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Ordering Results Between the Largest Claims Arising From Two General Heterogeneous Portfolios

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Abstract. This work is entirely devoted to compare the largest claims from two heterogeneous portfolios. It is assumed that the claim amounts in an insurance portfolio are nonnegative absolutely continuous random variables and belong to a general family of distributions. The largest claims have been compared based on various stochastic orderings. The established sufficient conditions are associated with the matrices and vectors of model parameters. Applications of the results are provided for the purpose of illustration.

1. Introduction

In survival analysis, models with nonmonotone failure rate play a vital role to fit the real life data sets. A large number of distributions exists in statistical theory, which have monotone failure rate. For example, the exponentiated Weibull and generalized gamma distributions have monotone failure rate. In this communication, a general family of distributions (exponentiated location-scale) is taken. It contains both monotone and nonmonotone failure rate models. Because of this, the general exponentiated location-scale (ELS) model is important from both practical and theoretical points of view. It is well-known that X belongs to the ELS model if $X \sim F^{\alpha}(\frac{x-\lambda}{\theta}),\ x > \lambda > 0$ and $\alpha,\ \theta > 0$. The functions $F(\cdot)$ and $f(\cdot)$ denote the baseline cumulative distribution and probability density functions of X, respectively. Here, we consider $F(\cdot)$ to be the absolutely continuous distribution function. The strictly positive real numbers $\alpha,\ \lambda$ and θ are respectively the shape, location and scale parameters. For $\lambda = 0$ and $\alpha = 1$, the exponentiated location-scale model reduces to the scale model. Further, we respectively get the proportional reversed hazard rate model and the location model, when $\lambda = 0, \theta = 1$ and $\alpha = 1, \theta = 1$.

For $i = 1, \ldots, n$, let X_i be the claim amount and J_i be the Bernoulli random variable. Further, $J_i = 0$, if the ith policy holder does not claim, and $J_i = 1$, if the ith policy holder makes random claim X_i. We assume that the claim is taken place with probability p_i, and is not taken place with probability $1 - p_i$. It is known that in an insurance portfolio consisting of n risks, the ith individual risk is a product of X_i and J_i. Throughout this paper, we consider that $\{X_1, \ldots, X_n\}$ and $\{Y_1, \ldots, Y_n\}$ are two collections of independent random claims of two portfolios with $X_i \sim F^{\alpha_i}(\frac{x-\lambda_i}{\theta_i})$ and $Y_i \sim F^{\beta_i}(\frac{x-\mu_i}{\delta_i})$, where $i = 1, \ldots, n$. Further, assume that $\{J_1, \ldots, J_n\}$

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and $\{J_1^*, \ldots, J_n^*\}$ are another two collections of independent Bernoulli random variables, independent of X_i's and Y_i's, respectively, $i = 1, \ldots, n$. Consider two vectors $\boldsymbol{U} = (U_1, \ldots, U_n)$ and $\boldsymbol{V} = (V_1, \ldots, V_n)$, such that for $i = 1, \ldots, n$, $U_i = J_i X_i$ and $V_i = J_i^* Y_i$ with $E(J_i) = p_i$ and $E(J_i^*) = q_i$. Denote $U_{n:n} = \max\{U_1, \ldots, U_n\}$ and $V_{n:n} = \max\{V_1, \ldots, V_n\}$ for the maximum claims arising from two insurance portfolios of n risks, where the ith individual risks are U_i and V_i, respectively, $i = 1, \ldots, n$. Besides this, $U_{n:n}$ has another interpretation in reliability theory. It represents the lifetime of a parallel system for which the components are equipped with starters. Here, the random variables X_i's can be treated as components' lifetimes and J_i's represent the status of the corresponding starters. Therefore, the present study of stochastic comparison is very important both from the mathematical research and real life applications.

Barmalzan and Najafabadi [4] and Barmalzan et al. [5] considered two collections of independent claims following heterogeneous Weibull distributions. Barmalzan and Najafabadi [4] addressed the comparisons between the minimum claims stochastically in the sense of the convex transform and right spread orders. They also derived upper and lower bounds of the coefficient of variations. Barmalzan et al. [5] discussed the sufficient conditions under which the likelihood ratio and dispersive orders hold between the smallest claim amounts. Barmalzan et al. [6] took scale model to compare the extreme claims with respect to the usual stochastic and the hazard rate orderings. Balakrishnan et al. [3] studied ordering properties of the largest claim amounts from two heterogeneous sets of portfolios. They proposed sufficient conditions to show various stochastic orderings between the largest claim amounts. Zhang et al. [20] established conditions to compare the extreme claims from two collections of insurance portfolios. Barmalzan et al.[1] developed stochastic comparisons results of extreme claim amounts having location-scale claim severities. To the best of our knowledge, stochastic comparisons of the largest claim amounts when random claims have ELS models have not been addressed in the literature so far. However, some generalized models to study ordering properties of extreme order statistics in the context of reliability studies can be found in [7–11, 16]. In this paper, we address this problem and derive sufficient conditions for the stochastic comparison of the largest claim amounts in the sense of various stochastic orderings.

This article is organized as follows. In section 2, we provide some basic definitions and results. Section 3 is emphasized on some ordering results based on the matrix chain majorization order, when heterogeneity presents in two parameters. Section 4 addresses comparisons between the largest claims with respect to the usual stochastic and reversed hazard rate orders. Here, we consider that the heterogeneity is presented in one parameter. Section 5 is devoted to illustrations of the results. Generalized linear failure rate and Pareto distributions are considered. Finally, we present some concluding remarks in Section 6.

Throughout this article, we assume that the random variables are nonnegative and absolutely continuous. The integrations and differentiations are well defined. Further, 'increasing' and 'decreasing' terms are employed in non-strict sense. For any function $h(\cdot)$, $h'(x) = \frac{dh(x)}{dx}$. We assume componentwise comparison when comparing two vectors.

2. Preliminaries

This section is concerned with some basic definitions and important lemmas, which are used to prove the results in the subsequent sections. Let U and V be two nonnegative absolutely continuous random variables. Assume that $f_U(\cdot)$ and $f_V(\cdot)$, $F_U(\cdot)$ and $F_V(\cdot)$, $\bar{F}_U(\cdot)$ and $\bar{F}_V(\cdot)$ are the probability density functions, the cumulative distribution functions and the survival functions of U and V, respectively. The following definition is for some concepts of stochastic orders. For comprehensive discussions on the properties and applications of the following stochastic orders, one may refer to Shaked and Shanthikumar [18].

Definition 2.1. *U is smaller than V in the*

(a) usual stochastic order, abbreviated by $U \leq_{st} V$, if $\bar{F}_U(x) \leq \bar{F}_V(x)$, for every $x \in \mathbb{R}$;

(b) reversed hazard rate order, abbreviated by $U \leq_{rh} V$, if the ratio $F_V(x)/F_U(x)$ is increasing with respect to x.

Next definition describes the concept of majorization and related orders. Prior to this, we consider vectors $\boldsymbol{x} = (x_1, \ldots, x_n)$ and $\boldsymbol{y} = (y_1, \ldots, y_n)$. Further, $x_{1:n} \leq \ldots \leq x_{n:n}$ and $y_{1:n} \leq \ldots \leq y_{n:n}$ respectively denote the components of $\boldsymbol{x}$ and $\boldsymbol{y}$.

Definition 2.2. $\boldsymbol{x}$ *is known to be*

(a) *weakly supermajorized by* $\boldsymbol{y}$, *abbreviated by* $\boldsymbol{x} \preceq^{w} \boldsymbol{y}$, *if* $\sum_{k=1}^{j} x_{k:n} \geq \sum_{k=1}^{j} y_{k:n}$, *for* $j = 1, \ldots, n$;

(b) *weakly submajorized by* $\boldsymbol{y}$, *abbreviated by* $\boldsymbol{x} \preceq_{w} \boldsymbol{y}$, *if* $\sum_{k=i}^{n} x_{k:n} \leq \sum_{k=i}^{n} y_{k:n}$, *for* $i = 1, \ldots, n$;

(c) *majorized by* $\boldsymbol{y}$, *abbreviated by* $\boldsymbol{x} \preceq^{m} \boldsymbol{y}$, *if* $\sum_{k=1}^{n} x_k = \sum_{k=1}^{n} y_k$ *and* $\sum_{k=1}^{j} x_{k:n} \geq \sum_{k=1}^{j} y_{k:n}$, *for* $j = 1, \ldots, n-1$;

(d) *p-larger than* $\boldsymbol{y}$, *denoted by* $\boldsymbol{x} \succeq^{p} \boldsymbol{y}$, *if* $\prod_{i=1}^{k} x_{i:n} \leq \prod_{i=1}^{k} y_{i:n}$, *for* $k = 1, \ldots, n$;

(e) *reciprocally majorized by* $\boldsymbol{y}$, *denoted by* $\boldsymbol{x} \preceq^{rm} \boldsymbol{y}$, *if* $\sum_{i=1}^{l} x_{i:n}^{-1} \leq \sum_{i=1}^{l} y_{i:n}^{-1}$, *for all* $l = 1, \ldots, n$.

It is easy to see that $\boldsymbol{x} \preceq^{m} \boldsymbol{y}$ implies both $\boldsymbol{x} \preceq^{w} \boldsymbol{y}$ and $\boldsymbol{x} \preceq_{w} \boldsymbol{y}$. Further, $\boldsymbol{x} \preceq^{w} \boldsymbol{y} \Rightarrow \boldsymbol{x} \preceq^{p} \boldsymbol{y}$. We may refer to Marshall et al. [15] for brief and extensive details on the majorization and their applications.

Definition 2.3. *Let* $\varphi : \mathbb{B}\ (\subseteq \mathbb{R}^n) \to \mathbb{R}$ *be a function. It is said to be Schur-convex (Schur-concave) on* $\mathbb{B}$ *if* $\boldsymbol{x} \preceq^{m} \boldsymbol{y} \Rightarrow \varphi(\boldsymbol{x}) \leq (\geq) \varphi(\boldsymbol{y})$, *for any* $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{B}$.

The following notations are used throughout the paper. We denote $\mathbf{1}_n = (1, \ldots, 1)$.

$$\mathcal{D}_{+} = \{(t_1, \ldots, t_n) : t_1 \geq \ldots \geq t_n > 0\};\ \mathcal{E}_{+} = \{(t_1, \ldots, t_n) : 0 < t_1 \leq \ldots \leq t_n\}.$$

Now, we move our attention to the notion of the matrix majorization. We say that a square matrix π is said to be a permutation matrix, if each row and column have a single entry, and except that, all entries are zero. One can easily find out that $n!$ number of such matrices arise after interchanging rows (or columns) of the $n \times n$ order identity matrix I_n. Let T_w denote a T-transform matrix, with the form

$$T_w = wI_n + (1-w)\pi,\ 0 \leq w \leq 1, \tag{1}$$

where π is a permutation matrix obtained by interchanging two rows or columns in identity matrix. Consider the T-transform matrices T_{w_1} and T_{w_2} such that $T_{w_1} = w_1 I_n + (1-w_1)\pi_1$ and $T_{w_2} = w_2 I_n + (1-w_2)\pi_2$, where π_1 and π_2 are the permutation matrices obtained by interchanging two rows or columns in identity matrix and $0 \leq w_1,\ w_2 \leq 1$. We say that T_{w_1} and T_{w_2} have the same structure, if $\pi_1 = \pi_2$, and have different structures, if $\pi_1 \neq \pi_2$. The definition given below describes the concept of multivariate majorization.

Definition 2.4. *Let us take two matrices* $C = [c_{ij}]$ *and* $D = [d_{ij}]$ *of order* $m \times n$, *where* $i = 1, \ldots, m$ *and* $j = 1, \ldots, n$. *Let* $T_{w_1}, \ldots, T_{w_k}$ *be a finite set of* $n \times n$ *T-transform matrices. Then, C is said to chain majorize D, abbreviated by* $C >> D$, *if* $D = CT_{w_1} \ldots T_{w_k}$.

Henceforth, $(\boldsymbol{r_1}, \ldots, \boldsymbol{r_m}; n)$ represents a matrix of order $m \times n$, where each real-valued vector $\boldsymbol{r_i}$ containing n elements denotes the ith row, for $i = 1, \ldots, m$. For $i, j = 1, \ldots, n$ and $x_i,\ y_j > 0$, let us consider

$$M_n = \left\{(\boldsymbol{x}, \boldsymbol{y}; n) :\ (x_i - x_j)(y_i - y_j) \geq 0\right\};$$
$$Q_n = \left\{(\boldsymbol{x}, \boldsymbol{y}; n) :\ (x_i - x_j)(y_i - y_j) \leq 0\right\}.$$

The next consecutive lemmas are helpful to prove few of the proposed ordering results. Interested readers are referred to the work by Balakrishnan et al. [2] for detailed idea on the proofs.

Lemma 2.5. *A differentiable function* $\varpi : \mathbb{R}^{+^4} \to \mathbb{R}^+$ *satisfies*

$$\varpi(C) \geq (\leq) \varpi(D) \text{ for all } C, D \in M_2 \text{ or } Q_2 \text{ and } C \gg D \tag{2}$$

if and only if for all $C \in M_2$ *or* Q_2 *and C satisfies*

(a) $\varpi(C) = \varpi(C\pi)$, *for all permutation matrices* π *and*

(b) $\sum_{i=1}^{2} \left(c_{ik} - c_{ij}\right)\left(\varpi_{ik}(C) - \varpi_{ij}(C)\right) \geq (\leq) 0$, *for all* $j, k = 1, 2$, *where* $\varpi_{ij}(C) = \frac{\partial \varpi(C)}{\partial c_{ij}}$.

Lemma 2.6. *Consider a differentiable function $\upsilon : \mathbb{R}^{+^2} \to \mathbb{R}^+$ and a function $\zeta_n : \mathbb{R}^{+^{2n}} \to \mathbb{R}^+$ such that*

$$\zeta_n(C) = \prod_{k=1}^{n} \upsilon\,(c_{1k}, c_{2k})\,. \tag{3}$$

Let ζ_2 satisfy (2). *Then, for all $C \in M_n$ or Q_n and $D = CT_\omega$, we have $\zeta_n(C) \geq (\leq)\zeta_n(D)$.*

Before proceeding to the next section, we introduce two lemmas, which deal with the analytical behavior of two mathematical functions. The proofs are omitted since these are simple. The first lemma is useful for the derivation of the results of Theorems 3.6, 3.7, 3.9, and the second one is used in the proof of Theorem 4.1.

Lemma 2.7. *Consider a function $k_1 : (0,\infty) \times (0,1) \times (0,1) \to (0,\infty)$ such that $k_1(\alpha, t, p) = \frac{(1-t^\alpha)}{1-p(1-t^\alpha)}$. Then, for all*

(i) $t \in (0,1)$, $k_1(\alpha, t, p)$ is increasing in to α and p;

(ii) $\alpha \in (0,\infty)$ and $p \in (0,1)$, $k_1(\alpha, t, p)$ is decreasing in t.

Lemma 2.8. *Let $k_2 : (0,\infty) \times (0,1) \to (0,\infty)$ be defined as $k_2(\alpha, p) = \frac{pt^\alpha \ln(t)}{1-p(1-t^\alpha)}$. Then, for all*

(i) $p \in (0,1)$, $k_2(\alpha, p)$ is increasing in α;

(ii) $\alpha \in (0,\infty)$, $k_2(\alpha, p)$ is decreasing in p.

3. Matrix chain majorization

In this section, we establish some ordering results between the largest claims when a matrix of parameters is related to another matrix of parameters in some mathematical senses. To begin with, let us write the respective cumulative distribution functions of $U_{n:n}$ and $V_{n:n}$ as

$$F_{n:n}(t) = \prod_{i=1}^{n} \left[1 - p_i\left[1 - F^{\alpha_i}\left(\frac{t-\lambda_i}{\theta_i}\right)\right]\right],\ t > \max\{\lambda_1, \ldots, \lambda_n\} \tag{4}$$

and

$$G_{n:n}(t) = \prod_{i=1}^{n} \left[1 - q_i\left[1 - F^{\beta_i}\left(\frac{t-\mu_i}{\delta_i}\right)\right]\right],\ t > \max\{\mu_1, \ldots, \mu_n\}. \tag{5}$$

First, we consider the following assumptions. These will be called in the main results when necessary.

($A1$) Let $F(\cdot)$ be the baseline distribution function. Again, let $\{X_1, \ldots, X_n\}$ and $\{Y_1, \ldots, Y_n\}$ be two sets of nonnegative independent random variables. For $i = 1, \ldots, n$, assume that $X_i \sim F^{\alpha_i}(\frac{t-\lambda_i}{\theta_i})$ and $Y_i \sim F^{\beta_i}(\frac{t-\mu_i}{\delta_i})$.

($A2$) Let $\{J_1, \ldots, J_n\}$ and $\{J_1^*, \ldots, J_n^*\}$ be two collections of independent Bernoulli random variables, independent of X_i's and Y_i's, respectively. Further, $E(J_i) = p_i$ and $E(J_i^*) = q_i$, for $i = 1, \ldots, n$.

Let $r(\cdot)$ be the hazard rate function of the baseline distribution $F(\cdot)$, where $r(x) = f(x)/(1 - F(x))$. We provide the following conditions, which are also required for the smooth presentation of the results.

($C1$) $r(x)$ is decreasing.

($C2$) $xr(x)$ is decreasing.

($C3$) $x^2r(x)$ is decreasing.

(C4) $\frac{r'(x)}{r(x)}$ is increasing.

(C5) $xr(x)$ is convex.

(C6) $x^3r^2(x)$ is decreasing.

(C7) $x^2[xr(x)]'$ is increasing.

(C8) $r(x)$ is convex.

The function $\psi : (0, 1) \to (0, \infty)$ is taken to be differentiable throughout the paper.

(C9) $\psi(w)$ is convex and increasing.

(C10) $\psi(w)$ is convex and decreasing.

The following result establishes conditions, under which multivariate chain majorization between two matrices of parameters implies the usual stochastic order between the largest claim amounts. In the sequel, we take a common shape parameter vector for both sets. It is equal to a scalar α, which lies in the interval $(0, 1]$. Note that the following result contains two parts, of which the second part generalizes Theorem 1 of Barmalzan et al. [6].

Theorem 3.1. *For $n = 2$, let (A1), (A2) and (C1) hold. Also, assume $\boldsymbol{\alpha} = \boldsymbol{\beta} = \alpha\mathbf{1}_2$ ($\alpha \le 1$).*

(*i*) *Suppose the function ψ satisfies (C9). If $\boldsymbol{\theta} = \boldsymbol{\delta} = \theta\mathbf{1}_2$ and $(\boldsymbol{\psi}(\boldsymbol{p}), \boldsymbol{\lambda}; 2) \in M_2$, then $(\boldsymbol{\psi}(\boldsymbol{p}), \boldsymbol{\lambda}; 2) \gg (\boldsymbol{\psi}(\boldsymbol{q}), \boldsymbol{\mu}; 2) \Rightarrow U_{2:2} \ge_{st} V_{2:2}$;*

(*ii*) *Suppose the function ψ satisfies (C10). If $\boldsymbol{\lambda} = \boldsymbol{\mu} = \mu\mathbf{1}_2$ and $(\boldsymbol{\psi}(\boldsymbol{p}), 1/\boldsymbol{\theta}; 2) \in M_2$, then $(\boldsymbol{\psi}(\boldsymbol{p}), 1/\boldsymbol{\theta}; 2) \gg (\boldsymbol{\psi}(\boldsymbol{q}), 1/\boldsymbol{\delta}; 2) \Rightarrow U_{2:2} \ge_{st} V_{2:2}$.*

Proof. (*i*) We have

$$F_{2:2}(t) = \prod_{i=1}^{2} \left[1 - \psi^{-1}(w_i)\left[1 - F^{\alpha}\left(\frac{t - \lambda_i}{\theta}\right)\right]\right], \tag{6}$$

where $\psi(p_i) = w_i$, for $i = 1, 2$. Note that $F_{2:2}(t)$ is permutation invariant in (w_i, λ_i). Hence, the first condition of Lemma 2.5 is fulfilled. Further, the partial derivatives of (6) with respect to w_i and λ_i are respectively obtained as

$$\frac{\partial F_{2:2}(t)}{\partial w_i} = -\frac{\partial \psi^{-1}(w_i)}{\partial w_i} \frac{[1 - F^{\alpha}\left(\frac{t-\lambda_i}{\theta}\right)]}{\left[1 - \psi^{-1}(w_i)\left[1 - F^{\alpha}\left(\frac{t-\lambda_i}{\theta}\right)\right]\right]} F_{2:2}(t), \tag{7}$$

and

$$\frac{\partial F_{2:2}(t)}{\partial \lambda_i} = -\frac{\alpha F^{\alpha-1}\left(\frac{t-\lambda_i}{\theta}\right) f\left(\frac{t-\lambda_i}{\theta}\right)}{\theta\left[1 - F^{\alpha}\left(\frac{t-\lambda_i}{\theta}\right)\right]} \frac{\psi^{-1}(w_i)[1 - F^{\alpha}\left(\frac{t-\lambda_i}{\theta}\right)]}{1 - \psi^{-1}(w_i)\left[1 - F^{\alpha}\left(\frac{t-\lambda_i}{\theta}\right)\right]} F_{2:2}(t). \tag{8}$$

We define

$$\phi_1(\boldsymbol{w}, \boldsymbol{\lambda}) = (w_i - w_j)\left[\frac{\partial F_{2:2}(t)}{\partial w_i} - \frac{\partial F_{2:2}(t)}{\partial w_j}\right] + (\lambda_i - \lambda_j)\left[\frac{\partial F_{2:2}(t)}{\partial \lambda_i} - \frac{\partial F_{2:2}(t)}{\partial \lambda_j}\right], \tag{9}$$

where the partial derivatives are given by (7) and (8). Together with Lemma 3 of Balakrishnan et al. [2] and the assumptions made, we can show that $\phi_1(\boldsymbol{w}, \boldsymbol{\lambda})$ is nonpositive. Thus, clearly, the second argument of Lemma 2.5 is verified, and the proof is completed. By adopting the arguments of the proof of the first part, the second part follows easily. □

Remark 3.2. *On using Theorem 3.1(ii), one can easily find out a lower bound for the reliability function of the largest claims having heterogeneous portfolios of risks in the form of reliability of the largest claims having homogeneous portfolio of risks. Consider a T- transform matrix* $T_{0.5}$ *of order* 2×2*, where the first and second rows are same and equal to* $(1/2, 1/2)$*. Let* $(\psi(p_1), \psi(p_2)) = (e^{-p_1}, e^{-p_2})$ *and* $1/\boldsymbol{\theta} = (1/\theta_1, 1/\theta_2)$*. Further, assume* $(\psi(q_1), \psi(q_2)) = ((e^{-p_1} + e^{-p_2})/2, (e^{-p_1} + e^{-p_2})/2)$ *and* $(1/\delta_1, 1/\delta_2) = ((\theta_1 + \theta_2)/(2\theta_1\theta_2), (\theta_1 + \theta_2)/(2\theta_1\theta_2))$*. It is easy to check that*

$$\begin{pmatrix} \psi(q_1) & \psi(q_2) \\ 1/\delta_1 & 1/\delta_2 \end{pmatrix} = \begin{pmatrix} \psi(p_1) & \psi(p_2) \\ 1/\theta_1 & 1/\theta_2 \end{pmatrix} T_{0.5}.$$

As a result, $\begin{pmatrix} \psi(p_1) & \psi(p_2) \\ 1/\theta_1 & 1/\theta_2 \end{pmatrix} \gg \begin{pmatrix} \psi(q_1) & \psi(q_2) \\ 1/\delta_1 & 1/\delta_2 \end{pmatrix}$*. Therefore, by Theorem 3.1(ii), we can propose a lower bound of the reliability function of* $U_{2:2}$ *as*

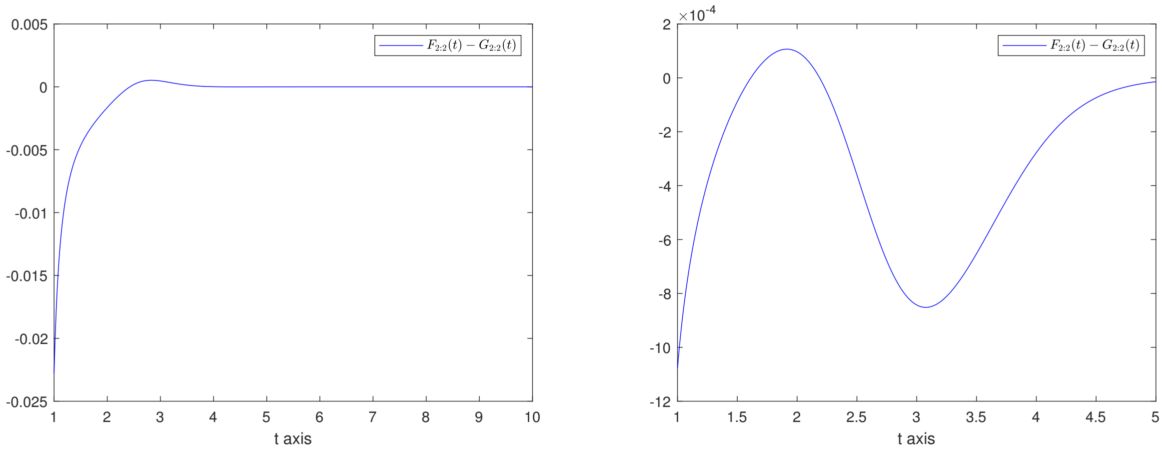
$$\bar{F}_{2:2}(t) \geq 1 - \left\{1 + \ln\left(\frac{e^{-p_1} + e^{-p_2}}{2}\right)\left[1 - F^{\alpha}\left(\frac{(t-\lambda)(\theta_1 + \theta_2)}{2\theta_1\theta_2}\right)\right]\right\}^2.$$

Now, it might be of interest to investigate whether the decreasing and convexity property of the function ψ is a must or not in Theorem 3.1(*ii*). The following numerical counterexample states that this assumption is required to get the usual stochastic order in Theorem 3.1(*ii*).

Counterexample 3.1. *Let* $F(t) = 1 - (1 + t^5)^{-4}$*,* $t > 0$ *and* $\psi(p) = 1 - p^3$*. It is easy to check that* $\psi(p)$ *is decreasing and concave. So, the condition in* (C10) *is relaxed. Let us take* $(\alpha_1, \alpha_2) = (\beta_1, \beta_2) = (0.01, 0.01)$*,* $(1/\theta_1, 1/\theta_2) = (0.7, 0.6)$*,* $(1/\delta_1, 1/\delta_2) = (0.66, 0.64)$*,* $(\lambda_1, \lambda_2) = (\mu_1, \mu_2) = (0.9, 0.9)$*,* $(p_1, p_2) = ((0.2)^{1/3}, (0.5)^{1/3})$ *and* $(q_1, q_2) = ((0.32)^{1/3}, (0.38)^{1/3})$*. Consider a T-transform matrix* $T_{0.6} = \begin{pmatrix} 0.6 & 0.4 \\ 0.4 & 0.6 \end{pmatrix}$*. It can be shown that*

$$\begin{pmatrix} \psi(q_1) & \psi(q_2) \\ 1/\delta_1 & 1/\delta_2 \end{pmatrix} = \begin{pmatrix} \psi(p_1) & \psi(p_2) \\ 1/\theta_1 & 1/\theta_2 \end{pmatrix} T_{0.6},$$

which produces $\begin{pmatrix} \psi(p_1) & \psi(p_2) \\ 1/\theta_1 & 1/\theta_2 \end{pmatrix} \gg \begin{pmatrix} \psi(q_1) & \psi(q_2) \\ 1/\delta_1 & 1/\delta_2 \end{pmatrix}$*. Further,* $\begin{pmatrix} \psi(p_1) & \psi(p_2) \\ 1/\theta_1 & 1/\theta_2 \end{pmatrix} \in M_2$*. Based on this present setup, we have* $F_{2:2}(1.5) - G_{2:2}(1.5) = 1.5928 \times e^{-5}$ (> 0) *and* $F_{2:2}(1.6) - G_{2:2}(1.6) = -1.9324 \times e^{-4}$ (< 0)*. For graphical view, please see Figure* 1(*a*)*. It establishes that Theorem 3.1(ii) does not hold, if* (C10) *is taken out.*



(a) (b)

Figure 1: (a) Graph of $F_{2:2}(t) - G_{2:2}(t)$ for Counterexample 3.1. (b) Graph of $F_{2:2}(t) - G_{2:2}(t)$ for Counterexample 3.2.

We next consider a counterexample to show that if the condition "$\begin{pmatrix} \psi(p_1) & \psi(p_2) \\ 1/\theta_1 & 1/\theta_2 \end{pmatrix} \notin M_2$" is removed, then Theorem 3.1(*ii*) may not hold.

Counterexample 3.2. *Consider the same baseline distribution function as in Counterexample 3.1. Suppose* $\psi(p) = -\ln p$. *Here, one can check that* (C1) *and* (C10) *are satisfied. Set* $(\alpha_1, \alpha_2) = (\beta_1, \beta_2) = (0.01, 0.01)$, $(1/\theta_1, 1/\theta_2) = (0.5, 0.3)$, $(1/\delta_1, 1/\delta_2) = (0.32, 0.48)$, $(\lambda_1, \lambda_2) = (\mu_1, \mu_2) = (0.9, 0.9)$, $(\psi(p_1), \psi(p_2)) = (0.23, 0.69)$ *and* $(\psi(q_1), \psi(q_2)) = (0.644, 0.276)$. *For a T-transform matrix* $T_{0.1} = \begin{pmatrix} 0.1 & 0.9 \\ 0.9 & 0.1 \end{pmatrix}$, *we have*

$$\begin{pmatrix} \psi(q_1) & \psi(q_2) \\ 1/\delta_1 & 1/\delta_2 \end{pmatrix} = \begin{pmatrix} \psi(p_1) & \psi(p_2) \\ 1/\theta_1 & 1/\theta_2 \end{pmatrix} T_{0.1}.$$

Thus, from Definition 2.4, $\begin{pmatrix} \psi(p_1) & \psi(p_2) \\ 1/\theta_1 & 1/\theta_2 \end{pmatrix} \gg \begin{pmatrix} \psi(q_1) & \psi(q_2) \\ 1/\delta_1 & 1/\delta_2 \end{pmatrix}$. *Again,* $\begin{pmatrix} \psi(p_1) & \psi(p_2) \\ 1/\theta_1 & 1/\theta_2 \end{pmatrix}$ *does not belong to* M_2. *Now, considering the assumed numerical values, we get* $F_{2:2}(1.6) - G_{2:2}(1.6) = -9.2138 \times e^{-6}$ (< 0) *and* $F_{2:2}(1.7) - G_{2:2}(1.7) = 5.0560 \times e^{-5}$ (> 0), *which negates the usual stochastic order, stated in Theorem 3.1(ii). See Figure 1(b) for the graph of* $F_{2:2}(t) - G_{2:2}(t)$.

The following three consecutive results can be thought of a generalization of the above result to arbitrary $n \geq 3$. The cases in which the chain majorization order holds between two matrices $(\boldsymbol{\psi}(\boldsymbol{q}), \boldsymbol{\mu}; n)$ and $(\boldsymbol{\psi}(\boldsymbol{p}), \boldsymbol{\lambda}; n)$, or $(\boldsymbol{\psi}(\boldsymbol{q}), 1/\boldsymbol{\delta}; n)$ and $(\boldsymbol{\psi}(\boldsymbol{p}), 1/\boldsymbol{\theta}; n)$ are considered. First, we use a T-transform matrix to obtain the matrices $(\boldsymbol{\psi}(\boldsymbol{q}), \boldsymbol{\mu}; n)$ and $(\boldsymbol{\psi}(\boldsymbol{q}), 1/\boldsymbol{\delta}; n)$ from $(\boldsymbol{\psi}(\boldsymbol{p}), \boldsymbol{\lambda}; n)$ and $(\boldsymbol{\psi}(\boldsymbol{p}), 1/\boldsymbol{\theta}; n)$, respectively, so that $(\boldsymbol{\psi}(\boldsymbol{p}), \boldsymbol{\lambda}; n) \gg (\boldsymbol{\psi}(\boldsymbol{q}), \boldsymbol{\mu}; n)$ and $(\boldsymbol{\psi}(\boldsymbol{p}), 1/\boldsymbol{\theta}; n) \gg (\boldsymbol{\psi}(\boldsymbol{q}), 1/\boldsymbol{\delta}; n)$ hold.

Theorem 3.3. *Let us assume that* (A1), (A2) *and* (C1) *hold. Further,* $\boldsymbol{\alpha} = \boldsymbol{\beta} = \alpha \mathbf{1}_n$ $(\alpha \leq 1)$. *Suppose* T_w *is a T-transform matrix.*

(i) *Let* ψ *satisfy* (C9). *If* $\boldsymbol{\theta} = \boldsymbol{\delta} = \theta \mathbf{1}_n$ *and* $(\boldsymbol{\psi}(\boldsymbol{p}), \boldsymbol{\lambda}; n) \in M_n$, *then* $(\boldsymbol{\psi}(\boldsymbol{q}), \boldsymbol{\mu}; n) = (\boldsymbol{\psi}(\boldsymbol{p}), \boldsymbol{\lambda}; n) T_w \Rightarrow U_{n:n} \geq_{st} V_{n:n}$;

(ii) *Let* ψ *satisfy* (C10). *If* $\boldsymbol{\lambda} = \boldsymbol{\mu} = \mu \mathbf{1}_n$ *and* $(\boldsymbol{\psi}(\boldsymbol{p}), 1/\boldsymbol{\theta}; n) \in M_n$, *then* $(\boldsymbol{\psi}(\boldsymbol{q}), 1/\boldsymbol{\delta}; n) = (\boldsymbol{\psi}(\boldsymbol{p}), 1/\boldsymbol{\theta}; n) T_w \Rightarrow U_{n:n} \geq_{st} V_{n:n}$.

Proof. Here, we present the proof of the first part. The second part can be proved with the similar arguments. Denote $\zeta_n(\boldsymbol{w}, \boldsymbol{\lambda}) = \prod_{i=1}^{n} \left[1 - \psi^{-1}(w_i)\left[1 - F^{\alpha}\left(\frac{t-\lambda_i}{\theta}\right)\right]\right] = \prod_{i=1}^{n} \zeta_i(w_i, \lambda_i)$, where $\zeta_i(w_i, \lambda_i) = 1 - \psi^{-1}(w_i)\left[1 - F^{\alpha}\left(\frac{t-\lambda_i}{\theta}\right)\right]$, for $i = 1, \ldots, n$. By Theorem 3.1(*i*), one can easily check that ζ_2 satisfies (2). Rest of the proof is completed from Lemma 2.6. □

Suppose $T_{w_1}, \ldots, T_{w_k}$ are k (finite) number of T-transform matrices. Also, assume that all these T-transform matrices have the same structure. Since, the product of a finite number of T-transform matrices having the same structure is again a T-transform matrix. Hence, the above result can be extended. This is addressed in the following corollary, which is a direct consequence of Theorem 3.3. Here, $(\boldsymbol{\psi}(\boldsymbol{p}), \boldsymbol{\lambda}; n) \gg (\boldsymbol{\psi}(\boldsymbol{q}), \boldsymbol{\mu}; n)$ and $(\boldsymbol{\psi}(\boldsymbol{p}), 1/\boldsymbol{\theta}; n) \gg (\boldsymbol{\psi}(\boldsymbol{q}), 1/\boldsymbol{\delta}; n)$ hold, since $(\boldsymbol{\psi}(\boldsymbol{q}), \boldsymbol{\mu}; n)$ and $(\boldsymbol{\psi}(\boldsymbol{q}), 1/\boldsymbol{\delta}; n)$ are respectively obtained from $(\boldsymbol{\psi}(\boldsymbol{p}), \boldsymbol{\lambda}; n)$ and $(\boldsymbol{\psi}(\boldsymbol{p}), 1/\boldsymbol{\theta}; n)$ using a finite number of T-transform matrices. Denote $T_w^* = T_{w_1} \ldots T_{w_k}$.

Corollary 3.4. *Suppose* (A1), (A2) *and* (C1) *hold. Also, let* $\boldsymbol{\alpha} = \boldsymbol{\beta} = \alpha \mathbf{1}_n$ $(\alpha \leq 1)$.

(i) *Let* ψ *satisfy* (C9). *If* $\boldsymbol{\theta} = \boldsymbol{\delta} = \theta \mathbf{1}_n$ *and* $(\boldsymbol{\psi}(\boldsymbol{p}), \boldsymbol{\lambda}; n) \in M_n$, *then* $(\boldsymbol{\psi}(\boldsymbol{q}), \boldsymbol{\mu}; n) = (\boldsymbol{\psi}(\boldsymbol{p}), \boldsymbol{\lambda}; n) T_w^* \Rightarrow U_{n:n} \geq_{st} V_{n:n}$;

(ii) *Let* ψ *satisfy* (C10). *If* $\boldsymbol{\lambda} = \boldsymbol{\mu} = \mu \mathbf{1}_n$ *and* $(\boldsymbol{\psi}(\boldsymbol{p}), 1/\boldsymbol{\theta}; n) \in M_n$, *then* $(\boldsymbol{\psi}(\boldsymbol{q}), 1/\boldsymbol{\delta}; n) = (\boldsymbol{\psi}(\boldsymbol{p}), 1/\boldsymbol{\theta}; n) T_w^* \Rightarrow U_{n:n} \geq_{st} V_{n:n}$.

Note that a finite product of T-transform matrices may not produce a T-transform matrix, when they have different structures. Thus, it is natural to face the question whether Corollary 3.4 still holds for differently structured T-transform matrices. The next result shows that it is possible under some new assumptions.

Theorem 3.5. *Let* (A1), (A2) *and* (C1) *hold. Assume* $\boldsymbol{\alpha} = \boldsymbol{\beta} = \alpha \mathbf{1}_n$ $(\alpha \leq 1)$. *Further, let* $T_{w_1}, \ldots, T_{w_j}$ *be the T-transform matrices with different structures for* $j = 1, \ldots, k-1$, $(k \geq 2)$.

(*i*) *Suppose* ψ *satisfies* (C9). *If* $\boldsymbol{\theta} = \boldsymbol{\delta} = \theta\mathbf{1}_n$ *and* $(\boldsymbol{\psi}(\boldsymbol{p}), \boldsymbol{\lambda}; n) \in M_n$, *then* $(\boldsymbol{\psi}(\boldsymbol{q}), \boldsymbol{\mu}; n) = (\boldsymbol{\psi}(\boldsymbol{p}), \boldsymbol{\lambda}; n)T_{w_1} \ldots T_{w_k} \Rightarrow U_{n:n} \geq_{st} V_{n:n}$;

(*ii*) *Suppose* ψ *satisfies* (C10). *If* $\boldsymbol{\lambda} = \boldsymbol{\mu} = \mu\mathbf{1}_n$ *and* $(\boldsymbol{\psi}(\boldsymbol{p}), 1/\boldsymbol{\theta}; n) \in M_n$, *then* $(\boldsymbol{\psi}(\boldsymbol{q}), 1/\boldsymbol{\delta}; n) = (\boldsymbol{\psi}(\boldsymbol{p}), 1/\boldsymbol{\theta}; n)T_{w_1} \ldots T_{w_k} \Rightarrow U_{n:n} \geq_{st} V_{n:n}$.

Proof. We provide the proof of the first part. The second part follows similarly. Let us consider $(\boldsymbol{\psi}(\boldsymbol{p}^{(j)}), \boldsymbol{\lambda}^{(j)}; n) = (\boldsymbol{\psi}(\boldsymbol{p}), \boldsymbol{\lambda}; n)T_{w_1} \ldots T_{w_j}$, where $j = 1, \ldots, k-1$. Further, let $W_1^{(j)}, \ldots, W_n^{(j)}$ be n independent random variables having the distribution function of $W_i^{(j)}$ as

$$F_{W_i}^{(j)}(t) = 1 - \psi^{-1}(w_i^{(j)})\left[1 - F^{\alpha}\left(\frac{t - \lambda_i^{(j)}}{\theta}\right)\right],$$

for $i = 1, \ldots, n$ and $j = 1, \ldots, k-1$. Using the assumptions made, we have $(\boldsymbol{\psi}(\boldsymbol{p}^{(j)}), \boldsymbol{\lambda}^{(j)}; n) \in M_n$, for $j = 1, \ldots, k-1$. Now,

$$\begin{aligned}\left(\boldsymbol{\psi}(\boldsymbol{q}), \boldsymbol{\mu}; n\right) &= (\boldsymbol{\psi}(\boldsymbol{p}), \boldsymbol{\lambda}; n)\, T_{w_1} \ldots T_{w_k} \\ &= \left[(\boldsymbol{\psi}(\boldsymbol{p}), \boldsymbol{\lambda}; n)\, T_{w_1} \ldots T_{w_{k-1}}\right] T_{w_k} \\ &= \left(\boldsymbol{\psi}(\boldsymbol{p}^{(k-1)}), \boldsymbol{\lambda}^{(k-1)}; n\right) T_{w_k},\end{aligned}$$

which implies $W_{n:n}^{(k-1)} \geq_{st} V_{n:n}$. Similarly, $(\boldsymbol{\psi}(\boldsymbol{p}^{(k-1)}), \boldsymbol{\lambda}^{(k-1)}; n) = (\boldsymbol{\psi}(\boldsymbol{p}^{(k-2)}), \boldsymbol{\lambda}^{(k-2)}; n)\; T_{w_{k-1}}$ implies $W_{n:n}^{(k-2)} \geq_{st} W_{n:n}^{(k-1)}$. Applying similar arguments, we obtain

$$U_{n:n} \geq_{st} W_{n:n}^{(1)} \geq_{st} \ldots \geq_{st} W_{n:n}^{(k-2)} \geq_{st} W_{n:n}^{(k-1)} \geq_{st} V_{n:n}.$$

This completes the proof of the first part. □

Till now, we have derived conditions under which the usual stochastic order holds between the largest claim amounts arising from two heterogeneous portfolios of risks. Now, one may wonder whether the above results can be upgraded to other stochastic orders. Below, we answer this question affirmatively. In this part, it is shown that under some new conditions, the usual stochastic order can be extended to the reversed hazard rate order. The reversed hazard rate of $U_{n:n}$ is given by

$$\tilde{r}_{n:n}(t) = \sum_{i=1}^{n} \frac{\alpha_i}{\theta_i} r\left(\frac{t - \lambda_i}{\theta_i}\right)\left[\frac{p_i F^{(\alpha_i - 1)}\left(\frac{t-\lambda_i}{\theta_i}\right)\left[1 - F\left(\frac{t-\lambda_i}{\theta_i}\right)\right]}{1 - p_i\left[1 - F^{\alpha_i}\left(\frac{t-\lambda_i}{\theta_i}\right)\right]}\right]. \tag{10}$$

The reversed hazard rate of $V_{n:n}$, denoted by $\tilde{s}_{n:n}(\cdot)$ can be obtained by substituting q_i, δ_i, μ_i, β_i in the place of p_i, θ_i, λ_i, α_i, respectively in (10). First, we consider two heterogeneous insurance portfolios each consisting of two individual risks.

Theorem 3.6. *For* $n = 2$, *let* (A1) *and* (A2) *hold. Further, take* $\boldsymbol{\alpha} = \boldsymbol{\beta} = \mathbf{1}_n$.

(*i*) *If* $\boldsymbol{p} = \boldsymbol{q}$ $(= p\mathbf{1}_2)$ *and* $(\boldsymbol{\lambda}, 1/\boldsymbol{\theta}; 2) \in Q_2$, *then we have* $(\boldsymbol{\lambda}, 1/\boldsymbol{\theta}; 2) \gg (\boldsymbol{\mu}, 1/\boldsymbol{\delta}; 2) \Rightarrow U_{2:2} \geq_{rh} V_{2:2}$, *provided* (C2), *C*(3), (C4) *and* (C5) *are satisfied;*

(*ii*) *Let* ψ *satisfy* (C9). *If* $\boldsymbol{\theta} = \boldsymbol{\delta}$ $(= \theta\mathbf{1}_2)$ *and* $(\boldsymbol{\lambda}, \boldsymbol{\psi}(\boldsymbol{p}); 2) \in M_2$, *then* $(\boldsymbol{\lambda}, \boldsymbol{\psi}(\boldsymbol{p}); 2) \gg (\boldsymbol{\mu}, \boldsymbol{\psi}(\boldsymbol{q}); 2) \Rightarrow U_{2:2} \geq_{rh} V_{2:2}$, *provided* (C1) *and* (C8) *hold;*

(*iii*) *Let* ψ *satisfy* (C9). *If* $\boldsymbol{\lambda} = \boldsymbol{\mu}$ $(= \lambda\mathbf{1}_2)$ *and* $(1/\boldsymbol{\theta}, \boldsymbol{\psi}(\boldsymbol{p}); 2) \in Q_2$, *then* $(1/\boldsymbol{\theta}, \boldsymbol{\psi}(\boldsymbol{p}); 2) \gg (1/\boldsymbol{\delta}, \boldsymbol{\psi}(\boldsymbol{q}); 2) \Rightarrow U_{2:2} \geq_{rh} V_{2:2}$, *provided* (C2) *and* (C5)*are satisfied.*

Proof. (*i*) Under the set up, Equation (10) can be expressed as

$$\tilde{r}_{2:2}(t) = \sum_{i=1}^{2} m_i r\left((t-\lambda_i)m_i\right)\left[\frac{p\left[1-F\left((t-\lambda_i)m_i\right)\right]}{1-p\left[1-F\left((t-\lambda_i)m_i\right)\right]}\right], \tag{11}$$

where $m_i = 1/\theta_i$, for $i = 1, \ldots, n$. On differentiating (11) with respect to m_i and λ_i partially, we respectively get

$$\begin{aligned}\frac{\partial \tilde{r}_{2:2}(t)}{\partial m_i} = {} & \frac{\partial}{\partial x}\left[xr(x)\right]_{x=((t-\lambda_i)m_i)}\left[\frac{p[1-F((t-\lambda_i)m_i)]}{1-p[1-F((t-\lambda_i)m_i)]}\right] \\ & -\left[xr^2(x)\right]_{x=((t-\lambda_i)m_i)}\left[\frac{p[1-F((t-\lambda_i)m_i)]}{\left[1-p[1-F((t-\lambda_i)m_i)]\right]^2}\right]\end{aligned} \tag{12}$$

and

$$\begin{aligned}\frac{\partial \tilde{r}_{2:2}(t)}{\partial \lambda_i} = {} & -\frac{\left[x^2 r(x)\right]_{x=((t-\lambda_i)m_i)}}{(t-\lambda_i)^2}\left[\frac{r'(x)}{r(x)}\right]_{x=((t-\lambda_i)m_i)}\left[\frac{p[1-F((t-\lambda_i)m_i)]}{1-p[1-F((t-\lambda_i)m_i)]}\right] \\ & +\frac{1}{(t-\lambda_i)^2}\left[xr(x)\right]^2_{x=((t-\lambda_i)m_i)}\left[\frac{p[1-F((t-\lambda_i)m_i)]}{\left[1-p[1-F((t-\lambda_i)m_i)]\right]^2}\right].\end{aligned} \tag{13}$$

Now, consider the following

$$\phi_2(\boldsymbol{m}, \boldsymbol{\lambda}) = (m_i - m_j)\left[\frac{\partial \tilde{r}_{2:2}(t)}{\partial m_i} - \frac{\partial \tilde{r}_{2:2}(t)}{\partial m_j}\right] + (\lambda_i - \lambda_j)\left[\frac{\partial \tilde{r}_{2:2}(t)}{\partial \lambda_i} - \frac{\partial \tilde{r}_{2:2}(t)}{\partial \lambda_j}\right]. \tag{14}$$

Under the assumptions and Lemma 2.7, it can be shown that the right hand side of (14) is greater than or equals to zero. Thus, the result follows from Lemma 2.5. Using similar approach, other two parts can be proved. □

The following counterexample shows that the condition in ($C9$) is necessary to obtain the reversed hazard rate order in Theorem 3.6(ii).

Counterexample 3.3. *Consider the baseline distribution function as $F(t) = 1-(1+5t)^{-1/5}$, $t > 0$, for which the hazard rate function $r(x)$ is decreasing and convex. Let $\psi(p) = 1 - p^2$. Clearly, $\psi(p)$ does not satisfy* (C9). *Set $(\alpha_1, \alpha_2) = (\beta_1, \beta_2) = (1, 1)$, $(\theta_1, \theta_2) = (\delta_1, \delta_2) = (0.5, 0.5)$, $(\lambda_1, \lambda_2) = (0.9, 0.6)$, $(\mu_1, \mu_2) = (0.81, 0.69)$, $(p_1, p_2) = (\sqrt{0.2}, \sqrt{0.3})$ and $(q_1, q_2) = (\sqrt{0.23}, \sqrt{0.27})$. Consider a T-transform matrix $T_{0.7} = \begin{pmatrix} 0.7 & 0.3 \\ 0.3 & 0.7 \end{pmatrix}$. It can be shown that*

$$\begin{pmatrix} \mu_1 & \mu_2 \\ \psi(q_1) & \psi(q_2) \end{pmatrix} = \begin{pmatrix} \lambda_1 & \lambda_2 \\ \psi(p_1) & \psi(p_2) \end{pmatrix} T_{0.7},$$

which yields $\begin{pmatrix} \lambda_1 & \lambda_2 \\ \psi(p_1) & \psi(p_2) \end{pmatrix} \gg \begin{pmatrix} \mu_1 & \mu_2 \\ \psi(q_1) & \psi(q_2) \end{pmatrix}$. Further, $\begin{pmatrix} \lambda_1 & \lambda_2 \\ \psi(p_1) & \psi(p_2) \end{pmatrix} \in M_2$. Denote $\eta(t) = \tilde{r}_{2:2}(t) - \tilde{s}_{2:2}(t)$. Then, $\eta(1.7) = 1.3522 \times e^{-4}$ (> 0) and $\eta(1.8) = -2.0998 \times e^{-4}$ (< 0), which shows that $\eta(t)$ changes sign, when t travels from 0 to ∞. Graph is presented in Figure 2(a) for clear view. Thus, $U_{2:2} \ngeq_{rh} V_{2:2}$.

In analogy to Theorem 3.3, the result in Theorem 3.6 can also be generalized for arbitrary $n \geq 3$. Here, we use a single T-transform matrix to get $(\boldsymbol{\mu}, 1/\boldsymbol{\delta}; n)$, $(\boldsymbol{\mu}, \boldsymbol{\psi}(\boldsymbol{q}); n)$ and $(1/\boldsymbol{\delta}, \boldsymbol{\psi}(\boldsymbol{q}); n)$ from $(\boldsymbol{\lambda}, 1/\boldsymbol{\theta}; n)$, $(\boldsymbol{\lambda}, \boldsymbol{\psi}(\boldsymbol{p}); n)$ and $(1/\boldsymbol{\theta}, \boldsymbol{\psi}(\boldsymbol{p}); n)$, respectively. That is, $(\boldsymbol{\lambda}, 1/\boldsymbol{\theta}; n) \gg (\boldsymbol{\mu}, 1/\boldsymbol{\delta}; n)$, $(\boldsymbol{\lambda}, \boldsymbol{\psi}(\boldsymbol{p}); n) \gg (\boldsymbol{\mu}, \boldsymbol{\psi}(\boldsymbol{q}); n)$ and $(1/\boldsymbol{\theta}, \boldsymbol{\psi}(\boldsymbol{p}); n) \gg (1/\boldsymbol{\delta}, \boldsymbol{\psi}(\boldsymbol{q}); n)$ hold.

Theorem 3.7. *Let* ($A1$) *and* ($A2$) *hold, and $\boldsymbol{\alpha} = \boldsymbol{\beta} = \mathbf{1}_n$. Let T_w be a T-transform matrix.*

- (i) *If $\boldsymbol{p} = \boldsymbol{q}$ $(= p\mathbf{1}_n)$ and $(\boldsymbol{\lambda}, 1/\boldsymbol{\theta}; n) \in Q_n$ hold, then we have $(\boldsymbol{\mu}, 1/\boldsymbol{\delta}; n) = (\boldsymbol{\lambda}, 1/\boldsymbol{\theta}; n)T_w \Rightarrow U_{n:n} \geq_{rh} V_{n:n}$, provided* (C2), (C3), (C4) *and* (C5) *are satisfied;*
- (ii) *Suppose ψ satisfies* (C9). *Then, for $\boldsymbol{\theta} = \boldsymbol{\delta}$ $(= \theta\mathbf{1}_n)$ and $(\boldsymbol{\lambda}, \boldsymbol{\psi}(\boldsymbol{p}); n) \in M_n$, we have $(\boldsymbol{\mu}, \boldsymbol{\psi}(\boldsymbol{q}); n) = (\boldsymbol{\lambda}, \boldsymbol{\psi}(\boldsymbol{p}); n)T_w \Rightarrow U_{n:n} \geq_{rh} V_{n:n}$, provided* (C1), (C8) *hold;*

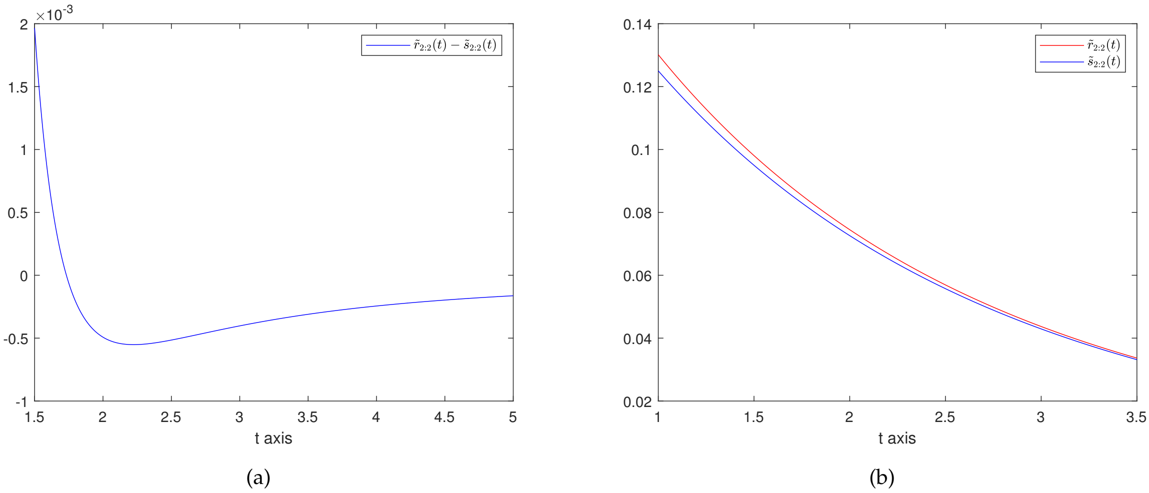


Figure 2: (a) Graph of $\tilde{r}_{2:2}(t) - \tilde{s}_{2:2}(t)$ for Counterexample 3.3. (b) Graphs of $\tilde{r}_{3:3}(t)$ and $\tilde{s}_{3:3}(t)$ for Example 5.3.

(*iii*) *Suppose* ψ *satisfies* (C9). *Then, for* $\boldsymbol{\lambda} = \boldsymbol{\mu}$ $(= \lambda \mathbf{1}_n)$ *and* $(1/\boldsymbol{\theta}, \boldsymbol{\psi}(\boldsymbol{p}); n) \in Q_n$, *we have* $(1/\boldsymbol{\delta}, \boldsymbol{\psi}(\boldsymbol{q}); n) = (1/\boldsymbol{\theta}, \boldsymbol{\psi}(\boldsymbol{p}); n)T_w \Rightarrow U_{n:n} \geq_{rh} V_{n:n}$, *provided* (C2), (C5) *hold.*

Proof. Using Lemma 2.6, the proof of the theorem follows similar to Theorem 3.3. Hence, it is omitted for the sake of brevity. □

Similar to Corollary 3.4, the above result in Theorem 3.7 can be extended from a T-transform matrix to a finite number of T-transform matrices having same structure. This is presented in the following corollary. In this sequel, we denote $T^*_w = T_{w_1} \dots T_{w_k}$, where k is finite.

Corollary 3.8. *Under the assumptions in* (A1) *and* (A2), *let* $\boldsymbol{\alpha} = \boldsymbol{\beta} = \mathbf{1}_n$.

(*i*) *Then, for* $\boldsymbol{p} = \boldsymbol{q}$ $(= p\mathbf{1}_n)$ *and* $(\boldsymbol{\lambda}, 1/\boldsymbol{\theta}; n) \in Q_n$, *we get* $(\boldsymbol{\mu}, 1/\boldsymbol{\delta}; n) = (\boldsymbol{\lambda}, 1/\boldsymbol{\theta}; n)T^*_w \Rightarrow U_{n:n} \geq_{rh} V_{n:n}$, *provided* (C2), (C3), (C4) *and* (C5) *hold;*

(*ii*) *Suppose* ψ *satisfies* (C9). *Let* (C1) *and* (C8) *hold. Then, for* $\boldsymbol{\theta} = \boldsymbol{\delta}$ $(= \theta\mathbf{1}_n)$ *and* $(\boldsymbol{\lambda}, \boldsymbol{\psi}(\boldsymbol{p}); n) \in M_n$, *we get* $(\boldsymbol{\mu}, \boldsymbol{\psi}(\boldsymbol{q}); n) = (\boldsymbol{\lambda}, \boldsymbol{\psi}(\boldsymbol{p}); n)\ T^*_w \Rightarrow U_{n:n} \geq_{rh} V_{n:n}$;

(*iii*) *Suppose* ψ *satisfies* (C9). *Let* (C2) *and* (C5) *be satisfied. Further, assume* $\boldsymbol{\lambda} = \boldsymbol{\mu}$ $(= \lambda\mathbf{1}_n)$ *and* $(1/\boldsymbol{\theta}, \boldsymbol{\psi}(\boldsymbol{p}); n) \in Q_n$. *Then,* $(1/\boldsymbol{\delta}, \boldsymbol{\psi}(\boldsymbol{q}); n) = (1/\boldsymbol{\theta}, \boldsymbol{\psi}(\boldsymbol{p}); n)T^*_w \Rightarrow U_{n:n} \geq_{rh} V_{n:n}$.

Next result states that the results in Corollary 3.8 also hold, when we have T-transform matrices with different structures instead of the same structure. The proof is similar to that of Theorem 3.5, and therefore, it is omitted.

Theorem 3.9. *Under* (A1) *and* (A2), *let* $\boldsymbol{\alpha} = \boldsymbol{\beta} = \mathbf{1}_n$. *Further, let* $T_{w_1}, \dots, T_{w_j}$ *be the T-transform matrices with different structures, for* $j = 1, \dots, k-1$, *where* $k \geq 2$.

(*i*) *Then, for* $\boldsymbol{p} = \boldsymbol{q}$ $(= p\mathbf{1}_n)$ *and* $(\boldsymbol{\lambda}, 1/\boldsymbol{\theta}; n) \in Q_n$, *we get* $(\boldsymbol{\mu}, 1/\boldsymbol{\delta}; n) = (\boldsymbol{\lambda}, 1/\boldsymbol{\theta}; n)T_{w_1} \dots T_{w_k} \Rightarrow U_{n:n} \geq_{rh} V_{n:n}$, *provided* (C2), (C3), (C4) *and* (C5) *hold;*

(*ii*) *Suppose* ψ *satisfies* (C9). *Let* (C1) *and* (C8) *hold. Then, for* $\boldsymbol{\theta} = \boldsymbol{\delta}$ $(= \theta\mathbf{1}_n)$ *and* $(\boldsymbol{\lambda}, \boldsymbol{\psi}(\boldsymbol{p}); n) \in M_n$, *we get* $(\boldsymbol{\mu}, \boldsymbol{\psi}(\boldsymbol{q}); n) = (\boldsymbol{\lambda}, \boldsymbol{\psi}(\boldsymbol{p}); n)\ T_{w_1} \dots T_{w_k} \Rightarrow U_{n:n} \geq_{rh} V_{n:n}$;

(*iii*) *Suppose* ψ *satisfies* (C9). *For* $\boldsymbol{\lambda} = \boldsymbol{\mu}$ $(= \lambda\mathbf{1}_n)$ *and* $(1/\boldsymbol{\theta}, \boldsymbol{\psi}(\boldsymbol{p}); n) \in Q_n$, *we have* $(1/\boldsymbol{\delta}, \boldsymbol{\psi}(\boldsymbol{q}); n) = (1/\boldsymbol{\theta}, \boldsymbol{\psi}(\boldsymbol{p}); n)\ T_{w_1} \dots T_{w_k} \Rightarrow U_{n:n} \geq_{rh} V_{n:n}$, *provided* (C2) *and* (C5) *hold.*

4. Vector majorization

This section is devoted to derive sufficient conditions for the comparison results between the largest claim amounts of two heterogeneous portfolios of risks in the sense of the usual stochastic and the reversed hazard rate orderings. First, we present the results for the usual stochastic ordering. The following theorem shows that under some conditions, the weakly supermajorized vector of shape parameters leads to larger largest claim amount in the sense of the usual stochastic order. Here, we assume that both the location and scale parameters are same and fixed.

Theorem 4.1. *Let* $(A1)$ *and* $(A2)$ *hold. Also, for* $i = 1, \ldots, n$, *assume* $\theta_i = \delta_i = \theta$, $p_i = q_i$, $\lambda_i = \mu_i = \lambda$ *and* $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathcal{E}_+(\mathcal{D}_+)$, $\boldsymbol{p} \in \mathcal{D}_+(\mathcal{E}_+)$. *Then,* $\boldsymbol{\alpha} \succeq^{w} \boldsymbol{\beta}$ *implies* $U_{n:n} \leq_{st} V_{n:n}$.

Proof. First, we consider the case $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathcal{E}_+$ and $\boldsymbol{p} \in \mathcal{D}_+$. Under the given set up, the distribution function of $U_{n:n}$ is obtained as

$$F_{n:n}(t) = \prod_{i=1}^{n} \left[1 - p_i \left[1 - F^{\alpha_i}\left(\frac{t-\lambda}{\theta}\right)\right]\right]. \tag{15}$$

After taking the partial derivative of $F_{n:n}(t)$ with respect to α_i, we get

$$\frac{\partial F_{n:n}(t)}{\partial \alpha_i} = G_i F_{n:n}(t), \tag{16}$$

where $G_i = (p_i \ln\left[F(\frac{t-\lambda}{\theta})\right] F^{\alpha_i}\left(\frac{t-\lambda}{\theta}\right))/(1 - p_i\left[1 - F^{\alpha_i}\left(\frac{t-\lambda}{\theta}\right)\right])$. For $i \leq j$, we have $p_i \geq p_j$, $\alpha_i \leq \alpha_j$. Thus, by Lemma 2.8, we obtain $G_i \leq G_j$. Hence, the difference $\left[\frac{\partial F_{n:n}(t)}{\partial \alpha_i} - \frac{\partial F_{n:n}(t)}{\partial \alpha_j}\right]$ can be shown to be at most zero. This implies that $F_{n:n}(t)$ is Schur-convex with respect to $\boldsymbol{\alpha} \in \mathcal{E}_+$ by Lemma 3.3 of Kundu et al. [14]. Also, it is decreasing with respect to $\boldsymbol{\alpha} \in \mathcal{E}_+$. Thus, the rest of the proof is completed by Theorem $A.8$ of Marshall et al. [15]. The proof is similar when $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathcal{D}_+$ and $\boldsymbol{p} \in \mathcal{E}_+$. □

In the following theorem, we consider that the location, scale and shape parameters are same, but vector valued.

Theorem 4.2. *Suppose* ψ *is a differentiable function. Let* $(A1)$ *and* $(A2)$ *hold. Further, take* $\alpha_i = \beta_i$, $\theta_i = \delta_i$ *and* $\lambda_i = \mu_i$, *for* $i = 1, \ldots, n$. *If* $\psi(.)$ *satisfies* (C9), *then* $\psi(\boldsymbol{p}) \succeq_w \psi(\boldsymbol{q}) \Rightarrow U_{n:n} \geq_{st} V_{n:n}$, *provided* $\boldsymbol{\theta}, \boldsymbol{\lambda}, \boldsymbol{\alpha}, \boldsymbol{p}, \boldsymbol{q} \in \mathcal{E}_+(\mathcal{D}_+)$.

Proof. We take $\boldsymbol{\theta}, \boldsymbol{\lambda}, \boldsymbol{\alpha}, \boldsymbol{p}, \boldsymbol{q} \in \mathcal{E}_+$. The proof of the other case is similar. Note that the distribution function of $U_{n:n}$ is

$$F_{n:n}(t) = \prod_{i=1}^{n} \left[1 - \psi^{-1}(w_i)\left[1 - F^{\alpha_i}\left(\frac{t-\lambda_i}{\theta_i}\right)\right]\right], \tag{17}$$

where $p_i = \psi^{-1}(w_i)$, for $i = 1, \ldots, n$. Differentiating $F_{n:n}(t)$ with respect to w_i partially, we obtain

$$\frac{\partial F_{n:n}(t)}{\partial w_i} = -\frac{\partial \psi^{-1}(w_i)}{\partial w_i} C_i F_{n:n}(t), \tag{18}$$

where $C_i = ([1 - F^{\alpha_i}(\frac{t-\lambda_i}{\theta_i})])/(1 - \psi^{-1}(w_i)[1 - F^{\alpha_i}(\frac{t-\lambda_i}{\theta_i})])$. Under the assumptions made, for $i \leq j$, we have $w_i \leq w_j$, $\lambda_i \leq \lambda_j$, $\theta_i \leq \theta_j$, $\alpha_i \leq \alpha_j$ and $\psi(w)$ is increasing, convex. Therefore, $\psi^{-1}(\cdot)$ is also increasing and convex. This implies $\psi^{-1}(w_i) \leq \psi^{-1}(w_j)$. Since $\psi^{-1}(\cdot)$ is convex, we can write $\frac{\partial \psi^{-1}(w_i)}{\partial w_i} \leq \frac{\partial \psi^{-1}(w_j)}{\partial w_j}$. Further, by Lemma 2.7, we have $C_i \leq C_j$. Combining these two inequalities, it can be checked that Equation (18) is negative, and the difference $\left[\frac{\partial F_{n:n}(t)}{\partial w_i} - \frac{\partial F_{n:n}(t)}{\partial w_j}\right]$ is greater than or equals to zero. Hence, by using Lemma 3.3 of Kundu et al. [14], $F_{n:n}(t)$ is Schur-concave with respect to $\boldsymbol{w} \in \mathcal{E}_+$. Also, $F_{n:n}(t)$ is decreasing. Now, applying Theorem $A.8$ of Marshall et al. [15], we get the required result. □

Remark 4.3. *Theorem 4.2 demonstrates that more heterogeneity among transformed occurrence probabilities with respect to the weakly submajorization order provides better tail function of the largest claim amount.*

The following theorem shows that the usual stochastic ordering holds between the largest claim amounts, when the reciprocal of the scale parameters are associated with the p-majorization and reciprocal majorization orders. We assume that the shape parameters are less than or equal to 1.

Theorem 4.4. *Suppose (A1) and (A2) hold. For $i = 1, \dots, n$, let $\alpha_i = \beta_i = \alpha \le 1$, $p_i = q_i$ and $\lambda_i = \mu_i$. Take $\boldsymbol{p}, \boldsymbol{\theta}, \boldsymbol{\delta}, \boldsymbol{\lambda} \in \mathcal{E}_+(\mathcal{D}_+)$.*

- (*i*) *If (C2) holds, then $1/\boldsymbol{\theta} \ge^{p} 1/\boldsymbol{\delta}$ implies $U_{n:n} \ge_{st} V_{n:n}$;*
- (*ii*) *If (C3) holds, then $1/\boldsymbol{\theta} \ge^{rm} 1/\boldsymbol{\delta}$ implies $U_{n:n} \ge_{st} V_{n:n}$.*

Proof. (*i*) We prove the result when $\boldsymbol{p}, \boldsymbol{\theta}, \boldsymbol{\delta}, \boldsymbol{\lambda} \in \mathcal{E}_+$. The proof for $\boldsymbol{p}, \boldsymbol{\theta}, \boldsymbol{\delta}, \boldsymbol{\lambda} \in \mathcal{D}_+$ is analogous. Under the present set up, the distribution function of $U_{n:n}$ is

$$F_{n:n}(t) = \prod_{i=1}^{n} \left[1 - p_i \left[1 - F^{\alpha}\left((t-\lambda_i)e^{s_i}\right)\right]\right], \tag{19}$$

where $s_i = -\ln \theta_i$, for $i = 1, \dots, n$. To prove the required result, we consider

$$\chi_1(e^{s}) = \prod_{i=1}^{n} \left[1 - p_i \left[1 - F^{\alpha}\left((t-\lambda_i)e^{s_i}\right)\right]\right].$$

Differentiating $\chi_1(e^{s})$ with respect to s_i partially, we have

$$\frac{\partial \chi_1(e^{s})}{\partial s_i} = \left[\frac{\alpha F^{(\alpha-1)}(x)[1-F(x)]}{1-F^{\alpha}(x)}\right]_{x=(t-\lambda_i)e^{s_i}} [xr(x)]_{x=(t-\lambda_i)e^{s_i}} A_i \chi_1(e^{s}), \tag{20}$$

where $A_i = ([1 - F^{\alpha}((t-\lambda_i)e^{s_i})])/(1-p_i[1-F^{\alpha}((t-\lambda_i)e^{s_i})])$. First, we consider the case $s_i \ge s_j$, $\lambda_i \le \lambda_j$, $p_i \le p_j$. Using the given assumptions and Lemma 2.7, the inequality $A_i \le A_j$ holds. Combining this with Lemma 3 of Balakrishnan et al. [2] and the decreasing property of $xr(x)$, we obtain

$$\left[\frac{\partial \chi_1(e^{s})}{\partial s_i} - \frac{\partial \chi_1(e^{s})}{\partial s_j}\right] \le 0. \tag{21}$$

From Lemma 3.1 of Kundu et al. [14], (21) implies that $\chi_1(e^{s})$ is Schur-concave with respect to $\boldsymbol{s} \in \mathcal{D}_+$. Further, $\chi_1(e^{s})$ is increasing with respect to $\boldsymbol{s}$. Thus, by Lemma 2.1 of Khaledi and Kochar [13], the rest of the proof is completed.

(*ii*) The cumulative distribution function of $U_{n:n}$ is

$$F_{n:n}(t) = \prod_{i=1}^{n} \left[1 - p_i \left[1 - F^{\alpha}\left(\frac{t-\lambda_i}{\theta_i}\right)\right]\right]. \tag{22}$$

Denote $\chi_2(1/\boldsymbol{\theta}) = \prod_{i=1}^{n}[1 - p_i[1 - F^{\alpha}(\frac{t-\lambda_i}{\theta_i})]]$. Differentiating, $\chi_2(1/\boldsymbol{\theta})$ with respect to θ_i partially, we get

$$\frac{\partial \chi_2(1/\boldsymbol{\theta})}{\partial \theta_i} = -\left[\frac{\alpha F^{(\alpha-1)}(x)[1-F(x)]}{1-F^{\alpha}(x)}\right]_{x=\left(\frac{t-\lambda_i}{\theta_i}\right)} \frac{\left[x^2 r(x)\right]_{x=\left(\frac{t-\lambda_i}{\theta_i}\right)}}{(t-\lambda_i)} B_i \chi_2(1/\boldsymbol{\theta}), \tag{23}$$

where $B_i = ([1 - F^{\alpha}(\frac{t-\lambda_i}{\theta_i})])/(1 - p_i[1 - F^{\alpha}(\frac{t-\lambda_i}{\theta_i})])$. Now, similar to the arguments used in the proof of Part (*i*) and by Lemma 1 of Hazra et al. [12], it can be established that $\chi_2(1/\boldsymbol{\theta})$ is decreasing and Schur-concave with respect to $\boldsymbol{\theta} \in \mathcal{E}_+(\mathcal{D}_+)$. Hence, the proof is completed. □

In the next, we prove ordering result between two largest claim amounts in the sense of the usual stochastic order, when the vectors of the location parameters are connected with the weak submajorization order. We consider that the scale and risk parameters are same, but vector valued. The proof is omitted since it follows similarly to that of Theorem 4.4(i).

Theorem 4.5. *Let* $(A1)$, $(A2)$ *and* $(C2)$ *hold. If* $\boldsymbol{\theta}, \boldsymbol{\lambda}, \boldsymbol{\mu}, \boldsymbol{p} \in \mathcal{D}_+(\mathcal{E}_+)$, *then we have* $\boldsymbol{\lambda} \succeq_w \boldsymbol{\mu} \Rightarrow U_{n:n} \geq_{st} V_{n:n}$, *provided* $\boldsymbol{\alpha} = \boldsymbol{\beta} = \alpha\mathbf{1}_n$ $(\alpha \leq 1)$, $\boldsymbol{\theta} = \boldsymbol{\delta}$ *and* $\boldsymbol{p} = \boldsymbol{q}$.

Below, we obtain two different sets of sufficient conditions for the existence of the usual stochastic order between the largest claim amounts arising from heterogeneous portfolios of risks for the case of different location, scale and risk parameters.

Theorem 4.6. *Let a function* ψ *be differentiable. Suppose* $(A1)$ *and* $(A2)$ *hold. Further, let* $\boldsymbol{\alpha} = \boldsymbol{\beta} = \alpha\mathbf{1}_n$ $(\alpha \leq 1)$ *and* $\boldsymbol{\theta}, \boldsymbol{\lambda}, \boldsymbol{\mu}, \boldsymbol{\delta}, \boldsymbol{p}, \boldsymbol{q} \in \mathcal{E}_+(\mathcal{D}_+)$.

(i) *If* $(C2)$ *and* $(C9)$ *hold, then* $1/\boldsymbol{\theta} \succeq^{p} 1/\boldsymbol{\delta}$, $\psi(\boldsymbol{p}) \succeq_w \psi(\boldsymbol{q})$ *and* $\boldsymbol{\lambda} \succeq_w \boldsymbol{\mu}$ *imply* $U_{n:n} \geq_{st} V_{n:n}$;

(ii) *If* $(C2)$, $(C3)$ *and* $(C9)$ *hold, then* $1/\boldsymbol{\theta} \succeq^{rm} 1/\boldsymbol{\delta}$, $\psi(\boldsymbol{p}) \succeq_w \psi(\boldsymbol{q})$ *and* $\boldsymbol{\lambda} \succeq_w \boldsymbol{\mu}$ *imply* $U_{n:n} \geq_{st} V_{n:n}$.

Proof. The proof of the first (second) part of the theorem follows from Theorem 4.2, Theorem 4.4(i) ((ii)) and Theorem 4.5. So, it is omitted. □

The following theorem states that the kth largest claim amounts can be comparable with respect to the usual stochastic order.

Theorem 4.7. *Let* $(A1)$ *and* $(A2)$ *hold, and* ψ *be a differentiable function. Also, let* $\lambda_i = \mu_i = \lambda$, $\theta_i = \delta_i = \theta$, *for* $i = 1, \ldots, n$. *Then,* $\boldsymbol{\alpha} \succeq^{m} \boldsymbol{\beta} \Rightarrow U_{k:n} \leq_{st} V_{k:n}$, *provided* $\psi(\boldsymbol{p}) = \psi(\boldsymbol{q}) = v$.

Proof. Consider the function $F(\alpha_i, t) = 1 - \psi^{-1}(v)\left[1 - F^{\alpha_i}\left(\frac{t-\lambda}{\theta}\right)\right]$, which can be shown to be increasing and log-convex with respect to α_i, for $i = 1, \ldots, n$. Thus, the result follows from Theorem 3.5 of Pledger and Proschan [17]. □

In this part, we study the conditions under which the reversed hazard rate order holds between the largest claim amounts from two heterogeneous insurance portfolios of risks.

Theorem 4.8. *Assume that* $(A1)$ *and* $(A2)$ *hold. Further, let* $\boldsymbol{\alpha} = \boldsymbol{\beta} = \mathbf{1}_n$, $\boldsymbol{p} = \boldsymbol{q}$, $\boldsymbol{\theta} = \boldsymbol{\delta}$ *and* $\boldsymbol{\lambda}, \boldsymbol{\theta}, \boldsymbol{\mu}, \boldsymbol{p} \in \mathcal{E}_+(\mathcal{D}_+)$. *Then,* $\boldsymbol{\lambda} \succeq_w \boldsymbol{\mu} \Rightarrow U_{n:n} \geq_{rh} V_{n:n}$ *if* $(C2)$, $(C3)$ *and* $(C4)$ *are satisfied.*

Proof. Under the set up, the reversed hazard rate function of $U_{n:n}$ is

$$\tilde{r}_{n:n}(t) = \sum_{i=1}^{n} \frac{1}{\theta_i} r\left(\frac{t-\lambda_i}{\theta_i}\right)\left[\frac{p_i\left[1 - F\left(\frac{t-\lambda_i}{\theta_i}\right)\right]}{1 - p_i\left[1 - F\left(\frac{t-\lambda_i}{\theta_i}\right)\right]}\right]. \tag{24}$$

The partial derivative of $\tilde{r}_{n:n}(t)$ given by (24) with respect to λ_i, is given by

$$\begin{aligned}\frac{\partial \tilde{r}_{n:n}(t)}{\partial \lambda_i} = & -\frac{\left[x^2 r(x)\right]_{x=\left(\frac{t-\lambda_i}{\theta_i}\right)}}{(t-\lambda_i)^2}\left[\frac{r'(x)}{r(x)}\right]_{x=\left(\frac{t-\lambda_i}{\theta_i}\right)}\left[\frac{p_i\left[1-F\left(\frac{t-\lambda_i}{\theta_i}\right)\right]}{1-p_i\left[1-F\left(\frac{t-\lambda_i}{\theta_i}\right)\right]}\right] \\ & + \frac{1}{(t-\lambda_i)^2}\left[x r(x)\right]^2_{x=\left(\frac{t-\lambda_i}{\theta_i}\right)}\left[\frac{p_i\left[1-F\left(\frac{t-\lambda_i}{\theta_i}\right)\right]}{\left[1-p_i\left[1-F\left(\frac{t-\lambda_i}{\theta_i}\right)\right]\right]^2}\right].\end{aligned} \tag{25}$$

It is easy to see that $\tilde{r}_{n:n}(t)$ is increasing with respect to λ_i, $i = 1, \ldots, n$, since $r(x)$ is decreasing. For any $1 \leq i \leq j \leq n$, we have $\frac{t-\lambda_i}{\theta_i} \geq (\leq) \frac{t-\lambda_j}{\theta_j}$. Applying the property of $\frac{r'(x)}{r(x)}$ (≤ 0) and $xr(x)$, we obtain

$$-\left[\frac{r'(x)}{r(x)}\right]_{x=\frac{t-\lambda_i}{\theta_i}} \leq (\geq) -\left[\frac{r'(x)}{r(x)}\right]_{x=\frac{t-\lambda_j}{\theta_j}} \text{ and } \left[xr(x)\right]_{x=\frac{t-\lambda_i}{\theta_i}} \leq (\geq)\left[xr(x)\right]_{x=\frac{t-\lambda_j}{\theta_j}}.$$

With these results, Lemma 2.7 yields that $[\frac{\partial \tilde{r}_{n:n}(t)}{\partial \lambda_i} - \frac{\partial \tilde{r}_{n:n}(t)}{\partial \lambda_j}]$ is less than or equals (greater than or equals) to zero. Thus, $\tilde{r}_{n:n}(t)$ is Schur-convex with respect to $\boldsymbol{\lambda} \in \mathcal{E}_+(\mathcal{D}_+)$ by using Lemma 3.3 (Lemma 3.1) of Kundu et al. [14]. The rest of the proof is completed by Theorem $A.8$ of Marshall et al. [15]. □

In the next theorem, we consider that the shape parameter vectors are equal to $\mathbf{1}_n$ and the location parameter vectors are same.

Theorem 4.9. *Suppose* $(A1)$ *and* $(A2)$ *hold. Further, we assume* $\boldsymbol{\alpha} = \boldsymbol{\beta} = \mathbf{1}_n, \boldsymbol{p} = \boldsymbol{q}, \boldsymbol{\lambda} = \boldsymbol{\mu}$ *and* $\boldsymbol{\lambda}, \boldsymbol{\theta}, \boldsymbol{\delta}, \boldsymbol{p} \in \mathcal{E}_+(\mathcal{D}_+)$.

- (i) *If* $(C2)$ *and* $(C5)$ *hold, then* $1/\boldsymbol{\theta} \succeq^{w} 1/\boldsymbol{\delta} \Rightarrow U_{n:n} \geq_{rh} V_{n:n}$;
- (ii) *If* $(C2)$, $(C6)$ *and* $(C7)$ *hold, then* $1/\boldsymbol{\theta} \succeq^{rm} 1/\boldsymbol{\delta} \Rightarrow U_{n:n} \geq_{rh} V_{n:n}$.

Proof. (i) Under the given set up, we have

$$\tilde{r}_{n:n}(t) = \sum_{i=1}^{n} m_i r\left((t-\lambda_i)m_i\right)\left[\frac{p_i\left[1-F\left((t-\lambda_i)m_i\right)\right]}{1-p_i\left[1-F\left((t-\lambda_i)m_i\right)\right]}\right], \tag{26}$$

where $m_i = 1/\theta_i$, for $i = 1, \ldots, n$. Taking derivative of (26) with respect to m_i partially, we obtain

$$\begin{aligned}\frac{\partial \tilde{r}_{n:n}(t)}{\partial m_i} = \quad & \frac{\partial}{\partial x}\left[xr(x)\right]_{x=((t-\lambda_i)m_i)}\left[\frac{p_i[1-F((t-\lambda_i)m_i)]}{1-p_i[1-F((t-\lambda_i)m_i)]}\right] \\ & -\left[xr^2(x)\right]_{x=((t-\lambda_i)m_i)}\left[\frac{p_i[1-F((t-\lambda_i)m_i)]}{\left[1-p_i[1-F((t-\lambda_i)m_i)]\right]^2}\right].\end{aligned} \tag{27}$$

From (27), it is clear that $\tilde{r}_{n:n}(t)$ is decreasing with respect to m_i, for $i = 1, \ldots, n$. Now, let us take $1 \leq i \leq j \leq n$. Then, $(t-\lambda_i)m_i \geq (\leq)(t-\lambda_j)m_j$. Moreover, $xr(x)$ is decreasing and convex. Hence, we get $\frac{\partial}{\partial x}\left[xr(x)\right]_{x=((t-\lambda_i)m_i)} \geq (\leq) \frac{\partial}{\partial x}\left[xr(x)\right]_{x=\left((t-\lambda_j)m_j\right)}$ and $[xr(x)]_{x=(t-\lambda_i)m_i} \leq (\geq)[xr(x)]_{x=(t-\lambda_j)m_j}$. Utilizing Lemma 2.7, we can show that for any $i \leq j$, $\frac{\partial \tilde{r}_{n:n}(t)}{\partial m_i} - \frac{\partial \tilde{r}_{n:n}(t)}{\partial m_j} \geq (\leq) 0$. Thus, $\tilde{r}_{n:n}(t)$ is Schur-convex with respect to $\boldsymbol{m} \in \mathcal{D}_+(\mathcal{E}_+)$. The rest of the proof follows from Lemma 3.1 (Lemma 3.3) of Kundu et al. [14] and Theorem $A.8$ of Marshall et al. [15]. The second part of the theorem can be proved in a similar manner by using Lemma 1 of Hazra et al. [12]. Thus, it is omitted. □

Theorem 4.10. *Let* $(A1)$, $(A2)$ *and* $(C2)$ *hold. Then,*

- (i) $\{\boldsymbol{\alpha} = \boldsymbol{\beta} = \mathbf{1}_n, \boldsymbol{\theta} \geq \boldsymbol{\delta}, \boldsymbol{\lambda} \geq \boldsymbol{\mu}, \boldsymbol{p} \geq \boldsymbol{q}\} \Rightarrow U_{n:n} \geq_{rh} V_{n:n}$;
- (ii) $\{\boldsymbol{\alpha} = \boldsymbol{\beta} = \alpha < 1, \boldsymbol{\theta} \geq \boldsymbol{\delta}, \boldsymbol{\lambda} \geq \boldsymbol{\mu}, \boldsymbol{p} \geq \boldsymbol{q}\} \Rightarrow U_{n:n} \geq_{rh} V_{n:n}$.

Proof. To prove the first part, it suffices to show that

$$\sum_{i=1}^{n} \frac{p_i}{\theta_i} r\left(\frac{t-\lambda_i}{\theta_i}\right) s_i \geq \sum_{i=1}^{n} \frac{q_i}{\delta_i} r\left(\frac{t-\mu_i}{\delta_i}\right) t_i, \tag{28}$$

where $s_i = (p_i[1-F(\frac{t-\lambda_i}{\theta_i})])/(1-p_i[1-F(\frac{t-\lambda_i}{\theta_i})]$ and $t_i = (q_i[1-F(\frac{t-\mu_i}{\delta_i})])(1-q_i[1-F(\frac{t-\mu_i}{\delta_i})])$. Note that the inequality given by (28) holds if

$$\frac{1}{\delta_i} r\left(\frac{t-\mu_i}{\delta_i}\right) \leq \frac{1}{\theta_i} r\left(\frac{t-\lambda_i}{\theta_i}\right) \tag{29}$$

and $s_i \geq t_i$, for all $i = 1, \ldots, n$. Thus, the proof is completed by the given assumptions and Lemma 2.7. The second part of the theorem can be proved by Lemmas $3(i), 3(ii)$ of Balakrishnan et al. [2]. □

Next theorem shows that $V_{n:n}$ is dominated by $U_{n:n}$ with respect to the reversed hazard rate order under some conditions. Here, we take that the location parameter and scale parameter vectors are equal. The shape parameters are taken fixed and less than or equal to 1.

Theorem 4.11. *Suppose $\psi : (0,1) \to (0,\infty)$ is a differentiable function satisfying* (C9). *Let* $(A1)$, $(A2)$ *and* $(C2)$ *hold. Again,* $\boldsymbol{\theta} = \boldsymbol{\delta}$, $\boldsymbol{\lambda} = \boldsymbol{\mu}$, $\boldsymbol{\alpha} = \boldsymbol{\beta} = \alpha \mathbf{1}_n$ (≤ 1) *and* $\boldsymbol{\lambda}$, $\boldsymbol{\theta}$, $\boldsymbol{p}$, $\boldsymbol{q} \in \mathcal{E}_+(\mathcal{D}_+)$. *Then,* $\psi(\boldsymbol{p}) \geq_w \psi(\boldsymbol{q}) \Rightarrow U_{n:n} \geq_{rh} V_{n:n}$.

Proof. The reversed hazard rate of $U_{n:n}$ is

$$\tilde{r}_{n:n}(t) = \sum_{i=1}^{n} \frac{1}{\theta_i} r\left(\frac{t-\lambda_i}{\theta_i}\right) D_i G_i, \tag{30}$$

where $D_i = ([1 - F^{\alpha}(\frac{t-\lambda_i}{\theta_i})])/(1 - \psi^{-1}(w_i)[1 - F^{\alpha}(\frac{t-\lambda_i}{\theta_i})])$ and $G_i = (\alpha F^{\alpha-1}(\frac{t-\lambda_i}{\theta_i})[1 - F(\frac{t-\lambda_i}{\theta_i})])/([1 - F^{\alpha}(\frac{t-\lambda_i}{\theta_i})])$. On differentiating (30) with respect to w_i partially, we get

$$\frac{\partial \tilde{r}_{n:n}(t)}{\partial w_i} = \frac{\partial \psi^{-1}(w_i)}{\partial w_i} \frac{D_i G_i}{1 - \psi^{-1}(w_i)\left[1 - F^{\alpha}\left(\frac{t-\lambda_i}{\theta_i}\right)\right]} \frac{1}{\theta_i} r\left(\frac{t-\lambda_i}{\theta_i}\right). \tag{31}$$

Since $\psi(w)$ is increasing, $\psi^{-1}(w)$ is also increasing. Therefore, $\tilde{r}_{n:n}(t)$ is increasing with respect to w_i, $i = 1, \ldots, n$. Now, under the assumptions made, we have $\frac{t-\lambda_i}{\theta_i} \geq (\leq) \frac{t-\lambda_j}{\theta_j}$. Using (C2), we can write $[xr(x)]_{x=\frac{t-\lambda_i}{\theta_i}} \leq (\geq) [xr(x)]_{x=\frac{t-\lambda_j}{\theta_j}}$. Further, by Lemma 2.7, and Lemma 3 of Balakrishnan et al. [2], we get $\frac{\partial \tilde{r}_{n:n}(t)}{\partial w_i} - \frac{\partial \tilde{r}_{n:n}(t)}{\partial w_j} \leq (\geq) 0$. Thus, $\tilde{r}_{n:n}(t)$ is Schur-convex with respect to $\boldsymbol{w} \in \mathcal{E}_+(\mathcal{D}_+)$, and the desired result readily follows from Lemma 3.3 (Lemma 3.1) of Kundu et al. [14] and Theorem A.8 of Marshall et al. [15]. □

The following theorem is an extension of Theorem 4.6. Proof of the first part of the theorem follows from Theorem 4.8, Theorem 4.9(i) and Theorem 4.11. The second part follows from Theorem 4.8, Theorem 4.9(ii) and Theorem 4.11.

Theorem 4.12. *Let ψ be a differentiable function. Further, let* $(A1)$ *and* $(A2)$ *hold. Also, assume* $\boldsymbol{\alpha} = \boldsymbol{\beta} = \mathbf{1}_n$ *and* $\boldsymbol{\theta}$, $\boldsymbol{\lambda}$, $\boldsymbol{\mu}$, $\boldsymbol{\delta}$, $\boldsymbol{p}$, $\boldsymbol{q} \in \mathcal{E}_+(\mathcal{D}_+)$.

(i) *Suppose* (C2), (C5) *and* (C9) *hold. Then,* $1/\boldsymbol{\theta} \geq^{w} 1/\boldsymbol{\delta}$, $\psi(\boldsymbol{p}) \geq_w \psi(\boldsymbol{q})$ *and* $\boldsymbol{\lambda} \geq_w \boldsymbol{\mu}$ *imply* $U_{n:n} \geq_{rh} V_{n:n}$;

(ii) *Suppose* (C2), (C3), (C6), (C7) *and* (C9) *hold. Then,* $1/\boldsymbol{\theta} \geq^{rm} 1/\boldsymbol{\delta}$, $\psi(\boldsymbol{p}) \geq_w \psi(\boldsymbol{q})$ *and* $\boldsymbol{\lambda} \geq_w \boldsymbol{\mu}$ *imply* $U_{n:n} \geq_{rh} V_{n:n}$.

Remark 4.13. *On using Theorem* 2.3 *of Zardasht [19], the usual stochastic ordering implies the incomplete cumulative residual entropy ordering. Further, the reversed hazard rate ordering implies the usual stochastic ordering. Therefore, the results obtained in this paper also compare two largest claim amounts arising from two heterogeneous portfolios of risks in the sense of the incomplete cumulative residual entropy ordering.*

5. Applications

In this section, we consider two special probability models and show the applicability of the established results. We take generalized linear failure rate and pareto distributions. For these distributions, we present few corollaries. However, one can easily obtain similar applications for other established results.

5.1 Generalized linear failure rate distribution

The distribution function of the generalized linear failure rate distribution is given by

$$F(x) = \left[1 - e^{-(ax + \frac{b}{2}x^2)}\right]^d, \; x \geq 0, \; a, \; b, \; d > 0. \tag{32}$$

One can easily check that for $d = 0.5, a = 1, b = 0$ and $d = 1, a = 1, b = 0$, the hazard rate function of (32) satisfies the conditions given in (C1) and $(C8)$. Here, we consider generalized linear failure rate distribution as the baseline distribution function.

Corollary 5.1. *Under the assumptions of Theorem 4.1 ,* $\boldsymbol{\alpha} \geq^{w} \boldsymbol{\beta} \Rightarrow U_{n:n} \leq_{st} V_{n:n}$.

Corollary 5.2. *Let the assumptions of Theorem 3.6(ii) hold. Then,* $(\boldsymbol{\lambda}, \psi(\boldsymbol{p}); 2) >> (\boldsymbol{\mu}, \psi(\boldsymbol{q}); 2) \Rightarrow U_{2:2} \geq_{rh} V_{2:2}$.

Next, we consider an example to illustrate Corollary 5.2.

Example 5.3. *Consider* $\{X_1, X_2\}$ *and* $\{Y_1, Y_2\}$ *are the collections of independent random variables such that* $X_i \sim F^{\alpha}(\frac{x-\lambda_i}{\theta})$ *and* $Y_i \sim F^{\alpha}(\frac{x-\mu_i}{\theta})$*, for* $i = 1, 2$*. Also, suppose that* $\{J_1, J_2\}$ *is a set of independent Bernoulli random variables, independent of* X_i*'s with* $E(J_i) = p_i$ *and* $\{J_1^*, J_2^*\}$ *is another set of independent Bernoulli random variables, independent of* Y_i*'s with* $E(J_i^*) = q_i$*, for* $i = 1, 2$*. Set* $\boldsymbol{\lambda} = (0.5, 0.3)$*,* $\boldsymbol{\mu} = (0.32, 0.48)$*,* $\boldsymbol{\theta} = (2, 2)$*,* $\boldsymbol{p} = (0.2, 0.1)$*,* $\boldsymbol{q} = (0.11, 0.19)$*,* $\alpha = 1$*. Here,* $\psi(p) = p$*, which is increasing and convex with respect to* p*. Let* $T_{0.1} = \begin{pmatrix} 0.1 & 0.9 \\ 0.9 & 0.1 \end{pmatrix}$*. Then,* $\begin{pmatrix} \lambda_1 & \lambda_2 \\ \psi(p_1) & \psi(p_2) \end{pmatrix} \gg \begin{pmatrix} \mu_1 & \mu_2 \\ \psi(q_1) & \psi(q_2) \end{pmatrix}$*. Also* $(C1)$ *and* $(C8)$ *hold, for* $d = 1, a = 1, b = 0$*. Thus, as an application of Corollary 5.2, one can write* $U_{2:2} \geq_{rh} V_{2:2}$*. The graphs of* $\tilde{r}_{2:2}(t)$ *and* $\tilde{s}_{2:2}(t)$ *are given in Figure 2(b) that verifies Corollary 5.2.*

5.1.1 Pareto distribution

The distribution function of the pareto distribution as

$$F(x) = 1 - x^{-a}, \; x \geq 1, \; a > 0. \tag{33}$$

For all a, the hazard rate function of the pareto distribution satisfies $(C1)$, $(C2)$, $(C4)$, $(C5)$, $(C7)$ and $(C8)$. Below, some corollaries are provided, which are the direct consequences of Theorem 3.1 (i). Consider the pareto distribution to be the baseline distribution function with $a = 0.002$.

Corollary 5.4. *Suppose the assumptions of Theorem 3.1* (i) *hold. Again, let* $\psi(p) = p^2$*. Then,*

$$\begin{pmatrix} \psi(p_1) & \psi(p_2) \\ \lambda_1 & \lambda_2 \end{pmatrix} \gg \begin{pmatrix} \psi(p_1^*) & \psi(p_2^*) \\ \mu_1 & \mu_2 \end{pmatrix} \Rightarrow U_{2:2} \geq_{st} V_{2:2}.$$

The example given below, illustrates Corollary 5.4.

Example 5.5. *Let* $\{X_1, X_2\}$ *and* $\{Y_1, Y_2\}$ *be two collections of independent random variables such that* $X_i \sim F^{\alpha}(\frac{x-\lambda_i}{\theta})$ *and* $Y_i \sim F^{\alpha}(\frac{x-\mu_i}{\theta})$*, for* $i = 1, 2$*. Set the baseline distribution function as pareto distribution. Also, let* $\{J_1, J_2\}$ *be a set of independent Bernoulli random variables, independent of* X_i*'s with* $E(J_i) = p_i$ *and* $\{J_1^*, J_2^*\}$ *be another set of independent Bernoulli random variables, independent of* Y_i*'s with* $E(J_i^*) = q_i$*, where* $i = 1, 2$*. Consider* $\boldsymbol{\lambda} = (5, 6.1)$*,* $\boldsymbol{\mu} = (5.44, 5.66)$*,* $\theta = 0.01$*,* $\alpha = 0.52$*,* $\psi(\boldsymbol{p}) = (0.2, 0.5)$*,* $\psi(\boldsymbol{q}) = (0.32, 0.38)$*. Here,* $\psi(p) = p^2$*, which is increasing and convex with respect to* p*. Let* $T_{0.6} = \begin{pmatrix} 0.6 & 0.4 \\ 0.4 & 0.6 \end{pmatrix}$*. Then,* $\begin{pmatrix} \psi(p_1) & \psi(p_2) \\ \lambda_1 & \lambda_2 \end{pmatrix} \gg \begin{pmatrix} \psi(q_1) & \psi(q_2) \\ \mu_1 & \mu_2 \end{pmatrix}$*. Thus, as an application of Corollary 5.4,* $U_{2:2} \geq_{st} V_{2:2}$*. The graph of* $F_{2:2}(t) - G_{2:2}(t)$ *is given in Figure 3(a) that verifies Corollary 5.4.*

6. Concluding remarks

Let us have two insurance portfolios of n individual risks. Assume that the portfolios are heterogeneous. The problem of comparison of the smallest and largest claim amounts arising from these portfolios of risks with respect to some well known stochastic orders is of recent interest from both theoretical and practical points of view. Here, under different conditions, we established stochastic comparisons between the largest claims in the sense of the usual stochastic and reversed hazard rate orderings. Both these orders are useful tools to a decision maker to choose better one among several risks. For example, for two risks X and Y, if $X \leq_{st} Y$, then a person will choose X over Y. Again, for the case of the reversed hazard rate ordering, a person should prefer a bond, which has smaller reversed hazard rate. The results have been developed using the concepts of the vector majorization and related orders, and the multivariate chain majorization order. Finally, the established results have been applied to two baseline distribution functions for explanation purpose.

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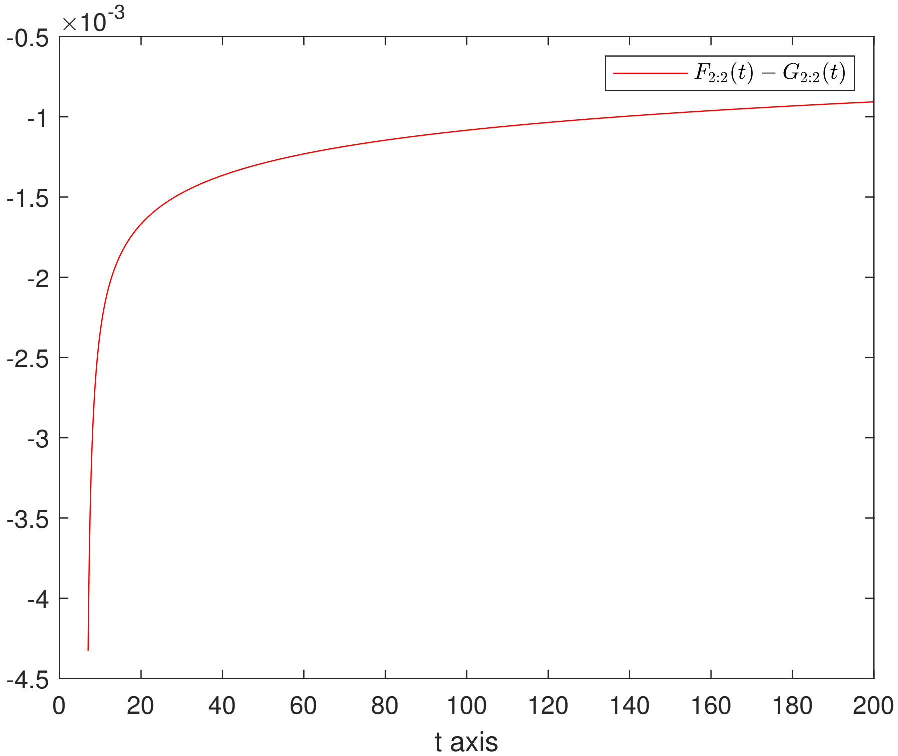

(a)

Figure 3: (a) Graph of $F_{2:2}(t) - G_{2:2}(t)$ for Example 5.5.

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