuous functions of two variables, Doklady Akademii nauk Soiuza Sovetskikh Sotsialisticheskikh Respublik. 115 (1957) 17–19.
[27] Z. Ye, X. Long, X.M. Zeng, Adjustment algorithms for Bézier curve and surface, International Conference on Computer Science and Education, Aug. 24-27 (2010), Hefei, China, 1712–1716.