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On the D_α Spectral Radius of Strongly Connected Digraphs

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Abstract. Let G be a strongly connected digraph with distance matrix $D(G)$ and let $Tr(G)$ be the diagonal matrix with vertex transmissions of G. For any real $\alpha \in [0, 1]$, define the matrix $D_\alpha(G)$ as

$$D_\alpha(G) = \alpha Tr(G) + (1 - \alpha)D(G).$$

The D_α spectral radius of G is the spectral radius of $D_\alpha(G)$. In this paper, we first give some upper and lower bounds for the D_α spectral radius of G and characterize the extremal digraphs. Moreover, for digraphs that are not transmission regular, we give a lower bound on the difference between the maximum vertex transmission and the D_α spectral radius. Finally, we obtain the D_α eigenvalues of the join of certain regular digraphs.

1. Introduction

Let $G = (V(G), E(G))$ be a digraph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and arc set $E(G)$. If there is an arc from v_i to v_j, we denote this by writing (v_i, v_j), call v_j and v_i the head and the tail of (v_i, v_j), respectively. The loop is an arc which starts and ends at a same vertex. The multiarcs are the arcs which start at a same vertex v_i and end at a same vertex v_j, where $v_i \neq v_j$. A digraph is simple if it has no loops and multiarcs. A digraph G is strongly connected if for every pair of vertices $v_i, v_j \in V(G)$, there exists a directed path from v_i to v_j. The complete digraph is a digraph in which every pair of vertices is connected by an arc. For a digraph G with vertex set $V(G)$, if $S \subset V(G)$, then we use $G[S]$ to denote the subdigraph of G induced by S. Let H be a subdigraph of G , if $G[V(H)]$ is a complete subdigraph of G, then H is called a clique of G. We follow [4, 5, 11] for terminology and notations. Throughout this paper, we only consider simple strongly connected digraphs.

For any vertex $v_i \in V(G)$, $N^+(v_i) = \{v_j : (v_i, v_j) \in E(G)\}$ is called the set of outneighbors of v_i. Let $d^+_G(v_i) = |N^+(v_i)|$ denote the outdegree of v_i, (we simply write $d^+(v_i)$ if it is clear from the context.) A digraph is called regular if each of its vertex has the same outdegree. For a strongly connected digraph G, the distance from v_i to v_j, denoted by $d_{v_iv_j}$ or simply d_{ij}, is defined as the length of the shortest directed path from v_i to v_j in G. The diameter of the strongly connected digraph G, denoted by $diam(G)$, is the maximum d_{ij} over all ordered pairs of vertices v_i, v_j.

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Let $A(G) = (a_{ij})_{n\times n}$ be the adjacency matrix of G, where $a_{ij} = 1$ if $(v_i, v_j) \in E(G)$ and $a_{ij} = 0$ otherwise. The signless Laplacian matrix of G is $Q(G) = Diag(G) + A(G)$, where $Diag(G) = \mathrm{diag}(d^+(v_1), d^+(v_2), \dots, d^+(v_n))$ is the diagonal matrix with outdegrees of vertices of G. For any real $\alpha \in [0, 1]$, the A_α matrix of G is introduced by J.P. Liu et al. [18] below. The matrix $A_\alpha(G) = \alpha Diag(G) + (1-\alpha)A(G)$ is called A_α matrix of G, which reduces to merging the adjacency spectral and signless Laplacian spectral theories.

The distance matrix of G, denoted by $D(G)$, is defined as $D(G) = (d_{ij})_{n\times n}$, where d_{ij} is defined as the length of the shortest directed path from v_i to v_j in the strongly connected digraph G. The transmission of a vertex v_i, denoted by $Tr_G(v_i)$ or Tr_i, is defined as the sum of distances from v_i to all other vertices in G, that is, $Tr_G(v_i) = Tr_i = \sum\limits_{j=1}^{n} d_{ij}$ $(i = 1, 2, \dots, n)$. In fact, for $1 \le i \le n$, the transmission of vertex v_i is just the i-th row sum of $D(G)$. A strongly connected digraph G is k-transmission regular if $Tr_i = k$ for each $v_i \in V(G)$; otherwise, G is non-transmission regular. The second transmission of v_i, denoted by T_i, is given by $T_i = \sum\limits_{t=1}^{n} d_{it}Tr_t$. Let $Tr(G) = \mathrm{diag}(Tr_1, Tr_2, \dots, Tr_n)$ be the diagonal matrix with vertex transmissions of G. Then $D^Q(G) = Tr(G) + D(G)$ is called the distance signless Laplacian matrix of G. The spectral radius of $D(G)$ is called the distance spectral radius of G, and the spectral radius of $D^Q(G)$ is called the distance signless Laplacian spectral radius of G. The distance spectral radius and distance signless Laplacian spectral radius of undirected graphs are well treated in the literature, see [1–3, 8, 14].

In [21], it was proposed to study the convex combinations $D_\alpha(G)$ of $Tr(G)$ and $D(G)$, defined by

$$D_\alpha(G) = \alpha Tr(G) + (1-\alpha)D(G), \quad 0 \le \alpha \le 1.$$

Since $D(G) = D_0(G)$, $Tr(G) = D_1(G)$ and $D^Q(G) = 2D_{\frac{1}{2}}(G)$, the matrices $D_\alpha(G)$ can underpin a unified theory of $D(G)$ and $D^Q(G)$. We call the eigenvalue with largest modulus of $D_\alpha(G)$ the D_α spectral radius of G, denoted by $\mu_\alpha(G)$. The collection of eigenvalues of $D_\alpha(G)$ together with multiplicities are called the D_α-spectrum of G. If $\alpha = 1$, $D_1(G) = Tr(G)$ the diagonal matrix with vertex transmissions of G which is not interesting. Unless stated otherwise, we assume that $0 \le \alpha < 1$ in the rest of this paper. Since G is a strongly connected digraph, then $D_\alpha(G)$ is a nonnegative irreducible matrix. It follows from the Perron Frobenius Theorem that $\mu_\alpha(G)$ is an eigenvalue of $D_\alpha(G)$, and there is a positive unit eigenvector corresponding to $\mu_\alpha(G)$. The positive unit eigenvector corresponding to $\mu_\alpha(G)$ is called the Perron vector of $D_\alpha(G)$. The D_α spectral radius of undirected graphs has been studied in the literature, see [6, 7, 9, 16]. We are interested in the D_α spectral radius of digraphs.

Recently, the distance spectral radius and distance signless Laplacian spectral radius of digraphs have been studied in some papers. For example, Lin et al. [17] characterized the extremal digraphs with the minimum distance spectral radius among all digraphs with given vertex connectivity. Lin and Shu [15] first gave sharp upper and lower bounds for the distance spectral radius of strongly connected digraphs, they then characterized the digraphs having the maximal and minimal distance spectral radii among all strongly connected digraphs, and they also determined the extremal digraphs with the minimal distance spectral radius among all strongly connected digraphs with given arc connectivity and dichromatic number, respectively. Xi and Wang [23] determined the strongly connected digraphs minimizing distance spectral radius among all strongly connected digraphs with given diameter d, for $d = 1, 2, 3, 4, 5, 6, 7, n-1$. Li et al. [12] gave sharp upper and lower bounds for the distance signless Laplacian spectral radius of strongly connected digraphs, they also determined the extremal digraph with the minimum distance signless Laplacian spectral radius among all strongly connected digraphs with given dichromatic number. Li et al. [13], Xi and Wang [20] independently characterized the digraph minimizes the distance signless Laplacian spectral radius among all strongly connected digraphs with given vertex connectivity. Xi et al. [22] characterized the extremal digraph achieving the minimum distance signless Laplacian spectral radius among all strongly connected digraphs with given arc connectivity.

Compared with the much studied on distance and distance signless Laplacian spectral radius of digraphs, the study of generalized distance spectral radius has just been proposed by Xi et al. in [21]. Some basic spectral properties of generalized distance matrix of strongly connected digraphs are established and bounds for the generalized distance spectral radius were obtained. In [21], Xi et al. also determined the digraphs which attain the minimum D_α spectral radius among all strongly connected digraphs with given parameters such as dichromatic number, vertex connectivity or arc connectivity.

In this paper, we first give some upper and lower bounds on D_α spectral radius of strongly connected digraphs. Moreover, for digraphs that are not transmission regular, we give a lower bound on the difference between the maximum vertex transmission and the D_α spectral radius. Finally, we obtain the D_α eigenvalues of the join of certain regular digraphs.

2. Bounds for the D_α spectral radius of strongly connected digraphs

In this section, we give some upper and lower bounds on the D_α spectral radius of digraphs.

Lemma 2.1. *([10]) Let $M = (m_{ij})$ be an $n \times n$ nonnegative matrix with spectral radius $\rho(M)$, and let $R_i(M)$ be the i-th row sum of M. Then*

$$\min\{R_i(M) : 1 \le i \le n\} \le \rho(M) \le \max\{R_i(M) : 1 \le i \le n\}.$$

Moreover, if M is irreducible, then any equality holds if and only if $R_1(M) = R_2(M) = \cdots = R_n(M)$.

From Lemma 2.1, we have the following corollary.

Corollary 2.2. *Let G be a strongly connected digraph with $V(G) = \{v_1, v_2, \ldots, v_n\}$. Then*

$$\min_{v_i \in V(G)} \sqrt{\alpha Tr_i^2 + (1-\alpha)T_i} \le \mu_\alpha(G) \le \max_{v_i \in V(G)} \sqrt{\alpha Tr_i^2 + (1-\alpha)T_i},$$

where $Tr_i = \sum_{j=1}^{n} d_{ij}$ and $T_i = \sum_{t=1}^{n} d_{it}Tr_t$ are the transmission and 2-transmission of the vertex v_i, respectively.

Proof. Since $D_\alpha(G) = \alpha Tr(G) + (1-\alpha)D(G)$, by a simple calculation, we have

$$R_i(D_\alpha(G)) = Tr_i,\ R_i(D(G)Tr(G)) = \sum_{t=1}^{n} d_{it}Tr_t = T_i,$$

$$R_i(D^2(G)) = \sum_{j=1}^{n}\sum_{t=1}^{n} d_{it}d_{tj} = \sum_{t=1}^{n}\sum_{j=1}^{n} d_{it}d_{tj} = \sum_{t=1}^{n} d_{it}\sum_{j=1}^{n} d_{tj} = \sum_{t=1}^{n} d_{it}Tr_t = T_i.$$

Then

$$\begin{aligned}
R_i(D_\alpha^2(G)) &= R_i(\alpha Tr(G)(\alpha Tr(G) + (1-\alpha)D(G))) \\
&\quad + R_i(\alpha(1-\alpha)D(G)Tr(G)) + R_i((1-\alpha)^2D^2(G)) \\
&= \alpha Tr_i R_i(D_\alpha(G)) + \alpha(1-\alpha)T_i + (1-\alpha)^2 T_i \\
&= \alpha Tr_i^2 + (1-\alpha)T_i.
\end{aligned}$$

Hence, by Lemma 2.1, we have

$$\mu_\alpha(G) \le \max_{v_i \in V(G)} \sqrt{\alpha Tr_i^2 + (1-\alpha)T_i}.$$

The proof of the other part is similar. □

Lemma 2.3. *Let $B = (b_{ij})$ be an $n \times n$ real matrix, and let μ be an eigenvalue of B with a left eigenvector X all of whose entries are nonnegative. Then*

$$\min\{R_i(B) : 1 \le i \le n\} \le \mu \le \max\{R_i(B) : 1 \le i \le n\},$$

where $R_i(B)$ is the i-th row sum of B.

Proof. Since $X^TB = \mu X^T$. Without loss of generality, we may assume that $\sum\limits_{j=1}^{n} x_j = 1$. Then

$$\mu = \mu\sum_{j=1}^{n} x_j = \sum_{j=1}^{n} \mu x_j = \sum_{j=1}^{n}\sum_{i=1}^{n} x_i b_{ij} = \sum_{i=1}^{n}\sum_{j=1}^{n} x_i b_{ij} = \sum_{i=1}^{n} x_i \sum_{j=1}^{n} b_{ij} = \sum_{i=1}^{n} x_i R_i(B).$$

In other words, since the entries of X are nonnegative and sum to 1, μ is a convex combination of the row sums of B. Therefore, $\min\{R_i(B) : 1 \le i \le n\} \le \mu \le \max\{R_i(B) : 1 \le i \le n\}$. □

From Lemma 2.3, we have the following corollary which is necessary for the proof of Theorem 2.5 (below).

Corollary 2.4. *Let G be a strongly connected digraph with $V(G) = \{v_1, v_2, \dots, v_n\}$. Let $p(x)$ be a polynomial on x, $\mu_\alpha(G)$ be the spectral radius of $D_\alpha(G)$. Then*

$$\min_{1\le i\le n}\{R_i(p(D_\alpha(G)))\} \le p(\mu_\alpha(G)) \le \max_{1\le i\le n}\{R_i(p(D_\alpha(G)))\},$$

where $R_i(p(D_\alpha(G)))$ is the i-th row sum of matrix $p(D_\alpha(G))$. Moreover, if the row sums of $p(D_\alpha(G))$ are not all equal, then both inequalities are strict.

Proof. Since $D_\alpha(G)$ is a nonnegative irreducible matrix, there exists a left positive vector $X = (x_1, x_2, \dots, x_n)^T$ such that $X^T D_\alpha(G) = \mu_\alpha(G)X^T$. Then

$$X^T p(D_\alpha(G)) = p(\mu_\alpha(G))X^T.$$

Hence, by Lemma 2.3, the result follows. □

Theorem 2.5. *Let $G = (V(G), E(G))$ be a strongly connected digraph on n vertices with transmission sequence $\{Tr_1, Tr_2, \dots, Tr_n\}$. Let $Tr_{\max}$ and $Tr_{\min}$ denote the maximum and minimum vertex transmission of G, respectively. Then*

$$\mu_\alpha(G) \ge \frac{(1-\alpha)(Tr_{\min}-1) + \sqrt{(1-\alpha)^2(Tr_{\min}-1)^2 + 4[\alpha Tr_{\min}^2 + (1-\alpha)W(G) - (1-\alpha)(n-1)Tr_{\min}]}}{2},$$

and if $\alpha \in [\frac{1}{2}, 1)$,

$$\mu_\alpha(G) \le \frac{(1-\alpha)(Tr_{\max}-1) + \sqrt{(1-\alpha)^2(Tr_{\max}-1)^2 + 4[\alpha Tr_{\max}^2 + (1-\alpha)W(G) - (1-\alpha)(n-1)Tr_{\max}]}}{2},$$

where $W(G) = \sum\limits_{i=1}^{n}\sum\limits_{j=1}^{n} d_{ij}$.

Proof. Since $D_\alpha(G) = \alpha Tr(G) + (1-\alpha)D(G)$, by a simple calculation, we have

$$D_\alpha^2(G) = \alpha^2 Tr^2(G) + \alpha(1-\alpha)Tr(G)D(G)) + \alpha(1-\alpha)D(G)Tr(G) + (1-\alpha)^2 D^2(G).$$

Therefore, from the proof of Corollary 2.2, the i-th row sum of $D_\alpha^2(G)$ is

$$R_i(D_\alpha^2(G)) = \alpha Tr_i^2 + (1-\alpha)T_i,$$

where $T_i = \sum\limits_{t=1}^{n} d_{it}Tr_t$. However, let $W(G) = \sum\limits_{i=1}^{n}\sum\limits_{j=1}^{n} d_{ij}$, then

$$\begin{aligned}
\sum_{t=1}^{n} d_{it}Tr_t = \sum_{t=1}^{n} Tr_t + \sum_{t=1}^{n}(d_{it}-1)Tr_t &= W(G) - Tr_i + \sum_{t=1, t\ne i}^{n}(d_{it}-1)Tr_t \\
&\ge W(G) - Tr_i + Tr_{\min}\sum_{t=1, t\ne i}^{n}(d_{it}-1) \\
&= W(G) - Tr_i + Tr_{\min}Tr_i - (n-1)Tr_{\min} \\
&= W(G) + (Tr_{\min}-1)Tr_i - (n-1)Tr_{\min},
\end{aligned}$$

and

$$\begin{aligned}\sum_{t=1}^{n} d_{it}Tr_t &= W(G) - Tr_i + \sum_{t=1,t\neq i}^{n}(d_{it}-1)Tr_t \\ &\leq W(G) - Tr_i + Tr_{\max}\sum_{t=1,t\neq i}^{n}(d_{it}-1) \\ &= W(G) - Tr_i + Tr_{\max}Tr_i - (n-1)Tr_{\max} \\ &= W(G) + (Tr_{\max}-1)Tr_i - (n-1)Tr_{\max}.\end{aligned}$$

Therefore, we have

$$R_i(D_\alpha^2(G)) = \alpha Tr_i^2 + (1-\alpha)\sum_{t=1}^{n} d_{it}Tr_t \geq \alpha Tr_i^2 + (1-\alpha)[W(G) + (Tr_{\min}-1)Tr_i - (n-1)Tr_{\min}], \tag{1}$$

and

$$R_i(D_\alpha^2(G)) \leq \alpha Tr_i^2 + (1-\alpha)[W(G) + (Tr_{\max}-1)Tr_i - (n-1)Tr_{\max}]. \tag{2}$$

Let $p(x) = x^2 - (1-\alpha)(Tr_{\min}-1)x$. Then the i-th row sum of $p(D_\alpha(G))$ is

$$\begin{aligned}R_i(p(D_\alpha(G))) &= R_i(D_\alpha^2(G) - (1-\alpha)(Tr_{\min}-1)D_\alpha(G)) \\ &= R_i(D_\alpha^2(G)) - (1-\alpha)(Tr_{\min}-1)R_i(D_\alpha(G)) \\ &= R_i(D_\alpha^2(G)) - (1-\alpha)(Tr_{\min}-1)Tr_i.\end{aligned}$$

Furthermore, combining (1), we have

$$\begin{aligned}R_i(p(D_\alpha(G))) &\geq \alpha Tr_i^2 + (1-\alpha)[W(G) + (Tr_{\min}-1)Tr_i - (n-1)Tr_{\min}] - (1-\alpha)(Tr_{\min}-1)Tr_i \\ &= \alpha Tr_i^2 + (1-\alpha)W(G) - (1-\alpha)(n-1)Tr_{\min}.\end{aligned}$$

From the above inequality, for $i = 1, 2, \ldots, n$, we have

$$R_i(p(D_\alpha(G))) \geq \alpha Tr_{\min}^2 + (1-\alpha)W(G) - (1-\alpha)(n-1)Tr_{\min}.$$

From this inequality and Corollary 2.4, we get

$$p(\mu_\alpha(G)) = \mu_\alpha^2(G) - (1-\alpha)(Tr_{\min}-1)\mu_\alpha(G) \geq \alpha Tr_{\min}^2 + (1-\alpha)W(G) - (1-\alpha)(n-1)Tr_{\min},$$

which can reduce that

$$\mu_\alpha(G) \geq \frac{(1-\alpha)(Tr_{\min}-1) + \sqrt{(1-\alpha)^2(Tr_{\min}-1)^2 + 4[\alpha Tr_{\min}^2 + (1-\alpha)W(G) - (1-\alpha)(n-1)Tr_{\min}]}}{2}.$$

Similarly, using the polynomial $p(x) = x^2 - (1-\alpha)(Tr_{\max}-1)x$ and (2), we can obtain the upper bound in the theorem. □

Now, we give a lower and upper bound on the D_α spectral radius of digraphs in terms of the maximum transmission, the maximum second transmission, the minimum transmission and the minimum second transmission of G.

Theorem 2.6. *Let G be a strongly connected digraph with $V(G) = \{v_1, v_2, \ldots, v_n\}$. Then*

$$\frac{\alpha Tr_{min} + \sqrt{\alpha^2 Tr_{min}^2 + 4(1-\alpha)T_{min}}}{2} \leq \mu_\alpha(G) \leq \frac{\alpha Tr_{max} + \sqrt{\alpha^2 Tr_{max}^2 + 4(1-\alpha)T_{max}}}{2},$$

where Tr_{max} and Tr_{min} are the maximum transmission and the minimum transmission of G, respectively, T_{max} and T_{min} are the maximum second transmission and the minimum second transmission of G, respectively.

Proof. Since $D_\alpha(G) = \alpha Tr(G) + (1-\alpha)D(G)$, from the proof of Corollary 2.2, we get

$$\begin{aligned} R_i(D_\alpha^2(G)) &= \alpha Tr_i R_i(D_\alpha(G)) + \alpha(1-\alpha)T_i + (1-\alpha)^2 T_i \\ &\leq \alpha Tr_{max} R_i(D_\alpha(G)) + (1-\alpha)T_{max}. \end{aligned}$$

So we have

$$R_i(D_\alpha^2(G) - \alpha Tr_{max} D_\alpha(G)) \leq (1-\alpha)T_{max}.$$

Since G is a strongly connected digraph, there is a positive left eigenvector X corresponding to $\mu_\alpha(G)$. Hence $D_\alpha^2(G) - \alpha Tr_{max} D_\alpha(G)$ has an eigenvalue $\mu_\alpha^2(G) - \alpha Tr_{max}\mu_\alpha(G)$ with positive left eigenvector X. By Lemma 2.3, we have

$$\mu_\alpha^2(G) - \alpha Tr_{max}\mu_\alpha(G) \leq \max_{1\leq i\leq n} R_i(D_\alpha^2(G) - \alpha Tr_{max} D_\alpha(G)) \leq (1-\alpha)T_{max},$$

that is

$$\mu_\alpha(G) \leq \frac{\alpha Tr_{max} + \sqrt{\alpha^2 Tr_{max}^2 + 4(1-\alpha)T_{max}}}{2}.$$

The proof of the other part is similar. □

Next, we present another upper bound on the D_α spectral radius of digraphs.

Theorem 2.7. *Let $G = (V(G), E(G))$ be a strongly connected digraph on $n \geq 2$ vertices with $V(G) = \{v_1, v_2, \ldots, v_n\}$, the diameter d and transmission sequence $\{Tr_1, Tr_2, \ldots, Tr_n\}$, where $Tr_1 \geq Tr_2 \geq \cdots \geq Tr_n$. Let $\phi_1 = Tr_1$ and for $2 \leq l \leq n$,*

$$\phi_l = \frac{\alpha Tr_1 + Tr_l - (1-\alpha)d + \sqrt{(Tr_l - \alpha Tr_1 + (1-\alpha)d)^2 + 4(1-\alpha)d\sum_{k=1}^{l-1}(Tr_k - Tr_l)}}{2}, \tag{3}$$

and $\phi_s = \min_{1\leq l\leq n}\{\phi_l\}$ for some $s \in \{1, 2, \ldots, n\}$. Then $\mu_\alpha(G) \leq \phi_s$. Moreover, $\mu_\alpha(G) = \phi_s$ if and only if G is a transmission regular digraph.

Proof. Firstly, we will show that $\mu_\alpha(G) \leq \phi_l$ for all $1 \leq l \leq n$.

Case 1: $l = 1$.

It is obvious that $\mu_\alpha(G) \leq Tr_1 = \phi_1$ by Lemma 2.1 and the definition of $D_\alpha(G)$.

Case 2: $2 \leq l \leq n$.

By (3), it is obvious that $\phi_l > \alpha Tr_1 - (1-\alpha)d$, and $(\phi_l - \alpha Tr_1 + (1-\alpha)d)(\phi_l - Tr_l) = (1-\alpha)d\sum_{k=1}^{l-1}(Tr_k - Tr_l)$.

Let $U = diag\{x_1, x_2, \ldots, x_{l-1}, 1, 1, \ldots, 1\}$ be an $n \times n$ diagonal matrix, where $x_i = 1 + \frac{Tr_i - Tr_l}{\phi_l - \alpha Tr_1 + (1-\alpha)d} \geq 1$ for $i \in \{1, 2, \ldots, l-1\}$. Then

$$x_1 \geq x_2 \geq \cdots \geq x_{l-1} \geq 1,$$

$$(1-\alpha)d\sum_{k=1}^{l-1}(x_k - 1) = (1-\alpha)d\sum_{k=1}^{l-1}\frac{Tr_k - Tr_l}{\phi_l - \alpha Tr_1 + (1-\alpha)d} = \phi_l - Tr_l, \tag{4}$$

and

$$U^{-1} = diag\{x_1^{-1}, x_2^{-1}, \ldots, x_{l-1}^{-1}, 1, 1, \ldots, 1\}.$$

Let $D_\alpha(G) = (\omega_{ij})_{n\times n} = \alpha Tr(G) + (1-\alpha)D(G)$ be the generalized distance matrix of G and $P = U^{-1}D_\alpha(G)U$. Obviously, P and $D_\alpha(G)$ have the same eigenvalues, thus $\mu_\alpha(G) = \rho(P)$, where $\rho(P)$ denotes the spectral radius of matrix P. Let $R_i(P)$ be the i-th row sum of P, $1 \leq i \leq n$. Now we will prove that $R_i(P) \leq \phi_l$ for any $1 \leq i \leq n$.

Subcase 2.1: $1 \le i \le l-1$.

$$\begin{aligned}
R_i(P) &= \sum_{k=1}^{l-1} \frac{x_k}{x_i}\omega_{ik} + \sum_{k=l}^{n} \frac{1}{x_i}\omega_{ik} = \frac{1}{x_i}\sum_{k=1}^{l-1}(x_k-1)\omega_{ik} + \frac{1}{x_i}\sum_{k=1}^{n}\omega_{ik} \\
&= \frac{1}{x_i}Tr_i + \frac{1}{x_i}\sum_{k=1}^{l-1}(x_k-1)\omega_{ik} \\
&= \frac{1}{x_i}Tr_i + \frac{1}{x_i}\left((x_i-1)\omega_{ii} + \sum_{k=1,k\neq i}^{l-1}(x_k-1)\omega_{ik}\right) \\
&\le \frac{1}{x_i}Tr_i + \frac{1}{x_i}\left((x_i-1)\alpha Tr_i + \sum_{k=1,k\neq i}^{l-1}(x_k-1)(1-a)d\right) \\
&\le \frac{1}{x_i}Tr_i + \frac{1}{x_i}\left((x_i-1)\alpha Tr_1 + \sum_{k=1,k\neq i}^{l-1}(x_k-1)(1-a)d\right) \\
&= \frac{1}{x_i}\left(Tr_i + (x_i-1)(\alpha Tr_1 - (1-a)d) + (1-a)d\sum_{k=1}^{l-1}(x_k-1)\right) \\
&= \frac{1}{x_i}\left(Tr_i + (\alpha Tr_1 - (1-a)d)\frac{Tr_i - Tr_l}{\phi_l - \alpha Tr_1 + (1-\alpha)d} + \phi_l - Tr_l\right) \text{ (using (4))} \\
&= \frac{\phi_l - \alpha Tr_1 + (1-\alpha)d}{\phi_l - \alpha Tr_1 + (1-\alpha)d + Tr_i - Tr_l} \cdot \frac{\phi_l(\phi_l - \alpha Tr_1 + (1-\alpha)d + Tr_i - Tr_l)}{\phi_l - \alpha Tr_1 + (1-\alpha)d} \\
&= \phi_l,
\end{aligned}$$

with equality if and only if (i) and (ii) hold:
(i) $x_i = 1$ or $\omega_{ii} = \alpha Tr_1$ for $x_i > 1$,
(ii) $x_k = 1$ or $\omega_{ik} = (1-\alpha)d$ for $x_k > 1$ if $1 \le k \le l-1$ with $k \neq i$.

Subcase 2.2: $l \le i \le n$.

$$\begin{aligned}
R_i(P) &= \sum_{k=1}^{l-1} x_k\omega_{ik} + \sum_{k=l}^{n}\omega_{ik} = \sum_{k=1}^{l-1}(x_k-1)\omega_{ik} + \sum_{k=1}^{n}\omega_{ik} \\
&= \sum_{k=1}^{l-1}(x_k-1)\omega_{ik} + Tr_i \\
&\le \sum_{k=1}^{l-1}(x_k-1)\omega_{ik} + Tr_l \\
&\le (1-a)d\sum_{k=1}^{l-1}(x_k-1) + Tr_l = \phi_l \text{ (using (4))},
\end{aligned}$$

with equality if and only if (iii) and (iv) hold:
(iii) $Tr_i = Tr_l$,
(iv) $x_k = 1$ or $\omega_{ik} = (1-\alpha)d$ for $x_k > 1$ if $1 \le k \le l-1$.

Hence, by Lemma 2.1, $\mu_\alpha(G) = \rho(P) \le \max_{1\le i\le n} R_i(P) \le \phi_l$ for any $2 \le l \le n$. Thus $\mu_\alpha(G) = \rho(P) \le \max_{1\le i\le n} R_i(P) \le \min_{2\le l\le n} \phi_l$.

Combining the above two cases, $\mu_\alpha(G) \le \min_{1\le l\le n} \phi_l$.

In the following, we will give the sufficient and necessary condition of the equality. Let $\phi_s = \min_{1\le l\le n} \phi_l$ for some $s \in \{1, 2, \ldots, n\}$. Then we consider the following two cases.

Case 1: $s = 1$.

It is obvious that $\mu_\alpha(G) = \phi_1 = Tr_1$ if and only if G is a transmission regular digraph by Lemma 2.1.

Case 2: $2 \le s \le n$.

Clearly, $D_\alpha(G)$ and P are irreducible nonnegative matrices because G is a strongly connected digraph. Then $\mu_\alpha(G) = \phi_s$ if and only if $\phi_1 \ge \phi_s$, $\rho(P) = \max\limits_{1\le i\le n} R_i(P)$ and $\max\limits_{1\le i\le n} R_i(P) = \phi_s$. Noting that $\rho(P) = \max\limits_{1\le i\le n} R_i(P)$ if and only if the row sums of P, $R_1(P), R_2(P), \ldots, R_n(P)$ are all equal by Lemma 2.1. Hence, we have $\mu_\alpha(G) = \phi_s$ if and only if $R_1(P) = R_2(P) = \cdots = R_n(P) = \phi_s$ and $\phi_1 \ge \phi_s$.

Noting that $R_1(P) = R_2(P) = \cdots = R_n(P) = \phi_s$ if and only if P satisfies the following four conditions:

(i) $x_i = 1$ or $\omega_{ii} = \alpha Tr_1$ for $x_i > 1$ holds for all $1 \le i \le s-1$,

(ii) $x_k = 1$ or $\omega_{ik} = (1-\alpha)d$ for $x_k > 1$ if $1 \le k \le s-1$ with $k \ne i$ holds for all $1 \le i \le s-1$,

(iii) $Tr_s = Tr_{s+1} = \cdots = Tr_n$,

(iv) $x_k = 1$ or $\omega_{ik} = (1-\alpha)d$ for $x_k > 1$ if $1 \le k \le s-1$ holds for all $s \le i \le n$.

Thus we only need to show that $(i)-(iv)$ hold if and only if G is a transmission regular digraph.

If $(i)-(iv)$ hold, we consider the following cases.

Subcase 2.1: $x_1 = 1$

Then $x_1 = x_2 = \cdots = x_{s-1} = 1$ by $x_1 \ge x_2 \ge \cdots \ge x_{s-1} \ge 1$, and thus $Tr_1 = Tr_2 = \cdots = Tr_{s-1} = Tr_s$. Furthermore, from (iii), we have G is a transmission regular digraph.

Subcase 2.2: $x_1 \ge x_2 \ge \cdots \ge x_{t-1} > 1$ and $x_t = \cdots = x_{s-1} = 1$ for some $t \in \{2, \ldots, s\}$.

Then $\omega_{ii} = \alpha Tr_i = \alpha Tr_1$ for $1 \le i \le t-1$ by (i), and $Tr_t = \cdots = Tr_{s-1} = Tr_s = \cdots = Tr_n$ by (iii). Thus $Tr_1 = Tr_2 = \cdots = Tr_{t-1} > Tr_t = \cdots = Tr_{s-1} = Tr_s = \cdots = Tr_n$. By (ii) and (iv), we have $\omega_{ik} = (1-\alpha)d$ for all $k \in \{1, 2, \ldots, t-1\}$ and $i \in \{1, 2, \ldots, n\}\backslash\{k\}$, which implies that $d_{ik} = d$ for all $k \in \{1, 2, \ldots, t-1\}$ and $i \in \{1, 2, \ldots, n\}\backslash\{k\}$. If $d \ge 2$, it is obvious that such digraphs are not exist. If $d = 1$, then G is a complete digraph, $Tr_1 = Tr_2 = \cdots = Tr_{t-1} = Tr_t = \cdots = Tr_{s-1} = Tr_s = \cdots = Tr_n$, which is a contradiction. Therefore, there are no digraphs in this case.

Conversely, if G is a transmission regular digraph, then $Tr_1 = Tr_2 = \cdots = Tr_n$ and $\phi_1 = \phi_2 = \cdots = \phi_n = Tr_1$, the result follows. □

In the next theorem, we show the equivalence between the A_α spectrum and the D_α spectrum of a strongly connected regular digraph of diameter two.

Lemma 2.8. *([10]) Let A be an $n \times n$ real matrix, which has eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ in any prescribed order, and let X be a unit vector such that $AX = \lambda_1 X$. Then, there is a unitary matrix $U = [X, \beta_2, \ldots, \beta_n]$ such that $U^*AU = T = (t_{ij})$ is an upper triangular matrix with diagonal entries $t_{ii} = \lambda_i$, $i = 1, 2, \ldots, n$.*

Theorem 2.9. *Let G be a strongly connected r-regular digraph on n vertices with diameter $d \le 2$. If $r, \lambda_\alpha^2, \ldots, \lambda_\alpha^n$ are the eigenvalues of the matrix $A_\alpha(G)$ of G, then the D_α eigenvalues of G are $2n-2-r$ and $2\alpha n - 2 - \lambda_\alpha^i$, $i = 2, 3, \ldots, n$.*

Proof. If $d = 1$, then G is a complete digraph, the result is trivial.

Now, let G be a digraph of diameter 2. Then, $A_\alpha(G) = \alpha rI + (1-\alpha)A(G)$, $D(G) = 2(J - I - A(G)) + A(G) = 2J - 2I - A(G)$. The transmission of each vertex $u \in V(G)$ is

$$Tr(u) = d^+(u) + 2(n-1-d^+(u)) = r + 2(n-1-r) = 2n-2-r.$$

Furthermore,

$$\begin{aligned} D_\alpha(G) &= \alpha Tr(G) + (1-\alpha)D(G) = \alpha(2n-r-2)I + (1-\alpha)(2J - 2I - A(G)) \\ &= (2\alpha n - \alpha r - 2)I + 2(1-\alpha)J - (1-\alpha)A(G) \\ &= (2\alpha n - 2)I + 2(1-\alpha)J - (\alpha rI + (1-\alpha)A(G)) \\ &= (2\alpha n - 2)I + 2(1-\alpha)J - A_\alpha(G). \end{aligned}$$

Since G is an r-regular digraph, $\mathbf{1} = (1, 1, \ldots, 1)^T$ is an eigenvector of the matrix $A_\alpha(G)$ corresponding to the eigenvalue r. Let $z = \frac{1}{\sqrt{n}}\mathbf{1}$, then by Lemma 2.8, there exists a unitary matrix $U = [z, z_2, \ldots, z_n]$ such that $U^*A_\alpha(G)U = T = (t_{ij})$ is an upper triangular matrix with diagonal entries $t_{11} = r$, $t_{ii} = \lambda_\alpha^i$, $i = 2, \ldots, n$. Therefore,

$$\begin{aligned} U^*D_\alpha(G)U &= U^*((2\alpha n - 2)I + 2(1-\alpha)J - A_\alpha(G))U \\ &= (2\alpha n - 2)I + 2(1-\alpha)U^*JU - U^*A_\alpha(G)U \\ &= (2\alpha n - 2)I + 2(1-\alpha)diag\{n, 0, \ldots, 0\} - T. \end{aligned}$$

Hence, the eigenvalues of $D_\alpha(G)$ are $2n - 2 - r$ and $2\alpha n - 2 - \lambda_\alpha^i$, $i = 2, 3, \ldots, n$. $\square$

Corollary 2.10. *Let G be a strongly connected r-regular digraph on n vertices with diameter 2. If $\{r, q_2, \cdots, q_n\}$ are the eigenvalues of the adjacency matrix $A(G)$ of G, then the D_α eigenvalues of G are $2n-2-r$ and $2\alpha n - \alpha r - 2 - (1-\alpha)q_i$, $i = 2, 3, \ldots, n$.*

Proof. From Theorem 2.9, we get

$$D_\alpha(G) = (2\alpha n - \alpha r - 2)I + 2(1-\alpha)J - (1-\alpha)A(G).$$

Hence, the eigenvalues of $D_\alpha(G)$ are $2n - 2 - r$ and $2\alpha n - \alpha r - 2 - (1-\alpha)q_i$, $i = 2, 3, \ldots, n$. $\square$

Theorem 2.11. *Let $G = (V(G), E(G))$ be a strongly connected digraph on $n \geq 2$ vertices with $V(G) = \{v_1, v_2, \ldots, v_n\}$, transmission sequence $\{Tr_1, Tr_2, \ldots, Tr_n\}$. if $S = \{v_1, v_2, \ldots, v_p\}$ is a clique of G such that $N^+(v_i)\backslash S = N^+(v_j)\backslash S$ for all $i, j \in \{1, 2, \ldots, p\}$, then $\nu = Tr_i = Tr_j$ for all $i, j \in \{1, 2, \ldots, p\}$ and $\alpha\nu - (1-\alpha)$ is an eigenvalue of $D_\alpha(G)$ with multiplicity at least $p - 1$.*

Proof. Since the vertices in S share the same outneighborhoods in $V(G)\backslash S$, any vertex v_i in S is at the same distance from v_i to all vertices in $V(G)\backslash S$, and any vertex in S is at distance 1 to any other vertex in S. Thus all vertices in S have the same transmission, say ν. To show that $\alpha\nu - (1-\alpha)$ is an eigenvalue of $D_\alpha(G)$ with multiplicity $p - 1$, it suffices to observe that the matrix $(\alpha\nu - (1-\alpha))I_n - D_\alpha(G)$ contains p identical rows. $\square$

3. Lower bound on the difference between the maximum vertex transmission and the D_α spectral radius

Lemma 3.1. *([19]) If $a, b > 0$, then $a(x-y)^2 + by^2 \geq abx^2/(a+b)$ with equality if and only if $y = ax/(a+b)$.*

Theorem 3.2. *Let $G = (V(G), E(G))$ be a strongly connected non-transmission irregular digraph on n vertices with transmission sequence $\{Tr_1, Tr_2, \ldots, Tr_n\}$, where $Tr_1 \geq Tr_2 \geq \cdots \geq Tr_n$. Let $Tr_i^- = \sum_{j=1}^{n} d_{ji}$. If $Tr_1 \geq Tr_i^-$, for all $i = 1, 2, \ldots, n$, then*

$$Tr_1 - \mu_\alpha(G) > \frac{(1-\alpha)(n+2)Tr_1(nTr_1 - W(G))}{4W(G)(nTr_1 - W(G)) + (1-\alpha)(n+2)nTr_1},$$

where $W(G) = \sum_{i=1}^{n}\sum_{j=1}^{n} d_{ij}$.

Proof. Let $x = (x_1, x_2, \ldots, x_n)^T$ be the Perron vector of $D_\alpha(G)$ corresponding to $\mu_\alpha(G)$, where x_i corresponding to the vertex v_i. Obviously, $x_1^2 + x_2^2 + \cdots + x_n^2 = 1$. Suppose that u, v are two vertices satisfying $x_u = \max_{1\leq i\leq n} x_i$ and $x_v = \min_{1\leq i\leq n} x_i$, respectively. Taking $W(G) = \sum_{i=1}^{n}\sum_{j=1}^{n} d_{ij}$. Since G is not transmission regular, we have $x_u > x_v$, and thus

$$\begin{aligned} \mu_\alpha(G) = x^T D_\alpha(G)x &= \alpha\sum_{i=1}^{n} Tr_i x_i^2 + (1-\alpha)\sum_{i=1}^{n}\sum_{j=1}^{n} d_{ij}x_i x_j \\ &< \alpha\sum_{i=1}^{n} Tr_i x_u^2 + (1-\alpha)\sum_{i=1}^{n}\sum_{j=1}^{n} d_{ij}x_u^2 \\ &= \alpha W(G)x_u^2 + (1-\alpha)W(G)x_u^2, \end{aligned}$$

which implies that $x_u^2 > \frac{\mu_\alpha(G)}{W(G)}$. And we have

$$\begin{aligned}
2Tr_1 - 2\mu_\alpha(G) &= 2Tr_1 - 2x^T D_\alpha(G)x \\
&= 2Tr_1 - 2\alpha \sum_{i=1}^{n} Tr_i x_i^2 - 2(1-\alpha)\sum_{i=1}^{n}\sum_{j=1}^{n} d_{ij}x_i x_j \\
&= 2Tr_1 \sum_{i=1}^{n} x_i^2 - 2\alpha \sum_{i=1}^{n} Tr_i x_i^2 + (1-\alpha)\sum_{i=1}^{n}\sum_{j=1}^{n} d_{ij}(x_i - x_j)^2 \\
&\quad - (1-\alpha)\sum_{i=1}^{n}\sum_{j=1}^{n} d_{ij}(x_i^2 + x_j^2).
\end{aligned}$$

Taking $Tr_i^- = \sum_{j=1}^{n} d_{ji}$, then $\sum_{i=1}^{n}\sum_{j=1}^{n} d_{ij}(x_i^2 + x_j^2) = \sum_{i=1}^{n} Tr_i x_i^2 + \sum_{i=1}^{n} Tr_i^- x_i^2$. Therefore,

$$\begin{aligned}
2Tr_1 - 2\mu_\alpha(G) &= \sum_{i=1}^{n}(2Tr_1 - (1+\alpha)Tr_i - (1-\alpha)Tr_i^-)x_i^2 + (1-\alpha)\sum_{i=1}^{n}\sum_{j=1}^{n} d_{ij}(x_i - x_j)^2 \\
&\geq \sum_{i=1}^{n}(2Tr_1 - (1+\alpha)Tr_i - (1-\alpha)Tr_i^-)x_v^2 + (1-\alpha)\sum_{i=1}^{n}\sum_{j=1}^{n} d_{ij}(x_i - x_j)^2 \\
&= (2nTr_1 - (1+\alpha)W(G) - (1-\alpha)W(G))x_v^2 + (1-\alpha)\sum_{i=1}^{n}\sum_{j=1}^{n} d_{ij}(x_i - x_j)^2 \\
&= (2nTr_1 - 2W(G))x_v^2 + (1-\alpha)\sum_{i=1}^{n}\sum_{j=1}^{n} d_{ij}(x_i - x_j)^2.
\end{aligned}$$

Suppose $P = v_1v_2\ldots v_{s+1}$ be the shortest directed path from v and u, where $v_1 = v$, $v_{s+1} = u$, and $s \geq 1$. Now we need to estimate $\sum_{i=1}^{n}\sum_{j=1}^{n} d_{ij}(x_i - x_j)^2$. Obviously,

$$\begin{aligned}
\sum_{i=1}^{n}\sum_{j=1}^{n} d_{ij}(x_i - x_j)^2 &= \sum_{v_i\in P}\sum_{j=1}^{n} d_{ij}(x_i - x_j)^2 + \sum_{v_i\in V(G)\backslash P}\sum_{j=1}^{n} d_{ij}(x_i - x_j)^2 \\
&> \sum_{v_i\in P}\sum_{v_j\in P} d_{ij}(x_i - x_j)^2 + \sum_{v_i\in V(G)\backslash P}\left(d_{iu}(x_i - x_u)^2 + d_{iv}(x_i - x_v)^2\right).
\end{aligned}$$

For any $v_i \in V(G)\backslash P$, by Cauchy-Schwarz inequality, we have

$$d_{iu}(x_i - x_u)^2 + d_{iv}(x_i - x_v)^2 \geq (x_i - x_u)^2 + (x_i - x_v)^2 \geq \frac{1}{2}(x_u - x_v)^2,$$

and then

$$\sum_{v_i\in V(G)\backslash P}(d_{iu}(x_i - x_u)^2 + d_{iv}(x_i - x_v)^2) \geq \sum_{v_i\in V(G)\backslash P}\frac{1}{2}(x_u - x_v)^2 = \frac{n-s-1}{2}(x_u - x_v)^2.$$

Then we consider the following two cases.

Case 1: $s = 1$. In this case, we have

$$
\begin{aligned}
2Tr_1 - 2\mu_\alpha(G) &\geq (2nTr_1 - 2W(G))x_v^2 + (1-\alpha)\sum_{i=1}^{n}\sum_{j=1}^{n} d_{ij}(x_i - x_j)^2 \\
&> 2(nTr_1 - W(G))x_v^2 + (1-\alpha)\sum_{v_i\in P}\sum_{v_j\in P} d_{ij}(x_i - x_j)^2 + (1-\alpha)\frac{n-2}{2}(x_u - x_v)^2 \\
&\geq 2(nTr_1 - W(G))x_v^2 + 2(1-\alpha)(x_u - x_v)^2 + (1-\alpha)\frac{n-2}{2}(x_u - x_v)^2 \\
&= 2(nTr_1 - W(G))x_v^2 + (1-\alpha)\frac{n+2}{2}(x_u - x_v)^2 \\
&\geq \frac{2(1-\alpha)(n+2)(nTr_1 - W(G))}{4(nTr_1 - W(G)) + (1-\alpha)(n+2)}x_u^2 \text{ (using Lemma 3.1).}
\end{aligned}
$$

Then we have $Tr_1 - \mu_\alpha(G) > \frac{(1-\alpha)(n+2)(nTr_1 - W(G))}{4(nTr_1 - W(G)) + (1-\alpha)(n+2)}x_u^2$. Recall that $x_u^2 > \frac{\mu_\alpha(G)}{W(G)}$. Therefore,

$$
Tr_1 - \mu_\alpha(G) > \frac{(1-\alpha)(n+2)(nTr_1 - W(G))}{4(nTr_1 - W(G)) + (1-\alpha)(n+2)} \cdot \frac{\mu_\alpha(G)}{W(G)},
$$

i.e.,

$$
Tr_1 - \mu_\alpha(G) > \frac{(1-\alpha)(n+2)Tr_1(nTr_1 - W(G))}{4W(G)(nTr_1 - W(G)) + (1-\alpha)(n+2)nTr_1}.
$$

Case 2: $s \geq 2$. For the vertex v_{i+1} $(1 \leq i \leq s-1)$ of the shortest directed path P from v and u, by Cauchy-Schwarz inequality, we get

$$
\begin{aligned}
d_{v_1v_{i+1}}(x_1 - x_{i+1})^2 + d_{v_{i+1}v_{s+1}}(x_{i+1} - x_{s+1})^2 &\geq \min\{i, s-i\}((x_1 - x_{i+1})^2 + (x_{i+1} - x_{s+1})^2) \\
&\geq \frac{1}{2}\min\{i, s-i\}(x_1 - x_{s+1})^2,
\end{aligned}
$$

and thus

$$
\begin{aligned}
\sum_{v_i\in P}\sum_{v_j\in P} d_{ij}(x_i - x_j)^2 &> d_{v_1v_2}(x_1 - x_2)^2 + d_{v_1v_3}(x_1 - x_3)^2 + \cdots + d_{v_1v_{s+1}}(x_1 - x_{s+1})^2 \\
&\quad + d_{v_2v_3}(x_2 - x_3)^2 + d_{v_2v_4}(x_2 - x_4)^2 + \cdots + d_{v_2v_{s+1}}(x_2 - x_{s+1})^2 \\
&\quad + d_{v_3v_4}(x_3 - x_4)^2 + d_{v_3v_5}(x_3 - x_5)^2 + \cdots + d_{v_3v_{s+1}}(x_3 - x_{s+1})^2 \\
&\quad + \cdots + d_{v_sv_{s+1}}(x_s - x_{s+1})^2 \\
&> d_{v_1v_{s+1}}(x_1 - x_{s+1})^2 + \sum_{i=1}^{s-1}\left(d_{v_1v_{i+1}}(x_1 - x_{i+1})^2 + d_{v_{i+1}v_{s+1}}(x_{i+1} - x_{s+1})^2\right) \\
&\geq s(x_1 - x_{s+1})^2 + \frac{1}{2}\sum_{i=1}^{s-1}\min\{i, s-i\}(x_1 - x_{s+1})^2.
\end{aligned}
$$

Subcase 2.1: $s \geq 2$ is even. Based on the above inequality, we have

$$
\begin{aligned}
\sum_{v_i\in P}\sum_{v_j\in P} d_{ij}(x_i - x_j)^2 &> s(x_1 - x_{s+1})^2 + \frac{(x_1 - x_{s+1})^2}{2}\left[(1 + 2 + \ldots + \frac{s}{2}) + (1 + 2 + \ldots + \frac{s-2}{2})\right] \\
&= \frac{s^2 + 8s}{8}(x_u - x_v)^2.
\end{aligned}
$$

Furthermore, we get

$$
\begin{aligned}
2Tr_1 - 2\mu_\alpha(G) &> 2(nTr_1 - W(G))x_v^2 + (1-\alpha)\sum_{v_i\in P}\sum_{v_j\in P} d_{ij}(x_i - x_j)^2 + (1-\alpha)\frac{n-s-1}{2}(x_u - x_v)^2 \\
&> (2nTr_1 - 2W(G))x_v^2 + (1-\alpha)\frac{s^2+8s}{8}(x_u - x_v)^2 + (1-\alpha)\frac{n-s-1}{2}(x_u - x_v)^2 \\
&= (2nTr_1 - 2W(G))x_v^2 + (1-\alpha)\frac{s^2+4s+4n-4}{8}(x_u - x_v)^2 \\
&> \frac{2(1-\alpha)(s^2+4s+4n-4)(nTr_1 - W(G))}{16(nTr_1 - W(G)) + (1-\alpha)(s^2+4s+4n-4)}x_u^2 \text{ (using Lemma 3.1).}
\end{aligned}
$$

Then we have

$$
Tr_1 - \mu_\alpha(G) > \frac{(1-\alpha)(s^2+4s+4n-4)(nTr_1 - W(G))}{16(nTr_1 - W(G)) + (1-\alpha)(s^2+4s+4n-4)}x_u^2.
$$

Since $x_u^2 > \frac{\mu_\alpha(G)}{W(G)}$, we have

$$
Tr_1 - \mu_\alpha(G) > \frac{(1-\alpha)(s^2+4s+4n-4)(nTr_1 - W(G))}{16(nTr_1 - W(G)) + (1-\alpha)(s^2+4s+4n-4)}\frac{\mu_\alpha(G)}{W(G)},
$$

i.e.,

$$
Tr_1 - \mu_\alpha(G) > \frac{(1-\alpha)(s^2+4s+4n-4)Tr_1(nTr_1 - W(G))}{16W(G)(nTr_1 - W(G)) + (1-\alpha)(s^2+4s+4n-4)nTr_1}.
$$

Suppose $f(t) = \frac{(1-\alpha)(4n+t)Tr_1(nTr_1 - W(G))}{16W(G)(nTr_1 - W(G)) + (1-\alpha)(4n+t)nTr_1}$. Then we can easily known that $f(t)$ is monotonically increasing function on $t > 0$. Therefore, we get

$$
\begin{aligned}
Tr_1 - \mu_\alpha(G) &> \frac{(1-\alpha)(4n+8)Tr_1(nTr_1 - W(G))}{16W(G)(nTr_1 - W(G)) + (1-\alpha)(4n+8)nTr_1} \\
&= \frac{(1-\alpha)(n+2)Tr_1(nTr_1 - W(G))}{4W(G)(nTr_1 - W(G)) + (1-\alpha)(n+2)nTr_1}.
\end{aligned}
$$

Subcase 2.2: $s \geq 3$ is odd. Then we have

$$
\begin{aligned}
\sum_{v_i\in P}\sum_{v_j\in P} d_{ij}(x_i - x_j)^2 &> s(x_1 - x_{s+1})^2 + \frac{(x_1 - x_{s+1})^2}{2}\left[2\times(1+2+\ldots+\frac{s-1}{2})\right] \\
&= \frac{s^2+8s-1}{8}(x_u - x_v)^2.
\end{aligned}
$$

Furthermore, we get

$$
\begin{aligned}
2Tr_1 - 2\mu_\alpha(G) &> 2(nTr_1 - W(G))x_v^2 + (1-\alpha)\sum_{v_i\in P}\sum_{v_j\in P} d_{ij}(x_i - x_j)^2 + (1-\alpha)\frac{n-s-1}{2}(x_u - x_v)^2 \\
&> 2(nTr_1 - W(G))x_v^2 + (1-\alpha)\frac{s^2+8s-1}{8}(x_u - x_v)^2 + (1-\alpha)\frac{n-s-1}{2}(x_u - x_v)^2 \\
&= 2(nTr_1 - W(G))x_v^2 + (1-\alpha)\frac{s^2+4s+4n-5}{8}(x_u - x_v)^2 \\
&> \frac{2(1-\alpha)(s^2+4s+4n-5)(nTr_1 - W(G))}{16(nTr_1 - W(G)) + (1-\alpha)(s^2+4s+4n-5)}x_u^2 \text{ (using Lemma 3.1).}
\end{aligned}
$$

Thus, as early, we get

$$\begin{aligned} Tr_1 - \mu_\alpha(G) &> \frac{(1-\alpha)(s^2+4s+4n-5)(nTr_1 - W(G))}{16(nTr_1 - W(G)) + (1-\alpha)(s^2+4s+4n-5)} x_u^2 \\ &> \frac{(1-\alpha)(s^2+4s+4n-4)(nTr_1 - W(G))}{16(nTr_1 - W(G)) + (1-\alpha)(s^2+4s+4n-4)} \frac{\mu_\alpha(G)}{W(G)}, \end{aligned}$$

which implies,

$$Tr_1 - \mu_\alpha(G) > \frac{(1-\alpha)(s^2+4s+4n-5)Tr_1(nTr_1 - W(G))}{16W(G)(nTr_1 - W(G)) + (1-\alpha)(s^2+4s+4n-5)nTr_1}.$$

By the monotonicity of $f(t) = \frac{(1-\alpha)(4n+t)Tr_1(nTr_1-W(G))}{16W(G)(nTr_1-W(G))+(1-\alpha)(4n+t)nTr_1}$, we obtain

$$\begin{aligned} Tr_1 - \mu_\alpha(G) &> \frac{(1-\alpha)(4n+16)Tr_1(nTr_1 - W(G))}{16W(G)(nTr_1 - W(G)) + (1-\alpha)(4n+16)nTr_1} \\ &= \frac{(1-\alpha)(n+4)Tr_1(nTr_1 - W(G))}{4W(G)(nTr_1 - W(G)) + (1-\alpha)(n+4)nTr_1} \\ &> \frac{(1-\alpha)(n+2)Tr_1(nTr_1 - W(G))}{4W(G)(nTr_1 - W(G)) + (1-\alpha)(n+2)nTr_1}. \end{aligned}$$

Therefore, the result follows by combining Cases 1 and 2. □

4. The D_α-spectrum of the join of digraphs

Let G_1 and G_2 be two disjoint digraphs, the join of G_1 and G_2, denoted by $G_1 \vee G_2$, is the digraph such that $V(G_1 \vee G_2) = V(G_1) \cup V(G_2)$ and $E(G_1 \vee G_2) = E(G_1) \cup E(G_2) \cup \{(u, v), (v, u) : u \in V(G_1) \text{ and } v \in V(G_2)\}$.

In this section, we give the D_α-spectrum of the join of two regular digraphs. For the rest of this section, I and J denote the unit and the all-one matrices of corresponding orders, respectively. Let M be an $n \times n$ matrix. The characteristic polynomial of M is defined as $p_M(\lambda) = det(\lambda I - M)$, where $det(*)$ denotes the determinant of $*$.

Theorem 4.1. *Let G_i be r_i-regular strongly connected digraph with order n_i, for $i = 1, 2$. Then the characteristic polynomial of $D_\alpha(G_1 \vee G_2)$ is*

$$P_{D_\alpha(G_1\vee G_2)}(\lambda) = (-1)^{n_1+n_2} \frac{P_{A_\alpha(G_1)}(\alpha n_2 + 2\alpha n_1 - 2 - \lambda) P_{A_\alpha(G_2)}(\alpha n_1 + 2\alpha n_2 - 2 - \lambda)}{(\lambda - \alpha n_2 - 2\alpha n_1 + r_1 + 2)(\lambda - \alpha n_1 - 2\alpha n_2 + r_2 + 2)} f(\lambda),$$

where $f(\lambda) = (\lambda - \alpha n_2 - 2n_1 + r_1 + 2)(\lambda - \alpha n_1 - 2n_2 + r_2 + 2) - (1-\alpha)^2 n_1 n_2$.

Proof. One can easily see

$$D_\alpha(G_1 \vee G_2) = \begin{pmatrix} \alpha n_2 I_{n_1} + M_1 & (1-\alpha) J_{n_1 \times n_2} \\ (1-\alpha) J_{n_2 \times n_1} & \alpha n_1 I_{n_2} + M_2 \end{pmatrix},$$

where $M_i = \alpha(2n_i - r_i - 2) I_{n_i} + 2(1-\alpha)(J_{n_i} - I_{n_i}) - (1-\alpha)A(G_i) = (2\alpha n_i - 2) I_{n_i} + 2(1-\alpha) J_{n_i} - A_\alpha(G_i)$.

Then the characteristic polynomial of $D_\alpha(G_1 \vee G_2)$ is

$$P_{D_\alpha(G_1\vee G_2)}(\lambda) = \begin{vmatrix} \lambda I_{n_1} - \alpha n_2 I_{n_1} - M_1 & -(1-\alpha) J_{n_1 \times n_2} \\ -(1-\alpha) J_{n_2 \times n_1} & \lambda I_{n_2} - \alpha n_1 I_{n_2} - M_2 \end{vmatrix}. \quad (5)$$

Let $m_1 = \lambda - \alpha n_2 - \alpha(2n_1 - r_1 - 2)$ and $m_2 = \lambda - \alpha n_1 - \alpha(2n_2 - r_2 - 2)$. We use $A(G_1) = (a_{ij})_{n_1 \times n_1}$ denotes the adjacency matrix of G_1 and $A(G_2) = (a'_{ij})_{n_2 \times n_2}$ denotes the adjacency matrix of G_2. The determinant (5) can be written as

$$\left|\begin{array}{cccccccc}
m_1 & -(1-\alpha)(2-a_{12}) & \cdots & -(1-\alpha)(2-a_{1n_1}) & -(1-\alpha) & -(1-\alpha) & \cdots & -(1-\alpha) \\
-(1-\alpha)(2-a_{21}) & m_1 & \cdots & -(1-\alpha)(2-a_{2n_1}) & -(1-\alpha) & -(1-\alpha) & \cdots & -(1-\alpha) \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
-(1-\alpha)(2-a_{n_11}) & -(1-\alpha)(2-a_{n_12}) & \cdots & m_1 & -(1-\alpha) & -(1-\alpha) & \cdots & -(1-\alpha) \\
-(1-\alpha) & -(1-\alpha) & \cdots & -(1-\alpha) & m_2 & -(1-\alpha)(2-a'_{12}) & \cdots & -(1-\alpha)(2-a'_{1n_2}) \\
-(1-\alpha) & -(1-\alpha) & \cdots & -(1-\alpha) & -(1-\alpha)(2-a'_{21}) & m_2 & \cdots & -(1-\alpha)(2-a'_{2n_2}) \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
-(1-\alpha) & -(1-\alpha) & \cdots & -(1-\alpha) & -(1-\alpha)(2-a'_{n_21}) & -(1-\alpha)(2-a'_{n_22}) & \cdots & m_2
\end{array}\right|. \tag{6}$$

We now perform the number of transformations that leave the value of the determinant (6) unchanged. Subtract the row (n_1+1) from the rows $(n_1+2),(n_1+3),\ldots,(n_1+n_2)$ of (6). Later on, adding the columns $(n_1+2),(n_1+3),\ldots,(n_1+n_2)$ to the column (n_1+1) of the obtained matrix, we arrive at the following determinant

$$P_{D_\alpha(G_1\vee G_2)}(\lambda) = |W_1||R|, \tag{7}$$

where

$$|R| = \left|\begin{array}{ccccc}
m_1 & -(1-\alpha)(2-a_{12}) & \cdots & -(1-\alpha)(2-a_{1n_1}) & -(1-\alpha)n_2 \\
-(1-\alpha)(2-a_{21}) & m_1 & \cdots & -(1-\alpha)(2-a_{2n_1}) & -(1-\alpha)n_2 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
-(1-\alpha)(2-a_{n_11}) & -(1-\alpha)(2-a_{n_12}) & \cdots & m_1 & -(1-\alpha)n_2 \\
-(1-\alpha) & -(1-\alpha) & \cdots & -(1-\alpha) & m_2-(1-\alpha)(2n_2-r_2-2)
\end{array}\right|,$$

$$|W_1| = \left|\begin{array}{cccc}
m_2+(1-\alpha)(2-a'_{12}) & -(1-\alpha)(a'_{13}-a'_{23}) & \cdots & -(1-\alpha)(a'_{1n_2}-a'_{2n_2}) \\
-(1-\alpha)(a'_{12}-a'_{32}) & m_2+(1-\alpha)(2-a'_{13}) & \cdots & -(1-\alpha)(a'_{1n_2}-a'_{3n_2}) \\
\vdots & \vdots & \ddots & \vdots \\
-(1-\alpha)(a'_{12}-a'_{n_22}) & -(1-\alpha)(a'_{13}-a'_{n_23}) & \cdots & m_2+(1-\alpha)(2-a'_{1n_2})
\end{array}\right|.$$

For the determinant $|R|$ in (7), subtract the first row from the rows $2,3,\ldots,n_1$. Later on, adding columns $2,3,\ldots,n_1$ to the first column of the obtained matrix, then we get the following determinant:

$$|R| = \left|\begin{array}{ccccc}
m_1-(1-\alpha)(2n_1-r_1-2) & -(1-\alpha)(2-a_{12}) & \cdots & -(1-\alpha)(2-a_{1n_1}) & -(1-\alpha)n_2 \\
0 & m_1+(1-\alpha)(2-a_{12}) & \cdots & -(1-\alpha)(a_{1n_1}-a_{2n_1}) & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & -(1-\alpha)(a_{12}-a_{n_12}) & \cdots & m_1+(1-\alpha)(2-a_{1n_1}) & 0 \\
-(1-\alpha)n_1 & -(1-\alpha) & \cdots & -(1-\alpha) & m_2-(1-\alpha)(2n_2-r_2-2)
\end{array}\right|. \tag{8}$$

Expand the determinant in (8) along the last column to obtain

$$|R| = [(m_1-(1-\alpha)(2n_1-r_1-2))(m_2-(1-\alpha)(2n_2-r_2-2))-(1-\alpha)^2n_1n_2]|W_2|, \tag{9}$$

where

$$|W_2| = \left|\begin{array}{cccc}
m_1+(1-\alpha)(2-a_{12}) & -(1-\alpha)(a_{13}-a_{23}) & \cdots & -(1-\alpha)(a_{1n_1}-a_{2n_1}) \\
-(1-\alpha)(a_{12}-a_{32}) & m_1+(1-\alpha)(2-a_{13}) & \cdots & -(1-\alpha)(a_{1n_1}-a_{3n_1}) \\
\vdots & \vdots & \ddots & \vdots \\
-(1-\alpha)(a_{12}-a_{n_12}) & -(1-\alpha)(a_{13}-a_{n_13}) & \cdots & m_1+(1-\alpha)(2-a_{1n_1})
\end{array}\right|.$$

Furthermore, by (9), (7) can be written as

$$P_{D_\alpha(G_1\vee G_2)}(\lambda) = \Big[(m_1-(1-\alpha)(2n_1-r_1-2))(m_2-(1-\alpha)(2n_2-r_2-2))-(1-\alpha)^2n_1n_2\Big]|W_2||W_1|. \tag{10}$$

For the determinant $|W_2|$, we have

$$
|W_2| = \frac{1}{m_1 + (1-\alpha)(r_1+2)} \times
\begin{vmatrix}
m_1 + (1-\alpha)(r_1+2) & (1-\alpha)a_{12} & (1-\alpha)a_{13} & \cdots & (1-\alpha)a_{1n_1} \\
0 & m_1 + (1-\alpha)(2-a_{12}) & -(1-\alpha)(a_{13}-a_{23}) & \cdots & -(1-\alpha)(a_{1n_1}-a_{2n_1}) \\
0 & -(1-\alpha)(a_{12}-a_{32}) & m_1 + (1-\alpha)(2-a_{13}) & \cdots & -(1-\alpha)(a_{1n_1}-a_{3n_1}) \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & -(1-\alpha)(a_{12}-a_{n_1 2}) & -(1-\alpha)(a_{13}-a_{n_1 3}) & \cdots & m_1 + (1-\alpha)(2-a_{1n_1})
\end{vmatrix}. \tag{11}
$$

Therefore, by subtracting the columns $2, 3, \ldots, n_1$ of the determinant (11) from the first column, and later on, add the first row of the obtained matrix to the rows $2, 3, \ldots, n_1$, we get

$$
\begin{aligned}
|W_2| &= \frac{1}{m_1 + (1-\alpha)(r_1+2)} \times
\begin{vmatrix}
m_1 + 2(1-\alpha) & (1-\alpha)a_{12} & (1-\alpha)a_{13} & \cdots & (1-\alpha)a_{1n_1} \\
(1-\alpha)a_{21} & m_1 + 2(1-\alpha) & (1-\alpha)a_{23} & \cdots & (1-\alpha)a_{2n_1} \\
(1-\alpha)a_{31} & (1-\alpha)a_{32} & m_1 + 2(1-\alpha) & \cdots & (1-\alpha)a_{3n_1} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
(1-\alpha)a_{n_1 1} & (1-\alpha)a_{n_1 2} & (1-\alpha)a_{n_1 3} & \cdots & m_1 + 2(1-\alpha)
\end{vmatrix} \\
&= \frac{(-1)^{n_1}}{m_1 + (1-\alpha)(r_1+2)} \times
\begin{vmatrix}
-m_1 - 2(1-\alpha) & -(1-\alpha)a_{12} & -(1-\alpha)a_{13} & \cdots & -(1-\alpha)a_{1n_1} \\
-(1-\alpha)a_{21} & -m_1 - 2(1-\alpha) & -(1-\alpha)a_{23} & \cdots & -(1-\alpha)a_{2n_1} \\
-(1-\alpha)a_{31} & -(1-\alpha)a_{32} & -m_1 - 2(1-\alpha) & \cdots & -(1-\alpha)a_{3n_1} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
-(1-\alpha)a_{n_1 1} & -(1-\alpha)a_{n_1 2} & -(1-\alpha)a_{n_1 3} & \cdots & -m_1 - 2(1-\alpha)
\end{vmatrix} \\
&= \frac{(-1)^{n_1}}{m_1 + (1-\alpha)(r_1+2)} P_{A_\alpha(G_1)}(\alpha n_2 + 2\alpha n_1 - 2 - \lambda).
\end{aligned} \tag{12}
$$

Similar as the calculate of the determinant $|W_2|$, we get the determinant $|W_1|$ is

$$
|W_1| = \frac{(-1)^{n_2}}{m_2 + (1-\alpha)(r_2+2)} P_{A_\alpha(G_2)}(\alpha n_1 + 2\alpha n_2 - 2 - \lambda). \tag{13}
$$

Substituting (12), (13), m_1 and m_2 in (10), we get the desired result. $\square$

For two strongly connected regular digraphs G_1 and G_2, the previous theorem establishes the relationship between the D_α eigenvalues of $G_1 \vee G_2$ and the A_α eigenvalues of G_1, G_2. Since G_i is r_i-regular, it follows that $A_\alpha(G_i) = \alpha r_i I_{n_i} + (1-\alpha)A(G_i)$, which implies that λ is an eigenvalue of $A(G_i)$ if and only if $\alpha r_i + (1-\alpha)\lambda$ is an eigenvalue of $A_\alpha(G_i)$. Combining this observation and Theorem 4.1, we can easily obtain the following result.

Corollary 4.2. *Let G_i be r_i-regular strongly connected digraph with order n_i, for $i = 1, 2$. Then the D_α eigenvalues of $G_1 \vee G_2$ are as follows:*

$$(2n_1 + n_2 - r_1)\alpha - 2 - (1-\alpha)\lambda_k(A(G_1)) \text{ for } 2 \le k \le n_1,$$

$$(2n_2 + n_1 - r_2)\alpha - 2 - (1-\alpha)\lambda_k(A(G_2)) \text{ for } 2 \le k \le n_2,$$

and the remaining two D_α eigenvalues of $G_1 \vee G_2$ are the two roots of the equation $(\lambda - \alpha n_2 - 2n_1 + r_1 + 2)(\lambda - \alpha n_1 - 2n_2 + r_2 + 2) - (1-\alpha)^2 n_1 n_2 = 0$, where $\lambda_k(A(G_i))$ for $2 \le k \le n_i$ denote the eigenvalues of the adjacency matrix $A(G_i)$ except for r_i.

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