Filomat 35:4 (2021), 1305–1313
https://doi.org/10.2298/FIL2104305P

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

On Coefficient Inequalities of Certain Subclasses of Bi-Univalent Functions Involving the Sălăgean Operator

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Abstract. In the present investigation, with motivation from the pioneering work of Srivastava et al. [28], which in recent years actually revived the study of analytic and bi-univalent functions, we introduce the subclasses $\mathcal{T}^*_\Sigma(n,\beta)$ and $\mathcal{T}_\Sigma(n,\alpha)$ of analytic and bi-univalent function class Σ defined in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$ and involving the Sălăgean derivative operator D^n. Moreover, we derive estimates on the initial coefficients $|a_2|$ and $|a_3|$ for functions in these subclasses and pointed out connections with some earlier known results.

1. Introduction

Let a function f is defined in the open unit disk $\mathbb{U}$ and have the form:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \tag{1}$$

which is normalized by the conditions $f(0) = 0$ and $f'(0) = 1$. We define the class $\mathcal{A}$ and its subclass $\mathcal{S}$ (see [6, 16]) as follows:

$$\mathcal{A} = \{f : \mathbb{U} \to \mathbb{C} : f \text{ is of the form (1) and analytic in } \mathbb{U}\},$$

$$\mathcal{S} = \{f : \mathbb{U} \to \mathbb{C} : f \in \mathcal{A} \text{ and univalent in } \mathbb{U}\}.$$

It is well known that the inverse function of every $f \in \mathcal{S}$ is defined by

$$f^{-1}(f(z)) = z, \ (z \in \mathbb{U}) \text{ and } f(f^{-1}(w)) = w, \ (|w| < r_0(f), r_0(f) \geq 1/4),$$

which may have an analytic continuation to $\mathbb{U}$ as follows:

$$g(w) = f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \cdots . \tag{2}$$

The set of functions $\{f \in \mathcal{S} : \text{both } f \text{ and } f^{-1} \text{ are univalent in } \mathbb{U}\}$ forms the bi-univalent function class Σ, which has been investigated by Lewin [10] in 1967 and proved that $|a_2| < 1.51$. Afterverse, Brannan and

2010 *Mathematics Subject Classification*. 30C45, 30C50
Keywords. analytic function, univalent function, bi-univalent function, coefficient estimate, Sălăgean differential operator.
Received: 02 April 2020; revised: 14 January 2021; Accepted: 06 February 2021
Communicated by Hari M. Srivastava
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Clunie [3] conjectured that $|a_2| \leq \sqrt{2}$ and Netanyahu [12] showed that $max\,|a_2| = 4/3$ for $f \in \Sigma$. Further in 1981, Styer and Wright [33] concluded that there exist functions in Σ for which $|a_2| > 4/3$ and then in 1984 Tan [34] proved that $|a_2| \leq 1.485$ for $f \in \Sigma$. But the coefficient estimate problem for $|a_n|$, $(n \in \mathbb{N} \setminus \{1, 2\})$ is still open.

The present paper is due to motivation from the pioneering work of Srivastava et al. [28], which has actually revived the study of analytic and bi-univalent functions in recent years. In fact, it is [28], which has led to a remarkably large number of its sequels by many different authors such as [20, 22–26, 29–32, 36, 37].

The results of Xu et al. [36] generalizes and improves the work of Srivastava et al. [28], whereas the results of Xu et al. [37] generalizes and improves the work of Frasin and Aouf [7], Xu et al. [36] etc. Also, the results of Srivastava et al. [22] generalizes and improves the work of Cağlar et al. [5], Xu et al. [37] etc. Srivastava et al. [26] obtained initial coefficient estimates for a general subclass of analytic and bi-univalent functions of the Ma–Minda type. Further sequels of [28] such as [25, 31, 32] consists of various subclasses of analytic and m-fold symmetric bi-univalent functions defined in the open unit disk $\mathbb{U}$. Srivastava et al. [23, 24, 29, 30] obtained upper bounds for the coefficients of functions belong to various subclasses of Σ by using the Faber polynomial expansions. Srivastava et al. [20] considered certain subclasses of Σ associated with the Horadam polynomials and solved the Fekete–Szegö problem for the functions in them. In addition, many researchers viz [1, 2, 5, 7, 14, 15, 17, 21, 35] investigated various interesting subclasses of Σ and obtained estimates on first two to three Taylor–Maclaurin coefficients for functions in them.

In 1983, Sălăgean [18] introduced the following derivative operator:

$$
\begin{aligned}
&D^0 f(z) = f(z), \\
&D^1 f(z) = Df(z) = zf'(z), \\
&D^n f(z) = D(D^{n-1} f(z)), \quad (f \in \mathcal{A},\ n \in \mathbb{N}).
\end{aligned}
$$

Further, for $f(z)$ given by (1), we have $D^n f(0) = 0$ and

$$
D^n f(z) = z + \sum_{k=2}^{\infty} k^n a_k z^k, \quad (n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}). \tag{3}
$$

In the year 1972, Ozaki and Nunokawa [13] proved that, if for $f \in \mathcal{A}$

$$
\left| \frac{z^2 f'(z)}{(f(z))^2} - 1 \right| < 1 \quad (z \in \mathbb{U}),
$$

then f is univalent in $\mathbb{U}$ and hence $f \in \mathcal{S}$. We denote by $\mathcal{T}(\beta)$ the class of functions $f \in \mathcal{A}$, which satisfies the following condition:

$$
\left| \frac{z^2 f'(z)}{(f(z))^2} - 1 \right| < (1 - \beta) \quad (z \in \mathbb{U}),
$$

where $0 \leq \beta < 1$ and $\mathcal{T}(0) = \mathcal{T}$. Clearly, $\mathcal{T}(\beta) \subset \mathcal{T} \subset \mathcal{S}$. Further, we have (see Kuroki et al. [9]):

$$
\Re \left(\frac{z^2 f'(z)}{(f(z))^2} \right) > \beta \quad (z \in \mathbb{U}).
$$

In the present paper, we derive estimates on the initial coefficients $|a_2|$ and $|a_3|$ for functions belong to the new subclasses $\mathcal{T}^*_\Sigma(n, \beta)$ and $\mathcal{T}_\Sigma(n, \alpha)$ of Σ. Also, we consider some of the related subclasses of Σ and mention connections with the earlier known results.

We need the following lemma (see [6, 16]) to prove our results.

Lemma 1.1. *Let $h(z) \in \mathcal{P}$ where, $\mathcal{P}$ denote the class of functions analytic in $\mathbb{U}$ with $\Re\,(h(z)) > 0$ and have the form $h(z) = 1 + h_1 z + h_2 z^2 + h_3 z^3 + \cdots$, $(z \in \mathbb{U})$. Then, $|h_n| \leq 2$ for each $n \in \mathbb{N}$.*

2. Coefficient Estimates for the Class $\mathcal{T}^*_\Sigma(n,\beta)$

Definition 2.1. *A function $f(z) \in \Sigma$ given by* (1) *is said to be in the class $\mathcal{T}^*_\Sigma(n,\beta)$ if the following conditions are satisfied:*

$$\Re\left(\frac{z^2\left(D^n f(z)\right)'}{\left(D^n f(z)\right)^2}\right) > \beta \qquad (z \in \mathbb{U})$$

and

$$\Re\left(\frac{w^2\left(D^n g(w)\right)'}{\left(D^n g(w)\right)^2}\right) > \beta \qquad (w \in \mathbb{U}),$$

where $n \in \mathbb{N}_0$, $0 \le \beta < 1$ and the function $g = f^{-1}$ is given by (2).

For $n = 0$, this class reduces to the class $\mathcal{T}_\Sigma(\beta)$ introduced by Naik and Patil [11], who proved the following result:

Corollary 2.2. *Let the function $f(z) \in \mathcal{T}_\Sigma(\beta)$; $(0 \le \beta < 1)$ be of the form* (1). *Then, $|a_2| \le 1$, $|a_3| \le 2\,(1-\beta)$ and $|a_4| \le 3\,(1-\beta)$.*

Theorem 2.3. *Let the function $f(z) \in \mathcal{T}^*_\Sigma(n,\beta)$; $(n \in \mathbb{N}, 0 \le \beta < 1)$ be of the form* (1). *Then,*

$$|a_2| \le \sqrt{\frac{2\,(1-\beta)}{4^n - 3^n}}, \tag{4}$$

$$|a_3| \le \frac{2\,(1-\beta)}{4^n - 3^n} \tag{5}$$

and at $n = 0$, we get an improvement in $|a_2|$ of Corollary 2.2 as follows:

$$|a_2| \le \begin{cases} 1, & \left(0 \le \beta \le \frac{1}{2}\right) \\ \sqrt{2\,(1-\beta)}, & \left(\frac{1}{2} \le \beta < 1\right) \end{cases} \quad \textit{and} \;\; |a_3| \le 2\,(1-\beta).$$

Proof. Using Definition 2.1 we can write:

$$\frac{z^2\left(D^n f(z)\right)'}{\left(D^n f(z)\right)^2} = \beta + (1-\beta)\,h(z) \tag{6}$$

and

$$\frac{w^2\left(D^n g(w)\right)'}{\left(D^n g(w)\right)^2} = \beta + (1-\beta)\,t(w), \tag{7}$$

where $h(z), t(w) \in \mathcal{P}$ have the form:

$$h(z) = 1 + h_1 z + h_2 z^2 + h_3 z^3 + \cdots, \; (z \in \mathbb{U}) \tag{8}$$

and

$$t(w) = 1 + t_1 w + t_2 w^2 + t_3 w^3 + \cdots, \; (w \in \mathbb{U}). \tag{9}$$

Hence we have:

$$\beta + (1-\beta)\,h(z) = 1 + (1-\beta)h_1 z + (1-\beta)h_2 z^2 + \cdots$$

and

$$\beta + (1-\beta)\, t(w) = 1 + (1-\beta)t_1 w + (1-\beta)t_2 w^2 + \cdots .$$

Using (1) and (2), we obtain:

$$\frac{z^2 \left(D^n f(z)\right)'}{\left(D^n f(z)\right)^2} = 1 + \left(3^n a_3 - 4^n a_2^2\right) z^2 + 2\left[2^n 4^n a_2^3 + 4^n a_4 - 2\cdot 2^n 3^n a_2 a_3\right] z^3 + \cdots ,$$

and

$$\frac{w^2 \left(D^n g(w)\right)'}{\left(D^n g(w)\right)^2} = 1 + \left[3^n \left(2a_2^2 - a_3\right) - 4^n a_2^2\right] w^2 + 2\left[2\cdot 2^n 3^n a_2 \left(2a_2^2 - a_3\right) - 4^n \left(5a_2^3 - 5a_2 a_3 + a_4\right) - 2^n 4^n a_2^3\right] w^3 + \cdots .$$

Equating the coefficients in (6) and (7) we get $h_1 = t_1 = 0$ and also:

$$\left(3^n a_3 - 4^n a_2^2\right) = (1-\beta) h_2, \tag{10}$$

$$2\left[2^n 4^n a_2^3 + 4^n a_4 - 2\cdot 2^n 3^n a_2 a_3\right] = (1-\beta) h_3, \tag{11}$$

$$\left[3^n \left(2a_2^2 - a_3\right) - 4^n a_2^2\right] = (1-\beta) t_2, \tag{12}$$

$$2\left[2\cdot 2^n 3^n a_2 \left(2a_2^2 - a_3\right) - 4^n \left(5a_2^3 - 5a_2 a_3 + a_4\right) - 2^n 4^n a_2^3\right] = (1-\beta) t_3. \tag{13}$$

Adding (10) in (12), we get:

$$2\left(3^n - 4^n\right) a_2^2 = (1-\beta)\left(h_2 + t_2\right) \tag{14}$$

or

$$a_2^2 = \frac{(1-\beta)\left(h_2 + t_2\right)}{2\left(3^n - 4^n\right)} \tag{15}$$

which, by using Lemma 1.1 gives:

$$\left|a_2^2\right| \le \frac{2(1-\beta)}{\left|3^n - 4^n\right|} \implies |a_2| \le \sqrt{\frac{2(1-\beta)}{4^n - 3^n}}.$$

Next, subtracting (12) from (10), we get:

$$2\cdot 3^n \left(a_3 - a_2^2\right) = (1-\beta)\left(h_2 - t_2\right)$$

or

$$a_3 = a_2^2 + \frac{(1-\beta)\left(h_2 - t_2\right)}{2\cdot 3^n}.$$

By using (15), this becomes:

$$a_3 = \frac{(1-\beta)\left(h_2 + t_2\right)}{2\left(3^n - 4^n\right)} + \frac{(1-\beta)\left(h_2 - t_2\right)}{2\cdot 3^n} \tag{16}$$

or

$$a_3 = \frac{(1-\beta)\left[\left(2 \cdot 3^n - 4^n\right) h_2 + 4^n t_2\right]}{2 \cdot 3^n \left(3^n - 4^n\right)}. \tag{17}$$

Using Lemma 1.1, it gives:

$$|a_3| \leq \frac{2(1-\beta)}{|3^n - 4^n|} \quad \Longrightarrow \quad |a_3| \leq \frac{2(1-\beta)}{4^n - 3^n}.$$

Finally, for $n = 0$, the equations (10) to (13) reduces to:

$$\left(a_3 - a_2^2\right) = (1-\beta)h_2, \tag{18}$$

$$2\left[a_2^3 + a_4 - 2a_2a_3\right] = (1-\beta)h_3, \tag{19}$$

$$\left(a_2^2 - a_3\right) = (1-\beta)t_2, \tag{20}$$

$$-2\left[2a_2^3 + a_4 - 3a_2a_3\right] = (1-\beta)t_3. \tag{21}$$

In light of Lemma 1.1, equation (18) and (20) yields:

$$\left|a_3 - a_2^2\right| = \left|a_2^2 - a_3\right| \leq 2(1-\beta). \tag{22}$$

Using the triangle inequality, we obtain

$$|a_3| - \left|a_2^2\right| \leq \left|a_3 - a_2^2\right| \leq 2(1-\beta) \quad \Longrightarrow \quad |a_3| \leq 2(1-\beta) \tag{23}$$

and also,

$$\left|a_2^2\right| - |a_3| \leq \left|a_2^2 - a_3\right| \leq 2(1-\beta) \quad \Longrightarrow \quad \left|a_2^2\right| \leq 2(1-\beta). \tag{24}$$

Now, adding (19) to (21), we get

$$2a_2\left(a_3 - a_2^2\right) = (1-\beta)(h_3 + t_3). \tag{25}$$

Which, on using Lemma 1.1, yields

$$|a_2|\left|\left(a_3 - a_2^2\right)\right| \leq 2(1-\beta). \tag{26}$$

This, along with (22), gives that

$$|a_2| \leq 1. \tag{27}$$

Equation (24) and (27) together implies that

$$|a_2| \leq min\left\{\sqrt{2(1-\beta)}, 1\right\}. \tag{28}$$

Which, with respect to the specific range of β shows that:

$$|a_2| \leq \begin{cases} 1, & \left(0 \leq \beta \leq \frac{1}{2}\right) \\ \sqrt{2(1-\beta)}, & \left(\frac{1}{2} \leq \beta < 1\right). \end{cases}$$

Hence the proof. □

For $n = 1$, Definition 2.1 reduces to the following class which is closely related to the bi-convex function class $\mathcal{K}_\Sigma(\beta)$ of order β (Brannan and Taha [4]) and the class $\mathcal{H}_\Sigma(\beta)$ (Srivastava et al. [28]).

Definition 2.4. *A function $f(z) \in \Sigma$ given by* (1) *is said to be in the class $\mathcal{T}^*_\Sigma(1,\beta) \equiv \mathcal{T}^*_\Sigma(\beta)$ if the following conditions are satisfied:*

$$\Re\left(\frac{f'(z) + zf''(z)}{(f'(z))^2}\right) = \Re\left(\frac{1 + \frac{zf''(z)}{f'(z)}}{f'(z)}\right) > \beta \qquad (z \in \mathbb{U})$$

and

$$\Re\left(\frac{g'(w) + wg''(w)}{(g'(w))^2}\right) = \Re\left(\frac{1 + \frac{wg''(w)}{g'(w)}}{g'(w)}\right) > \beta \qquad (w \in \mathbb{U}),$$

where $0 \le \beta < 1$ and the function $g = f^{-1}$ is given by (2).

For this class, Theorem 2.3 produces the following result.

Corollary 2.5. *Let the function $f(z) \in \mathcal{T}^*_\Sigma(1,\beta)$; $(0 \le \beta < 1)$. Then,*

$$|a_2| \le \sqrt{2(1-\beta)}$$

and

$$|a_3| \le 2(1-\beta).$$

3. Coefficient Estimates for the Class $\mathcal{T}_\Sigma(n,\alpha)$

Definition 3.1. *A function $f(z) \in \Sigma$ given by* (1) *is said to be in the class $\mathcal{T}_\Sigma(n,\alpha)$ if the following conditions are satisfied:*

$$\left|arg\left(\frac{z^2\,(D^n f(z))'}{(D^n f(z))^2}\right)\right| < \frac{\alpha\pi}{2} \qquad (z \in \mathbb{U})$$

and

$$\left|arg\left(\frac{w^2\,(D^n g(w))'}{(D^n g(w))^2}\right)\right| < \frac{\alpha\pi}{2} \qquad (w \in \mathbb{U}),$$

where $n \in \mathbb{N}_0$, $0 < \alpha \le 1$ and the function $g = f^{-1}$ is given by (2).

For $n = 0$, this class reduces to the class $\mathcal{T}^\alpha_\Sigma$ introduced by Naik and Patil [11], who proved the following result:

Corollary 3.2. *Let the function $f(z) \in \mathcal{T}^\alpha_\Sigma$; $(0 < \alpha \le 1)$ be of the form* (1). *Then, $|a_2| \le 1$, $|a_3| \le 2\alpha$ and $|a_4| \le 3\alpha$.*

Theorem 3.3. *Let the function $f(z) \in \mathcal{T}_\Sigma(n,\alpha)$; $(n \in \mathbb{N}, 0 < \alpha \le 1)$ be of the form* (1). *Then,*

$$|a_2| \le \sqrt{\frac{2\alpha}{4^n - 3^n}}, \tag{29}$$

$$|a_3| \le \frac{2\alpha}{4^n - 3^n} \tag{30}$$

and at $n = 0$, we get an improvement in $|a_2|$ of Corollary 3.2 as follows:

$$|a_2| \leq \begin{cases} \sqrt{2\alpha}, & \left(0 < \alpha \leq \frac{1}{2}\right) \\ 1, & \left(\frac{1}{2} \leq \beta \leq 1\right) \end{cases} \quad \textit{and} \ \ |a_3| \leq 2\alpha.$$

Proof. Definition 3.1 implies that there exist functions $h(z), t(w) \in \mathcal{P}$ given by (8) and (9) respectively, such that:

$$\frac{z^2 \left(D^n f(z)\right)'}{\left(D^n f(z)\right)^2} = [h(z)]^\alpha \tag{31}$$

and

$$\frac{w^2 \left(D^n g(w)\right)'}{\left(D^n g(w)\right)^2} = [t(w)]^\alpha . \tag{32}$$

we have:

$$[h(z)]^\alpha = 1 + \alpha h_1 z + \left[\alpha h_2 + \frac{\alpha(\alpha-1)}{2} h_1^2\right] z^2 + \left[\alpha h_3 + \alpha(\alpha-1) h_1 h_2 + \frac{\alpha(\alpha-1)(\alpha-2)}{6} h_1^3\right] z^3 + \cdots$$

and

$$[t(w)]^\alpha = 1 + \alpha t_1 w + \left[\alpha t_2 + \frac{\alpha(\alpha-1)}{2} t_1^2\right] w^2 + \left[\alpha t_3 + \alpha(\alpha-1) t_1 t_2 + \frac{\alpha(\alpha-1)(\alpha-2)}{6} t_1^3\right] w^3 + \cdots .$$

Equating the coefficients in (31) and (32) we get $h_1 = t_1 = 0$ and also:

$$\left(3^n a_3 - 4^n a_2^2\right) = \alpha\, h_2,$$

$$2\left[2^n 4^n a_2^3 + 4^n a_4 - 2 \cdot 2^n 3^n a_2 a_3\right] = \alpha\, h_3,$$

$$\left[3^n \left(2a_2^2 - a_3\right) - 4^n a_2^2\right] = \alpha\, t_2,$$

$$2\left[2 \cdot 2^n 3^n a_2 \left(2a_2^2 - a_3\right) - 4^n \left(5a_2^3 - 5a_2 a_3 + a_4\right) - 2^n 4^n a_2^3\right] = \alpha\, t_3.$$

Now, continuing as in Theorem 2.3, we can complete the further proof. □

For $n = 1$, Definition 3.1 reduces to the following class which is closely related to the strongly bi-convex function class $\mathcal{K}_\Sigma[\alpha]$ of order α (Brannan and Taha [4]) and the class $\mathcal{H}_\Sigma^\alpha$ (Srivastava et al. [28]).

Definition 3.4. *A function $f(z) \in \Sigma$ given by* (1) *is said to be in the class $\mathcal{T}_\Sigma(1, \alpha) \equiv \mathcal{T}_\Sigma(\alpha)$ if the following conditions are satisfied:*

$$\left| arg\left(\frac{f'(z) + zf''(z)}{\left(f'(z)\right)^2}\right)\right| = \left| arg\left(\frac{1 + \frac{zf''(z)}{f'(z)}}{f'(z)}\right)\right| < \frac{\alpha\pi}{2} \qquad (z \in \mathbb{U})$$

and

$$\left| arg\left(\frac{g'(w) + wg''(w)}{\left(g'(w)\right)^2}\right)\right| = \left| arg\left(\frac{1 + \frac{wg''(w)}{g'(w)}}{g'(w)}\right)\right| < \frac{\alpha\pi}{2} \qquad (w \in \mathbb{U}),$$

where $0 < \alpha \leq 1$ and the function $g = f^{-1}$ is given by (2).

For this class, Theorem 3.3 produces the following result.

Corollary 3.5. *Let the function* $f(z) \in \mathcal{T}_\Sigma(1, \alpha)$; $(0 < \alpha \leq 1)$. *Then,*

$$|a_2| \leq \sqrt{2\alpha}$$

and

$$|a_3| \leq 2\alpha.$$

4. Conclusion

In the present paper, we obtain estimates on initial coefficients $|a_2|$ and $|a_3|$ for functions belong to the subclasses $\mathcal{T}^*_\Sigma(n, \beta)$ and $\mathcal{T}_\Sigma(n, \alpha)$ of Σ, that are defined by using the Sălăgean derivative operator D^n. According to our main results, we can conclude that the geometrical similarities in the classes $\mathcal{T}^*_\Sigma(n, \beta)$ and $\mathcal{T}_\Sigma(n, \alpha)$ are also reflects in their initial coefficient estimations, which clearly shows the existence of interrelationship between the geometric behavior and the analytic characterization of the functions belong to these subclasses. The study of such interrelationship is actually the main key interest of the researchers in this field since from more than hundred years.

In recent years, the basic (or q-) extensions of the various developments on the bi-univalent functions are being investigated by many authors such as Khan et al. [8], Srivastava et al. [27] etc. Researchers can refer to these papers for current development in this field.

Motivated by a recently-published survey-cum-expository review article by Srivastava [19], the interested reader's attention is drawn toward the possibility of investigating the basic (or q-) extensions of the results which are presented in this paper. However, as already pointed out by Srivastava [19], their further extensions using the so-called (p, q)-calculus will be rather trivial and inconsequential variations of the suggested extensions which are based upon the classical q-calculus, the additional parameter p being redundant or superfluous (see, for details, [19, p. 340]).

Acknowledgment

The authors wish to express their sincere thanks to the referee/referees of this paper for valuable comments and suggestions.

References

[1] R. M. Ali, S. K. Lee, V. Ravichandran, S. Supramaniam, Coefficient estimates for bi-univalent Ma-Minda starlike and convex functions, Appl. Math. Lett. 25 (3) (2012) 344–351.

[2] Ş. Altinkaya, S. Yalçin, Coefficient estimates for two new subclasses of bi-univalent functions with respect to symmetric points, J. funct. spaces Article ID 145242 (2015) 1–5.

[3] D. A. Brannan, J. G. Clunie (Eds.), Aspects of Contemporary Complex Analysis, (Proceedings of the NATO Advanced Study Institute held at the University of Durham, Durham; July 1–20, 1979), Academic Press, New York and London, 1980.

[4] D. A. Brannan, T. S. Taha, On some classes of bi-univalent functions, Studia Univ. Babes-Bolyai Math. 31 (2) (1986) 70–77.

[5] M. Cağlar, H. Orhan, N. Yağmur, Coefficient bounds for new subclasses of bi-univalent functions, Filomat 27 (7) (2013) 1165–1171.

[6] P. L. Duren, Univalent Functions, Grundlehren der Mathematischen Wissenschaften, Springer, New York, 1983.

[7] B. A. Frasin, M. K. Aouf, New subclasses of bi-univalent functions, Appl. Math. Lett. 24 (2011) 1569–1573.

[8] B. Khan, H. M. Srivastava, M. Tahir, M. Darus, Q.Z. Ahmad, N. Khan, Applications of a certain q-integral operator to the subclasses of analytic and bi-univalent functions, AIMS Mathematics 6 (2021) 1024–1039.

[9] K. Kuroki, T. Hayami, N. Uyanik, S. Owa, Some properties for a certain class concerned with univalent functions, Comput. Math. Appl. 63 (2012) 1425–1432.

[10] M. Lewin, On a coefficient problem for bi-univalent functions, Proc. Amer. Math. Soc. 18 (1967) 63–68.

[11] U. H. Naik, A. B. Patil, On initial coefficient inequalities for certain new subclasses of bi-univalent functions, J. Egyptian Math. Soc. 25 (3) (2017) 291–293.

[12] E. Netanyahu, The minimal distance of the image boundary from the origin and the second coefficient of a univalent function in $|z| < 1$, Arch. Ration. Mech. Anal. 32 (1969) 100–112.

[13] S. Ozaki, M. Nunokawa, The Schwarzian derivative and univalent functions, Proc. Amer. Math. Soc. 33 (1972) 392–394.

[14] A. B. Patil, U. H. Naik, On subclasses of bi-univalent functions associated with the Răducanu-Orhan differential operator, Acta Univ. Apulensis Math. Inform. 52 (2017) 129–137.
[15] A. B. Patil, U. H. Naik, Bounds on initial coefficients for a new subclass of bi-univalent functions, New Trends Math. Sci. 6 (1) (2018) 85–90.
[16] C. Pommerenke, Univalent functions, Vandenhoeck and Rupercht, Göttingen, 1975.
[17] S. Porwal, M. Darus, On a new subclass of bi-univalent functions, J. Egyptian Math. Soc. 21 (3) (2013) 190–193.
[18] G. S. Sălăgean, Subclasses of univalent functions, Complex Analysis - Fifth Romanian Finish Seminar, Bucharest vol. 1 (1983) 362–372.
[19] H. M. Srivastava, Operators of basic (or q-) calculus and fractional q-calculus and their applications in geometric function theory of complex analysis, Iran. J. Sci. Technol. Trans. A: Sci. 44 (2020) 327–344.
[20] H. M. Srivastava, Ş. Altinkaya, S. Yalçin, Certain subclasses of bi-univalent functions associated with the Horadam polynomials, Iran. J. Sci. Technol. Trans. A: Sci. 43 (2019) 1873–1879.
[21] H. M. Srivastava, D. Bansal, Coefficient estimates for a subclass of analytic and bi-univalent functions, J. Egyptian Math. Soc. 23 (2) (2015) 242–246.
[22] H. M. Srivastava, S. Bulut, M. Cağlar, N. Yağmur, Coefficient estimates for a general subclass of analytic and bi-univalent functions, Filomat 27 (5) (2013) 831–842.
[23] H. M. Srivastava, S. S. Eker, R. M. Ali, Coefficient bounds for a certain class of analytic and bi-univalent functions, Filomat 29 (8) (2015) 1839–1845.
[24] H. M. Srivastava, S. S. Eker, S. G. Hamidi, J. M. Jahangiri, Faber polynomial coefficient estimates for bi-univalent functions defined by the Tremblay fractional derivative operator, Bull. Iran. Math. Soc. 44 (2018) 149–157.
[25] H. M. Srivastava, S. Gaboury, F. Ghanim, Initial coefficient estimates for some subclasses of m-fold symmetric bi-univalent functions, Acta Math. Sci. Ser. B Engl. Ed. 36 (2016) 863–871.
[26] H. M. Srivastava, S. Gaboury, F. Ghanim, Coefficient estimates for a general subclass of analytic and bi-univalent functions of the Ma–Minda type, Rev. Real Acad. Cienc. Exactas Fís. Natur. Ser. A Mat. (RACSAM) 112 (2018) 1157–1168.
[27] H. M. Srivastava, S. Khan, Q. Z. Ahmad, N. Khan, S. Hussain, The Faber polynomial expansion method and its application to the general coefficient problem for some subclasses of bi-univalent functions associated with a certain q-integral operator, Stud. Univ. Babeş-Bolyai Math. 63 (2018) 419–436.
[28] H. M. Srivastava, A. K. Mishra, P. Gochhayat, Certain subclasses of analytic and bi-univalent functions, Appl. Math. Lett. 23 (2010) 1188–1192.
[29] H. M. Srivastava, A. Motamednezhad, E. A. Adegani, Faber polynomial coefficient estimates for bi-univalent functions defined by using differential subordination and a certain fractional derivative operator, Mathematics 8 Article ID 172 (2020) 1–12.
[30] H. M. Srivastava, F. M. Sakar, H. O. Guney, Some general coefficient estimates for a new class of analytic and bi-univalent functions defined by a linear combination, Filomat 32 (4) (2018) 1313–1322.
[31] H. M. Srivastava, S. Sivasubramanian, R. Sivakumar, Initial coefficient bounds for a subclass of m-fold symmetric bi-univalent functions, Tbilisi Math. J. 7 (2) (2014) 1–10.
[32] H. M. Srivastava, A. K. Wanas, Initial Maclaurin coefficient bounds for new subclasses of analytic and m-fold symmetric bi-univalent functions defined by a linear combination, Kyungpook Math. J. 59 (2019) 493–503.
[33] D. Styer, D. J. Wright, Results on bi-univalent functions, Proc. Amer. Math. Soc. 82 (2) (1981) 243–248.
[34] D. –L. Tan, Coefficient estimates for bi-univalent functions, Chinese Ann. Math. Ser. A 5 (1984) 559–568.
[35] H. Tang, G. –T. Deng, N. Magesh, S. –H. Li, Coefficient estimates for new subclasses of Ma-Minda bi-univalent functions, J. Inequal. Appl. (2013) 2013:317.
[36] Q. –H. Xu, Y. –C. Gui, H. M. Srivastava, Coefficient estimates for a certain subclass of analytic and bi-univalent functions, Appl. Math. Lett. 25 (2012) 990–994.
[37] Q. –H. Xu, H. –G. Xiao, H. M. Srivastava, A certain general subclass of analytic and bi-univalent functions and associated coefficient estimate problems, Appl. Math. Comput. 218 (2012) 11461–11465.