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Convergence Results of a Faster Iterative Scheme Including Multi-Valued Mappings in Banach Spaces

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Abstract. In this paper, we study M-iterative scheme in the new context of multi-valued generalized α-nonexpansive mappings. A uniformly convex Banach space is used as underlying setting for our approach. We also provide a new example of generalized α-nonexpasive mappings. We connect M iterative scheme and other well known schemes with this example, to show the numerical efficiency of our results. Our results improve and extend many existing results in the current literature.

1. Introduction and Preliminaries

A point $y \in \mathcal{B}$ is said to be a fixed point for a selfmap f of $\mathcal{B}$ if $y = f(y)$. In 1922, Banach [8] proved that every contraction map defined on a complete metric space admits a unique fixed point. Moreover, Banach's result [8] suggests a Picard iterative scheme $u_{m+1} = f(u_m)$ for the approximation of the unique fixed point of contraction maps. However, an extensively and widely studied class of nonlinear mappings is the class of nonexpansive mappings: a selfmap f on a subset $\mathcal{B}$ of a Banach space is called nonexpansive if for any pair of points $u, v \in \mathcal{B}$, the inequality $\|f(u) - f(v)\| \leq \|u - v\|$ holds. In 1965, Kirk [22], Browder [11] and Göhde [14] independently proved a fixed point theorem for nonexpansive mappings. In what the Picard iterative scheme may fail to converge for this class of mappings. The main reason for such behavior is that, unlike contraction mappings, successive iterates for nonexpansive mappings could fail to converge to a fixed point (see Berinde [9]). To overcome such problems and to get better rate of convergence, widely and extensively studied iterative schemes are the Mann [24], Ishikawa [18], Noor [26], Agarwal [5], Abbas [1], Picard-S [16].

In 2008, Suzuki [39] introduced the notion of generalized nonexpansive mappings which is a condition called condition (C): a selfmap f on a subset $\mathcal{B}$ of a Banach space is said to satisfy condition (C) (or said to be a Suzuki mapping) if for all $u, v \in \mathcal{B}$, the nonexpansive condition $\|f(u) - f(v)\| \leq \|u - v\|$ holds whenever the inequality $\frac{1}{2}\|u - f(u)\| \leq \|u - v\|$ is satisfied. It is obvious that every nonexpansive mapping satisfies the condition (C). In 2011, Aoyama and Kohsaka [7] introduced the class of α-nonexpansive mappings: a self mapping f on a subset $\mathcal{B}$ of a Banach space is called α-nonexpansive if for any pair of points $u, v \in \mathcal{B}$, there

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is a real number $\alpha \in [0,1)$ such that the inequality $\|f(u)-f(v)\|^2 \leq \alpha\|u-f(v)\|^2+\alpha\|v-f(u)\|^2+(1-2\alpha)\|u-v\|^2$ holds. Recently in 2017, Pant and Shukla [28] introduced the class of generalized α-nonexpansive mappings: a selfmap f on a subset $\mathcal{B}$ of a Banach space is called generalized α-nonexpansive if for all $u, v \in \mathcal{B}$, the condition $\|f(u)-f(v)\| \leq \alpha\|u-f(v)\|+\alpha\|v-f(u)\|+(1-2\alpha)\|u-v\|$ holds whenever the inequality $\frac{1}{2}\|u-f(u)\| \leq \|u-v\|$ is satisfied. The class of generalized α-nonexpansive mappings is important, because it properly includes the class of Suzuki mappings and partially extend the class of α-nonexpansive mappings.

Fixed point theory of multi-valued mappings has many useful applications in physics, game theory, optimization and mathematical economics (see e.g., [15] and references therein). Therefore it is very natural to extend the fixed points results from the case of single-valued to the case of multi-valued. In 1969, Nadler [25] (cf. also [4, 12]) extended the Banach [8] result to the case of multi-valued mappings. In 1974, Lim [23] obtained a multi-valued version of Kirk-Browder-Gohde result. In 2011, Abkar and Eslamian [3] extended the notion of condition (C) to the case of multi-valued mappings. In 2019, Hajisharifi [17] studied the multi-valued version of α-nonexpansive mappings while Iqbal et al. [19] studied the multi-valued version of generalized α-nonexpansive mappings. The first convergence to a fixed point of a multi-valued nonexpansive mapping was proved by Sastry and Babu [33] in the setting of Hilbert spaces using Mann and Ishikawa iterative schemes. Panyanak [29] extended the results of Sastry and Babu [33] to the general setting of uniformly convex [13] Banach spaces. Song and Wang [38] improved the results of Panyanak [29] using the endpoint condition. To avoid the endpoint condition, Shahzad and Zegeye [36] introduced another Ishikawa iterative scheme using $P_T(u) = \{v \in Tu : \|u-v\| = d(u, Tu)\}$ where T is a given multi-valued map. For more details in this direction, we refer the reader to [2, 20, 21, 30, 37].

Recently, in 2018, Ullah and Arshad [40] introduced a new iterative scheme so-called M-iterative scheme in the context of uniformly convex Banach spaces. They proved that M-iterative scheme converges faster than the leading two-step Agarwal [5] and leading three-step Picard-S [16] iterative schemes for the class of Suzuki mappings. Ullah et al. [41] extended all of the results of Ullah and Arshad [40] to the general setting of generalized α-nonexpansive mappings. For more details and literature of M-iterative scheme, we refer the reader [6, 32, 42].

Motivated by the above, we define the M-iterative scheme for multi-valued mappings as follows.

$$\begin{cases} u_1 \in \mathcal{B}, \\ w_m = (1-a_m)u_m + a_m z_m, \\ v_m = z'_m \\ u_{m+1} = z''_m, m \geq 1, \end{cases} \tag{1}$$

where $z_m \in P_T(u_m)$, $z'_m \in P_T(w_m)$, $z''_m \in P_T(v_m)$ and $a_m \in (0,1)$. The purpose of this work is to prove weak and strong convergence of (1) for the class of multi-valued generalized α-nonexpansive mappings.

Let $X = (X, \|.\|)$ be a Banach space and $\mathcal{B}$ a nonempty subset of X. The set $\mathcal{B}$ is called proximinal if for each $u \in X$, there exists some $b \in \mathcal{B}$ such that $d(a,u) = d(a,\mathcal{B})$, where $d(a,\mathcal{B}) = \inf\{\|a-b\| : b \in \mathcal{B}\}$. From now on, we will denote by $\mathbb{P}_{px}(\mathcal{B})$, $\mathbb{P}_{cb}(\mathcal{B})$ and $\mathbb{P}(\mathcal{B})$ the set of nonempty proximinal subsets, closed bounded subsets and all possible subsets of $\mathcal{B}$ respectively. A point $y \in \mathcal{B}$ is called a fixed point of $T : \mathcal{B} \to \mathbb{P}(\mathcal{B})$ if $y \in T(y)$ and is called an endpoint of T if $\{y\} = T(y)$. From now on, we will denote by $F(T)$, the set of all fixed points of T. A multi-valued mapping T is said to satisfy the endpoint condition if $\{y\} = T(y)$ for all $y \in F(T)$. The Pompeiu-Hausdorff metric [10] on the set $\mathbb{P}_{cb}(\mathcal{B})$ is defined as:

$$\mathcal{H}(\mathcal{K}, \mathcal{M}) = \max\left\{\sup_{a\in\mathcal{K}}(a, \mathcal{M}), \sup_{b\in\mathcal{M}}(b, \mathcal{K})\right\}, \text{ for all } \mathcal{K}, \mathcal{M} \in \mathbb{P}_{cb}(\mathcal{B}).$$

Definition 1.1. Let $T : \mathcal{B} \to \mathbb{P}(\mathcal{B})$ be a multi-valued mapping. Then T is said to be

(s_1) nonexpansive if and only if $\mathcal{H}(Tu, Tv) \leq \|u-v\|$ for each $u, v \in \mathcal{B}$;

(s_2) quasi-nonexpansive if and only if $\mathcal{H}(Tu, Ty) \leq \|u-y\|$ for each $u \in \mathcal{B}$ and $y \in F(T)$;

(s_3) Suzuki mapping if and only if $\frac{1}{2}d(u, Tu) \leq \|u-v\| \Rightarrow \mathcal{H}(Tu, Tv) \leq \|u-v\|$ for each $u, v \in \mathcal{B}$;

(s_4) α-nonexpansive if there is a real number $\alpha \in [0,1)$ such that for each $u, v \in \mathcal{B}$, $\mathcal{H}^2(Tu, Tv) \le \alpha d^2(v, Tu) + \alpha d^2(u, Tv) + (1-2\alpha)\|u-v\|^2$;

(a_5) generalized α-nonexpansive if and only if there is a real number $\alpha \in [0,1)$ such that for each $u, v \in \mathcal{B}$, $\frac{1}{2}d(u, Tu) \le \|u-v\| \Rightarrow \mathcal{H}(Tu, Tv) \le \alpha d(v, Tu) + \alpha d(u, Tv) + (1-2\alpha)\|u-v\|$.

Definition 1.2. ([27]) A Banach space X is said to have Opial property if and only if for each weakly convergent sequence $\{u_m\}$ in X with weak limit $u \in X$, we have

$$\limsup_{m\to\infty} \|u_m - u\| < \limsup_{m\to\infty} \|u_m - v\| \text{ for each } v \in X - \{u\}.$$

Lemma 1.3. ([37]) *Let $T : \mathcal{B} \to \mathbb{P}_{px}(\mathcal{B})$ and $P_T(u) = \{v \in Tu : \|u-v\| = d(u, Tu)\}$. Then the following statements are equivalent:*

(s_1) $y \in F(T)$.

(s_2) $P_T(y) = \{y\}$.

(s_3) $y \in F(P_T(y))$.

Moreover, $F(T) = F(P_T)$.

The following lemma gives many examples of multi-valued generalized α-nonexpansive mappings.

Lemma 1.4. ([19]) *Let $\mathcal{B}$ be a nonempty subset of a Banach space and $T : \mathcal{B} \to \mathbb{P}(\mathcal{B})$. If T is Suzuki mapping, then T is generalized α-nonexpansive.*

Lemma 1.5. ([19]) *Let $\mathcal{B}$ be a nonempty subset of a Banach space and $T : \mathcal{B} \to \mathbb{P}(\mathcal{B})$. If T is generalized α-nonexpansive, then T is quasi-nonexpansive.*

Lemma 1.6. ([19]) *Let $\mathcal{B}$ be a nonempty subset of a Banach space X and $T : \mathcal{B} \to \mathbb{P}_{cb}(\mathcal{B})$ is generalized α-nonexpansive. Then*

$$d(u, Tv) \le \left(\frac{3+\alpha}{1-\alpha}\right) d(u, Tu) + \|u-v\|,$$

for each $u, v \in \mathcal{B}$.

The following result is a characterization of uniform convexity, which was proved by Schu [34].

Lemma 1.7. *Let X be a uniformly convex Banach space and $0 < u \le \delta_m \le v < 1$ for all $m \ge 1$. If $\{s_m\}$ and $\{t_m\}$ are any sequences in X such that $\limsup_{m\to\infty} \|s_m\| \le k$, $\limsup_{m\to\infty} \|t_m\| \le k$ and $\lim_{m\to\infty} \|\delta_m s_m + (1-\delta_m)t_m\| = k$ for some $k \ge 0$, then $\lim_{m\to\infty} \|s_m - t_m\| = 0$.*

2. Main Results

We begin this section by proving a key lemma.

Lemma 2.1. *Let X be a uniformly convex Banach space and $\mathcal{B}$ be a nonempty closed convex subset of X. Let $T : \mathcal{B} \to \mathbb{P}_{px}(\mathcal{B})$ be such that P_T is generalized α-nonexpansive and $F(T) \neq \emptyset$. Let $\{u_m\}$ be the sequence generated by (1). Then $\lim_{m\to\infty} \|u_m - y\|$ exists $\forall y \in F(T)$.*

Proof. Let $y \in F(T)$. By lemma 1.3, $P_T(y) = \{y\}$. By Lemma 1.5, we have

$$\begin{aligned}
\|u_{m+1} - y\| &= \|z''_m - y\| \leq \mathcal{H}(P_T(v_m), P_T(y)) \leq \|v_m - y\| \\
&= \|z'_m - y\| \leq \mathcal{H}(P_T(w_m), P_T(y)) \leq \|w_m - y\| \\
&= \|(1 - a_m)u_m + a_m z_m - y\| \\
&\leq (1 - a_m)\|u_m - y\| + a_m\|z_m - y\| \\
&\leq (1 - a_m)\|u_m - y\| + a_m\mathcal{H}(P_T(u_m), P_T(y)) \\
&\leq (1 - a_m)\|u_m - y\| + a_m\|u_m - y\| \\
&= \|u_m - y\|.
\end{aligned}$$

So $\{\|u_m - y\|\}$ is bounded and nonincreasing and so $\lim_{m\to\infty} \|u_m - y\|$ exists $\forall y \in F(T)$. □

The following result is helpful for the next results.

Lemma 2.2. *Let X be a uniformly convex Banach space and $\mathcal{B}$ be a nonempty closed convex subset of X. Let $T : \mathcal{B} \to \mathbb{P}_{px}(\mathcal{B})$ be a such that P_T is a generalized α-nonexpansive and $F(T) \neq \emptyset$. Let $\{u_m\}$ be the sequence generated by (1). Then $\lim_{m\to\infty} d(u_m, P_T(u_m)) = 0$.*

Proof. Fix $y \in F(T)$. By Lemma 2.1, $\lim_{n\to\infty} \|u_m - y\|$ exists. Put

$$\lim_{m\to\infty} \|u_m - y\| = k. \tag{2}$$

When $k = 0$ then we have nothing to prove. We consider the case $k > 0$. In the view of proof of Lemma 2.1, we

$$\|w_m - y\| \leq \|u_m - y\|$$

$$\Rightarrow \limsup_{m\to\infty} \|w_m - y\| \leq \limsup_{m\to\infty} \|u_m - y\| = k. \tag{3}$$

Since $y \in P_T(y)$ and noting $\|z_m - y\| = d(z_m, P_T(y))$. By Lemma 1.5, P_T is quasi-nonexpansive. Hence

$$\|z_m - y\| = d(z_m, P_T(y)) \leq \mathcal{H}(P_T(u_m), P_T(y)) \leq \|u_m - y\|.$$

$$\Rightarrow \limsup_{m\to\infty} \|z_m - y\| \leq \limsup_{m\to\infty} \|u_m - y\| = k. \tag{4}$$

Again in the proof of Lemma 2.1, we have

$$\|u_{m+1} - y\| \leq \|w_m - y\|$$

$$\Rightarrow k = \liminf_{m\to\infty} \|u_{m+1} - y\| \leq \liminf_{m\to\infty} \|w_m - y\|. \tag{5}$$

From (3), and (5), we obtain

$$\lim_{m\to\infty} \|w_m - y\| = k. \tag{6}$$

From (6), we have

$$k = \lim_{m\to\infty} \|w_m - y\| = \lim_{m\to\infty} \{(1 - a_m)(u_m - y) - a_m(z_m - y)\}.$$

By Lemma 1.7, we obain

$$\lim_{m\to\infty} \|u_m - z_m\| = 0 \tag{7}$$

which yields

$$\lim_{m\to\infty} d(u_m, P_T(u_m)) = 0. \tag{8}$$

□

We prove a strong convergence of $\{u_m\}$ generated by (1) for multi-valued generalized α-nonexpansive.

Theorem 2.3. *Let X be a uniformly convex Banach space and $\mathcal{B}$ be a nonempty compact convex subset of X. Let $T : \mathcal{B} \to \mathbb{P}_{px}(\mathcal{B})$ be such that P_T is a generalized α-nonexpansive and $F(T) \neq \emptyset$. Let $\{u_m\}$ be the sequence defined by (1). Then $\{u_m\}$ converges strongly to a fixed point of T.*

Proof. By Lemma 2.2, $\lim_{m\to\infty} d(u_m, P_T(u_m)) = 0$. By compactness of $\mathcal{B}$ we can find a subsequence $\{u_{m_j}\}$ of $\{u_m\}$ such that $\{u_{m_j}\}$ converges to some z in $\mathcal{B}$. In the view of Lemma 1.6, we have

$$\begin{aligned} d(z, P_T(z)) &\leq \|z - u_{m_j}\| + d(u_{m_j}, P_T(z)) \\ &\leq \|z - u_{m_j}\| + \left(\frac{3+\alpha}{1-\alpha}\right) d(u_{m_j}, P_T(u_{m_j})) + \|u_{m_j} - z\| \\ &= 2\|z - u_{m_j}\| + \left(\frac{3+\alpha}{1-\alpha}\right) d(u_{m_j}, P_T(u_{m_j})) \longrightarrow 0. \end{aligned}$$

Hence $z \in P_T(z)$. By Lemma 1.3, $z \in F(P_T) = F(T)$. By Lemma 2.1, $\lim_{m\to\infty} \|u_m - z\|$ exists. Hence z is the strong limit of $\{u_m\}$. □

The proof of the following result is elementary and hence omitted.

Theorem 2.4. *Let X be a uniformly convex Banach space and $\mathcal{B}$ be a nonempty closed convex subset of X. Let $T : \mathcal{B} \to \mathbb{P}_{px}(\mathcal{B})$ be such that P_T is a generalized α nonexpansive. If $F(T) \neq \emptyset$ and $\liminf_{m\to\infty} d(u_m, F(T)) = 0$. Then $\{u_m\}$ generated by (1) converges strongly to a fixed point of T.*

Before proving strong convergence of $\{u_m\}$ using condition I, we first recall the definition of multi-valued version of condition I.

Definition 2.5. ([35]) A map $T : \mathcal{B} \to \mathbb{P}(\mathcal{B})$ is said to satisfy condition I (where $\mathcal{B}$ is a nonempty subset of a Banach space) if there is a function $\lambda : [0,\infty) \to [0,\infty)$ with the properties $\lambda(0) = 0$ and $\lambda(q) > 0$ for every $q \in (0,\infty)$ and $d(u, Tu) \geq \lambda(d(u, F(T)))$ for every $u \in \mathcal{B}$.

Theorem 2.6. *Let X be a uniformly convex Banach space and $\mathcal{B}$ be a nonempty closed convex subset of X. Let $T : \mathcal{B} \to \mathbb{P}_{px}(\mathcal{B})$ be a multivalued mapping with $F(T) \neq \emptyset$. If P_T is a generalized α-nonexpansive mapping. Then $\{u_m\}$ generated by (1) converges strongly to a fixed point of T provided that T satisfies the condition I.*

Proof. By Lemma 2.1, $\lim_{m\to\infty} \|u_m - y\|$ exists for all $y \in F(T)$. Put $k = \lim_{m\to\infty} \|u_m - y\|$ for some $k \geq 0$. If $k = 0$ then the result follows. Suppose that $k > 0$. Then,

$$\|u_{m+1} - y\| \leq \|u_m - y\|.$$

It follows that

$$d(u_{m+1}, F(T)) \leq d(u_m, F(T)).$$

Hence $\lim_{m\to\infty} d(u_m, F(T))$ exists. We show that $\lim_{m\to\infty} d(u_m, F(T)) = 0$. From Lemma 2.2, it follows that $\lim_{m\to\infty} d(u_m, P_T(u_m)) = 0$. Also from Lemma 1.3, $F(T) = F(P_T)$. Using these facts and condition I, we have

$$\lim_{m\to\infty} \lambda(d(u_m, F(T)) = 0.$$

Since λ is nondecreasing and $\lambda(0) = 0$. We get $\lim_{m\to\infty} d(u_m, F(T)) = 0$. By Theorem 2.4, we obtain the required conclusions. □

Finally we prove a weak convergence of $\{u_m\}$.

Theorem 2.7. *Let X be a uniformly convex Banach space satisfying Opial's condition and $\mathcal{B}$ be a nonempty closed cobvex subset of X. Let $T : \mathcal{B} \to \mathbb{P}_{px}(\mathcal{B})$ be a multivalued mapping with $F(T) \neq \emptyset$. Suppose that P_T is a generalized α-nonexpansive mapping and $I - P_T$ is demiclosed with respect to zero. If $\{u_m\}$ is a sequence generated by (1). Then $\{u_m\}$ converges weakly to a fixed point of T.*

Proof. By the proof of Lemma 2.1 $\{u_m\}$ is bounded. Since X is uniformly convex, so X is reflexive by Milman-Pettis's Theorem. By Eberlin's Theorem, every bounded sequence in X has a weakly convergent subsequence. Thus we can find a weakly convergent subsequence $\{u_{m_j}\}$ of $\{u_m\}$ with weak limit say y_1 in $\mathcal{B}$. By demicloseness of $I - P_T$ at 0, $y_1 \in F(P_T) = F(T)$. We prove that y_1 is the unique weak limit of $\{u_m\}$. Let one can find another weakly convergent subsequence $\{u_{m_k}\}$ of $\{u_m\}$ with weak limit say $y_2 \in \mathcal{B}$ and $y_2 \neq y_1$. Again $y_2 \in F(P_T) = F(T)$. By the Opial property and Lemma 2.1, we have

$$\begin{aligned}
\lim_{m\to\infty} \|u_m - y_1\| &= \lim_{i\to\infty} \|u_{m_i} - y_1\| \\
&< \lim_{i\to\infty} \|u_{m_i} - y_2\| \\
&= \lim_{m\to\infty} \|u_m - y_2\| \\
&< \lim_{k\to\infty} \|u_{m_k} - y_2\| \\
&< \lim_{k\to\infty} \|u_{m_k} - y_1\| \\
&= \lim_{m\to\infty} \|u_m - y_1\|.
\end{aligned}$$

This is a contradiction. Hence $\{u_m\}$ converges weakly to y_1. □

3. Example

In this section we provide an example of multi-valued mapping for which P_T is generalized α-nonexpansive mapping. Also, using this example, we compare various well known iterative schemes such as Picard-S [16], Abbas [1], Agarwal [5], Noor [26], Ishikawa [18] and Mann [24] iteration process with our M-iterative schemes to show the numerical efficiency of our results.

Example 3.1. Let $\mathcal{B}$ be endowed with the usual norm. Define $T : \mathcal{B} \to \mathbb{P}_{px}(\mathcal{B})$ be defined by

$$Tu = \begin{cases} \{0\} & \text{if } u \in \left[0, \frac{1}{9000}\right) \\ [0, \frac{1}{2}u] & \text{if } u \in \left[\frac{1}{9000}, 1\right]. \end{cases}$$

If $u \in [0, \frac{1}{9000})$, then $P_T(u) = \{0\}$. For $u \in [\frac{1}{9000}, 1]$, then $P_T(u) = \{\frac{1}{2}u\}$. We show that P_T is generalized $\frac{1}{3}$-nonexpansive mapping with $F(T) \neq \emptyset$. We shall devide the proof into the following three cases.

Case (i). For $u, v \in [0, \frac{1}{9000})$, we have

$$\frac{1}{3}d(v, P_T(u)) + \frac{1}{3}d(u, P_T(v)) + (1 - 2(\frac{1}{3}))\|u - v\| \geq 0 = \mathcal{H}(P_T(u), P_T(v)).$$

Case (ii). For $u, v \in [\frac{1}{9000}, 1]$, we have

$$\begin{aligned}
\frac{1}{3}d(v, P_T(u)) + \frac{1}{3}d(u, P_T(v)) + (1 - 2(\frac{1}{3}))\|u - v\| &= \frac{1}{3}d\left(v, \left\{\frac{u}{2}\right\}\right) + \frac{1}{3}d\left(u, \left\{\frac{v}{2}\right\}\right) + \frac{1}{3}|u - v| \\
&\geq \frac{1}{3}d\left(v, \left\{\frac{u}{2}\right\}\right) + \frac{1}{3}d\left(u, \left\{\frac{v}{2}\right\}\right) \\
&= \frac{1}{3}|v - \frac{u}{2}| + \frac{1}{3}|u - \frac{v}{2}| \\
&\geq \frac{1}{3}|\frac{3u - 3v}{2}| \\
&= \left|\frac{u - v}{2}\right| \\
&= \mathcal{H}(P_T(u), P_T(v))
\end{aligned}$$

Case (iii). For $u \in [\frac{1}{9000}, 1]$ and $v \in [0, \frac{1}{9000})$, we have

$$\begin{aligned}
\frac{1}{3}d(v, P_T(u)) + \frac{1}{3}d(u, P_T(v)) + (1 - 2(\frac{1}{3}))\|u - v\| &= \frac{1}{3}d\left(v, \left\{\frac{u}{2}\right\}\right) + \frac{1}{3}d(u, \{0\}) + \frac{1}{3}|u - v| \\
&\geq \frac{1}{3}d\left(v, \left\{\frac{u}{2}\right\}\right) + \frac{1}{3}|u - v| \\
&= \frac{1}{3}|v - \frac{u}{2}| + \frac{1}{3}|u - v| \\
&\geq \frac{1}{3}|\frac{u + 2u}{2}| \\
&= |\frac{u}{2}| = \mathcal{H}(P_T(u), P_T(v))
\end{aligned}$$

Hence, for all $u, v \in \mathcal{B}$, we get $\mathcal{H}(P_T(u), P_T(v)) \leq \frac{1}{3}d(v, P_T(u)) + \frac{1}{3}d(u, P_T(v)) + (1 - 2(\frac{1}{3}))\|u - v\|$. As a result P_T is generalized $\frac{1}{3}$-nonexpansive mapping. Moreover $y = 0$ is a fixed point of T. When $u = \frac{1}{16000}$ and $v = \frac{1}{9000}$. Then $\frac{1}{2}d(u, P_T(u)) < |u - v|$ and $\mathcal{H}(P_T(u), P_T(v)) > |u - v|$. Hence P_T is not Suzuki mapping. The values generated by M [40], Picard-S [16], Abbas [1], Agarwal [5], Noor [26], Ishikawa [18] and Mann [24] iterations under $a_m = 0.70$, $b_m = 0.65$ and $c_m = 0.90$ are given in Table 1 and Figure 1.

Table 1: Sequences defined by M, Picard-S, Abbas, Agarwal, Noor, Ishikawa and Mann iterative scheme.

u_m	**M**	**Picard-S**	**Abbas**	**Agarwal**	**Noor**	**Ishikawa**	**Mann**
u_1	0.8	0.8	0.8	0.8	0.8	0.8	0.8
u_2	0.1300	0.1545	0.1964	0.3090	0.3880	0.4290	0.5200
u_3	0.2112	0.0298	0.0482	0.1193	0.1882	0.2300	0.3380
u_4	0.0034	0.0057	0.0118	0.0460	0.0913	0.1233	0.2197
u_5	0.0005	0.0011	0.0029	0.0178	0.0442	0.0661	0.1428
u_6	**0**	0.0002	0.0007	0.0068	0.0214	0.0354	0.0928
u_7	0	**0**	0.0001	0.0026	0.0104	0.0190	0.0603
u_8	0	0	**0**	0.0010	0.0050	0.0102	0.0392
u_9	0	0	0	0.0003	0.0024	0.0054	0.0254
u_{10}	0	0	0	0.0001	0.0011	0.0029	0.0165
u_{11}	0	0	0	**0**	0.0005	0.0015	0.0107
u_{12}	0	0	0	0	0.0002	0.0008	0.0070
u_{13}	0	0	0	0	0.0001	0.0004	0.0045
u_{14}	0	0	0	0	**0**	0.0002	0.0029
u_{15}	0	0	0	0	0	0.0001	0.0019
u_{16}	0	0	0	0	0	**0**	0.0012

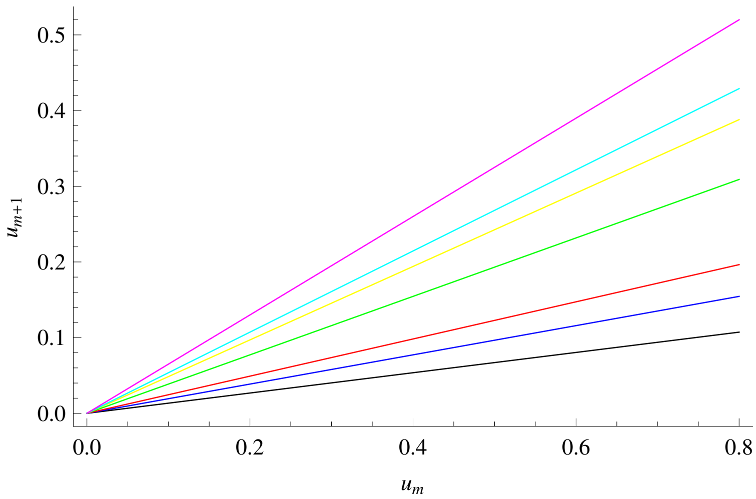

Figure 1: Convergence behavior of M (black line), Picard-S (blue line), Abbas (red line), Agarwal (green line), Noor (yellow line), Ishikawa (cyan line) and Mann (magenta line) iterative schemes under ($a_m = 0.70$, $b_m = 0.65$, $c_m = 0.90$ and $u_1 = 0.8$).

Now using the above example, we make different choices of parameters a_m and b_m and also we get $\|u_m - u^*\| < 10^{-10}$ as our stopping criterion where $u^* = 0$ is a fixed point of T. We obtained the influence of initial points for the M-iterative scheme by $a_m = \frac{2m}{(7m+11)^{\frac{1}{2}}}$ and $b_m = \frac{1}{(3m+9)^{\frac{1}{2}}}$. The results are compared with the leading three-step Picard-S [16] and leading two-step Agarwal [5] iterative schemes. The Items in bold in the Tables 2 and 3 show that M- iterative scheme converges faster than the other iterative schemes.

Table 2: Influence of initial points for various iterative iterative schemes.

Number of iterations required to obtain fixed point.			
Initial points	Agarwal iterates	Picard-S iterates	M iterates
0.2	10	6	**5**
0.4	11	6	**5**
0.6	12	7	**5**
0.8	12	7	**5**
1.0	12	7	**5**

Table 3: Influence of parameters: comparison of various iterative schemes.

Iterations	Initial points					
	0.1	0.3	0.5	0.6	0.8	0.9
For $a_m = \left(\frac{m+1}{5m+2}\right)^{\frac{1}{11}}$, $b_m = \frac{2m}{(5m+100)^{\frac{1}{2}}}$						
Agarwal	11	12	13	13	14	14
Picard-S	6	6	7	7	7	7
M	**4**	**5**	**5**	**5**	**5**	**5**
for $a_m = \frac{1}{m}$, $b_m = \frac{1}{(m+25)^{\frac{1}{2}}}$						
Agarwal	11	12	13	13	14	14
Picard-S	6	7	7	7	7	7
M	**5**	**6**	**6**	**6**	**6**	**6**
for $a_m = 1 - (\frac{1}{5m+3})^{\frac{1}{5}}, b_m = \frac{1}{m^5}$						
Agarwal	11	13	13	14	14	14
Picard-S	6	7	7	7	7	7
M	**5**	**6**	**6**	**6**	**6**	**6**
for $a_m = \left(\frac{m+1}{(5m+2)}\right)^{\frac{1}{2}}$, $b_m = \frac{1}{(4m+8)^{\frac{4}{3}}}$						
Agarwal	11	13	14	14	14	14
Picard-S	6	7	7	7	7	7
M	**5**	**6**	**6**	**6**	**6**	**6**

4. Conclusions

In the view of previous discussion, the cases when the mapping is multi-valued Suzuki or else is multi-valued nonexpansive are now special cases of our obtained results. Tables 1, 2, 3 and Figure 1 show the numerical efficiency of our results. Our results extend and improve the corresponding single-valued results of Abbas and Nazir [1], Gursoy and Karakaya [16], Pant and Shukla [28], Phuengrattana [31], Ullah and Arshad [40] and many others. Moreover, our results improve and extend the corresponding multi-valued results of Khan et al. [20], Khan and Yildirim [21], Panyanak [29], Shahzad and Zegeye [36], Song and Wang [38] and many others. More precisely, our results extend the recent results of Ullah et al. [41] from the setting of single-valued generalized α-nonexpansive mappings to the general setting of multi-valued generalized α-nonexpansive mappings.

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References

[1] M. Abbas, T. Nazir, A new faster iteration process applied to constrained minimization and feasibility problems, Mat. Vesnik 66 (2014) 223–234.
[2] T. Abdeljawad, K. Ullah, J. Ahmad, N. Mlaiki, Iterative approximation of endpoints for multivalued mappings in Banach spaces. J. Function Spaces, 2020, Article ID 2179059.
[3] A. Abkar, M. Eslamian, A fixed point theorem for generalized nonexpansive multivalued mappings, Fixed Point Theory 12 (2011) 241–246.
[4] R.P. Agarwal, D. O'Regon, D.R. Sahu, Fixed Point Theory for Lipschitzian-type Mappings with Applications Series. Topological Fixed Point Theory and Its Applications, vol. 6, Springer, New York, 2009.
[5] R.P. Agarwal, D. O'Regon, D.R. Sahu, Iterative construction of fixed points of nearly asymtotically nonexpansive mappings, J. Nonlinear Convex Anal. 8 (2007) 61–79.
[6] O. Alagoz, G. Birol, G. Sezgin, Numerical reckoning fixed points for Barinde mappings via a faster iteration process, Facta Universitatis, Ser. Math. Inform. 33 (2018) 295–305.

[7] K. Aoyama, F. Kohsaka, Fixed point theorem for α-nonexpansive mappings in Banach spaces, Nonlinear Anal. 74 (2011) 4387–4391.
[8] S. Banach, Sur les operations dans les ensembles abstraits et leur application aux equations integrales, Fund. Math. 3 (1922) 133–181.
[9] V. Berinde, Iterative Approximation of Fixed Points, Springer, Berlin, (2007).
[10] V. Berinde, M. Pacurar, The role of the Pompeiu-Hausdorff metric in fixed point theory, Creat. Math. Inform. 22 (2013) 143–150.
[11] F.E. Browder, Nonexpansive nonlinear operators in a Banach space, Proc. Nat. Acad. Sci. USA 54 (1965) 1041–1044.
[12] H. Covitz, S.B. Nadler, Multi-valued contraction mappings in generalized metric spaces, Israel J. Math. 8 (1970) 5–11.
[13] J. A. Clarkson, Uniformly convex spaces, Trans. Amer. Math. Soc. 40 (1936) 396–414.
[14] D. Göhde, Zum Prinzip der Kontraktiven Abbildung, Math. Nachr. 30 (1965) 251–258.
[15] L. Gorniewicz, Topological Fixed Point Theory of Multivalued Mappings, Kluwer Academic Pub., Dordrecht, Netherlands, 1999.
[16] F. Gursoy, V. Karakaya, A Picard-S hybrid type iteration method for solving a differential equation with retarded argument, arXiv:1403.2546v2 (2014).
[17] H.M. Hajisharifi, On some generalization of multivalued nonexpansive mappings, Bull. Iranian Math. Soc. 46 (2020) 557–571.
[18] S. Ishikawa, Fixed points by a new iteration method, Proc. Amer. Math. Soc. 44 (1974) 147–150.
[19] H. Iqbal, M. Abbas, S.M. Husnine, Existence and approximation of fixed points of multivalued generalized α-nonexpansive mappings in Banach spaces, Numer. Algor. (2019), DOI: 10.1007/s11075-019-00854-z.
[20] S.H. Khan, M. Abbas, S. Ali, Fixed points of multi-valued quasi nonexpansive mappings using a faster iterative process, Acta Math. Sinica, Engl. Ser. 30 (2014) 1231–1241.
[21] S.H. Khan, I. Yildirim, Fixed points of multivalued nonexpansive mappings in Banach spaces, Fixed Point Theory Appl. 73 (2011) 1–9.
[22] W.A. Kirk, A fixed point theorem for mappings which do not increase distance, Amer. Math. Monthly 72 (1965) 1004–1006.
[23] T.C. Lim, A fixed point theorem for multivalued nonexpansive mappings in a uniformly convex Banach spaces, Bull. Amer. Math. Soc. 80 (1974) 1123–1126.
[24] W.R. Mann, Mean value methods in iteration, Proc. Amer. Math. Soc. 4 (1953) 506–510.
[25] S.B. Nadler, Multivalued contraction mappings, Pracific J. Math. 30 (1969) 745–488.
[26] M.A. Noor, New approximation schemes for general variational inequalities, J. Math. Anal. Appl. 251 (2000) 217–229.
[27] Z. Opial, Weak and strong convergence of the sequence of successive approximations for nonexpansive mappings, Bull. Amer. Math. Soc. 73 (1967) 591–597.
[28] R. Pant, R. Shukla, Approximating fixed points of generalized α-nonexpansive mappings in Banach spaces, Numer. Funct. Anal. Optim. 38 (2017) 248–266.
[29] B. Panyanak, Mann and Ishikawa iterative processes for multivalued mappings in Banach spaces, Comp. Math. Appl. 54 (2007) 872–877.
[30] R.P. Pathak, S. Dashputre, S.D. Diwan, R. Gupta, On Noor-type iteration schemes for multivalued mappings in CAT(0) spaces, Fixed Point Theory Appl. 133 (2015) 1–12.
[31] W. Phuengrattana, Approximating fixed points of Suzuki generalized nonexpansive mappings, Nonlinear Anal. Hybrid Syst. 5 (2011) 583–590.
[32] A. Sahin, Some new results of M-iteration process in hyperbolic spaces, Carpathian J. Math. 35 (2019) 221–232.
[33] K.P.R. Sastry, G.V.R. Babu, Convergence of Ishikawa iterates for multivalued mappings with a fixed point, Czechoslovak Math. J. 55 (2005) 817–826.
[34] J. Schu, Weak and strong convergence to fixed points of asymptotically nonexpansive mappings, Bull. Austral. Math. Soc. 43 (1991) 153–159.
[35] H.F. Senter, W.G. Dotson, Approximating fixed points of nonexpansive mappings, Proc. Amer. Math. Soc. 44 (1974) 375–380.
[36] N. Shahzad, H. Zegeye, On Mann and Ishikawa iteration schemes for multivalued maps in Banach spaces, Nonlinear Anal. 71 (2009) 838–844.
[37] Y. Song, Y.J. Cho, Some notes on Ishikawa iteration for multi-valued mappings. Bull. Korean Math. Soc. 48 (2011) 575–584.
[38] Y. Song, H. Wang, Erratum to "Mann and Ishikawa iterative processes for multivalued mappings in Banach spaces", Comput. Math. Appl. 52 (2008) 2999–3002.
[39] T. Suzuki, Fixed point theorems and convergence theorems for some generalized nonexpansive mapping, J. Math. Anal. Appl. 340 (2008) 1088–1095.
[40] K. Ullah, M. Arshad, Numerical reckoning fixed points for Suzuki's generalized nonexpansive mappings via new iteration process, Filomat 32 (2018) 187–196.
[41] K. Ullah, F. Ayaz, J. Ahmad, Some convergence results of M iterative process in Banach spaces, Asian-Europian J. Math. (2019), Article ID 2150017.
[42] K. Ullah, H.N. Khan, M. Arshad, Numerical reckoning fixed points in CAT(0) spaces, Sahand Commun. Math. Anal. 12 (2018) 97–111.