Filomat 35:4 (2021), 1393–1403
https://doi.org/10.2298/FIL2104393Z

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Some Inequalities for the (p, q)-Mixed Geominimal Surface Areas and L_p Radial Blaschke-Minkowski Homomorphisms

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Abstract. Wang et al. introduced L_p radial Blaschke-Minkowski homomorphisms based on Schuster's radial Blaschke-Minkowski homomorphisms. In 2018, Feng and He gave the concept of (p, q)-mixed geominimal surface area according to the Lutwak, Yang and Zhang's (p, q)-mixed volume. In this article, associated with the (p, q)-mixed geominimal surface areas and the L_p radial Blaschke-Minkowski homomorphisms, we establish some inequalities including two Brunn-Minkowski type inequalities, a cyclic inequality and two monotonic inequalities.

1. Introduction

We use $\mathcal{K}^n$ to denote the set of convex bodies, that is compact, convex subsets with nonempty interiors in Euclidean space $\mathbb{R}^n$. For the set of convex bodies containing the origin in their interiors, we write $\mathcal{K}^n_o$. For the set of star bodies (about the origin) in $\mathbb{R}^n$, we write $\mathcal{S}^n_o$. As usual, $V(K)$ denotes the n-dimensional volume of a body K, B the standard unit ball and S^{n-1} the unit sphere in $\mathbb{R}^n$.

For each $K \in \mathcal{S}^n_o$, the intersection body, IK, of K is a star body symmetric with respect to origin whose radial function on S^{n-1} is given by (see [16]):

$$\rho(IK, u) = v_{n-1}(K \cap u^{\perp}),$$

for all $u \in S^{n-1}$. Here v_{n-1} is $(n-1)$-dimensional volume and $K \cap u^{\perp}$ denotes the intersection of K with the subspace $u^{\perp}$ that passes through the origin and is orthogonal to u.

Based on the properties of intersection bodies, Schuster ([20]) introduced the notion of radial Blaschke-Minkowski homomorphisms as follows:

Definition 1.1. *A map Ψ: $\mathcal{S}^n_o \to \mathcal{S}^n_o$ is called a radial Blaschke-Minkowski homomorphism if it satisfies the following conditions:*

2010 *Mathematics Subject Classification*. Primary 52A40; Secondary 52A20
Keywords. L_p radial Blaschke-Minkowski homomorphism, the (p, q)-mixed geominimal surface area, Brunn-Minkowski type inequality, cyclic inequality, monotonic inequality.
Received: 06 April 2020; Accepted: 06 July 2020
Communicated by Dragan S. Djordjević
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Research is supported in part by the National Natural Science Foundation of China (Grant No.11371224) and Innovation Foundation of Graduate Student of China Three Gorges University (Grant No.2019SSPY145).
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(a) Ψ is continuous.
(b) For all $K, L \in \mathcal{S}_o^n$,

$$\Psi(K\hat{+}L) = \Psi K\tilde{+}\Psi L,$$

where $\Psi K\tilde{+}\Psi L$ denotes the radial Minkowski addition of ΨK and ΨL (see (11)), $K\hat{+}L$ denotes the radial Blaschke addition of star bodies K and L (see (12)).
(c) For all $K \in \mathcal{S}_o^n$ and every $\vartheta \in SO(n)$, $\Psi(\vartheta K) = \vartheta\Psi K$.
Here, $SO(n)$ is the group of rotations in n dimensions.

In 2011, Wang, Liu and He ([24]) introduced the notion of L_p radial Blaschke-Minkowski homomorphisms as follows:

Definition 1.2. *A map Ψ_p: $\mathcal{S}_o^n \to \mathcal{S}_o^n$ is called an L_p radial Blaschke-Minkowski homomorphism if it satisfies the following conditions:*
(a) Ψ_p is continuous.*
(b) For all $K, L \in \mathcal{S}_o^n$,*

$$\Psi_p(K\hat{+}_pL) = \Psi_pK\tilde{+}_p\Psi_pL,$$

where $K\hat{+}_pL$ denotes the L_p radial Blaschke addition of star bodies K and L, and $\Psi_pK\tilde{+}_p\Psi_pL$ denotes the L_p radial Minkowski addition of Ψ_pK and Ψ_pL.
(c) For all $K \in \mathcal{S}_o^n$ and every $\vartheta \in SO(n)$, $\Psi_p(\vartheta K) = \vartheta\Psi_pK$.*

Remark 1.1. The L_p intersection body is a special case of the L_p radial Blaschke-Minkowski homomorphism, it was first introduced by Haberl and Ludwig (see [10]): For $K \in \mathcal{S}_o^n$ and $0 < p < 1$, the L_p-intersection body, I_pK, of K is the origin-symmetric star body whose radial function was defined by

$$\rho(I_pK, u)^p = \frac{1}{2(n-p)} \int_{S^{n-1}} | u \cdot v |^{-p} \rho(K, v)^{n-p} dS(v),$$

for all $u \in S^{n-1}$. Here $u \cdot x$ denotes the standard inner product of u and x.

Schuster ([20]) also introduced the notion of Blaschke-Minkowski homomorphisms, Wang ([23]) extended this notion to L_p version later. Regarding the studies of L_p radial Blaschke-Minkowski homomorphisms and L_p Blaschke-Minkowski homomorphisms, many results have been found in [1, 5–7, 13, 14, 26, 31–36].

In 2016, Huang, Lutwak, Yang and Zhang ([11]) constructed the dual curvature measures in the dual Brunn-Minkowski theory. These measures are dual to Federer's curvature measures which are fundamental in the classical Brunn-Minkowski theory. In 2018, Lutwak, Yang and Zhang ([18]) took a further major step and introduced the L_p dual curvature measures. Based on this concept, they defined the following (p, q)-mixed volumes.

For $p, q \in \mathbb{R}$, $K, Q \in \mathcal{K}_o^n$ and $L \in \mathcal{S}_o^n$, the (p, q)-mixed volume, $\widetilde{V}_{p,q}(K, Q, L)$, is defined by

$$\widetilde{V}_{p,q}(K, Q, L) = \frac{1}{n} \int_{S^{n-1}} \left(\frac{h_Q}{h_K}\right)(\alpha_K(u))^p \rho_K(u)^q \rho_L(u)^{n-q} du, \tag{1}$$

where α_K is the radial Gauss map.

By (1), they also gave the following special cases:

$$\widetilde{V}_{p,q}(K, Q, K) = V_p(K, Q), \tag{2}$$

$$\widetilde{V}_{p,n}(K, Q, L) = V_p(K, Q). \tag{3}$$

Using the (p, q)-mixed volumes, Feng and He ([3]) introduced the concept of (p, q)-mixed geominimal surface areas as follows:

Definition 1.3. *For $p, q \in \mathbb{R}$, $K \in \mathcal{K}_o^n$ and $L \in \mathcal{S}_o^n$, the (p,q)-mixed geominimal surface area, $\widetilde{G}_{p,q}(K,L)$, of K and L is defined by*

$$\omega_n^{\frac{p}{n}} \widetilde{G}_{p,q}(K,L) = \inf\{n\widetilde{V}_{p,q}(K,Q,L)V(Q^*)^{\frac{p}{n}} : Q \in \mathcal{K}_o^n\}. \tag{4}$$

If $L = K$ or $q = n$ in (4), then from (2) or (3) we see that the definition is just Lutwak's L_p geominimal surface area for $p \geq 1$ (see [17]). For the studies of L_p geominimal surface areas, some results have been obtained in these articles (see e.g., [2, 4, 12, 21, 22, 25, 27–30, 37–39]).

In this paper, associated with the (p,q)-mixed geominimal surface areas, we sequentially research the L_p radial Blaschke-Minkowski homomorphisms. Firstly, we establish the following two related Brunn-Minkowski type inequalities.

Theorem 1.1. *For $K \in \mathcal{K}_o^n$ and $L_1, L_2 \in \mathcal{S}_o^n$, let $\Psi_p : \mathcal{S}_o^n \to \mathcal{S}_o^n$ be an L_p radial Blaschke-Minkowski homomorphism. If $0 < n - q < p$, then*

$$\widetilde{G}_{p,q}(K, \Psi_p(L_1 \hat{+}_p L_2))^{\frac{p}{n-q}} \geq \widetilde{G}_{p,q}(K, \Psi_p L_1)^{\frac{p}{n-q}} + \widetilde{G}_{p,q}(K, \Psi_p L_2)^{\frac{p}{n-q}}, \tag{5}$$

with equality if and only if L_1 and L_2 are dilates. Here $\hat{+}_p$ denotes the L_p-radial Blaschke addition.

Theorem 1.2. *For $K \in \mathcal{K}_o^n$ and $L_1, L_2 \in \mathcal{S}_o^n$, let $\Psi_p : \mathcal{S}_o^n \to \mathcal{S}_o^n$ be an L_p radial Blaschke-Minkowski homomorphism. If $0 < p < n$ and $0 < (n-p)(n-q) < p(n+p)$, then*

$$\frac{\widetilde{G}_{p,q}(K, \Psi_p(L_1 \mp_p L_2))^{\frac{p(n+p)}{(n-p)(n-q)}}}{V(L_1 \mp_p L_2)} \geq \frac{\widetilde{G}_{p,q}(K, \Psi_p L_1)^{\frac{p(n+p)}{(n-p)(n-q)}}}{V(L_1)} + \frac{\widetilde{G}_{p,q}(K, \Psi_p L_2)^{\frac{p(n+p)}{(n-p)(n-q)}}}{V(L_2)}, \tag{6}$$

with equality if and only if L_1 and L_2 are dilates. Here $\mp_p$ denotes the L_p-harmonic Blaschke addition.

Then, we give a cyclic inequality for L_p radial Blaschke-Minkowski homomorphisms as follows:

Theorem 1.3. *Let $\Psi_p : \mathcal{S}_o^n \to \mathcal{S}_o^n$ be an L_p radial Blaschke-Minkowski homomorphism. If $K \in \mathcal{K}_o^n$, $L \in \mathcal{S}_o^n$ and $1 \leq r < s < t$, then*

$$\widetilde{G}_{p,s}(K, \Psi_s L)^{t-r} \leq \widetilde{G}_{p,r}(K, \Psi_r L)^{t-s} \widetilde{G}_{p,t}(K, \Psi_t L)^{s-r}, \tag{7}$$

with equality if and only if $\Psi_r L$, $\Psi_s L$ and $\Psi_t L$ are dilates each other.

Finally, together with the L_p radial Blaschke-Minkowski homomorphisms, we obtain two monotonic inequalities for the (p,q)-mixed geominimal surface areas.

Theorem 1.4. *Let $\Psi_p : \mathcal{S}_o^n \to \mathcal{S}_o^n$ be an L_p radial Blaschke-Minkowski homomorphism. For $K \in \mathcal{K}_o^n$, $L_1, L_2 \in \mathcal{S}_o^n$ and $0 < q < n$, if $L_1 \subseteq L_2$, then*

$$\widetilde{G}_{p,q}(K, \Psi_p L_1) \leq \widetilde{G}_{p,q}(K, \Psi_p L_2), \tag{8}$$

equality holds when $L_1 = L_2$.

Theorem 1.5. *Let $\Psi_p : \mathcal{S}_o^n \to \mathcal{S}_o^n$ be an L_p radial Blaschke-Minkowski homomorphism. For $K, L_1, L_2 \in \mathcal{K}_o^n$ and $0 < q < n$, if $L_1 \subseteq L_2$, then*

$$\widetilde{G}_{p,q}(K, \Psi_p^* L_1) \geq \widetilde{G}_{p,q}(K, \Psi_p^* L_2), \tag{9}$$

equality holds when $L_1 = L_2$. Here Ψ_p^* denotes the polar of L_p radial Blaschke-Minkowski homomorphisms.

2. Preliminaries

Our work belongs to the new and developed rapidly dual Brunn-Minkowski theory, we collect some interrelated backgrounds and notations.

For $K \in \mathcal{K}^n$, then the support function of K, $h_K = h(K, \cdot) : \mathbb{R}^n \to \mathbb{R}$, is defined by (see [8])

$$h(K, x) = \max\{x \cdot y : y \in K\}, \quad x \in \mathbb{R}^n,$$

where $x \cdot y$ denotes the standard inner product of x and y in $\mathbb{R}^n$.

If K is a compact star-shaped (about the origin) in $\mathbb{R}^n$, its radial function of K, $\rho_K = \rho(K, \cdot) : \mathbb{R}^n \setminus \{0\} \to [0, +\infty)$, is given by (see [19])

$$\rho(K, x) = \max\{\lambda \geq 0 : \lambda x \in K\}, \quad x \in \mathbb{R}^n \setminus \{0\}.$$

If ρ_K is positive and continuous, K will be called a star body (with respect to the origin). Two star bodies K and L are said to be dilates (of one another) if $\rho_K(u)/\rho_L(u)$ is independent of any $u \in S^{n-1}$.

If E is a nonempty subset in $\mathbb{R}^n$, then the polar set, E^*, of E is defined by (see [8, 19])

$$E^* = \{x \in \mathbb{R}^n : x \cdot y \leq 1, \ y \in E\}.$$

Meanwhile, it is easy to get that $(K^*)^* = K$ for all $K \in \mathcal{K}_o^n$.

From the above definitions, we know that if $K \in \mathcal{K}_o^n$, then (see [8, 19])

$$h(K^*, \cdot) = \frac{1}{\rho(K, \cdot)}, \quad \rho(K^*, \cdot) = \frac{1}{h(K, \cdot)}. \tag{10}$$

Associated with (10), if $K, L \in \mathcal{K}_o^n$ and $K \subseteq L$, then $K^* \supseteq L^*$.

For $K, L \in \mathcal{S}_o^n$, $p > 0$ and $\lambda, \mu \geq 0$ (not both zero), the L_p-radial Minkowski combination, $\lambda K \widetilde{+}_p \mu L$, of K and L is given by (see [9])

$$\rho(\lambda K \widetilde{+}_p \mu L, \cdot)^p = \lambda \rho(K, \cdot)^p + \mu \rho(L, \cdot)^p, \tag{11}$$

where λK denotes the L_p-radial Minkowski scalar multiplication. When $p = 1$, it is just the classical counterpart.

For $K, L \in \mathcal{S}_o^n$, $n > p > 0$ and $\lambda, \mu \geq 0$ (not both zero), the L_p-radial Blaschke combination, $\lambda \circ K \hat{+}_p \mu \circ L$, of K and L is given by (see [9])

$$\rho(\lambda \circ K \hat{+}_p \mu \circ L, \cdot)^{n-p} = \lambda \rho(K, \cdot)^{n-p} + \mu \rho(L, \cdot)^{n-p}, \tag{12}$$

where $\hat{+}_p$ denotes the L_p-radial Blaschke addition, $\lambda \circ K$ denotes the L_p-radial Blaschke scalar multiplication and $\lambda \circ K = \lambda^{\frac{1}{n-p}} K$. When $\lambda = \mu = 1$, $K \hat{+}_p L$ is called L_p-radial Blaschke sum. When $p = 1$, it's the classic case.

From the definitions of above two combinations, we easily see

$$\lambda K \widetilde{+}_{n-p} \mu L = \lambda \circ K \hat{+}_p \mu \circ L. \tag{13}$$

For $K, L \in \mathcal{S}_o^n$, $p > 0$ and $\lambda, \mu \geq 0$ (not both zero), the L_p-harmonic Blaschke combination, $\lambda * K \mp_p \mu * L \in \mathcal{S}_o^n$, of K and L is defined by (see [15])

$$\frac{\rho(\lambda * K \mp_p \mu * L, \cdot)^{n+p}}{V(\lambda * K \mp_p \mu * L)} = \lambda \frac{\rho(K, \cdot)^{n+p}}{V(K)} + \mu \frac{\rho(L, \cdot)^{n+p}}{V(L)}, \tag{14}$$

where the operation '$\mp_p$' is called L_p-harmonic Blaschke addition, $\lambda * K$ denotes L_p-harmonic Blaschke scalar multiplication and $\lambda * K = \lambda^{\frac{1}{p}} K$. When $\lambda = \mu = 1$, $K \mp_p L$ is called L_p-harmonic Blaschke sum.

3. Proofs of Theorems

In this section, we will prove Theorems 1.1-1.5. To complete the proof of Theorem 1.1, we require the following lemma.

Lemma 3.1. *Let $\Psi_p : \mathcal{S}_o^n \to \mathcal{S}_o^n$ be an L_p radial Blaschke-Minkowski homomorphism. For $K \in \mathcal{K}_o^n$ and $L_1, L_2 \in \mathcal{S}_o^n$, if $0 < n - q < p$, then*

$$\widetilde{V}_{p,q}(K, Q, \Psi_p(L_1 \hat{+}_p L_2))^{\frac{p}{n-q}} \geq \widetilde{V}_{p,q}(K, Q, \Psi_p L_1)^{\frac{p}{n-q}} + \widetilde{V}_{p,q}(K, Q, \Psi_p L_2)^{\frac{p}{n-q}}, \tag{15}$$

with equality if and only if L_1 and L_2 are dilates.

Proof. Since $0 < n - q < p$, thus $0 < \frac{n-q}{p} < 1$. From the definition of L_p radial Blaschke-Minkowski homomorphism and definition (1), according to the Minkowski's integral inequality (see [16]), we get that for any $Q \in \mathcal{K}_o^n$,

$$\begin{aligned}
\widetilde{V}_{p,q}(K,Q,\Psi_p(L_1\hat{+}_pL_2))^{\frac{p}{n-q}} &= \Big[\frac{1}{n}\int_{S^{n-1}}(\frac{h_Q}{h_K})^p(\alpha_K(u))\rho_K^q(u)\rho(\Psi_p(L_1\hat{+}_pL_2),u)^{n-q}du\Big]^{\frac{p}{n-q}}\\
&= \Big[\frac{1}{n}\int_{S^{n-1}}(\frac{h_Q}{h_K})^p(\alpha_K(u))\rho_K^q(u)\Big(\rho(\Psi_pL_1\tilde{+}_p\Psi_pL_2,u)^p\Big)^{\frac{n-q}{p}}du\Big]^{\frac{p}{n-q}}\\
&= \Big[\frac{1}{n}\int_{S^{n-1}}(\frac{h_Q}{h_K})^p(\alpha_K(u))\rho_K^q(u)\Big(\rho(\Psi_pL_1,u)^p+\rho(\Psi_pL_2,u)^p\Big)^{\frac{n-q}{p}}du\Big]^{\frac{p}{n-q}}\\
&\geq \Big[\frac{1}{n}\int_{S^{n-1}}(\frac{h_Q}{h_K})^p(\alpha_K(u))\rho_K^q(u)\rho(\Psi_pL_1,u)^{n-q}du\Big]^{\frac{p}{n-q}}\\
&\quad+\Big[\frac{1}{n}\int_{S^{n-1}}(\frac{h_Q}{h_K})^p(\alpha_K(u))\rho_K^q(u)\rho(\Psi_pL_2,u)^{n-q}du\Big]^{\frac{p}{n-q}}\\
&= \widetilde{V}_{p,q}(K,Q,\Psi_pL_1)^{\frac{p}{n-q}}+\widetilde{V}_{p,q}(K,Q,\Psi_pL_2)^{\frac{p}{n-q}}.
\end{aligned}$$

This yields inequality (15).

By the equality condition of the Minkowski's integral inequality, we see that equality holds in (15) if and only if L_1 and L_2 are dilates. □

Proof of Theorem 1.1. For $K \in \mathcal{K}_o^n$, $L_1, L_2 \in \mathcal{S}_o^n$ and $0 < n - q < p$, then by (4) and (15), we have

$$\begin{aligned}
\Big[\omega_n^{\frac{p}{n}}\widetilde{G}_{p,q}(K,\Psi_p(L_1\hat{+}_pL_2))\Big]^{\frac{p}{n-q}} &= \Big[\inf\Big\{n\widetilde{V}_{p,q}(K,Q,\Psi_p(L_1\hat{+}_pL_2))V(Q^*)^{\frac{p}{n}}:Q\in\mathcal{K}_o^n\Big\}\Big]^{\frac{p}{n-q}}\\
&\geq \Big[\inf\Big\{n\widetilde{V}_{p,q}(K,Q,\Psi_pL_1)V(Q^*)^{\frac{p}{n}}:Q\in\mathcal{K}_o^n\Big\}\Big]^{\frac{p}{n-q}}\\
&\quad+\Big[\inf\Big\{n\widetilde{V}_{p,q}(K,Q,\Psi_pL_2)V(Q^*)^{\frac{p}{n}}:Q\in\mathcal{K}_o^n\Big\}\Big]^{\frac{p}{n-q}}\\
&= \Big[\omega_n^{\frac{p}{n}}\widetilde{G}_{p,q}(K,\Psi_pL_1)\Big]^{\frac{p}{n-q}}+\Big[\omega_n^{\frac{p}{n}}\widetilde{G}_{p,q}(K,\Psi_pL_2)\Big]^{\frac{p}{n-q}},
\end{aligned}$$

i.e.,

$$\widetilde{G}_{p,q}(K,\Psi_p(L_1\hat{+}_pL_2))^{\frac{p}{n-q}} \geq \widetilde{G}_{p,q}(K,\Psi_pL_1)^{\frac{p}{n-q}}+\widetilde{G}_{p,q}(K,\Psi_pL_2)^{\frac{p}{n-q}},$$

This gives inequality (5).

According to the equality condition of inequality (15), we see that the equality holds in (5) if and only if L_1 and L_2 are dilates. □

Notice that $\lambda K\widetilde{+}_{n-p}\mu L = \lambda\circ K\hat{+}_p\mu\circ L$ (see(13)), we obtain a Brunn-Minkowski inequality for the L_{n-p}-radial Minkowski combination.

Corollary 3.1. *For $K \in \mathcal{K}_o^n$ and $L_1, L_2 \in \mathcal{S}_o^n$, let $\Psi_p : \mathcal{S}_o^n \to \mathcal{S}_o^n$ be an L_p radial Blaschke-Minkowski homomorphism. If $0 < n - q < p$, then*

$$\widetilde{G}_{p,q}(K,\Psi_p(L_1\tilde{+}_{n-p}L_2))^{\frac{p}{n-q}} \geq \widetilde{G}_{p,q}(K,\Psi_pL_1)^{\frac{p}{n-q}}+\widetilde{G}_{p,q}(K,\Psi_pL_2)^{\frac{p}{n-q}},$$

with equality if and only if L_1 and L_2 are dilates.

Because of the L_p intersection body is a special example of the L_p radial Blaschke-Minkowski homomorphisms, from Theorem 1.1 we obtain the following result:

Corollary 3.2. *For $K \in \mathcal{K}_o^n$, $L_1, L_2 \in \mathcal{S}_o^n$ and $0 < n - q < p$, then*

$$\widetilde{G}_{p,q}(K, I_p(L_1 \hat{+}_p L_2))^{\frac{p}{n-q}} \geq \widetilde{G}_{p,q}(K, I_p L_1)^{\frac{p}{n-q}} + \widetilde{G}_{p,q}(K, I_p L_2)^{\frac{p}{n-q}},$$

with equality if and only if L_1 and L_2 are dilates.

Lemma 3.2 ([24]). *A map $\Psi_p : \mathcal{S}_o^n \to \mathcal{S}_o^n$ be an L_p radial Blaschke-Minkowski homomorphism if and only if there is a non-negative measure $\mu \in \mathcal{M}(S^{n-1}, \hat{e})$ such that*

$$\rho(\Psi_p K, \cdot)^p = \rho(K, \cdot)^{n-p} * \mu. \tag{16}$$

From (16), we easily know that $\Psi_p K = \Psi_p L$ if and only if $K = L$.

Lemma 3.3. *If $K, L \in \mathcal{S}_o^n$, $0 < p < n$, then for any $u \in S^{n-1}$,*

$$\frac{\rho(\Psi_p(K \mp_p L), u)^{\frac{p(n+p)}{n-p}}}{V(K \mp_p L)} \geq \frac{\rho(\Psi_p K, u)^{\frac{p(n+p)}{n-p}}}{V(K)} + \frac{\rho(\Psi_p L, u)^{\frac{p(n+p)}{n-p}}}{V(L)}, \tag{17}$$

with equality if and only if K and L are dilates.

Proof. Because of $0 < p < n$ implies $0 < \frac{n-p}{n+p} < 1$, thus by (16) and the Minkowski's integral inequality (see [16]), we have for any $u \in S^{n-1}$,

$$\begin{aligned}
\frac{\rho(\Psi_p(K \mp_p L), u)^{\frac{p(n+p)}{n-p}}}{V(K \mp_p L)} &= \frac{\left[\rho(\Psi_p(K \mp_p L), u)^p\right]^{\frac{n+p}{n-p}}}{V(K \mp_p L)} = \frac{\left[\rho(K \mp_p L, u)^{n-p} * \mu\right]^{\frac{n+p}{n-p}}}{V(K \mp_p L)} \\
&= \left[\left(\frac{\rho(K \mp_p L, u)^{n+p}}{V(K \mp_p L)}\right)^{\frac{n-p}{n+p}} * \mu\right]^{\frac{n+p}{n-p}} \\
&= \left[\left(\frac{\rho(K, u)^{n+p}}{V(K)} + \frac{\rho(L, u)^{n+p}}{V(L)}\right)^{\frac{n-p}{n+p}} * \mu\right]^{\frac{n+p}{n-p}} \\
&\geq \left[\frac{\rho(K, u)^{n-p} * \mu}{V(K)^{\frac{n-p}{n+p}}}\right]^{\frac{n+p}{n-p}} + \left[\frac{\rho(L, u)^{n-p} * \mu}{V(L)^{\frac{n-p}{n+p}}}\right]^{\frac{n+p}{n-p}} \\
&= \frac{\rho(\Psi_p K, u)^{\frac{p(n+p)}{n-p}}}{V(K)} + \frac{\rho(\Psi_p L, u)^{\frac{p(n+p)}{n-p}}}{V(L)}.
\end{aligned}$$

This deduces inequality (17).

From the equality condition of the Minkowski's integral inequality, we know that equality holds in (17) if and only if K and L are dilates. □

Lemma 3.4. *For $K \in \mathcal{K}_o^n$ and $L_1, L_2 \in \mathcal{S}_o^n$. If $0 < p < n$ and $0 < (n-p)(n-q) < p(n+p)$, then*

$$\frac{\tilde{V}_{p,q}(K, Q, \Psi_p(L_1 \mp_p L_2))^{\frac{p(n+p)}{(n-p)(n-q)}}}{V(L_1 \mp_p L_2)} \geq \frac{\tilde{V}_{p,q}(K, Q, \Psi_p L_1)^{\frac{p(n+p)}{(n-p)(n-q)}}}{V(L_1)} + \frac{\tilde{V}_{p,q}(K, Q, \Psi_p L_2)^{\frac{p(n+p)}{(n-p)(n-q)}}}{V(L_2)}, \tag{18}$$

with equality if and only if L_1 and L_2 are dilates.

Proof. Since $0 < p < n$ and $0 < (n-p)(n-q) < p(n+p)$, thus $0 < \frac{(n-p)(n-q)}{p(n+p)} < 1$. Using definition (1), inequality (17) and the Minkowski's integral inequality (see [16]), we have for any $Q \in \mathcal{K}_o^n$,

$$\begin{aligned}
\frac{\widetilde{V}_{p,q}(K,Q,\Psi_p(L_1 \mp_p L_2))^{\frac{p(n+p)}{(n-p)(n-q)}}}{V(L_1 \mp_p L_2)} &= \frac{\left[\frac{1}{n}\int_{S^{n-1}}(\frac{h_Q}{h_K})^p(\alpha_K(u))\rho_K^q(u)\rho(\Psi_p(L_1 \mp_p L_2),u)^{n-q}du\right]^{\frac{p(n+p)}{(n-p)(n-q)}}}{V(L_1 \mp_p L_2)} \\
&= \left[\frac{1}{n}\int_{S^{n-1}}(\frac{h_Q}{h_K})^p(\alpha_K(u))\rho_K^q(u)\Big(\frac{\rho(\Psi_p(L_1 \mp_p L_2),u)^{\frac{p(n+p)}{n-p}}}{V(L_1 \mp_p L_2)}\Big)^{\frac{(n-p)(n-q)}{p(n+p)}}du\right]^{\frac{p(n+p)}{(n-p)(n-q)}} \\
&\geq \left[\frac{1}{n}\int_{S^{n-1}}(\frac{h_Q}{h_K})^p(\alpha_K(u))\rho_K^q(u)\Big(\frac{\rho(\Psi_pL_1,u)^{\frac{p(n+p)}{n-p}}}{V(L_1)} + \frac{\rho(\Psi_pL_2,u)^{\frac{p(n+p)}{n-p}}}{V(L_2)}\Big)^{\frac{(n-p)(n-q)}{p(n+p)}}du\right]^{\frac{p(n+p)}{(n-p)(n-q)}} \\
&\geq \frac{\left[\frac{1}{n}\int_{S^{n-1}}(\frac{h_Q}{h_K})^p(\alpha_K(u))\rho_K^q(u)\rho(\Psi_pL_1,u)^{n-q}du\right]^{\frac{p(n+p)}{(n-p)(n-q)}}}{V(L_1)} \\
&\quad + \frac{\left[\frac{1}{n}\int_{S^{n-1}}(\frac{h_Q}{h_K})^p(\alpha_K(u))\rho_K^q(u)\rho(\Psi_pL_2,u)^{n-q}du\right]^{\frac{p(n+p)}{(n-p)(n-q)}}}{V(L_2)} \\
&= \frac{\widetilde{V}_{p,q}(K,Q,\Psi_pL_1)^{\frac{p(n+p)}{(n-p)(n-q)}}}{V(L_1)} + \frac{\widetilde{V}_{p,q}(K,Q,\Psi_pL_2)^{\frac{p(n+p)}{(n-p)(n-q)}}}{V(L_2)}.
\end{aligned}$$

From this, inequality (18) is obtained.

By the equality conditions of inequality (17) and the Minkowski integral inequality, we see that equality holds in (18) if and only if L_1 and L_2 are dilates. □

Proof of Theorem 1.2. From $0 < p < n$ and $0 < (n-p)(n-q) < p(n+p)$, we know that $0 < \frac{(n-p)(n-q)}{p(n+p)} < 1$. Thus by (4) and (18) we obtain that

$$\begin{aligned}
&\frac{\left[\omega_n^{\frac{p}{n}}\widetilde{G}_{p,q}(K,\Psi_p(L_1 \mp_p L_2))\right]^{\frac{p(n+p)}{(n-p)(n-q)}}}{V(L_1 \mp_p L_2)} \\
&= \frac{\left[\inf\left\{n\widetilde{V}_{p,q}(K,Q,\Psi_p(L_1 \mp_p L_2))V(Q^*)^{\frac{p}{n}} : Q \in \mathcal{K}_o^n\right\}\right]^{\frac{p(n+p)}{(n-p)(n-q)}}}{V(L_1 \mp_p L_2)} \\
&\geq \inf\left\{\Big[\frac{[n\widetilde{V}_{p,q}(K,Q,\Psi_pL_1)]^{\frac{p(n+p)}{(n-p)(n-q)}}}{V(L_1)} + \frac{[n\widetilde{V}_{p,q}(K,Q,\Psi_pL_2)]^{\frac{p(n+p)}{(n-p)(n-q)}}}{V(L_2)}\Big]V(Q^*)^{\frac{p^2(n+p)}{n(n-p)(n-q)}} : Q \in \mathcal{K}_o^n\right\} \\
&\geq \frac{\left[\inf\{n\widetilde{V}_{p,q}(K,Q,\Psi_pL_1)V(Q^*)^{\frac{p}{n}} : Q \in \mathcal{K}_o^n\}\right]^{\frac{p(n+p)}{(n-p)(n-q)}}}{V(L_1)} \\
&\quad + \frac{\left[\inf\{n\widetilde{V}_{p,q}(K,Q,\Psi_pL_2)V(Q^*)^{\frac{p}{n}} : Q \in \mathcal{K}_o^n\}\right]^{\frac{p(n+p)}{(n-p)(n-q)}}}{V(L_2)} \\
&= \frac{\left[\omega_n^{\frac{p}{n}}\widetilde{G}_{p,q}(K,\Psi_pL_1)\right]^{\frac{p(n+p)}{(n-p)(n-q)}}}{V(L_1)} + \frac{\left[\omega_n^{\frac{p}{n}}\widetilde{G}_{p,q}(K,\Psi_pL_2)\right]^{\frac{p(n+p)}{(n-p)(n-q)}}}{V(L_2)}.
\end{aligned}$$

This gives inequality (6). In addition, equality holds in inequality (6) if and only if L_1 and L_2 are dilates.
□

Together with L_p intersection bodies, we immediately have another Brunn-Minkowski type inequality.

Corollary 3.3. *For $K \in \mathcal{K}_o^n$ and $L_1, L_2 \in \mathcal{S}_o^n$. If $0 < p < n$ and $0 < (n-p)(n-q) < p(n+p)$, then*

$$\frac{\widetilde{G}_{p,q}(K, I_p(L_1 \mp_p L_2))^{\frac{p(n+p)}{(n-p)(n-q)}}}{V(L_1 \mp_p L_2)} \geq \frac{\widetilde{G}_{p,q}(K, I_p L_1)^{\frac{p(n+p)}{(n-p)(n-q)}}}{V(L_1)} + \frac{\widetilde{G}_{p,q}(K, I_p L_2)^{\frac{p(n+p)}{(n-p)(n-q)}}}{V(L_2)},$$

with equality if and only if L_1 and L_2 are dilates.

Lemma 3.5. *Let $\Psi_p : \mathcal{S}_o^n \to \mathcal{S}_o^n$ be an L_p radial Blaschke-Minkowski homomorphism. If $K \in \mathcal{K}_o^n$, $L \in \mathcal{S}_o^n$ and $1 \leq r < s < t$, then*

$$\widetilde{V}_{p,s}(K, Q, \Psi_s L)^{t-r} \leq \widetilde{V}_{p,r}(K, Q, \Psi_r L)^{t-s} \widetilde{V}_{p,t}(K, Q, \Psi_t L)^{s-r}, \tag{19}$$

with equality if and only if $\Psi_r L$, $\Psi_s L$ and $\Psi_t L$ are dilates each other.

Proof. Since $1 \leq r < s < t$ and $\Psi_r L, \Psi_t L \in \mathcal{S}_o^n$, there exists $\Psi_s L \in \mathcal{S}_o^n$ such that

$$\rho(\Psi_s L, u)^{(n-s)(t-r)} = \rho(\Psi_r L, u)^{(n-r)(t-s)} \rho(\Psi_t L, u)^{(n-t)(s-r)}. \tag{20}$$

Notice that $\frac{t-r}{t-s} > 1$, according to the Hölder's integral inequality (see [8]), (1) and (20), we arrive at

$$\begin{aligned}
\widetilde{V}_{p,r}(K, Q, \Psi_r L)^{\frac{t-s}{t-r}} \widetilde{V}_{p,t}(K, Q, \Psi_t L)^{\frac{s-r}{t-r}} &= \left[\frac{1}{n}\int_{S^{n-1}} \left(((\frac{h_Q}{h_K})^p(\alpha_K(u))\rho_K^r(u)\rho(\Psi_r L, u)^{n-r})^{\frac{t-s}{t-r}}\right)^{\frac{t-r}{t-s}} du\right]^{\frac{t-s}{t-r}} \\
&\quad \cdot \left[\frac{1}{n}\int_{S^{n-1}} \left(((\frac{h_Q}{h_K})^p(\alpha_K(u))\rho_K^t(u)\rho(\Psi_t L, u)^{n-t})^{\frac{s-r}{t-r}}\right)^{\frac{t-r}{s-r}} du\right]^{\frac{s-r}{t-r}} \\
&\geq \frac{1}{n}\int_{S^{n-1}} \left((\frac{h_Q}{h_K})^p(\alpha_K(u))\rho_K^r(u)\rho(\Psi_r L, u)^{n-r}\right)^{\frac{t-s}{t-r}} \\
&\quad \cdot \left((\frac{h_Q}{h_K})^p(\alpha_K(u))\rho_K^t(u)\rho(\Psi_t L, u)^{n-t}\right)^{\frac{s-r}{t-r}} du \\
&= \widetilde{V}_{p,s}(K, Q, \Psi_s L).
\end{aligned}$$

This yields inequality (19).

From the equality condition of Hölder's integral inequality, we know that equality holds in (19) if and only if $\Psi_r L$, $\Psi_s L$ and $\Psi_t L$ are dilates each other. □

Proof of Theorem 1.3. Since $1 \leq r < s < t$, hence by (4) and (19), we obtain

$$\begin{aligned}
\left[\omega_n^{\frac{p}{n}} \widetilde{G}_{p,r}(K, \Psi_r L)\right]^{t-s} \left[\omega_n^{\frac{p}{n}} \widetilde{G}_{p,t}(K, \Psi_t L)\right]^{s-r} &= \left[\inf\left\{n\widetilde{V}_{p,r}(K, Q, \Psi_r L) V(Q^*)^{\frac{p}{n}} : Q \in \mathcal{K}_o^n\right\}\right]^{t-s} \\
&\quad \cdot \left[\inf\left\{n\widetilde{V}_{p,t}(K, Q, \Psi_t L) V(Q^*)^{\frac{p}{n}} : Q \in \mathcal{K}_o^n\right\}\right]^{s-r} \\
&\geq \left[\inf\left\{n\widetilde{V}_{p,s}(K, Q, \Psi_s L) V(Q^*)^{\frac{p}{n}} : Q \in \mathcal{K}_o^n\right\}\right]^{t-r} \\
&= \left[\omega_n^{\frac{p}{n}} \widetilde{G}_{p,s}(K, \Psi_s L)\right]^{t-r}.
\end{aligned}$$

This gives

$$\widetilde{G}_{p,s}(K, \Psi_s L)^{t-r} \leq \widetilde{G}_{p,r}(K, \Psi_r L)^{t-s} \widetilde{G}_{p,t}(K, \Psi_t L)^{s-r}.$$

This deduces (7).

According to the equality condition of inequality (19), we see that the equality of the above inequality holds if and only if $\Psi_r L$, $\Psi_s L$ and $\Psi_t L$ are dilates each other. □

According to Theorem 1.3, we may obtain a related cyclic inequality for L_p intersection bodies.

Corollary 3.4. *If $K \in \mathcal{K}_o^n$, $L \in \mathcal{S}_o^n$ and $1 \le r < s < t$, then*

$$\widetilde{G}_{p,s}(K, I_sL)^{t-r} \le \widetilde{G}_{p,r}(K, I_rL)^{t-s}\widetilde{G}_{p,t}(K, I_tL)^{s-r},$$

with equality if and only if I_rL, I_sL and I_tL are dilates each other.

Proof of Theorem 1.4. For $K \in \mathcal{K}_o^n$, $L_1, L_2 \in \mathcal{S}_o^n$ and $0 < q < n$. From (16), then

$$L_1 \subseteq L_2 \Longleftrightarrow \Psi_pL_1 \subseteq \Psi_pL_2. \tag{21}$$

This means

$$\rho(\Psi_pL_1, \cdot) \le \rho(\Psi_pL_2, \cdot). \tag{22}$$

Thus, together with (1) and (22), we obtain that

$$\begin{aligned}\widetilde{V}_{p,q}(K, Q, \Psi_pL_1) &= \frac{1}{n}\int_{S^{n-1}} (\frac{h_Q}{h_K})^p(\alpha_K(u))\rho_K^q(u)\rho(\Psi_pL_1, u)^{n-q}du \\ &\le \frac{1}{n}\int_{S^{n-1}} (\frac{h_Q}{h_K})^p(\alpha_K(u))\rho_K^q(u)\rho(\Psi_pL_2, u)^{n-q}du \\ &= \widetilde{V}_{p,q}(K, Q, \Psi_pL_2).\end{aligned} \tag{23}$$

And equality holds in (23) if and only if $L_1 = L_2$.

Therefore, from definition (4) and (23), we get

$$\begin{aligned}\omega_n^{\frac{p}{n}}\widetilde{G}_{p,q}(K, \Psi_pL_1) &= \inf\left\{n\widetilde{V}_{p,q}(K, Q, \Psi_pL_1)V(Q^*)^{\frac{p}{n}} : Q \in \mathcal{K}_o^n\right\} \\ &\le \inf\left\{n\widetilde{V}_{p,q}(K, Q, \Psi_pL_2)V(Q^*)^{\frac{p}{n}} : Q \in \mathcal{K}_o^n\right\} \\ &= \omega_n^{\frac{p}{n}}\widetilde{G}_{p,q}(K, \Psi_pL_2),\end{aligned}$$

i.e.,

$$\widetilde{G}_{p,q}(K, \Psi_pL_1) \le \widetilde{G}_{p,q}(K, \Psi_pL_2).$$

This yields inequality (8).

By the equality condition of inequality (23), there exists equality in (8) when $L_1 = L_2$. □

Associated with the L_p intersection bodies, we have the following inequality.

Corollary 3.5. *For $K \in \mathcal{K}_o^n$, $L_1, L_2 \in \mathcal{S}_o^n$ and $0 < q < n$, if $L_1 \subseteq L_2$, then*

$$\widetilde{G}_{p,q}(K, I_pL_1) \le \widetilde{G}_{p,q}(K, I_pL_2),$$

equality holds when $L_1 = L_2$.

Proof of Theorem 1.5. For $K, L_1, L_2 \in \mathcal{K}_o^n$, $L_1 \subseteq L_2$ and $0 < q < n$. Using (21), we get that

$$h(\Psi_pL_1, \cdot) \le h(\Psi_pL_2, \cdot), \tag{24}$$

with equality if and only if $L_1 = L_2$.

Combined with (10), (1) and inequality (24), we have

$$\begin{aligned}\widetilde{V}_{p,q}(K,Q,\Psi_p^*L_1) &= \frac{1}{n}\int_{S^{n-1}}(\frac{h_Q}{h_K})^p(\alpha_K(u))\rho_K^q(u)\rho(\Psi_p^*L_1,u)^{n-q}du\\ &= \frac{1}{n}\int_{S^{n-1}}(\frac{h_Q}{h_K})^p(\alpha_K(u))\rho_K^q(u)h(\Psi_pL_1,u)^{-(n-q)}du\\ &\geq \frac{1}{n}\int_{S^{n-1}}(\frac{h_Q}{h_K})^p(\alpha_K(u))\rho_K^q(u)h(\Psi_pL_2,u)^{-(n-q)}du\\ &= \frac{1}{n}\int_{S^{n-1}}(\frac{h_Q}{h_K})^p(\alpha_K(u))\rho_K^q(u)\rho(\Psi_p^*L_2,u)^{n-q}du\\ &= \widetilde{V}_{p,q}(K,Q,\Psi_p^*L_2),\end{aligned}$$

i.e.,

$$\widetilde{V}_{p,q}(K,Q,\Psi_p^*L_1) \geq \widetilde{V}_{p,q}(K,Q,\Psi_p^*L_2). \tag{25}$$

According to the equality condition of inequality (24), we know that equality holds in (25) if and only if $L_1 = L_2$.

By (4) and (25), similar to the proof of Theorem 1.4, we obtain

$$\widetilde{G}_{p,q}(K,\Psi_p^*L_1) \geq \widetilde{G}_{p,q}(K,\Psi_p^*L_2).$$

This yields (9).

From the equality condition of inequality (24) and (25), we see that equality holds in (9) when $L_1 = L_2$. □

From Theorem 1.5, we may obtain another monotonic inequality for L_p intersection bodies.

Corollary 3.6. *For $K, L_1, L_2 \in \mathcal{K}_o^n$ and $0 < q < n$, if $L_1 \subseteq L_2$, then*

$$\widetilde{G}_{p,q}(K,I_p^*L_1) \geq \widetilde{G}_{p,q}(K,I_p^*L_2),$$

equality holds when $L_1 = L_2$.

Acknowledgment

The authors would like to strongly thank the referee for the very valuable comments and helpful suggestions that directly lead to improve the original manuscript.

References

[1] B. Chen, W. D. Wang, A type of Busemann-Petty problems for Blaschke-Minkowski homomorphisms, Wuhan University Journal of Natural Sciences 23 (2018) 289-294.
[2] H. P. Chen, W. D. Wang, L_p-dual mixed geominimal surface area, Wuhan University Journal of Natural Sciences 22 (2017) 307-312.
[3] Y. B. Feng, B. W. He, The (p,q)-mixed geominimal surface areas, Quaestiones Mathematicae 42 (2019) 1031-1043.
[4] Y. B. Feng, W. D. Wang, L_p dual mixed geominimal surface area, Glasgow Mathematical Journal 56 (2014) 229-239.
[5] Y. B. Feng, W. D. Wang, Blaschke-Minkowski homomorphisms and affine surface area, Publicationes Mathematicae Debrecen 85 (2014) 297-308.
[6] Y. B. Feng, W. D. Wang, J. Yuan, Differences of quermass- and dual quermassintegral of Blaschke-Minkowski and Radial Blaschke-Minkowski homomorphism, Bulletin of the Belgian Mathematical Society 21 (2014) 577-592.
[7] Y. B. Feng, W. D. Wang, J. Yuan, Inequalities of quermassintegral about mixed Blaschke-Minkowski homomorphisms, Tamkang Journal of Mathematics 46 (2015) 217-227.
[8] R. J. Gardner, Geometric Tomography, Second edn, Cambridge University Press, Cambridge, 2006.
[9] C. Haberl, L_p-intersection bodies, Advances in Mathematics 217 (2008) 2599-2624.
[10] C. Haberl, M. Ludwig, A characterization of L_p intersection bodies, International Mathematics Research Notices 2006 (2006) 1-30.
[11] Y. Huang, E. Lutwak, D. Yang, G. Y. Zhang, Geometric measures in the dual Brunn-Minkowski theory and their associated Minkowski problems, Acta Mathematica 216 (2016) 325-388.

[12] Y. N. Li, W. D. Wang, The L_p-dual mixed geominimal surface area for multiple star bodies, Journal of Inequalities and Applications 2014 (2014) 1-10.

[13] Y. N. Li, W. D. Wang, Monotonicity inequalities for L_p Blaschke-Minkowski homomorphisms, Journal of Inequalities and Applications 2014 (2014) 10 pages.

[14] Z. F. Li, W. D. Wang, Star dual of radial Blaschke-Minkowski homomorphism, Wuhan University Journal of Natural Sciences 23 (2018) 295-300.

[15] F. H. Lu, G. S. Leng, On L_p-Brunn-Minkowski type inequalities of convex bodies, Boletín de la Sociedad Matemática Mexicana 13 (2007) 167-176.

[16] E. Lutwak, Intersection bodies and dual mixed volumes, Advances in Mathematics 71 (1988) 232-261.

[17] E. Lutwak, The Brunn-Minkowski-Firey theory. II. Affine and geominimal surface areas, Advances in Mathematics 118 (1996) 244-294.

[18] E. Lutwak, D. Yang, G. Y. Zhang, L_p dual curvature measures, Advances in Mathematics 329 (2018) 85-132.

[19] R. Schneider, Convex Bodies: The Brunn-Minkowski Theory, Second edn, Cambridge University Press, New York, 2014.

[20] F. E. Schuster, Volume inequalities and additive maps of convex bodies, Mathematika, 149 (1999) 211-234.

[21] Z. H. Shen, Y. N. Li, W. D. Wang, L_p-dual geominimal surface areas for the general L_p-intersection bodies, Journal of Nonlinear Sciences and Applications 10 (2017) 3519-3529.

[22] X. Y. Wan, W. D. Wang, L_p-dual geominimal surface area, Journal of Wuhan University (Natural Science Edition), 59 (2013) 515-518. (In Chinese)

[23] W. Wang, L_p Blaschke-Minkowski homomorphisms, Journal of Inequalities and Applications 2013 (2013) 1-14.

[24] W. Wang, L. J. Liu, B. W. He, L_p radial Minkowski homomorphisms, Taiwanese Journal of Mathematics 15 (2011) 1183-1199.

[25] W. D. Wang, C. Qi, L_p-dual geominimal surface area, Journal of Inequalities and Applications 2011 (2011) 10 pages.

[26] B. Wei, W. D. Wang, F. H. Lu, Inequalities for radial Blaschke-Minkowski homomorphisms, Annales Polonici Mathematici 113 (2015) 243-253.

[27] L. Yan, W. D. Wang, L. Si, L_p-dual mixed geominimal surface areas, Journal of Nonlinear Sciences and Applications 9 (2016) 1143-1152.

[28] D. P. Ye, L_p geominimal surface areas and their inequalities, International Mathematics Research Notices 2015 (2015) 2465-2498.

[29] D. P. Ye, B. C. Zhu, J. Z. Zhou, The mixed L_p geominimal surface areas for multiple convex bodies, Indiana University Mathematics Journal 64 (2013) 1513-1552.

[30] J. Zhang, W. D. Wang, The Shephard type problems for general L_p centroid bodies, Communnications in Mathemtical Research 35 (2019) 27-34.

[31] C. J. Zhao, On Blaschke-Minkowski homomorphisms, Geometriae Dedicata 149 (2010) 373-378.

[32] C. J. Zhao, On radial Blaschke-Minkowski homomorphisms, Geometriae Dedicata 167 (2013) 1-10.

[33] C. J. Zhao, On radial and polar Blaschke-Minkowski homomorphisms, Proceedings of the American Mathematical Society 141 (2013) 667-676.

[34] C. J. Zhao, Volume sums of polar Blaschke-Minkowski homomorphism, Proceedings-Mathematical Sciences 125 (2015) 209-219.

[35] C. J. Zhao, Orlicz-Brunn-Minkowski inequality for radial Blaschke-Minkowski homomorphisms, Quaestiones Mathematicae (2018) 1-14.

[36] Y. Zhou, W. D. Wang, Some Brunn-Minkowski type inequalities for L_p radial Blaschke-Minkowski homomorphisms, Journal of Inequalities and Applications 2016 (2016) 1-11.

[37] B. C. Zhu, N. Li, J. Z. Zhou, Isoperimetric inequalities for L_p geominimal surface area, Glasgow Mathematical Journal 53 (2011) 717-726.

[38] B. C. Zhu, J. Z. Zhou, W. X. Xu, Affine isoperimetric inequalities for L_p geominimal surface area, Real and complex submanifolds, Springer Japan 106 (2014) 167-176.

[39] B. C. Zhu, J. Z. Zhou, W. X. Xu, L_p mixed geominimal surface area, Journal of Mathematical Analysis and Applications 422 (2015) 1247-1263.