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$N(\kappa)-$Contact Metric Manifolds with Generalized Tanaka-Webster Connection

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Abstract. In this paper, we characterize $N(\kappa)$-contact metric manifolds with generalized Tanaka-Webster connection.We obtain some curvature properties. It is proven that if an $N(\kappa)$-contact metric manifold with generalized Tanaka-Webster connection is K-contact then it is an example of generalized Sasakian space form. Also, we examine some flatness and symmetric conditions of concircular curvature tensor on an $N(\kappa)$-contact metric manifolds with generalized Tanaka-Webster connection.

1. Introduction

A nullity condition for an almost contact metric manifold $(M^{(2n+1)}, \phi, \xi, \eta, g)$ was defined with curvature identity $R(X_1, X_2)\xi = 0$ for all $X_1, X_2 \in \Gamma(TM)$ by Blair et al. in [6]. The tangent sphere bundle of a flat Riemannian manifold admits such a structure [8]. By apply $D-$homothetic deformations to this structure, a special class of contact manifolds is obtained. Such a manifold is called by a $(\kappa, \mu)-$space and it satisfies

$$R(X_1, X_2)\xi = (\kappa I + \mu h)(\eta(X_2)X_1 - \eta(X_1)X_2)$$

where κ and μ are constants and $2h$ is the Lie derivative of ξ in the direction ϕ. On the other hand $(\kappa, \mu)-$nullity distribution of an almost contact metric manifold is defined by

$$N_p(\kappa, \mu) = \{X_3 \in \Gamma(T_pM) : R(X_1, X_2)X_3 = (\kappa I + \mu h)[g(X_2, X_3)X_1 - g(X_1, X_3)X_2]$$

for all $X_1, X_2 \in \Gamma(TM)$ and $p \in M$. If $\xi \in N(\kappa, \mu)$ then the manifold is called $(\kappa, \mu)-$contact metric manifold. On a $(\kappa, \mu)-$contact metric manifold $\kappa \leq 1$ and if $\kappa = 1, \mu = 0$ (i.e μ is indeterminant) then the manifold is to be Sasakian. Also, it is known that for $\kappa \leq 1$ the $(\kappa, \mu)-$ nullity condition determines the curvature of M completely [6]. We get $\kappa-$nullity distribution if $\mu = 0$, such a structure as defined in [20]. $\kappa-$nullity distribution of a Riemann manifold M is determined by

$$N_p(\kappa) = \{X_3 \in \Gamma(T_pM) : R(X_1, X_2)X_3 = \kappa[g(X_2, X_3)X_1 - g(X_1, X_3)X_2]\}$$

for all $X_1, X_2 \in \Gamma(TM)$. If $\xi \in N(\kappa)$ then M is called $N(\kappa)-$contact metric manifold. This type of manifolds has been studied by many researchers such as [4, 7, 9, 10, 12, 14, 15, 17, 26]. In [5] Blair proved that an almost

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contact metric manifold with the condition $R(X_1, X_2)\xi = 0$ is locally the product of a flat $(n+1)-$dimensional manifold and an $n-$dimensional manifold of positive constant curvature 4. Thus it has been seen that an $N(0)-$contact metric manifold is locally isometric to $\mathbb{S}^n(4) \times \mathbb{E}^{2n+1}$.

In [7] Blair et al. studied on $N(\kappa)$-contact metric manifolds with concircular curvature tensor. They gave an example and proved that an $N(\kappa)$-contact metric manifold is locally isometric to $S^{(2n+1)}(1)$ under the condition $\mathcal{Z}(X_1, \xi)\mathcal{Z} = 0$ or $\mathcal{Z}(X_1, \xi)R = 0$ for concircular curvature tensor $\mathcal{Z}$, Riemann curvature tensor R and $X_1 \in \Gamma(TM)$. De et al. [9] worked on some flatness conditions of concircular curvature tensor and they presented an example. Also, they proved that an $N(\kappa)$-contact metric manifold satisfies $\mathcal{Z}(X_1, \xi)S = 0$ if and only if the manifold is an Einstein-Sasakian manifold.

A generalized Tanaka-Webster connection has been introduced by Tanno [21] as a generalization of Tabaka-Webster connection [23, 25]. Contact manifolds with generalized Tanaka-Webster connection were studied by many researchers [11, 13, 16, 18, 19]. The curvature tensors has been used for studying differential geometry of manifolds with structures since they determine most geometric properties of the related object. There are many works under the certain curvature conditions on the contact manifolds [2, 3, 22, 24].

This paper is on $N(\kappa)$-contact metric manifolds with generalized Tanaka-Webster connection. Firstly, we give the definition of a generalized Tanaka-Webster connection for an $N(\kappa)$-contact metric manifold. Then we obtain some basic results and curvature relations. We prove that an $N(\kappa)$-contact metric manifold with generalized Tanaka-Webster connection is an example of generalized Sasakian space forms. Secondly we consider the flatness conditions and some symmetry conditions of concircular curvature tensor related to generalized Tanaka-Webster connection on an $N(\kappa)$-contact metric manifold.

2. Preliminaries

In this section we give some basic facts about contact manifolds and $N(\kappa)$-contact metric manifolds. For detail, we refer to the reader [8].

Let M be a $(2n+1)-$dimensional smooth manifold. (ϕ, ξ, η) is called an almost contact structure on M if we have

$$\phi^2 X = X - \eta(X)\xi \ , \ \phi(\xi) = 0, \ \ , \ \eta \circ \phi = 0 \ \ , \ \eta(\xi) = 1$$

for a $(1,1)$ tensor field ϕ, a vector field ξ and a $1-$ form η on M. The kernel of η defines a non-integrable distribution on M, which is called by contact distribution. The rank of ϕ is $2m$. The Riemannian metric g is called associated metric if

$$g(\phi X_1, \phi X_2) = g(X_1, X_2) - \eta(X_1)\eta(X_2) \tag{1}$$

and it is called compatible metric if

$$d\eta(X_1, X_2) = g(\phi X_1, X_2)$$

for all $X_1, X_2 \in \Gamma(TM)$. The manifold M is called an almost contact metric manifold with the structure (ϕ, ξ, η) and associated metric g.

The $(1,1)-$tensor field $h = \frac{1}{2}\mathcal{L}_\xi\phi$ has an important role in the Riemannian geometry of contact manifolds, where $\mathcal{L}$ is the Lie derivative of ϕ in the direction ξ . We have the following properties for h [8];

Lemma 2.1. *On a contact metric manifold;*

1. *h is symmetric operator i.e $g(hX, Y) = g(X, hY)$.*
2. *The derivation of h in the direction ξ is given by*

$$\nabla_X \xi = -\phi X - hX \tag{2}$$

3. *h anticommutes with ϕ, i.e $h\phi + \phi h = 0$.*
4. *h is trace free.*

If the characteristic vector field ξ is a Killing vector field then M is called a $K-$contact manifold. That is in a $K-$contact manifold $h = 0$. An almost contact metric manifold M is said to be normal if ϕ is integrable. Also, when an almost contact metric manifold is normal then $h = 0$. If the second fundamental form Ω of an almost contact metric manifold M is given by $\Omega(X_1, X_2) = g(\phi X, Y)$ and M is normal then M is called Sasakian. A Sasakian manifold is a $K-$contact manifold, but the converse holds only if $dimM^{(2n+1)} = 3$. On the other hand, M is a Sasakian manifold if and only if one of the following conditions is satisfied;

$$\begin{aligned} \left(\nabla_{X_1}\phi\right) X_2 &= g(X_1, X_2)\xi - \eta(X_2)X_1 \\ R(X_1, X_2)\xi &= \eta(X_1)X_2 - \eta(X_2)X_1 \end{aligned} \tag{3}$$

for all $X_1, X_2 \in \Gamma(TM)$.

Similar to holomorphic sectional curvature of complex manifolds, we have $\phi-$sectional curvature in contact geometry. A Sasakian manifold is called Sasakian space form if the $\phi-$sectional curvature is constant. Alegre at al. [1] generalized the Sasakian space forms as in generalized complex space forms. An almost contact metric manifold is called a generalized Sasakian space form if its curvature has the following form;

$$\begin{aligned} R(X_1, X_2)X_3 = {} & F_1[g(X_2, X_3)X_1 - g(X_1, X_3)X_2] \\ & + F_2[g(X_1, \phi X_3)\phi X_2 - g(X_2, \phi X_3)\phi X_1 + 2g(X_1, \phi X_2)\phi X_3] \\ & + F_3[\eta(X_1)\eta(X_3)X_2 - \eta(X_2)\eta(X_3)X_1 + g(X_1, X_3)\eta(X_2)\xi - g(X_2, X_3)\eta(X_1)\xi] \end{aligned} \tag{4}$$

where F_1, F_2 and F_3 are real valued functions on M. This type of manifolds contain real space forms and some classes of contact space forms with special values of F_i, $i = 1, 2, 3$.

Let M be an $N(\kappa)$-contact metric manifold. Then for all X_1, X_2 vector fields on M we have the following properties [6, 7];

$$\left(\nabla_{X_1}\phi\right) X_2 = g(X_1 + hX_1, X_2)\xi - \eta(X_2)(X_1 + hX_1), \tag{5}$$

$$h^2 = (\kappa - 1)\phi^2, \tag{6}$$

$$(\nabla_{X_1}h)X_2 = [(1 - \kappa)g(X_1, \phi X_2) + g(X_1, h\phi X_2)]\xi, +\eta(X_2)[h(\phi X_1 + \phi hX_1)] \tag{7}$$

$$(\nabla_{X_1}\eta)X_2 = g(X_1 + hX_1, \phi X_2) \tag{8}$$

Also, we have

$$R(X_1, \xi)\xi = \kappa[X_1 - \eta(X_1)\xi] \tag{9}$$

$$R(X_1, X_2)\xi = \kappa\left[\eta(X_2)X_1 - \eta(X_1)X_2\right], \tag{10}$$

$$R(X_1, \xi)X_2 = -\kappa\left[g(X_1, X_2)\xi - \eta(X_2)X_1\right]. \tag{11}$$

The Ricci curvature S and scalar curvature τ of M is given by;

$$S(X_1, X_2) = 2(n - 1)g(X_1, X_2) + 2(n - 1)g(hX_1, X_2) + [2n\kappa - 2(n - 1)]\,\eta(X_1)\eta(X_2) \tag{12}$$

$$S(X_1, \xi) = 2\kappa n\eta(X_1), S(\xi, \xi) = 2\kappa n \tag{13}$$

$$\tau = 2n(2n - 2 + \kappa) \tag{14}$$

for all $X_1, X_2 \in \Gamma(TM)$. An $N(\kappa)$-contact metric manifold is called $\eta-$Einstein if $S(X_1, X_2) = Ag(X_1, X_2) + B\eta(X_1)\eta(X_2)$ for smooth functions A and B on the manifold. $\eta-$Einstein manifolds are generalization of the Einstein manifolds which arises from the general relativity.

We will use the following basic equalities from Riemann geometry

$$g(X_1, X_2) = \sum_{i=1}^{2n+1} g(X_1, E_i)g(E_i, X_2), \tag{15}$$

$$g(\phi X_1, \phi X_2) = \sum_{i=1}^{2n+1} g(X_1, \phi E_i)g(\phi E_i, X_2), \tag{16}$$

$$\sum_{i=1}^{2n+1} g(hE_i, Ei) = 0 \tag{17}$$

where $E_i \in \{E_1, ..., E_n, \phi E_1, ..., \phi E_n, \xi\}$ is the orthonormal basis of M and $X_1, X_2 \in \Gamma(TM)$.

In [6], Blair et al. showed that $(\kappa, \mu)-$nullity distribution is invariant under the following transformations

$$\bar{\kappa} = \frac{\kappa + a^2 - 1}{a}, \ \ \bar{\mu} = \frac{\mu + 2c - 2}{a} \tag{18}$$

where a and c are positive constants. In [28] Boeckx introduced the number $I_M = \frac{1-\frac{\mu}{2}}{\sqrt{1-k}}$ for non-Sasakian (k, μ)-contact metric manifolds. This number is called by Boeckx invariant. There are two classes in the classification of non-Sasakian $(\kappa, \mu)-$spaces. The first class is a manifold with constant sectional curvature c. In this case $\kappa = c(2 - c)$ and $\mu = -2c$ and by this we get an example of $N(\kappa)$-contact metric manifold. The second class is on 3−dimensional Lie groups. Boeckx proved that two Boeckx invariants of two non-Sasakian $(\kappa, \mu)-$spaces are equal if and only if these manifolds are locally isometric to contact metric manifolds. Blair, Kim and Tripathi [7] gave the following example of $N(\kappa)$-contact metric manifolds by using the Boeckx invariant for the first class.

Example 2.2. *The Boeckx invariant for a $N(1 - \frac{1}{n}, 0)$-manifold is $\sqrt{n} > -1$. By consider the tangent sphere bundle of an $(n + 1)$-dimensional manifold of constant curvature c, as the resulting D-homothetic deformation is $\kappa = c(2 - c)$, $\mu = -2c$ and from (18), we get*

$$c = \frac{(\sqrt{n} \pm 1)^2}{n - 1}, \qquad a = 1 + c$$

and taking c and a to be these values we obtain $N(1 - \frac{1}{n})$-contact metric manifold.

Blair et al. [7] proved that an $N(\kappa)$-contact metric manifold is locally isometric to Example 2.2 if we have $\mathcal{Z}(\xi, X).\mathcal{Z} = 0$ for the concircular curvature $\mathcal{Z}$. Also, De et al. [9] showed that $\xi-$ concircularly flat $N(\kappa)$-contact metric manifold is locally isometric to Example 2.2.

Blair et al. [6] classified 3−dimensional $(\kappa, \mu)-$spaces by Lie groups of 3−dimensional Riemann manifolds. They gave an example of 3−dimensional $(\kappa, \mu)-$space. By taking $\kappa \leq 1$ and $\mu = 0$ in their work, we get an example of $N(\kappa)$-contact metric manifold with $\kappa = 1 - \lambda^2$ for real constant λ. Also, De et al. [9] examined the example and they obtained some curvature properties. This example is given as follows.

Example 2.3. *Let $M = \{(x_1, x_2, x_3) \in \mathbb{R}^3\}$ be a subset of $\mathbb{R}^3$, where x_1, x_2, x_3 are standard coordinates in $\mathbb{R}^3$ and E_1, E_2, E_3 be 3-vector fields in $\mathbb{R}^3$ satisfies*

$$[E_1, E_2] = (1 - \lambda)E_3, \ [E_2, E_3] = 2E_1 \ \ \text{and} \ \ [E_3, E_1] = (1 - \lambda)E_2,$$
$$g(E_1, E_3) = g(E_2, E_3) = g(E_1, E_2) = 0, \ \ g(E_1, E_1) = g(E_2, E_2) = 1, \ \ \eta(U) = g(U, E_1),$$

where λ is a real constant, g is a Riemann metric and U is an arbitrary vector field on M. Take a $(1, 1)-$tensor field ϕ is defined by

$$\phi E_1 = 0, \ \phi E_2 = E_3, \ \phi E_3 = -E_2.$$

Using the linearity of ϕ and g we have

$$\eta(E_1) = 1, \ \ \phi^2(U) = -U + \eta(U)E_1$$

and

$$g(\phi X_1, \phi X_2) = g(X_1, X_2) - \eta(X_1)\eta(X_2)$$

for any $X_1, X_2 \in \Gamma(TM)$. Moreover,

$$hE_1 = 0, \ hE_2 = \lambda E_2, \ \text{and} \ hE_3 = -\lambda E_3.$$

In [9], it is shown that (M, ϕ, η, g) is a $N(1 - \lambda^2)$-contact metric manifold. The covariant derivation of orthonormal basis $\{E_1, E_2, E_3\}$ is given as the followings [9] :

$$\nabla_{E_1} E_1 = \nabla_{E_1} E_2 = \nabla_{E_1} E_3 = \nabla_{E_2} E_2 = \nabla_{E_3} E_3 = 0 \tag{19}$$
$$\nabla_{E_2} E_1 = -(1 + \lambda)E_3, \ \ , \nabla_{E_2} E_3 = (1 + \lambda)E_1,$$
$$\nabla_{E_3} E_1 = (1 - \lambda)E_2, \ \ \nabla_{E_3} E_2 = -(1 - \lambda)E_1.$$

3. $N(\kappa)$-contact metric manifolds with Generalized Tanaka Webster Connection

3.1. General results

Definition 3.1 ([21]). *Let (M, ϕ, η, ξ, g) be an almost contact metric manifold and ∇ be a Levi-Civita connection on M. For any vector fields $X_1, X_2 \in \Gamma(TM)$ the following map is called by generalized Tanaka-Webster connection on M:*

$$\mathring{\nabla} : \Gamma(TM) \times \Gamma(TM) \to \Gamma(TM)$$

$$\mathring{\nabla}_{X_1} X_2 = \nabla_{X_1} X_2 + (\nabla_{X_1}\eta)X_2\xi - \eta(X_2)\nabla_X\xi + \eta(X_1)\phi Y.$$

Then from (2) and (8), the generalized Tanaka-Webster connection on an $N(\kappa)$-contact metric manifold is given by

$$\mathring{\nabla}_{X_1} X_2 = \nabla_{X_1} X_2 + g(X_1 + hX_1, \phi X_2)\xi + \eta(X_1)\phi X_2 + \eta(X_2)\phi(hX_1 + X_1) \tag{20}$$

for all $X_1, X_2 \in \Gamma(TM)$, where ∇ is the Levi-Civita connection on M. It is easy to verify that $\mathring{\nabla}$ is a linear connection. Also, we get $(\mathring{\nabla}_{X_1} g)(X_2, X_3) = 0$. On the other hand, the torsion of $\mathring{\nabla}$ is given by

$$\mathring{T} = \quad (g(X_1 + hX_1, \phi X_2) - g(X_2 + hX_2, \phi X_1))\xi + \eta(X_2)(\phi X_1 + \phi hX_1) - \eta(X_1)(\phi X_2 + \phi hX_2)$$

for every $X_1, X_2 \in \Gamma(TM)$. As a result, we get:

Lemma 3.2. *The map is given by (20) is a linear, metric and non-symmetric connection on M.*

Using (2),(5), (7), (8), (20) and with some computations, on an $N(\kappa)$-contact metric manifold M with generalized Tanaka-Webster connection we have;

$$\begin{aligned}
\mathring{\nabla}_{X_1}\xi &= 0, \ \ \mathring{\nabla}_{\xi}\xi = 0 \\
(\mathring{\nabla}_{X_1}\eta)X_2 &= 0, \\
(\mathring{\nabla}_{X_1}\phi)X_2 &= (\nabla_{X_1}\phi)X_2 - g(X_1 + hX_1, X_2)\xi + \eta(X_2)hX_1 + \eta(X_2)X_1, \\
(\mathring{\nabla}_{X_1}h)X_2 &= [(\kappa - 1)g(\phi X_1, X_2) + g(hX_1, \phi X_2)]\xi + \eta(X_1)\phi(X_2 + hX_2)
\end{aligned}$$

for all $X_1, X_2 \in \Gamma(TM)$

Thus from (3) we get following corollary:

Corollary 3.3. *Let M be an $N(\kappa)$-contact metric manifold with generalized Tanaka-Webster connection. If M is Sasakian then $(\mathring{\nabla}_{X_1}\phi)X_2 = 0$, for all $X_1, X_2 \in \Gamma(TM)$.*

Remark 3.4. *In [11] De and Ghosh proved that on a Sasakian manifold M with generalized Tanaka-Webster connection $\mathring{\nabla}$ we have $(\mathring{\nabla}_{X_1}\phi)X_2 = 0$ for all $X_1, X_2 \in \Gamma(TM)$. Thus above corollary is compatible with this result.*

3.2. Curvature Properties

Let M be an $N(\kappa)$-contact metric manifold with generalized Tanaka-Webster connection. The Riemannian curvature of M is given by

$$\mathring{R}(X_1, X_2)X_3 = \mathring{\nabla}_{X_1}\mathring{\nabla}_{X_2}X_3 - \mathring{\nabla}_{X_2}\mathring{\nabla}_{X_1}X_3 - \mathring{\nabla}_{[X_1,X_2]}X_3 \tag{21}$$

for all $X_1, X_2, X_3 \in \Gamma(TM)$. By using (20), from (5), (7) and with a long computation we get

$$\begin{aligned}
\mathring{R}(X_1, X_2)X_3 \quad &= R(X_1, X_2)X_3 + \kappa\{(\eta(X_2)g(X_1, X_3) - \eta(X_1)g(X_2, X_3))\xi \\
&-\eta(X_2)\eta(X_3)X_1 + \eta(X_1)\eta(X_3)X_2\} \\
&-g(X_2 + hX_2, \phi X_3)[\phi X_1 + \phi hX_1] + g(X_1 + hX_1, \phi X_3)[\phi X_2 + \phi hX_2] \\
&+[g(X_1, \phi X_2 + \phi hX_2) + g(X_2, \phi X_1 + \phi hX_1)]\phi X_3
\end{aligned} \tag{22}$$

where R is the Riemann curvature of M with Levi-civita connection and $\mathring{R}$ is the Riemann curvature of M with generalized Tanaka-Webster connection. The symmetry properties of $\mathring{R}$ are given by

$$\begin{aligned}
\mathring{R}(X_1,X_2,X_3,X_4)+\mathring{R}(X_2,X_1,X_4,X_3) &= 0\\
\mathring{R}(X_1,X_2,X_3,X_4)+\mathring{R}(X_1,X_2,X_4,X_3) &= 0\\
\mathring{R}(X_1,X_2,X_3,X_4)+\mathring{R}(X_3,X_4,X_1,X_2) &= -2(g(\phi X_1,X_4)g(hX_2,\phi X_3)-g(X_2,\phi X_3)g(\phi hX_1,X_4)\\
&\quad -g(hX_1,\phi X_3)g(X_2,\phi X_4)+g(X_1,\phi X_3)g(\phi hX_2,X_4)\\
&\quad -g(X_3,\phi hX_4)g(\phi X_1,X_2))
\end{aligned}$$

Also we have

$$\mathring{R}(X_1,X_2)X_3+\mathring{R}(X_2,X_3)X_1+\mathring{R}(X_3,X_1)X_2 = 2(g(\phi X_1,X_2)\phi hX_3-g(\phi X_1,X_3)\phi hX_2+g(\phi X_2,X_3)\phi hX_1).$$

It is easy to see that the Bianchi identity of $\mathring{R}$ is satisfied when ξ is Killing.

On the other hand from (9),(10), (11) and (22) we get

$$\mathring{R}(X_1,X_2)\xi=\mathring{R}(\xi,X_1)X_2=\mathring{R}(X_1,\xi)\xi=0.$$

for all $X_1,X_2\in\Gamma(TM)$.

A generalized Sasakian space form is a class of almost contact metric manifolds which are defined on an almost contact metric manifold by the curvature relation (4). By the following theorem, we obtain a new example of the generalized Sasakian space forms.

Theorem 3.5. *Let M be an $N(\kappa)$-contact metric manifold with generalized Tanaka-Webster connection. If M is K-contact manifold, then M is a generalized Sasakian space form with $F_1=F_3=\kappa,\ F_2=1$.*

Proof. Let M be an $N(\kappa)$-contact metric manifold with generalized Tanaka-Webster connection . If ξ is Killing then from (22) we have

$$\begin{aligned}
\mathring{R}(X_1,X_2)X_3 \;=\; & \kappa\{g(X_2,X_3)X_1-g(X_1,X_3)X_2\}\\
&+g(X_1,\phi X_3)\phi X_2-g(X_2,\phi X_3)\phi X_1+2g(X_1,\phi X_2)\phi X_3\\
&+\kappa\{\eta(X_1)\eta(X_3)X_2-\eta(X_2)\eta(X_3)X_1+g(X_1,X_3)\eta(X_2)\xi-g(X_2,X_3)\eta(X_1)\xi\}.
\end{aligned}$$

This shows us M is a generalized Sasakian space form with $F_1=F_3=\kappa,\ F_2=1$. □

The Ricci curvature of an $N(\kappa)$-contact metric manifold with generalized Tanaka-Webster connection is defined by

$$\mathring{S}(X_1,X_4)=\sum_{i=1}^{2n+1}\mathring{R}(X_1,E_i,E_i,X_4)$$

where $E_i, 1\le i\le 2n+1$ are the orthonormal basis of M and $X_1,X_4\in\Gamma(TM)$. Thus from (1),(15), (16), (17) and (22) we have

$$\mathring{S}(X_1,X_4)=S(X_1,X_4)+(3-\kappa)g(X_1,X_4)+(-(2n-1)\kappa-3)\eta(X_1)\eta(X_4)-g(hX_1,hX_4).$$

On the other hand, since h is symmetric from (6), we get

$$g(hX_1,hX_4)=(\kappa-1)(-g(X_1,X_4)+\eta(X_1)\eta(X_4)).$$

Thus the Ricci curvature of a $N(\kappa)$-contact metric manifold with generalized Tanaka-Webster connection is obtained as

$$\mathring{S}(X_1,X_4)=S(X_1,X_4)+2g(X_1,X_4)-2(n\kappa+1)\eta(X_1)\eta(X_4). \tag{23}$$

and so, from (12) we get

$$\mathring{S}(X_1, X_4) = 2ng(X_1, X_2) + 2(n-1)g(hX_1, X_2) - 2n\eta(X_1)\eta(X_2).$$

Also we have

$$\mathring{S}(X_1, \xi) = \mathring{S}(\xi, \xi) = 0. \tag{24}$$

From (12), we know if an $N(\kappa)$-contact metric manifold is Sasakian then it is $\eta-$Einstein.

Corollary 3.6. *If an $N(\kappa)$-contact metric manifold with Levi Civita connection M is Sasakian then it is $\eta-$Einstein with generalized Tanaka-Webster connection.*

Proof. Let M be an $N(\kappa)$-contact metric manifold with Levi-Civita connection. If M is Sasakian, then from (12) we get

$$S(X_1, X_4) = 2(n-1)g(X_1, X_4) + 2\eta(X_1)\eta(X_4).$$

Thus from (23) we obtain

$$\mathring{S}(X_1, X_4) = 2ng(X_1, X_4) - 2n\eta(X_1)\eta(X_4)$$

which shows us M is $\eta-$Einstein. □

The scalar curvature of an $N(\kappa)$-contact metric manifold with generalized Tanaka-Webster connection M is obtained as

$$\mathring{\tau} = \tau + 4n - 2n\kappa.$$

From (14) we get

$$\mathring{\tau} = 4n^2. \tag{25}$$

4. Concircular Curvature Tensor on $N(\kappa)-$Contact Metric Manifolds with Generalized Tanaka-Webster Connection

The concircular curvature tensor was defined by Yano [27]. Blair et al. [7, 9] studied on $N(\kappa)$-contact metric manifold under certain curvature conditions via concircular curvature tensor. In this section, we study on concircular curvature tensor on an $N(\kappa)$-contact metric manifold M with generalized Tanaka-Webster connection.

Concircular curvature tensor $\mathcal{Z}$ on an $N(\kappa)$-contact metric manifold with generalized Tanaka-Webster connection is given by

$$\mathring{\mathcal{Z}}(X_1, X_2)X_3 = \mathring{R}(X_1, X_2)X_3 - \frac{\mathring{\tau}}{2n(2n+1)}[g(X_2, X_3)X_1 - g(X_1, X_3)X_2].$$

From (25) we get

$$\mathring{\mathcal{Z}}(X_1, X_2)X_3 = \mathring{R}(X_1, X_2)X_3 - \frac{2n}{2n+1}[g(X_2, X_3)X_1 - g(X_1, X_3)X_2]. \tag{26}$$

For all $X_1, X_2, X_3 \in \Gamma(TM)$, we obtain

$$\mathring{\mathcal{Z}}(X_1, \xi)\xi = K\phi^2 X_1 \tag{27}$$

$$\mathring{\mathcal{Z}}(X_1, X_2)\xi = K(\eta(X_2)X_1 - \eta(X_1)X_2) \tag{28}$$

$$\mathring{\mathcal{Z}}(X_1, \xi)X_2 = K(\eta(X_2)X_1 - g(X_1, X_2)\xi) \tag{29}$$

$$\eta(\mathring{\mathcal{Z}}(X_1, X_2)X_3) = K(\eta(X_3)g(X_1, X_2) - \eta(X_1)g(X_3, X_2))$$

where $K = -\frac{2n}{2n+1}$.

An $N(\kappa)$-contact metric manifold with Levi-Civita connection is called $\xi-$concircularly flat if $\mathcal{Z}(X_1, X_2)\xi = 0$. In [9] De et al. proved that a $\xi-$concircularly flat $N(\kappa)$-contact metric manifold is locally isometric to Example 2.2. We recall an $N(\kappa)$-contact metric manifold with generalized Tanaka-Webster connection by $\mathring{\xi}-$concircularly flat if $\mathring{\mathcal{Z}}(X_1, X_2)\xi = 0$. Thus from (28), we obtain:

Theorem 4.1. *An $N(\kappa)$-contact metric manifold with generalized Tanaka-Webster connection can not to be $\mathring{\xi}-$concircularly flat.*

An $N(\kappa)$-contact metric manifold is called $\phi-$concircularly flat with generalized Tanaka-Webster connection if $g(\mathring{\mathcal{Z}}(\phi X_1, \phi X_2)\phi X_3, \phi X_4) = 0$.

Theorem 4.2. *A $\phi-$concircularly flat $N(\kappa)$-contact metric manifold with generalized Tanaka-Webster connection is $\eta-$Einstein.*

Proof. Let M be a $\phi-$concircularly flat $N(\kappa)$-contact metric manifold with the generalized Tanaka-Webster connection . Thus from (26) we have

$$g(\mathring{R}(\phi X_1, \phi X_2)\phi X_3, \phi X_4) = \frac{2n}{2n+1}(g(\phi X_2, \phi X_3)g(\phi X_1, \phi X_4) - g(\phi X_2, \phi X_4)). \tag{30}$$

Let $S = \{E_1, ..., E_n, \phi E_1, ..., \phi E_n, \xi\}$ be an orthonormal ϕ-basis of the tangent space. Putting $X_2 = X_3 = E_i \in S$ in (30) and by taking summation over $i = 1$ to $2n + 1$ we get

$$\mathring{S}(X_1, X_4) = \frac{2n(2n-1)}{2n+1} g(\phi X_1, \phi X_4).$$

Replacing X_1 and X_4 by ϕX_1 and ϕX_4 and using (23), (24) we obtain

$$S(X_1, X_4) = (\frac{-2(2n^2+n+1)}{2n+1})g(X_1, X_4) + (\frac{2n(2n-1)}{2n+1} + 2(n\kappa+1))\eta(X_1)\eta(X_4).$$

Thus, M is $\eta-$Einstein. □

A Riemann manifold M is called locally symmetric or semi-symmetric if $R.R = 0$. Also if $R.S = 0$ then M is called Ricci semi-symmetric manifold. For two $(1,3)-$type tensors $\mathcal{T}_1, \mathcal{T}_2$ we have

$$\begin{aligned}(\mathcal{T}_1(X_1, X_2).\mathcal{T}_2)(X_3, X_4)X_5 &= \mathcal{T}_1(X_1, X_2)\mathcal{T}_2(X_3, X_4)X_5 - \mathcal{T}_2(\mathcal{T}_1(X_1, X_2)X_3, X_4)X_5 \\ &\quad -\mathcal{T}_2(X_3, \mathcal{T}_1(X_1, X_2)X_4)X_5 - \mathcal{T}_2(X_3, X_4)\mathcal{T}_1(X_1, X_2)X_5\end{aligned} \tag{31}$$

and for $(0,2)-$type tensor ω, we have

$$(\mathcal{T}_1(X_1, X_2).\omega)(X_3, X_4) = \omega(\mathcal{T}_1(X_1, X_2)X_3, X_4) + \omega(X_3, \mathcal{T}_1(X_1, X_2)X_4). \tag{32}$$

In [7] the authors proved that an $N(\kappa)$-contact metric manifold which satisfies $\mathcal{Z}(\xi, X_1).\mathcal{Z} = 0$ is locally isometric to $\mathbb{E}^{n+1} \times \mathbb{R}^n$. Also, an $N(\kappa)$-contact metric manifold satisfies $\mathcal{Z}(\xi, X_1).S = 0$ if and only if the manifold is an Einstein-Sasakian manifold. We consider an $N(\kappa)$-contact metric manifold with generalized Tanaka-Webster connection under the conditions $\mathring{\mathcal{Z}}(\xi, X_1).\mathring{\mathcal{Z}} = 0$ and $\mathring{\mathcal{Z}}(\xi, X_1).\mathring{S} = 0$ for all $X_1 \in \Gamma(TM)$.

Theorem 4.3. *An $N(\kappa)$-contact metric manifold with generalized Tanaka-Webster connection can not satisfy $\mathring{\mathcal{Z}}(\xi, X_1)\mathring{S} = 0$.*

Proof. Assume that an $N(\kappa)$-contact metric manifold with generalized Tanaka-Webster connection satisfies $\mathring{\mathcal{Z}}(\xi, X_1)\mathring{S} = 0$. Then from (32) we get

$$\mathring{S}(\mathring{\mathcal{Z}}(\xi, X_1)X_2, X_3) + \mathring{S}(X_2, \mathring{\mathcal{Z}}(\xi, X_1)X_3) = 0.$$

Let take $X_3 = \xi$ then by using (24) and (29) we obtain

$$K\mathring{S}(X_1, X_2) = 0.$$

Since $K \neq 0$ and also $\mathring{S}$ can not vanish, we have a contradiction. This completes the proof. □

Theorem 4.4. *An $N(\kappa)$-contact metric manifold with generalized Tanaka-Webster connection can not satisfy $\mathring{\mathcal{Z}}(\xi, X)\mathring{\mathcal{Z}} = 0$.*

Proof. From (31) we have

$$\begin{aligned}(\mathring{\mathcal{Z}}_1(\xi, X_2).\mathring{\mathcal{Z}}_2)(X_3, X_4)X_5 &= \mathring{\mathcal{Z}}_1(\xi, X_2)\mathring{\mathcal{Z}}_2(X_3, X_4)X_5 - \mathring{\mathcal{Z}}_2(\mathring{\mathcal{Z}}_1(\xi, X_2)X_3, X_4)X_5 \\ &\quad -\mathring{\mathcal{Z}}_2(X_3, \mathring{\mathcal{Z}}_1(\xi, X_2)X_4)X_5 - \mathring{\mathcal{Z}}_2(X_3, X_4)\mathring{\mathcal{Z}}_1(\xi, X_2)X_5\end{aligned}$$

for all $X_2, X_3, X_4, X_5 \in \Gamma(TM)$. Suppose that $\mathring{\mathcal{Z}}(\xi, X_2)\mathring{\mathcal{Z}} = 0$. Let take $X_5 = \xi$ then from (28), (29) and with a long computations we get

$$\mathring{\mathcal{Z}}(X_2, X_3)X_1 = K\{[2(\eta(X_3)g(X_1, X_2) - \eta(X_2)g(X_1, X_3))]\xi - g(\phi X_1, \phi X_2)X_3 - g(\phi X_1, \phi X_3)X_2\}$$

By setting $X_3 = \xi$ and taking inner product with ξ, we obtain

$$\eta(\mathring{\mathcal{Z}}(X_2, \xi)X_1) = -3Kg(\phi X_2, \phi X_1).$$

Thus from (29) we have $g(\phi X_1, \phi X_2) = 0$. So the condition $\mathring{\mathcal{Z}}(\xi, X_2).\mathring{\mathcal{Z}} = 0$ can not satisfy. □

5. Example

Let M be an $N(\kappa)$-contact metric manifold which is given in Example 2 with a generalized Tanaka-Webster connection. Then from (19) and by using (20) we get

$$\begin{aligned}&\mathring{\nabla}_{E_1}E_1 = \mathring{\nabla}_{E_2}E_1 = \mathring{\nabla}_{E_3}E_1 = \mathring{\nabla}_{E_2}E_2 = \mathring{\nabla}_{E_2}E_3 = \mathring{\nabla}_{E_3}E_2 = \mathring{\nabla}_{E_3}E_3 = 0\\ &\mathring{\nabla}_{E_1}E_2 = E_3,\ \mathring{\nabla}_{E_1}E_3 = -E_2.\end{aligned}$$

Thus from (21), the curvature of M is obtained as followings:

$$\begin{aligned}&\mathring{R}_{121} = \mathring{R}_{122} = \mathring{R}_{123} = \mathring{R}_{131} = \mathring{R}_{132} = \mathring{R}_{133} = \mathring{R}_{231} = 0,\\ &\mathring{R}_{232} = -2E_3, \mathring{R}_{233} = 2E_2\end{aligned}$$

where $\mathring{R}_{ijk} = \mathring{R}(E_i, E_j)E_k$. Also, we can obtain same results from (22). By using the definition of generalized Tanaka-Webster connection , we get $\mathring{\nabla}_{E_i}E_1 = 0$, $(\mathring{\nabla}_{E_i}\phi)E_j = 0$ and $(\mathring{\nabla}_{E_i}\eta)E_j = 0$ for $1 \leq i, j \leq 3$.

The Ricci and scalar curvature of M is obtained by

$$\mathring{S}(E_1, E_1) = 0,\ \mathring{S}(E_2, E_2) = 2,\ \mathring{S}(E_3, E_3) = 2$$

and thus $\mathring{\tau} = 4$. These results verify (23) and (25). Suppose that, the condition $\mathring{Z}(E_1, E_j).\mathring{S} = 0$ is satisfied on M . Then from (32), we get

$$\mathring{S}(\mathring{\mathcal{Z}}(E_1, E_j)E_k, E_r) + \mathring{S}(E_k, \mathring{\mathcal{Z}}(E_1, E_j)E_r) = 0.$$

Let choose $E_r = E_1$ then from (24), (27) and (29), we obtain

$$\frac{2}{3}\mathring{S}(E_k, E_j) = 0.$$

If $E_k = E_j = E_2$, then since $\mathring{S}(E_2, E_2) = 2$, we have a contradiction. So, $\mathring{Z}(E_1, E_j).\mathring{S} = 0$ can not satisfy on M. This is verified the Theorem 5.

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