Regularized Iterative Method for Ill-posed Linear Systems Based on Matrix Splitting

Ashish Kumar Nandi[a], Jajati Keshari Sahoo[a]

[a]Department of Mathematics, BITS Pilani, K.K. Birla Goa Campus, Goa, India

Abstract. In this paper, the concept of matrix splitting is introduced to solve a large sparse ill-posed linear system via Tikhonov's regularization. In the regularization process, we convert the ill-posed system to a well-posed system. The convergence of such a well-posed system is discussed by using different types of matrix splittings. Comparison analysis of both systems are studied by operating certain types of weak splittings. Further, we have extended the double splitting of [Song J. and Song Y, Calcolo 48(3), 245–260, 2011] to double weak splitting of type II for nonsingular symmetric matrices. In addition to that, some more comparison results are presented with the help of such weak double splittings of type I and type II.

1. Introduction

In the view of Hadamard [19], the discretization of Fredholm integral equations of the first kind [16] is formed an ill-posed linear system

$$Ax = b, \tag{1}$$

where $A \in \mathbb{R}^{m \times n}$, $x \in \mathbb{R}^n$ and $b \in \mathbb{R}^m$. In practice, this type of ill-posed system appears in several branches of science and engineering such as noisy image restoration [1], computer tomography [15] and inverse problems within electromagnetic [38]. Ill-posed problems were extensively studied in the context of an inverse problem [8, 12, 17] and image restorations [6]. In image restoration, the main objective is to establish a blurred free image that requires the approximate solution of the system (1). For more details one can refer [1, 6]. To find the approximate solution of the ill-posed system (1), several iterative methods such as Accelerated Landweber iterative method [20], GMRES and singular preconditioner method [11], conjugate gradient method are studied in the recent past. However, the utilization of the splittings method along with regularization is quite a new idea.

In order to solve the system $Ax = b$, i.e, find the least square solution $A^\dagger b$, we first normalize as $A^T Ax = A^T b$. This does not make the problem simple as most of the cases the matrix $A^T A$ is singular and ill-conditioned which is affected highly by round-off errors [14]. Thus we need to make the system $Ax = b$, well-posed by introducing a regularization parameter $\lambda(> 0)$, and the corresponding modified well-posed system based on Tikhonov's regularization [47] is given by

$$(A^T A + \lambda I)x = A^T b. \tag{2}$$

If we consider $B_\lambda = A^T A + \lambda I$, then the system (2) reduces to the following system

$$B_\lambda x = A^T b. \tag{3}$$

The above procedure is known as regularization and the parameter λ decides what extent the original ill-posed system (1) is changed. Further, the value of the regularization parameter λ determines how well the solution x_λ of (3) approximates the exact solution $A^\dagger b$. One can see [12] for discussions on this parameter choice method. A large number of heuristic parameter choice methods have been proposed in the literature due to the importance of being able to determine a suitable value of the regularization parameter when the discrepancy principle cannot be used. For more details one can refer [7, 18, 24, 25, 39].

There are several ways to regularize such type of ill-posed system. Among them, the most classical regularization is *Tikhonov's regularization* introduced by Tikhonov in 1963 [47]. Some iterative methods for the system (2) in framework of operator theory can be found in [21] and the references therein. The main motivation to analyze and compare the numerical solution of both system (1) and (3) comes from Barata and Hussein [3], where the authors have shown that $B_\lambda^{-1} A^T b \to A^\dagger b$ as $\lambda \to 0$.

On the other hand, the matrix splitting (A decomposition $A = U - V$ is called a splitting of the matrix A) methods are more significant and numerically stable in dealing with rectangular matrices. In this direction, Berman and Plemmons [4] first introduced the *proper splitting* (A splitting $A = U - V$ is called *proper* if the null space of A is equal to the null space of U and the range space of A is equal to range space U). If $A = U - V$ is a proper splitting of $A \in \mathbb{R}^{m \times n}$, then the associated iterative scheme for solving $Ax = b$, is given by

$$x^{k+1} = U^\dagger V x^k + U^\dagger b. \tag{4}$$

It is well known that the iterative scheme defined in (4) converges to $A^\dagger b$, for any initial vector x^0 if and only if the spectral radius of $U^\dagger V$ is less than 1. Further, if the system $Ax = b$ is consistent, then the above iterative process converges to a solution of (1). In [4], it was proved that if $A = U - V$ is a proper splitting such that $U^\dagger \geq 0$ (entry-wise) and $U^\dagger V \geq 0$, then $A^\dagger \geq 0$ if and only if the spectral radius of $U^\dagger V$ is less than 1.

In case of nonsingular coefficient matrix B_λ, if $B_\lambda = M_\lambda - N_\lambda$ is a splitting of $B_\lambda \in \mathbb{R}^{n \times n}$ such that M_λ is invertible, then the associated iterative scheme is given by

$$x^{k+1} = M_\lambda^{-1} N_\lambda x^k + M_\lambda^{-1} A^T b. \tag{5}$$

It is clear that this iterative method converges to $B_\lambda^{-1} A^T b$ ($= A^\dagger b$ as $\lambda \to 0$) for any initial vector x^0 if and only if the spectral radius of $M_\lambda^{-1} N_\lambda$ is less than 1. We call a splitting convergent if the associated iterative scheme convergent. Several types of splittings and numerous comparative studies can be found in the literature [26, 29, 30, 32, 45, 51] and the reference therein.

The main objective of this article is to introduce a new regularized nonsingular approach for the rectangular or singular system and study the convergence of the iterative method (5) associated with different types of splittings of B_λ. In case of the regularized iterative scheme, we can relax some strong conditions such as non-negativeness of A, $A^\dagger$ to assure the convergence. Besides that, we have introduced a new matrix splitting, called double weak splitting of type II. Further, a few comparative studies between the original system (1) and regularized system (3) are provided. The theoretical results provided show that the regularized iterative scheme convergence faster (in terms of spectral radius).

1.1. Outline

The paper is organized as follows: some useful notations and definitions are discussed in Section 2. In addition to these we review some basic theories of iterative methods, which will be used throughout this paper. The main results of this article are elaborated in Section 3. Numerous comparison results related to the systems (1) and (3) are established. Also, a double weak splitting of type II is newly introduced as well as few comparison theorems have been proved for the double weak splitting of type II. In Section 4, we have numerically validated the proposed regularized iterative scheme in comparison to an ill-posed system. The manuscript is concluded along with a few future research perspectives in Section 5.

2. Preliminaries

First, we elaborate on some notations and definitions which will be useful throughout the article. The set of all real rectangular matrices of order $m \times n$ is denoted by $\mathbb{R}^{m\times n}$. For matrices $A, B \in \mathbb{R}^{m\times n}$, a matrix B is said to be *nonnegative* ($B \geq 0$) if all entries of B are nonnegative and $A \geq B$ implies $A - B \geq 0$. If L and M are two complementary subspaces of $\mathbb{R}^n$, then $P_{L,M}$ is the projection on L along M. So, $P_{L,M}B = B$ if and only if $R(B) \subseteq L$ and $BP_{L,M} = B$ if and only if $N(B) \supseteq M$. Henceforth, $R(A)$ and $N(A)$ denotes the range space and null space of the matrix A. We denote the transpose of a matrix A by A^T. The *spectral radius* of a matrix $B \in \mathbb{R}^{n\times n}$ is denoted as $\rho(B)$ and defined by $\rho(B) = \max\limits_{1\leq i\leq n} |\sigma_i|$, where σ_i's are the eigenvalues of B. It is well known that for any square matrix B, $\rho(B^T) = \rho(B)$ and $\rho(AB) = \rho(BA)$ for well defined product of matrices A and B. We recall the Moore-Penrose inverse of a matrix B. The unique matrix $X \in \mathbb{R}^{n\times m}$, satisfying $BXB = B$, $XBX = X$, $(BX)^T = BX$ and $(XB)^T = XB$, is called the *Moore-Penrose inverse* of B and denoted by $B^\dagger$. A few properties of $B^\dagger$ which are frequently being used: $R(B^T) = R(B^\dagger)$; $N(B^T) = N(B^\dagger)$; $B^\dagger B = P_{R(B^T)}$ and $BB^\dagger = P_{R(B)}$. Further, a nonsingular matrix B is called *monotone* if $B^{-1} \geq 0$. Similarly, we called a matrix $B \in \mathbb{R}^{m\times n}$ is *semi-monotone* if $B^\dagger \geq 0$.

Next, we discuss some necessary results based on non-negativeness regularization and matrix splittings. The very first result is for nonnegative matrices.

Theorem 2.1 (Theorem 2.1.11, **[5]).** *Let $B \in \mathbb{R}^{n\times n}$, $B \geq 0$, $x \geq 0$ ($x \neq 0$) and α be a positive scalar. Then the followings hold.*

  (i) If $\alpha x \leq Bx$, then $\alpha \leq \rho(B)$.
  (ii) For $x > 0$, if $Bx \leq \alpha x$, then $\rho(B) \leq \alpha$.

We now collect a few parts of the classical Perron-Frobenius theorem. Perron proved it for positive matrices and Frobenius gave the extension to irreducible matrices.

Theorem 2.2 (Theorem 2.20, **[48]).** *Let $A \in \mathbb{R}^{n\times n}$ be a nonnegative matrix. Then*

  (i) A has a nonnegative real eigenvalue equal to its spectral radius.
  (ii) there exists a nonnegative eigenvector for its spectral radius.

Theorem 2.3 (Theorem 2.7, **[48]).** *Let $A \in \mathbb{R}^{n\times n}$ be a nonnegative matrix. If A is irreducible, then*

  (i) A has a positive real eigenvalue equal to its spectral radius.
  (ii) there exists a positive eigenvector for its spectral radius.

In connection to the spectral radius, the following result is collected from [48].

Theorem 2.4 (Theorem 2.21, **[48]).** *If $A, B \in \mathbb{R}^{n\times n}$ and $A \geq B \geq 0$, then $\rho(A) \geq \rho(B)$.*

In view of proper splitting, we state the following essential results.

Theorem 2.5 (Theorem 1, **[9]).** *Let $A = U - V$ be a proper splitting of $A \in \mathbb{R}^{m\times n}$. Then*

  (i) $A = (I - VU^\dagger)U = U(I - U^\dagger V)$,
  (ii) $I - VU^\dagger$ is nonsingular,
  (iii) $A^\dagger = U^\dagger(I - VU^\dagger)^{-1} = (I - U^\dagger V)^{-1}U^\dagger$.

Theorem 2.6 (Theorem 2.2, **[34]).** *Let $A = U - V$ be a proper splitting of $A \in \mathbb{R}^{m\times n}$. Then*

  (i) $UU^\dagger = AA^\dagger$ and $U^\dagger U = A^\dagger A$,
  (ii) $U^\dagger = (I + A^\dagger V)^{-1}A^\dagger = A^\dagger(I + VA^\dagger)^{-1}$,
  (iii) $U^\dagger VA^\dagger = A^\dagger VU^\dagger$.

Next, we recall the definition of weak proper splitting of the first type and the second type.

Definition 2.7 (Definition 2, [9]). *A proper splitting $A = U - V$ of $A \in \mathbb{R}^{m \times n}$ is called a weak proper splitting of the first type (respectively, the second type), if $U^\dagger V \geq 0$ (respectively, $VU^\dagger \geq 0$).*

In case of nonsingular matrices, the splittings defined in Definition 2.7 are called respectively as weak splitting of the first type and weak splitting of the second type (which were respectively introduced by Marek & Szyld [29]), and by Woźnicki [52]) and stated in the next definition.

Definition 2.8. *A splitting $A = U - V$ of $A \in \mathbb{R}^{n \times n}$ is called a weak splitting of the first type, if $U^{-1}V \geq 0$ and a weak splitting of the second type, if $VU^{-1} \geq 0$.*

The next result is a combination of Theorem 2 and Remark 2 of [9].

Theorem 2.9. *Let $A = U - V$ be a weak proper splitting of the first type (or second type) of $A \in \mathbb{R}^{m \times n}$. Then $A^\dagger V$ (or $VA^\dagger$) ≥ 0 if and only if $\rho(U^\dagger V) = \frac{\rho(A^\dagger V)}{1 + \rho(A^\dagger V)} < 1$ (respectively, $\rho(VU^\dagger) = \frac{\rho(VA^\dagger)}{1 + \rho(VA^\dagger)} < 1$).*

In a special case of the above result (Theorem 2.9), which was proved in [10] is stated in the next theorem.

Theorem 2.10 (Theorem 3 and Remark 4 of [10]). *Let $A \in \mathbb{R}^{n \times n}$ be nonsingular, and let $A = U - V$ be a weak splitting of the first type (respectively, the second type). Then $A^{-1}V \geq 0$ (respectively, $VA^{-1} \geq 0$) if and only if $\rho(U^{-1}V) = \frac{\rho(A^{-1}V)}{1 + \rho(A^{-1}V)} < 1$ (respectively, $\rho(VU^{-1}) = \frac{\rho(VA^{-1})}{1 + \rho(VA^{-1})} < 1$).*

Further, we recall one comparison theorem of [46] for two weak splittings of the second type.

Theorem 2.11 (Theorem 2.11, [46]). *Let $A = M_1 - N_1 = M_2 - N_2$ be two convergent weak splittings of the second type of $A \in \mathbb{R}^{n \times n}$. If $A^{-1} \geq 0$ and $M_1 \leq M_2$, then $\rho(N_1 M_1^{-1}) \leq \rho(N_2 M_2^{-1})$.*

The notion of double splitting was first introduced by Woźnicki [50] in 1993. Later, several characterizations of double splitting were investigated by many researchers (one can refer [31], [41], and [42]). In addition to these, Song and Song [44] introduced the double nonnegative splitting to discuss the iterative solution of the nonsingular system $Ax = b$. Further, the comparison results of [44] have been extended by the authors of [26], [27], and [30]. For convenience, we have renamed the double nonnegative splitting as the double weak splitting of type I. Hence the Definition 1.3 of [44] is restated as follows.

Definition 2.12. *The splitting $A = P - R + S$ is called double weak splitting of type I of a nonsingular matrix $A \in \mathbb{R}^{n \times n}$ if $P^{-1}R \geq 0$ and $-P^{-1}S \geq 0$.*

If $A = P - R + S$ be a double weak splitting of type I of a nonsingular matrix $A \in \mathbb{R}^{n \times n}$, then the iterative solution to the system $Ax = b$, can be easily obtained from the following iterative scheme

$$x^{k+1} = P^{-1}Rx^k - P^{-1}Sx^{k-1} + P^{-1}b.$$

Further, its block matrix representation is given by

$$\begin{pmatrix} x^{k+1} \\ x^k \end{pmatrix} = \begin{pmatrix} P^{-1}R & -P^{-1}S \\ I & 0 \end{pmatrix} \begin{pmatrix} x^k \\ x^{k-1} \end{pmatrix} + \begin{pmatrix} P^{-1}b \\ 0 \end{pmatrix} = \widehat{\mathbf{W}} \begin{pmatrix} x^k \\ x^{k-1} \end{pmatrix} + \begin{pmatrix} P^{-1}b \\ 0 \end{pmatrix}, \tag{6}$$

where I is an identity matrix of order n and the iteration matrix $\widehat{\mathbf{W}}$ is,

$$\widehat{\mathbf{W}} = \begin{pmatrix} P^{-1}R & -P^{-1}S \\ I & 0 \end{pmatrix}.$$

The convergence of the above iterative scheme which was proved by Song and Song [44], is given in the following theorem.

Theorem 2.13 ([44]). *Let $A = P - R + S$ be a double weak splitting of type I of a nonsingular matrix $A \in \mathbb{R}^{n \times n}$. Then the iterative scheme defined in equation (6) converges to $A^{-1}b$ if and only if $\rho(\widehat{\mathbf{W}}) < 1$.*

Further, an equivalent characterization for a double weak splitting of type I, is stated below.

Theorem 2.14 (Theorem 2.4, **[44]).** *Let $A = P - R + S$ be a double weak splitting of type I of a nonsingular matrix $A \in \mathbb{R}^{n \times n}$. Then the following conditions are equivalent:*

  (i) $\rho(\widehat{\mathbf{W}}) < 1$.
 (ii) $\rho(P^{-1}(R - S)) < 1$.
(iii) $A^{-1}P \geq 0$.
(iv) $A^{-1}P \geq I$.

Followed by the remarkable work of Neumann [37], Jena *et al.* [22] introduced double proper splitting as follows.

A decomposition $A = P - R + S$ of $A \in \mathbb{R}^{m \times n}$ is called *double proper splitting* if $R(A) = R(P)$ and $N(A) = N(P)$.

Applying the double proper splitting $A = P - R + S$ to the system (1), we get the following iterative scheme.

$$x^{k+1} = P^\dagger R x^k - P^\dagger S x^{k-1} + P^\dagger b, \;\; k > 0.$$

Further, its block matrix form is given by

$$\begin{pmatrix} x^{k+1} \\ x^k \end{pmatrix} = \begin{pmatrix} P^\dagger R & -P^\dagger S \\ I & 0 \end{pmatrix} \begin{pmatrix} x^k \\ x^{k-1} \end{pmatrix} + \begin{pmatrix} P^\dagger b \\ 0 \end{pmatrix}. \tag{7}$$

The authors of [22] have proved that the iterative scheme (7) converges to the unique least square solution $A^\dagger b$ of (1) for any initial vectors x^0 and x^1 if the spectral radius of the iteration matrix

$$\mathbf{W} = \begin{pmatrix} P^\dagger R & -P^\dagger S \\ I & 0 \end{pmatrix} \tag{8}$$

is less than one, i.e., $\rho(\mathbf{W}) < 1$. More on the convergence of the scheme (7) concerning different types of splittings and its comparison analysis can be found in [22],[33], and [49]. In addition to these, Mishra [33] introduced the double proper nonnegative splitting which we renamed as the double proper weak splitting of type I and defined as follows.

Definition 2.15. *A decomposition $A = P - R + S$ is called a double proper weak splitting of type I if $R(A) = R(P)$, $N(A) = N(P)$, $P^\dagger R \geq 0$ and $P^\dagger S \leq 0$.*

The convergence of double proper weak splitting have proved by Mishra [33] stated below.

Theorem 2.16 (Theorem 4.5, **[33]).** *Let $A^\dagger P \geq 0$. If $A = P - R + S$ is a double proper weak splitting of type I of $A \in \mathbb{R}^{m \times n}$, then $\rho(\mathbf{W}) < 1$.*

At the end of this section, we collect few results based on the existence and the convergence of regularized system.

Theorem 2.17 (Lemma 4.2, **[3]).** *For all $A \in \mathbb{R}^{m \times n}$,*

$$\lim_{\lambda \to 0}(A^T A + \lambda I)^{-1} A^T = \lim_{\lambda \to 0} B_\lambda^{-1} A^T \; exists.$$

Theorem 2.18 (Theorem 4.3, **[3]).** *For all $A \in \mathbb{R}^{m \times n}$,*

$$\lim_{\lambda \to 0}(A^T A + \lambda I)^{-1} A^T = A^\dagger = \lim_{\lambda \to 0} B_\lambda^{-1} A^T.$$

3. Main Results

This section has three parts. In the first part of this section, we discuss some convergence and comparison results related to the weak splitting of the first type and the second type. The concept of double weak splitting of type II is introduced in the second part. In addition, several results based on double weak splitting of type II has been discussed. In the last part, we study double proper weak splitting of type I and its comparison with respect to the double weak splitting of type II.

3.1. Convergence and comparison using weak splittings

We first study the convergence of regularized iterative scheme (5) for the well-posed system (3). In view of Theorem 2.18 and Theorem 2.10, it is clear that the iterative scheme (5) converges to $A^\dagger b$ and summarized in the next result.

Theorem 3.1. *Let $A \in \mathbb{R}^{m \times n}$. For $\lambda > 0$, if $B_\lambda = M_\lambda - N_\lambda$ is a weak splitting of the first type (respectively, second type) of $B_\lambda \in \mathbb{R}^{n \times n}$ with $\lim_{\lambda \to 0} B_\lambda^{-1} N_\lambda \geq 0$ (respectively, $\lim_{\lambda \to 0} N_\lambda B_\lambda^{-1} \geq 0$), then the iterative scheme (5) converges to $B_\lambda^{-1} A^T b = A^\dagger b$ as $\lambda \to 0$.*

Due to the fact that both iterative methods (4) and (5) converge to the same least square solution $A^\dagger b$, it is better to study and analyze the spectral radius of the respective iteration matrix. Motivated by Theorem 3.11 of [2], we have an affirmative answer to these spectral radii and stated below.

Theorem 3.2. *Let $A = M - N$ be a weak proper splitting of the first type of $A \in \mathbb{R}^{m \times n}$. For $\lambda > 0$, let $B_\lambda = M_\lambda - N_\lambda$ be a weak splitting of the first type of $B_\lambda \in \mathbb{R}^{n \times n}$. If $A^\dagger N \geq \lim_{\lambda \to 0} B_\lambda^{-1} N_\lambda \geq 0$, then $\lim_{\lambda \to 0} \rho(M_\lambda^{-1} N_\lambda) \leq \rho(M^\dagger N) < 1$.*

Proof. Let $A^\dagger N \geq 0$ and $\lim_{\lambda \to 0} (B_\lambda^{-1} N_\lambda) \geq 0$. Then by Theorem 2.9 and 2.10 we obtain $\rho(M^\dagger N) < 1$ and $\lim_{\lambda \to 0} \rho(M_\lambda^{-1} N_\lambda) < 1$, respectively. By Theorem 2.4, the inequality $\rho(A^\dagger N) \geq \lim_{\lambda \to 0} \rho(B_\lambda^{-1} N_\lambda)$ follows from the assumption $A^\dagger N \geq \lim_{\lambda \to 0} (B_\lambda^{-1} N_\lambda)$. Since $\frac{\sigma}{1+\sigma}$ is a strictly increasing function in $\sigma (\geq 0)$, so we have $\frac{\rho(A^\dagger N)}{1+\rho(A^\dagger N)} \geq \lim_{\lambda \to 0} \frac{\rho(B_\lambda^{-1} N_\lambda)}{1+\rho(B_\lambda^{-1} N_\lambda)}$. In view of Theorem 2.9 and Theorem 2.10, one can conclude that $\lim_{\lambda \to 0} \rho(M_\lambda^{-1} N_\lambda) \leq \rho(M^\dagger N) < 1$. $\square$

Note that, in the above theorem, we do not assume semi-monotone condition on A as considered in [2] while comparing two nonnegative splittings. Similarly, we can show the next theorem for a weak splitting of the second type.

Theorem 3.3. *Let $A = M - N$ be a weak proper splitting of the second type of a singular matrix $A \in \mathbb{R}^{n \times n}$. For $\lambda > 0$, let $B_\lambda = M_\lambda - N_\lambda$ be a weak splitting of the second type of $B_\lambda \in \mathbb{R}^{n \times n}$. If $NA^\dagger \geq \lim_{\lambda \to 0} N_\lambda B_\lambda^{-1} \geq 0$, then $\lim_{\lambda \to 0} \rho(M_\lambda^{-1} N_\lambda) \leq \rho(M^\dagger N) < 1$.*

Further, we discuss a few comparison results by considering weak splittings of alternate types.

Theorem 3.4. *Let $A = M - N$ be a weak proper splitting of the second type of a singular semi-monotone matrix $A \in \mathbb{R}^{n \times n}$ with $NA^\dagger \geq 0$. For $\lambda > 0$, let $B_\lambda = M_\lambda - N_\lambda$ be a weak splitting of the first type of the matrix $B_\lambda \in \mathbb{R}^{n \times n}$ with $\lim_{\lambda \to 0} B_\lambda^{-1} N_\lambda \geq 0$. If $\lim_{\lambda \to 0} M_\lambda^{-1} A^T \geq M^\dagger$, then $\lim_{\lambda \to 0} \rho(M_\lambda^{-1} N_\lambda) \leq \rho(M^\dagger N) < 1$.*

Proof. Using Theorem 2.9 and 2.10, we get $\rho(M^\dagger N) < 1$ and $\lim_{\lambda \to 0} \rho(M_\lambda^{-1} N_\lambda) < 1$, respectively. By Theorem 2.5, the condition $\lim_{\lambda \to 0} (M_\lambda^{-1} A^T) \geq M^\dagger$ yields the following inequality $\lim_{\lambda \to 0} (I - M_\lambda^{-1} N_\lambda) B_\lambda^{-1} A^T \geq A^\dagger (I - NM^\dagger)$. Applying $A^\dagger = \lim_{\lambda \to 0} B_\lambda^{-1} A^T$ (from Theorem 2.18), we obtain

$$\lim_{\lambda \to 0} (B_\lambda^{-1} A^T - M_\lambda^{-1} N_\lambda B_\lambda^{-1} A^T) \geq \lim_{\lambda \to 0} B_\lambda^{-1} A^T (I - NM^\dagger). \tag{9}$$

Since $M_\lambda^{-1}N_\lambda \geq 0$, by Theorem 2.2 there exists a nonnegative eigenvector x^T such that $x^T M_\lambda^{-1}N_\lambda = \rho(M_\lambda^{-1}N_\lambda)x^T$. Taking limit $\lambda \to 0$ both sides, further it leads

$$\lim_{\lambda \to 0} x^T M_\lambda^{-1}N_\lambda = \lim_{\lambda \to 0} \rho(M_\lambda^{-1}N_\lambda)x^T. \tag{10}$$

Pre-multiplying equation (9) by x^T, we get

$$\lim_{\lambda \to 0} x^T M_\lambda^{-1}N_\lambda B_\lambda^{-1}A^T \leq \lim_{\lambda \to 0} x^T B_\lambda^{-1}A^T N M^\dagger. \tag{11}$$

Equation (10) and (11) yields $\lim_{\lambda \to 0} \rho(M_\lambda^{-1}N_\lambda)z_\lambda^T \leq z_\lambda^T N M^\dagger$, where $z_\lambda^T = \lim_{\lambda \to 0} x^T B_\lambda^{-1}A^T$. Taking transpose, we obtain

$$\lim_{\lambda \to 0} \rho(M_\lambda^{-1}N_\lambda)z_\lambda \leq (NM^\dagger)^T z_\lambda. \tag{12}$$

Now $z_\lambda^T = x^T \lim_{\lambda \to 0} B_\lambda^{-1}A^T = x^T A^\dagger \geq 0$. If $z_\lambda^T = 0$, then $\lim_{\lambda \to 0} x^T B_\lambda^{-1}A^T = 0$. Further, $0 = \lim_{\lambda \to 0} A(B_\lambda^{-1})^T x = \lim_{\lambda \to 0} A^T A(B_\lambda^{-1})^T x = \lim_{\lambda \to 0}(A^T A + \lambda I)(B_\lambda^{-1})^T x = \lim_{\lambda \to 0}(A^T A + \lambda I)^T(B_\lambda^{-1})^T x = \lim_{\lambda \to 0} B_\lambda^T(B_\lambda^{-1})^T x = x$, which is a contradiction. Hence $z_\lambda^T > 0$. Applying Theorem 2.1 to equation (12), we conclude that

$$\lim_{\lambda \to 0} \rho(M_\lambda^{-1}N_\lambda) \leq \rho(NM^\dagger)^T = \rho(NM^\dagger) = \rho(M^\dagger N) < 1.$$

$\square$

The semi-monotone condition given in the Theorem 3.4 can be relaxed as discussed in the next result.

Theorem 3.5. *Let $A = M - N$ be a weak proper splitting of the second type with $NA^\dagger \geq 0$. For $\lambda > 0$, suppose $B_\lambda = M_\lambda - N_\lambda$ is a weak splitting of the first type of the matrix B_λ with $\lim_{\lambda \to 0} B_\lambda^{-1}N_\lambda \geq 0$. If $\lim_{\lambda \to 0} M_\lambda^{-1}N_\lambda A^\dagger \leq A^\dagger N M^\dagger$, then $\lim_{\lambda \to 0} \rho(M_\lambda^{-1}N_\lambda) \leq \rho(M^\dagger N) < 1$.*

The proof will go a similar way as in Theorem 3.4.

Similarly, we can show the following result for the same type of weak splittings.

Lemma 3.6. *Let $A = M - N$ be a weak proper splitting of the first type of a singular matrix $A \in \mathbb{R}^{n \times n}$ with $A^\dagger N \geq 0$. For $\lambda > 0$, let $B_\lambda = M_\lambda - N_\lambda$ be a weak splitting of the first type of the matrix B_λ with $\lim_{\lambda \to 0} B_\lambda^{-1}N_\lambda \geq 0$. If $\lim_{\lambda \to 0} M_\lambda^{-1}N_\lambda A^\dagger \leq A^\dagger M^\dagger N$, then $\lim_{\lambda \to 0} \rho(M_\lambda^{-1}N_\lambda) \leq \rho(M^\dagger N) < 1$.*

Next, we discuss another comparison theorem for different pair of weak splittings.

Theorem 3.7. *Let $A = M - N$ be a weak proper splitting of the first type of $A \in \mathbb{R}^{m \times n}$ with $A^\dagger N \geq 0$. For $\lambda > 0$, let $B_\lambda = M_\lambda - N_\lambda$ be a weak splitting of the second type of the matrix $B_\lambda \in \mathbb{R}^{n \times n}$ with $\lim_{\lambda \to 0} N_\lambda B_\lambda^{-1} \geq 0$. If $\lim_{\lambda \to 0} N_\lambda M_\lambda^{-1} \leq M^\dagger N$, then $\lim_{\lambda \to 0} \rho(M_\lambda^{-1}N_\lambda) \leq \rho(M^\dagger N) < 1$.*

Proof. By Theorem 2.9 and Theorem 2.10, $\rho(M^\dagger N) < 1$ and $\lim_{\lambda \to 0} \rho(M_\lambda^{-1}N_\lambda) < 1$, respectively. The splitting of the matrix $B_\lambda = M_\lambda - N_\lambda$ gives $M_\lambda = (I + N_\lambda B_\lambda^{-1})B_\lambda$. Hence $M_\lambda^{-1} = B_\lambda^{-1}(I + N_\lambda B_\lambda^{-1})^{-1}$. Applying Theorem 2.6 (*ii*) to the proper splitting $A = M - N$, we get $M^\dagger = (I + A^\dagger N)^{-1}A^\dagger$. Therefore, the condition $\lim_{\lambda \to 0} N_\lambda M_\lambda^{-1} \leq M^\dagger N$ implies

$$\lim_{\lambda \to 0}(N_\lambda B_\lambda^{-1}(I + N_\lambda B_\lambda^{-1})^{-1}) \leq (I + A^\dagger N)^{-1}A^\dagger N. \tag{13}$$

Since $I + A^\dagger N \geq 0$ and $\lim_{\lambda \to 0}(I + N_\lambda B_\lambda^{-1}) \geq 0$, so pre-multiplying $I + A^\dagger N$ and post-multiplying $\lim_{\lambda \to 0}(I + N_\lambda B_\lambda^{-1})$ to the equation (13), we obtain

$$\lim_{\lambda \to 0}(I + A^\dagger N)N_\lambda B_\lambda^{-1} \leq \lim_{\lambda \to 0} A^\dagger N(I + N_\lambda B_\lambda^{-1}). \tag{14}$$

Equation (14) leads to $\lim_{\lambda \to 0} N_\lambda B_\lambda^{-1} \leq \lim_{\lambda \to 0} A^\dagger N = A^\dagger N$. By Theorem 2.4, we have $\rho(A^\dagger N) \geq \lim_{\lambda \to 0} \rho(N_\lambda B_\lambda^{-1}) \geq 0$. As $\frac{\gamma}{\gamma+1}$ is a strictly increasing function for every $\gamma \geq 0$, hence $\frac{\rho(A^\dagger N)}{1+\rho(A^\dagger N)} \geq \lim_{\lambda \to 0} \frac{\rho(N_\lambda B_\lambda^{-1})}{1+\rho(N_\lambda B_\lambda^{-1})}$. Again, by Theorem 2.9 and Theorem 2.10, we get $\lim_{\lambda \to 0} \rho(M_\lambda^{-1}N_\lambda) \leq \rho(M^\dagger N) < 1$. $\square$

3.2. Double weak splitting of type II

Motivated by the work of the authors [26], [27], [30] and [44], we have introduced the double weak splitting of type II for symmetric matrices. In connection to double weak splitting of type II, we have extended a few results of [44]. Further, some comparison theorems for double weak splitting of type I and type II have been established in this subsection. First, we define the double weak splitting of type II as follows.

Definition 3.8. *The splitting $A = P - R + S$ is called double weak splitting of type II of a symmetric nonsingular matrix $A \in \mathbb{R}^{n \times n}$ if $RP^{-1} \geq 0$ and $-SP^{-1} \geq 0$.*

Suppose $A = P - R + S$ be a double weak splitting of type II of a symmetric nonsingular matrix $A \in \mathbb{R}^{n \times n}$. The iterative scheme of (3) corresponding to such type of splitting is

$$x^{k+1} = (RP^{-1})^T x^k - (SP^{-1})^T x^{k-1} + (P^{-1})^T b,$$

and its block matrix representation is given by

$$
\begin{aligned}
\begin{pmatrix} x^{k+1} \\ x^k \end{pmatrix} &= \begin{pmatrix} (RP^{-1})^T & -(SP^{-1})^T \\ I & 0 \end{pmatrix} \begin{pmatrix} x^k \\ x^{k-1} \end{pmatrix} + \begin{pmatrix} (P^{-1})^T b \\ 0 \end{pmatrix} \\
&= \widetilde{\mathbf{W}} \begin{pmatrix} x^k \\ x^{k-1} \end{pmatrix} + \begin{pmatrix} (P^{-1})^T b \\ 0 \end{pmatrix},
\end{aligned}
\tag{15}
$$

where I is an identity matrix of order n and the iteration matrix $\widetilde{\mathbf{W}}$ is,

$$\widetilde{\mathbf{W}} = \begin{pmatrix} (RP^{-1})^T & -(SP^{-1})^T \\ I & 0 \end{pmatrix}.$$

Next, we recall the following result from Song and Song [44].

Theorem 3.9 (Theorem 2.2, [44]). *Let $A = P - R + S$ be a double weak splitting of type I of the nonsingular matrix $A \in \mathbb{R}^{n \times n}$. Then the double splitting is convergent if and only if $\rho(P^{-1}(R - S)) < 1$.*

In regard to Theorem 3.9, we have the following convergence theorem for double weak splitting of type II.

Theorem 3.10. *Let $A = P - R + S$ be a double weak splitting of type II of a symmetric nonsingular matrix $A \in \mathbb{R}^{n \times n}$. Then $\rho(\widetilde{\mathbf{W}}) < 1$ if and only if $\rho((R - S)P^{-1}) = \rho(P^{-1}(R - S)) < 1$.*

Proof. Let $A = P - R + S$ be a double weak splitting of type II. Then $(RP^{-1})^T \geq 0$ and $-(SP^{-1})^T \geq 0$. Hence $\widetilde{\mathbf{W}} \geq 0$. By proceeding similar lines of the proof of Theorem 3.9, we can prove the theorem. □

In view of Lemma 2.7 of [46] and Theorem 3.10, we can show the following result.

Theorem 3.11. *Let $A = P - R + S$ be a double weak splitting of type II of the symmetric nonsingular matrix $A \in \mathbb{R}^{n \times n}$. Then the following conditions are equivalent:*

(i) $\rho(\widetilde{\mathbf{W}}) < 1$.
(ii) $PA^{-1} \geq 0 \ (A^{-1}P^T \geq 0)$.
(iii) $PA^{-1} \geq I \ (A^{-1}P^T \geq I)$.
(iv) $(R - S)A^{-1} \geq 0 \ (A^{-1}(R - S)^T \geq 0)$.
(v) $(R - S)A^{-1} \geq -I \ (A^{-1}(R - S)^T \geq -I)$.
(vi) $(I - (R - S)^T (P^{-1})^T)^{-1} \geq 0$.
(vii) $(I - (R - S)^T (P^{-1})^T)^{-1} \geq I$.

If we consider $A = P_1 - R_1 + S_1 = P_2 - R_2 + S_2$ be two double weak splittings of a symmetric nonsingular matrix $A \in \mathbb{R}^{n \times n}$, then the respective iteration block matrices are

$$\widetilde{\mathbf{W}}_1 = \begin{pmatrix} (R_1 P_1^{-1})^T & -(S_1 P_1^{-1})^T \\ I & 0 \end{pmatrix} \text{ and } \widetilde{\mathbf{W}}_2 = \begin{pmatrix} (R_2 P_2^{-1})^T & -(S_2 P_2^{-1})^T \\ I & 0 \end{pmatrix}.$$

To analyze the spectral radius of both iteration matrices $\widetilde{\mathbf{W}}_1$ and $\widetilde{\mathbf{W}}_2$, we follow analogous to Song and Song [44]. We first define $\mathbb{A} = \begin{pmatrix} A & -I \\ 0 & I \end{pmatrix}$. Then it is easy to verify $\mathbb{A}^{-1} = \begin{pmatrix} A^{-1} & A^{-1} \\ 0 & I \end{pmatrix}$. Further, consider $\mathbb{A} = \mathbb{M}_i - \mathbb{N}_i$ be two splitting of $\mathbb{A} \in \mathbb{R}^{2n \times 2n}$. If we take

$$\mathbb{M}_i = \begin{pmatrix} P_i & 0 \\ -S_i & I \end{pmatrix} \text{ and } \mathbb{N}_i = \begin{pmatrix} R_i - S_i & I \\ -S_i & 0 \end{pmatrix}, \tag{16}$$

then we can show that

$$\widetilde{\mathbf{W}}_i = (\mathbb{N}_i \mathbb{M}_i^{-1})^T, \text{ for } i = 1, 2. \tag{17}$$

We recall the comparison theorem of [44] which was proved for the double weak splitting of type I.

Theorem 3.12 (Theorem 3.3,**[44]).** *Let $A = P_1 - R_1 + S_1 = P_2 - R_2 + S_2$ be two convergent and double weak splittings of type I of the monotone matrix $A \in \mathbb{R}^{n \times n}$. If $P_1 \leq P_2$ and $S_2 \leq S_1$, then $\rho(\widehat{\mathbf{W}}_1) \leq \rho(\widehat{\mathbf{W}}_2) < 1$.*

Using similar lines of Theorem 3.12, we can show the following result for double weak splitting of type II.

Theorem 3.13. *Let $A = P_1 - R_1 + S_1 = P_2 - R_2 + S_2$ be two convergent double weak splittings of type II of a symmetric monotone matrix $A \in \mathbb{R}^{n \times n}$. If $P_1 \leq P_2$ and $S_2 \leq S_1$, then $\rho(\widetilde{\mathbf{W}}_1) \leq \rho(\widetilde{\mathbf{W}}_2) < 1$.*

Theorem 2.11 of [46] motivated us to study the above comparison theorem without considering the monotone condition and established the following result.

Theorem 3.14. *Let $A = P_1 - R_1 + S_1 = P_2 - R_2 + S_2$ be two double weak splittings of type II of a symmetric matrix $A \in \mathbb{R}^{n \times n}$. If $P_2 A^{-1} \geq P_1 A^{-1} \geq 0$ and $S_2 A^{-1} \leq S_1 A^{-1} \leq 0$, then $\rho(\widetilde{\mathbf{W}}_1) \leq \rho(\widetilde{\mathbf{W}}_2) < 1$.*

Proof. From the conditions $P_1 A^{-1} \geq 0, P_2 A^{-1} \geq 0$ and Theorem 3.11, it is trivial that $\rho(\widetilde{\mathbf{W}}_1) < 1$ and $\rho(\widetilde{\mathbf{W}}_2) < 1$. Since $P_2 A^{-1} \geq P_1 A^{-1} \geq 0$ and $S_2 A^{-1} \leq S_1 A^{-1} \leq 0$, so we have

$$\begin{pmatrix} P_2 A^{-1} & P_2 A^{-1} \\ -S_2 A^{-1} & -S_2 A^{-1} + I \end{pmatrix} \geq \begin{pmatrix} P_1 A^{-1} & P_1 A^{-1} \\ -S_1 A^{-1} & -S_1 A^{-1} + I \end{pmatrix} \geq 0.$$

Further, we can write as

$$\begin{pmatrix} P_2 & 0 \\ -S_2 & I \end{pmatrix} \begin{pmatrix} A^{-1} & A^{-1} \\ 0 & I \end{pmatrix} \geq \begin{pmatrix} P_1 & 0 \\ -S_1 & I \end{pmatrix} \begin{pmatrix} A^{-1} & A^{-1} \\ 0 & I \end{pmatrix} \geq 0.$$

Hence by equation (16), $\mathbb{M}_2 \mathbb{A}^{-1} \geq \mathbb{M}_1 \mathbb{A}^{-1} \geq 0$. Therefore, by Theorem 2.11, the splittings $\mathbb{A} = \mathbb{M}_1 - \mathbb{N}_1 = \mathbb{M}_2 - \mathbb{N}_2$ yield $\rho(\mathbb{N}_1 \mathbb{M}_1^{-1}) \leq \rho(\mathbb{N}_2 \mathbb{M}_2^{-1}) < 1$. Thus by equation (17), $\rho(\widetilde{\mathbf{W}}_1) \leq \rho(\widetilde{\mathbf{W}}_2) < 1$. $\square$

In support of Theorem 3.14, the following example is worked-out.

Example 3.15. Consider

$$A = \begin{bmatrix} 10 & -4 \\ -4 & 6 \end{bmatrix} = \begin{bmatrix} 12 & 0 \\ 0 & 8 \end{bmatrix} - \begin{bmatrix} 2 & 2 \\ 4 & 2 \end{bmatrix} + \begin{bmatrix} 0 & -2 \\ 0 & 0 \end{bmatrix} = P_1 - R_1 + S_1$$

$$= \begin{bmatrix} 16 & 0 \\ 0 & 10 \end{bmatrix} - \begin{bmatrix} 6 & 2 \\ 0 & 4 \end{bmatrix} + \begin{bmatrix} 0 & -2 \\ -4 & 0 \end{bmatrix} = P_2 - R_2 + S_2$$

be two convergent double weak splitting of type II of the matrix A. One can verify that

$$P_2 A^{-1} = \begin{bmatrix} 2.1818 & 1.4545 \\ 0.9091 & 2.2727 \end{bmatrix} > \begin{bmatrix} 1.6364 & 1.0909 \\ 0.7273 & 1.8182 \end{bmatrix} = P_1 A^{-1} > 0,$$

$$S_2 A^{-1} = \begin{bmatrix} -0.1818 & -0.4545 \\ -0.5455 & -0.3636 \end{bmatrix} \le \begin{bmatrix} -0.1818 & -0.4545 \\ 0 & 0 \end{bmatrix} = S_1 A^{-1} \le 0, \text{ and}$$

$0.6667 = \rho(\widetilde{W}_1) < \rho(\widetilde{W}_2) = 0.7729 < 1$.

Another comparison theorem for symmetric nonsingular matrices presented below.

Theorem 3.16. *Let $A = P_1 - R_1 + S_1 = P_2 - R_2 + S_2$ be two convergent double weak splitting of type II of a symmetric matrix $A \in \mathbb{R}^{n \times n}$. If $R_1 P_1^{-1} \ge R_2 P_2^{-1}$ and $A P_1^{-1} \ge A P_2^{-1}$, then $\rho(\widetilde{W}_1) \le \rho(\widetilde{W}_2) < 1$.*

Proof. If $\rho(\widetilde{W}_1) = 0$, then it is trivial. Assume that $0 < \rho(\widetilde{W}_1) < 1$. Since $\widetilde{W}_1 \ge 0$, so by Theorem 2.2, there exists an eigenvector $x = (x_1, x_2)^T$ such that $\widetilde{W}_1 x = \rho(\widetilde{W}_1) x$. Which implies

$$(R_1 P_1^{-1})^T x_1 - (S_1 P_1^{-1})^T x_2 = \rho(\widetilde{W}_1) x_1 \text{ and } x_1 = \rho(\widetilde{W}_1) x_2. \tag{18}$$

Now

$$(R_2 P_2^{-1})^T x_1 - (S_2 P_2^{-1})^T x_2 - \rho(\widetilde{W}_1) x_1$$

$$= (R_2 P_2^{-1})^T x_1 - \frac{1}{\rho(\widetilde{W}_1)} (S_2 P_2^{-1})^T x_1 - (R_1 P_1^{-1})^T x_1 + \frac{1}{\rho(\widetilde{W}_1)} (S_1 P_1^{-1})^T x_1$$

$$\ge \frac{1}{\rho(\widetilde{W}_1)} [(R_2 P_2^{-1})^T - (R_1 P_1^{-1})^T + (S_1 P_1^{-1})^T - (S_2 P_2^{-1})^T] x_1$$

$$= \frac{1}{\rho(\widetilde{W}_1)} [(P_2^{-1})^T (R_2^T - S_2^T) + (P_1^{-1})^T (S_1^T - R_1^T)] x_1$$

$$= \frac{1}{\rho(\widetilde{W}_1)} [(P_2^{-1})^T (P_2^T - A) + (P_1^{-1})^T (A - P_1^T)] x_1$$

$$= \frac{1}{\rho(\widetilde{W}_1)} [(P_1^{-1})^T A - (P_2^{-1})^T A] x_1 \ge 0. \tag{19}$$

Using equations (18) and (19), we obtain

$$\widetilde{W}_2 x - \rho(\widetilde{W}_1) x = \begin{pmatrix} (R_2 P_2^{-1})^T x_1 - (S_2 P_2^{-1})^T x_2 - \rho(\widetilde{W}_1) x_1 \\ x_1 - \rho(\widetilde{W}_1) x_2 \end{pmatrix} \ge 0.$$

Hence by Theorem 2.1, $\rho(\widetilde{W}_1) \le \rho(\widetilde{W}_2) < 1$. $\square$

The converse of the Theorem 3.16 need not true in general as shown in the below example.

Example 3.17. Consider the matrices A, P_1, R_1, S_1, P_2, R_2, and S_2 as given in Example 3.15. Clearly $0.6667 = \rho(\widetilde{W}_1) < \rho(\widetilde{W}_2) = 0.7729 < 1$ but

$$R_1 P_1^{-1} - R_2 P_2^{-1} = \begin{bmatrix} -0.2083 & 0.0500 \\ 0.3333 & -0.1500 \end{bmatrix} \ngeq 0, A P_1^{-1} - A P_2^{-1} = \begin{bmatrix} 0.2083 & -0.1000 \\ -0.0833 & 0.1500 \end{bmatrix} \ngeq 0$$

On account of Theorem 3.6 [44] and equations (16) and (17), the following comparison theorem similarly follows.

Theorem 3.18. *Let $A = P_1 - R_1 + S_1 = P_2 - R_2 + S_2$ be two convergent double weak splittings of type II of the nonsingular symmetric matrix $A \in \mathbb{R}^{n \times n}$. If any one of the following conditions*

(i) $P_2 P_1^{-1} \geq I$ *and* $S_1 P_1^{-1} \geq S_2 P_1^{-1}$,
(ii) $P_1 P_2^{-1} \leq I$ *and* $S_1 P_2^{-1} \geq S_2 P_2^{-1}$,

holds, then $\rho(\widetilde{\mathbf{W}}_1) \leq \rho(\widetilde{\mathbf{W}}_2) < 1$.

Analogous to the Theorem 3.1 of [31], Corollary 4.10 of [33], and the proof of Theorem 3.16, we obtain the below result for double weak splittings of type II.

Theorem 3.19. *Let* $A_1 = P_1 - R_1 + S_1$ *and* $A_2 = P_2 - R_2 + S_2$ *be two convergent double weak splitting of type II of the nonsingular symmetric matrices* $A_1 \in \mathbb{R}^{n \times n}$ *and* $A_2 \in \mathbb{R}^{n \times n}$, *respectively. If* $R_1 P_1^{-1} \geq R_2 P_2^{-1}$ *and* $A_1 P_1^{-1} \geq A_2 P_2^{-1}$, *then* $\rho(\widetilde{\mathbf{W}}_1) \leq \rho(\widetilde{\mathbf{W}}_2) < 1$.

Similarly, the following results can be proved by considering a double weak splitting of type I for the matrices A_1 and A_2.

Theorem 3.20. *Let* $A_1 = P_1 - R_1 + S_1$ *be a convergent double weak splitting of type II of the nonsingular symmetric matrix* $A_1 \in \mathbb{R}^{n \times n}$ *and* $A_2 = P_2 - R_2 + S_2$ *be a convergent double weak splitting of type I of the nonsingular symmetric matrix* $A_2 \in \mathbb{R}^{n \times n}$. *If* $P_2^{-1} R_2 \leq (R_1 P_1^{-1})^T$ *and* $P_2^{-1} A_2 \leq (P_1^{-1})^T A_1$, *then* $\rho(\widetilde{\mathbf{W}}_1) \leq \rho(\widehat{\mathbf{W}}_2) < 1$.

Exchanging the splitting type in Theorem 3.20, we obtain the below result.

Theorem 3.21. *Let* $A_1 = P_1 - R_1 + S_1$ *be a convergent double weak splitting of type I of the nonsingular symmetric matrix* $A_1 \in \mathbb{R}^{n \times n}$ *and* $A_2 = P_2 - R_2 + S_2$ *be a convergent double weak splitting of type II of the nonsingular symmetric matrix* $A_2 \in \mathbb{R}^{n \times n}$. *If* $(R_2 P_2^{-1})^T \leq P_1^{-1} R_1$ *and* $(P_2^{-1})^T A_2 \leq P_1^{-1} A_1$, *then* $\rho(\widehat{\mathbf{W}}_1) \leq \rho(\widetilde{\mathbf{W}}_2) < 1$.

The convergence and some of the comparison results of the double weak splitting of type II can be found in [40].

3.3. Convergence and comparison using double weak splittings

In this subsection, we introduce a regularized iterative scheme for the well-posed system (3) based on double weak splittings. In addition, the convergence of the regularized scheme is established. Further, a few comparison theorems for the systems (1) and (3) are analyzed with the help of double weak splittings.

Let $B_\lambda = P_\lambda - R_\lambda + S_\lambda$ be a double splitting (introduced by Woźnicki [50]) of the nonsingular matrix $B_\lambda \in \mathbb{R}^{n \times n}$. If P_λ is invertible, then regularized iterative scheme corresponding to the double splitting $B_\lambda = P_\lambda - R_\lambda + S_\lambda$ is given by

$$x^{k+1} = P_\lambda^{-1} R_\lambda x^k - P_\lambda^{-1} S_\lambda x^{k-1} + P_\lambda^{-1} A^T b.$$

Further, its block matrix form is

$$\begin{pmatrix} x^{k+1} \\ x^k \end{pmatrix} = \begin{pmatrix} P_\lambda^{-1} R_\lambda & -P_\lambda^{-1} S_\lambda \\ I & 0 \end{pmatrix} \begin{pmatrix} x^k \\ x^{k-1} \end{pmatrix} + \begin{pmatrix} P_\lambda^{-1} A^T b \\ 0 \end{pmatrix} = \mathbf{W}_\lambda \begin{pmatrix} x^k \\ x^{k-1} \end{pmatrix} + \begin{pmatrix} P_\lambda^{-1} A^T b \\ 0 \end{pmatrix}, \tag{20}$$

where I is an identity matrix of order n and $\mathbf{W}_\lambda = \begin{pmatrix} P_\lambda^{-1} R_\lambda & -P_\lambda^{-1} S_\lambda \\ I & 0 \end{pmatrix}$ is the iteration matrix.

Ensuing the idea of Golub *et al.* [13] and [41], the iterative scheme (20) for the system $B_\lambda x = A^T b$, converges to $B_\lambda^{-1} A^T b$ (= $A^\dagger b$ as $\lambda \to 0$) for any initial vectors x^0 and x^1 if and only if $\lim_{\lambda \to 0} \rho(\mathbf{W}_\lambda) < 1$.

In case of convergent double weak splittings of type I or type II and by virtue of Theorem 2.16, the convergence of (5) follows easily and presented below.

Theorem 3.22. *For any matrix* $A \in \mathbb{R}^{m \times n}$ *and* $\lambda > 0$, *let* $B_\lambda = P_\lambda - R_\lambda + S_\lambda$ *be a double weak splitting of type I (respectively, type II) with* $\lim_{\lambda \to 0} B_\lambda^{-1} P_\lambda \geq 0$ *(respectively,* $\lim_{\lambda \to 0} P_\lambda B_\lambda^{-1} \geq 0$*), then the iterative scheme (20) converges to* $B_\lambda^{-1} A^T b$ (= $A^\dagger b$ *as* $\lambda \to 0$*).*

Under the suitable sufficient condition, the following theorem signifies that the splitting of B_λ will converge faster (in terms of spectral radius) than the splitting of the original matrix A.

Theorem 3.23. *Let $A = P - R + S$ be a double proper weak splitting of type I of $A \in \mathbb{R}^{m \times n}$ with $A^\dagger P \geq 0$. For $\lambda > 0$, let $B_\lambda = P_\lambda - R_\lambda + S_\lambda$ be a double weak splitting of type I with $B_\lambda^{-1} P_\lambda \geq 0$. If any one of the following conditions holds*

(i) $P^\dagger R \geq \lim_{\lambda \to 0} P_\lambda^{-1} R_\lambda$ and $\lim_{\lambda \to 0} P_\lambda^{-1} S_\lambda \geq P^\dagger S$,

(ii) $P^\dagger (R - S) \geq I$,

then $\lim_{\lambda \to 0} \rho(\mathbf{W}_\lambda) \leq \rho(\mathbf{W}) < 1$.

Proof. By Theorem 2.14 and Theorem 2.16, it is clear that $\lim_{\lambda \to 0} \rho(\mathbf{W}_\lambda) < 1$ and $\rho(\mathbf{W}) < 1$. If $\lim_{\lambda \to 0} \rho(\mathbf{W}_\lambda) = 0$, then the theorem is trivial. Let us assume that $0 < \lim_{\lambda \to 0} \rho(\mathbf{W}_\lambda) < 1$. Since $\lim_{\lambda \to 0} \mathbf{W}_\lambda > 0$, by Theorem 2.2 there exists a vector $x(\neq 0) \in \mathbb{R}^{2n}$ such that $\lim_{\lambda \to 0} \mathbf{W}_\lambda x = \lim_{\lambda \to 0} \rho(\mathbf{W}_\lambda) x$. This implies

$$\lim_{\lambda \to 0} P_\lambda^{-1} R_\lambda x_1 - \lim_{\lambda \to 0} P_\lambda^{-1} S_\lambda x_2 = \lim_{\lambda \to 0} \rho(\mathbf{W}_\lambda) x_1 \text{ and } x_1 = \lim_{\lambda \to 0} \rho(\mathbf{W}_\lambda) x_2.$$

Now

$$\mathbf{W}x - \lim_{\lambda \to 0} \rho(\mathbf{W}_\lambda) x = \begin{pmatrix} P^\dagger R x_1 - P^\dagger S x_2 - \lim_{\lambda \to 0} \rho(\mathbf{W}_\lambda) x_1 \\ x_1 - \lim_{\lambda \to 0} \rho(\mathbf{W}_\lambda) x_2 \end{pmatrix} = \begin{pmatrix} \Delta_1 \\ 0 \end{pmatrix},$$

where $\Delta_1 = P^\dagger R x_1 - P^\dagger S x_2 - \lim_{\lambda \to 0} \rho(\mathbf{W}_\lambda) x_1$.
Case-I: Let $P^\dagger R \geq \lim_{\lambda \to 0} P_\lambda^{-1} R_\lambda$ and $\lim_{\lambda \to 0} P_\lambda^{-1} S_\lambda \geq P^\dagger S$. Then

$$
\begin{aligned}
\Delta_1 &= P^\dagger R x_1 - P^\dagger S x_2 - \lim_{\lambda \to 0} \rho(\mathbf{W}_\lambda) x_1 \\
&= P^\dagger R x_1 - \lim_{\lambda \to 0} \frac{1}{\rho(\mathbf{W}_\lambda)} P^\dagger S x_1 - \lim_{\lambda \to 0} P_\lambda^{-1} R_\lambda x_1 + \lim_{\lambda \to 0} \frac{1}{\rho(\mathbf{W}_\lambda)} P_\lambda^{-1} S_\lambda x_1 \\
&= [(P^\dagger R - \lim_{\lambda \to 0} P_\lambda^{-1} R_\lambda) + \lim_{\lambda \to 0} \frac{1}{\rho(\mathbf{W}_\lambda)} (P_\lambda^{-1} S_\lambda - P^\dagger S)] x_1 \\
&\geq 0.
\end{aligned}
$$

Hence $\mathbf{W}x - \lim_{\lambda \to 0} \rho(\mathbf{W}_\lambda) x \geq 0$.
Case-II: Let $P^\dagger (R - S) \geq I$. Then

$$
\begin{aligned}
\Delta_1 &= P^\dagger R x_1 - P^\dagger S x_2 - \lim_{\lambda \to 0} \rho(\mathbf{W}_\lambda) x_1 \\
&= \lim_{\lambda \to 0} \rho(\mathbf{W}_\lambda) P^\dagger R x_2 - P^\dagger S x_2 - \lim_{\lambda \to 0} \rho(\mathbf{W}_\lambda)^2 x_2 \\
&\geq \lim_{\lambda \to 0} \rho(\mathbf{W}_\lambda)^2 P^\dagger R x_2 - \lim_{\lambda \to 0} \rho(\mathbf{W}_\lambda)^2 P^\dagger S x_2 - \lim_{\lambda \to 0} \rho(\mathbf{W}_\lambda)^2 x_2 \\
&= \lim_{\lambda \to 0} \rho(\mathbf{W}_\lambda)^2 [P^\dagger (R - S) - I] x_2 \\
&\geq 0
\end{aligned}
$$

In both cases, we obtain $\mathbf{W}x - \lim_{\lambda \to 0} \rho(\mathbf{W}_\lambda) x \geq 0$. Thus by Theorem 2.1, $\lim_{\lambda \to 0} \rho(\mathbf{W}_\lambda) \leq \rho(\mathbf{W}) < 1$. $\square$

In the next result, we discuss a comparison theorem for considering a double weak splitting of type II for the regularized matrix B_λ.

Theorem 3.24. *Let $A = P - R + S$ be a convergent double proper weak splitting of type I of the singular symmetric matrix $A \in \mathbb{R}^{n \times n}$. For $\lambda > 0$, let $B_\lambda = P_\lambda - R_\lambda + S_\lambda$ be a convergent double weak splitting of type II of the nonsingular matrix B_λ. If $\lim_{\lambda \to 0} (R_\lambda P_\lambda^{-1})^T \geq P^\dagger R$, $\lim_{\lambda \to 0} (P_\lambda^{-1})^T B_\lambda^T \geq P^\dagger A$, $P^\dagger R > 0$ and $-P^\dagger S > 0$, then $\lim_{\lambda \to 0} \rho(\widetilde{\mathbf{W}}_\lambda) \leq \rho(\mathbf{W}) < 1$.*

Proof. If $\rho(\mathbf{W}) = 0$, then it is trivial. Assume that $0 < \rho(\mathbf{W}) < 1$. From $P^\dagger R > 0$ and $-P^\dagger S > 0$ and Lemma 3.1 of [41], $\mathbf{W}$ is irreducible. By Theorem 2.3 there exists a positive vector $x = (x_1, x_2)^T \in \mathbb{R}^{2n}$ such that $\mathbf{W}x = \rho(\mathbf{W})x$. This leads to

$$P^\dagger R x_1 - P^\dagger S x_2 = \rho(\mathbf{W})x_1 \text{ and } x_1 = \rho(\mathbf{W})x_2. \tag{21}$$

From equation (21), it is clear that $x_1 \in R(P^\dagger) = R(P^T)$. Now the iteration matrix corresponding to the double weak splitting of type II ($B_\lambda = P_\lambda - R_\lambda + S_\lambda$) is

$$\widetilde{\mathbf{W}}_\lambda = \begin{pmatrix} (R_\lambda P_\lambda^{-1})^T & -(S_\lambda P_\lambda^{-1})^T \\ I & 0 \end{pmatrix}.$$

Using $\widetilde{\mathbf{W}}_\lambda$ and equation(21), we have

$$\lim_{\lambda \to 0} \widetilde{\mathbf{W}}_\lambda x - \rho(\mathbf{W})x = \begin{pmatrix} \lim_{\lambda \to 0}(R_\lambda P_\lambda^{-1})^T x_1 - \lim_{\lambda \to 0}(S_\lambda P_\lambda^{-1})^T x_2 - \rho(\mathbf{W})x_1 \\ x_1 - \rho(\mathbf{W})x_2 \end{pmatrix}$$

$$= \begin{pmatrix} (\lim_{\lambda \to 0}(R_\lambda P_\lambda^{-1})^T - P^\dagger R)x_1 - \frac{1}{\rho(\mathbf{W})}(\lim_{\lambda \to 0}(S_\lambda P_\lambda^{-1})^T - P^\dagger S)x_1 \\ 0 \end{pmatrix}.$$

Applying the condition $\lim_{\lambda \to 0}(R_\lambda P_\lambda^{-1})^T \geq P^\dagger R$ and $P^\dagger A - \lim_{\lambda \to 0}(P_\lambda^{-1})^T B_\lambda^T \leq 0$, we obtain

$$\lim_{\lambda \to 0} \widetilde{\mathbf{W}}_\lambda x - \rho(\mathbf{W})x \leq \frac{1}{\rho(\mathbf{W})}(\lim_{\lambda \to 0}(R_\lambda P_\lambda^{-1})^T - P^\dagger R)x_1 - \frac{1}{\rho(\mathbf{W})}(\lim_{\lambda \to 0}(S_\lambda P_\lambda^{-1})^T - P^\dagger S)x_1$$

$$= \frac{1}{\rho(\mathbf{W})} \lim_{\lambda \to 0}[(P_\lambda^{-1})^T(P_\lambda^T - B_\lambda^T) + P^\dagger(A - P)]x_1$$

$$= \frac{1}{\rho(\mathbf{W})}(x_1 - \lim_{\lambda \to 0}(P_\lambda^{-1})^T B_\lambda^T x_1 + P^\dagger A x_1 - x_1)$$

$$= \frac{1}{\rho(\mathbf{W})}(P^\dagger A - \lim_{\lambda \to 0}(P_\lambda^{-1})^T B_\lambda^T)x_1 \leq 0.$$

Hence by Theorem 2.1, we get $\lim_{\lambda \to 0} \rho(\widetilde{\mathbf{W}}_\lambda) \leq \rho(\mathbf{W}) < 1$. $\quad\square$

4. Numerical Example

In this section, we discuss the solution of an elliptic partial differential equation by using the proposed regularized iterative methods. The performance which measures are the number of iterations (IT), residue and the error bounds. The following stopping criteria is used to terminate the iteration: $\|x_k - x_{k-1}\|_2 \leq \epsilon$ or the maximum allowed iterations 4000. The symbol $(-)$ signifies the iterative scheme does not converge within the maximum allowed iteration, whereas $O(A)$ means the order of the matrix A. In Table 1, $C(B_\lambda)$ and $C(A)$ represents the condition number of the respective matrices.

Example 4.1. ([43], [28]) Let us assume a two-dimensional Poisson's equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x, y), \ (x, y) \in \Omega = (0, 1) \times (0, 1)$$

with Neumann boundary conditions. Applying the central difference scheme to the above problem yields a linear system $Ax = b$, where A is singular matrix of order $(n+1)^2 \times (n+1)^2$. The construction of the matrix A can be found in [28]. By using such A, the matrix B_λ is constructed for $\lambda = 10^{-4}$ and $\lambda = 10^{-8}$. First, we consider a convergent splittings of $B_\lambda = M_\lambda - N_\lambda$ corresponds to the iterative scheme (5), which converges to $B_\lambda^{-1} A^T b$ ($= A^\dagger b$ as $\lambda \to 0$). Secondly, we consider a convergent double splitting of $B_\lambda = P_\lambda - R_\lambda + S_\lambda$ corresponding to the iterative method (20), which also converges to $B_\lambda^{-1} A^T b$ ($= A^\dagger b$ as $\lambda \to 0$). Next, we

denote $e_\lambda = \|B_\lambda^{-1}A^T b - x_k\|_2$ is the error generated from the iterates of the iterative schemes (5) and (20), and $e = \|A^\dagger b - x_k\|_2$. Similarly, the residue is denoted by $r_\lambda = \|B_\lambda x_k - A^T b\|_2$ and $r = \|Ax_k - b\|_2$. The main objective of this example is to compare the error and residue generated from original system (1) and the modified system (3) for different values of λ. So from Table 1, one can observe that when $\lambda = 10^{-4}$ and $\lambda = 10^{-8}$, both the iterative scheme converges to $A^\dagger b$, i.e., the errors e_λ of the well-posed system are very close to 0. Also it can be seen that error e_λ is all most equal to the actual error e. The residue r_λ obtained for different order of matrices of the well-posed system is better than the residue r of original system $Ax = b$ (see Table 1). Further, the comparison between two iterative schemes (5) and (20) along with the condition number is presented in the Table 1. It shows that the condition number of B_λ is small as compared to the condition number of the ill-posed matrix A.

Table 1: Comparison of two iterative schemes for $\epsilon = 10^{-7}$

λ	O(A)	Scheme	ITE	e_λ	e	r_λ	r	$C(B_\lambda)$	$C(A)$
10^{-4}	100	(5)	2885	$4.5723e^{-4}$	$4.5723e^{-4}$	$2.6686e^{-5}$	0.9809	$6.5219e^5$	$3.1295e^{17}$
		(20)	1918	$8.8737e^{-5}$	$8.8737e^{-5}$	$4.5074e^{-6}$	0.9809	$6.5219e^5$	$3.1295e^{17}$
10^{-4}	600	(5)	–	0.0016	0.0016	$1.6454e^{-4}$	0.9833	$6.4637e^5$	$2.1059e^{16}$
		(20)	2141	$2.3132e^{-4}$	$2.3132e^{-4}$	$4.3275e^{-6}$	0.9832	$6.4637e^5$	$2.1059e^{16}$
10^{-4}	1600	(5)	–	0.0027	0.0027	$1.6823e^{-4}$	0.9836	$6.4606e^5$	$1.2112e^{16}$
		(20)	3061	$2.1542e^{-4}$	$2.1542e^{-4}$	$4.3727e^{-6}$	0.9835	$6.4606e^5$	$1.2112e^{16}$
10^{-4}	3600	(5)	2605	0.0026	0.0026	$1.4761e^{-4}$	0.9839	$6.4595e^5$	$1.0797e^{16}$
		(20)	–	$2.5100e^{-4}$	$2.5100e^{-4}$	$4.5487e^{-6}$	0.9838	$6.4595e^5$	$1.0797e^{16}$
10^{-8}	100	(5)	2883	$4.5790e^{-4}$	$4.5791e^{-4}$	$2.6686e^{-5}$	0.9809	$6.5219e^9$	$3.1295e^{17}$
		(20)	1920	$8.8921e^{-5}$	$8.8924e^{-5}$	$4.5077e^{-6}$	0.9809	$6.5219e^9$	$3.1295e^{17}$
10^{-8}	600	(5)	–	0.0016	0.0016	$1.6458e^{-4}$	0.9833	$6.4637e^9$	$2.1059e^{16}$
		(20)	2142	$2.3234e^{-4}$	$2.3234e^{-4}$	$4.3305e^{-6}$	0.9832	$6.4637e^9$	$2.1059e^{16}$
10^{-8}	1600	(5)	–	0.0027	0.0027	$1.6810e^{-4}$	0.9836	$6.4606e^9$	$1.2112e^{16}$
		(20)	3065	$2.1665e^{-4}$	$2.1665e^{-4}$	$4.3714e^{-6}$	0.9835	$6.4606e^9$	$1.2112e^{16}$
10^{-8}	3600	(5)	2604	0.0026	0.0026	$1.4756e^{-4}$	0.9839	$6.4595e^9$	$1.0797e^{16}$
		(20)	–	$2.5269e^{-4}$	$2.5269e^{-4}$	$4.5484e^{-6}$	0.9838	$6.4595e^9$	$1.0797e^{16}$

5. Conclusion

The notion of double weak splitting of type II was introduced along with regularized iterative scheme via Tikhonov's regularization. In regards to the double weak splitting of type II, a few convergence and comparison theorems have been proved. The results in Section 3 show that if we consider the regularized iterative scheme based on the splitting of B_λ with some prescribed conditions, it converges faster (in terms of spectral radius) than the iterative scheme generated by the splitting of the original matrix A. Further, some comparison results are established with the help of weak splitting of the first type and second type, where we do not assume the monotone condition. Several equivalent comparison theorems of various combinations of weak splittings are also demonstrated. The proposed scheme as an application to a partial differential differential equation is presented.

For future research perspectives, it is interesting to study the following points.

1. The results derived in subsection 3.2 can be extended to singular symmetric matrices.
2. The three-step alternating iterative schemes derived in [35, 36], confirms that further extension can be possible by considering the alternating regularized iterative scheme.
3. The same idea can be developed for P-proper splittings [23].
4. As tensors are natural extensions to matrices, one possible research could be to consider the multilinear system of tensor equations.

References

[1] H.C. Andrews, B.R. Hunt, Digital Image Restoration, Prentice-Hall, Englewood Cliffs, NJ, 1977.
[2] A.K. Baliarsingh, D. Mishra, Comparison results for proper nonnegative splittings of matrices, Results in Mathematics 71 (2017) 93–109.
[3] J.C. A. Barata, M.S. Hussein, The Moore–Penrose pseudoinverse: A tutorial review of the theory, Brazilian Journal of Physics 42 (2012) 146–165.
[4] A. Berman, R.J. Plemmons, Cones and iterative methods for best least squares solutions of linear systems, SIAM Journal on Numerical Analysis 11(1) (1974) 145–154.
[5] A. Berman, R.J. Plemmons, Nonnegative Matrices in the Mathematical Sciences, SIAM, Philadelphia, 1994.
[6] M. Bertero, P. Boccacci, Introduction to Inverse Problems in Imaging, CRC press, 1998.
[7] F. Bauer, M. A. Lukas, Comparing parameter choice methods for regularization of ill-posed problem, Mathematics and Computers in Simulation 81 (2011),1795–1841.
[8] D. Calvetti, L. Reichel, Q. Zhang, Iterative solution methods for large linear discrete ill-posed problems, Applied and Computational Control, Signals, and Circuits 1 (1999) 313–367.
[9] J.J. Climent, A. Devesa, C. Perea, Convergence Results for Proper Splittings, Recent Advances in Applied Theory Math, World Scientific and Engineering Society Press, Singapore, 2000.
[10] J.J. Climent, C. Perea, Some comparison theorems for weak nonnegative splittings of bounded operators, Linear Algebra and its Applications 275 (1998) 77–106.
[11] L. Eldén, V. Simoncini, Solving ill-posed linear systems with GMRES and a singular preconditioner, SIAM Journal on Matrix Analysis and Applications 33(4) (2012) 1369–1394.
[12] H. W. Engl, M. Hanke, A. Neubauer, Regularization of Inverse Problems, Springer Science & Business Media, 1996.
[13] G. H. Golub, R. S. Varga, Chebyshev semi-iterative methods, successive overrelaxation iterative methods, and second order Richardson iterative methods, Numerische Mathematik 3 (1961) 147–156.
[14] G. Golub, Numerical methods for solving linear least squares problems, Numerische Mathematik 7(3) (1965) 206–216.
[15] P. Grangeat, Mathematical Methods in Tomography, Springer-Verlag, Berlin, 1990.
[16] C. W. Groetsch, The Theory of Tikhonov Regularization for Fredholm Integral Equations of the First Kind, Pitman, Boston, MA, 1984.
[17] C. W. Groetsch, Inverse Problems in the Mathematical Sciences, Braunschweig Vieweg, 1993.
[18] M. E. Hochstenbach, L. Reichel, G. Rodriguez, Regularization parameter determination for discrete ill-posed problems, Journal of Computational and Applied Mathematics 273 (2015), 132–149.
[19] J. Hadamard, Lectures on Cauchy's Problem in Linear Partial Differential Equations, Yale University Press, New Haven, 1923.
[20] M. Hanke, Accelerated Landweber iterations for the solution of ill-posed equations, Numerische Mathematik 60(1) (1991) 341–373.
[21] M. Hanke, C. W. Groetsch, Non tationary iterated Tikhonov regularization, Journal of Optimization Theory and Applications 98 (1998) 37–53.
[22] L. Jena, D. Mishra, S. Pani, Convergence and comparison theorems for single and double decompositions of rectangular matrices, Calcolo 51(1) (2014) 141–149.
[23] M. R. Kannan, P-proper splittings, Aequationes Mathematicae 91(4) (2017) 619–633.
[24] S. Kindermann, Convergence analysis of minimization-based noise-level-free parameter choice rules for ill-posed problems, Electronic Transactions on Numerical Analysis 38 (2011), 233–257.
[25] S. Kindermann, Discretization independent convergence rates for noise level-free parameter choice rules for the regularization of ill-conditioned problems, Electronic Transactions on Numerical Analysis 40 (2013), 58–81.
[26] C. X. Li, S. H. Li, Comparison theorems of spectral radius for splittings of matrices, Journal of Applied Mathematics, Volume 2014, Article ID 573024, (2014). http://dx.doi.org/10.1155/2014/573024.
[27] C. X. Li, S. L. Wu, Some new comparison theorems for double splittings of matrices, Applied Mathematics & Information Sciences 8(5) (2014) 2523–2526.
[28] H. Ma, C. Xiao, Convergence of nonstationary iterative methods for solving singular linear equations with index one, Numerical Functional Analysis and Optimization 38(11) (2017) 1507–1525.
[29] I. Marek, D.B. Szyld, Comparison theorems for weak splittings of bounded operators, Numerische Mathematik 58(1) (1990) 387–397.
[30] S. X. Miao, Comparison theorems for nonnegative double splittings of different monotone matrices, Journal of Information and Computing Science 9(6) (2012) 1421–1428.
[31] S. X. Miao, B. Zheng, A note on double splittings of different monotone matrices, Calcolo 46(4) (2009) 261–266.
[32] D. Mishra, Further study of alternating iterations for rectangular matrices, Linear and Multilinear Algebra 65(8) (2017) 1566–1580.
[33] D. Mishra, Nonnegative splittings for rectangular matrices, Computers & Mathematics with Applications 67(1) (2014) 136–144.
[34] D. Mishra, K. C. Sivakumar, Comparison theorems for a subclass of proper splittings of matrices, Applied Mathematics Letters 25(12) (2012) 2339–2343.
[35] A. K. Nandi, J. K. Sahoo, P. Ghosh, Three-step alternating and preconditioned scheme for rectangular matrices, Journal of Applied Mathematics and Computing 60 (2019) 485–515.
[36] A. K. Nandi, J. K. Sahoo, D. Mishra, Three-step alternating iterations for index 1 and non-singular matrices, Numerical Algorithms 84 (2020) 457–483.
[37] M. Neumann, 3-part splittings for singular and rectangular linear systems, Journal of Mathematical Analysis and Applications 64(2) (1978) 297–318.
[38] V. G. Romanov, S. I. Kabanikhin, Inverse Problems for Maxwell's Equations, VSP, Utrecht, 1994.

[39] L. Reichel, G. Rodriguez, Old and new parameter choice rules for discrete ill-posed problems, Numerical Algorithms 63 (2013), 65—87.
[40] V. Shekhar, C. K. Giri, D. Mishra, A note on double weak splittings of type II, Linear and Multilinear Algebra (2020) 1–21. https://doi.org/10.1080/03081087.2020.1795057.
[41] S. Q. Shen, T. Z. Huang, Convergence and comparison theorems for double splittings of matrices, Computers & Mathematics with Applications 51(12) (2006) 1751–1760.
[42] S. Q. Shen, T. Z. Huang, J. L. Shao, Convergence and comparison results for double splittings of Hermitian positive definite matrices, Calcolo 44(3) (2007) 127–135.
[43] A. Sidi, DGMRES: A GMRES-type algorithm for Drazin inverse solution of singular nonsymmetric linear systems, Linear Algebra and its Applications 335 (2001) 189–204.
[44] J. Song, Y. Song, Convergence for nonnegative double splittings of matrices, Calcolo 48(3) (2011) 245–260.
[45] Y. Song, Comparisons of nonnegative splittings of matrices, Linear Algebra and its Applications 154 (1991) 433–455.
[46] Y. Song, Comparison theorems for splittings of matrices, Numerische Mathematik 92(3) (2002) 563–591.
[47] A. N. Tikhonov, Solution of incorrectly formulated problems and the regularization method, Soviet Mathematics Doklady 4 (1963) 1035–1038.
[48] R. S. Varga, Matrix Iterative Analysis, Springer, New York, 2000.
[49] X. Z. Wang, Convergence of H-double splitting for H-matrices, Results in Mathematics 66(1) (2014) 125–135.
[50] Z. I. Woźnicki, Estimation of the optimum relaxation factors in partial factorization iterative methods, SIAM Journal on Matrix Analysis and Applications 14(1) (1993) 59–73.
[51] Z. I. Woźnicki, Matrix splitting principles, International Journal of Mathematics and Mathematical Sciences 28(5) (2001) 251–284.
[52] Z. I. Woźnicki, Nonnegative splitting theory, Japan Journal of Industrial and Applied Mathematics 11(2) (1994) 289–342.