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The Categorical Relationships Between Neighborhood Spaces, $\top$-Neighborhood Spaces and Stratified L-Neighborhood Spaces

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Abstract. In this paper, for a complete residuated lattice L, we present the categorical properties of $\top$-neighborhood spaces and their categorical relationships to neighborhood spaces and stratified L-neighborhood spaces. The main results are: (1) the category of $\top$-neighborhood spaces is a topological category; (2) neighborhood spaces can be embedded in $\top$-neighborhood spaces as a reflective subcategory, and when L is a meet-continuous complete residuated lattice, $\top$-neighborhood spaces can be embedded in stratified L-neighborhood spaces as a reflective subcategory; (3) when L is a continuous complete residuated lattice, neighborhood spaces (resp., $\top$-neighborhood spaces) can be embedded in $\top$-neighborhood spaces (resp., stratified L-neighborhood spaces) as a simultaneously reflective and coreflective subcategory.

1. Introduction

By a neighborhood space [15], we mean a pair $(X, \mathscr{U})$, where X is a non-empty set and $\mathscr{U} := \{\mathscr{U}^x\}_{x\in X}$ satisfies: (1) $\mathscr{U}^x$ is a filter on X, and (2) $\mathscr{U}^x \subseteq \dot{x} := \{A \subseteq X | x \in A\}$. In this case, $\mathscr{U}$ is also called a neighborhood system on X. A neighborhood space $(X, \mathscr{U})$ is called topological if it satisfies moreover (NT) $A \in \mathscr{U}^x \Longrightarrow \exists B \in \mathscr{U}^x$ s.t. $\forall y \in B$, $A \in \mathscr{U}^y$. It is well known that one can establish a bijection between topological spaces and topological neighborhood spaces by takeing that "A is an open set of a topological space X ifand only if $A \in \mathscr{U}^x$ for any $x \in A$". Therefore, neighborhood space is an extension of topological space and is often called the pretopological space. Moreover, it is easy to see that pretopological space is also equivalent to pretopological convergence space [13, 35].

Since fuzzy sets were applied in topology, various types of lattice-valued neighborhood spaces were proposed as the extension of some types lattice-valued topological spaces, see Fang [9], Shi [39], Zhang [46, 48] and Höhel-Šostak [16], and so on. Quite recently, types of lattice-valued neighborhood spaces are closely related to rough sets [29, 49, 50]. It is well known that the categorical properties and the categorical relationships have always been the focus of lattice-valued topology theory. Thus, as the extension

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of lattice-valued topological spaces, it is natural and interesting to study the categorical properties and categorical relationships about lattice-valued neighborhood spaces.

In recent years, given a complete residuated lattice L as the membership of fuzzy sets, the notions of stratified L-topology and strong L-topology have attracted much attention of fuzzy topologist, see Fang and Yue [8, 9, 45], Jäger [21, 22], Pang [33, 34], Ahsanullah [4, 5], Han and Yao [12, 43], Reid and Richardson [36, 37], Li [26, 27], Lai and Zhang [23], and so on. Particularly:

- $\diamond$ When L is a complete MV-algebra, Fang and Yue [9] proved that $\top$-neighborhood spaces are equivalent to pretopological $\top$-convergence spaces, and topological $\top$-neighborhood spaces are equivalent to strong L-topological spaces.
- $\diamond$ When L is meet-continuous (MV-algebra is meet-continuous), the author and co-author [27] showed that $\top$-neighborhood spaces are equivalent to conical L-neighborhood spaces in [23].
- $\diamond$ When L is a complete Heyting algebra, Jäger [21] verified that stratified L-neighborhood spaces are equivalent to pretopological stratified L-convergence spaces, and topological stratified L-neighborhood spaces are equivalent to stratified L-topological spaces, later the author and co-author [26] generalized Jäger's results from the lattice-context (precisely, from complete Heyting algebra to meet-continuous complete residuated lattice).

To sum up, $\top$-neighborhood space is an extension of strong L-topological space, and stratified L-neighborhood space is an extension of stratified L-topological space.

The concept of topological groups is defined as a group equipped with a topology such that the group operations are continuous with respect to the topology. Since topological spaces and continuous functions can be described by neighborhood systems, topological groups can also be described by neighborhood systems. For L a complete Heyting algebra, as an extension of topological groups, Al-Mufarrij and Ahsanullah [3] introduced a notion of stratified L-neighborhood topological groups, which is defined as a group equipped with a topological stratified L-neighborhood space such that the group operations are continuous. Note that topological stratified L-neighborhood spaces describe precisely stratified L-topological spaces [3, Proposition 1.10] and the continuous functions between stratified L-topological spaces can be characterized by stratified L-neighborhood systems [3, Remark 1.8], hence stratified L-neighborhood topological groups characterize precisely stratified L-topological groups. Later, by dropping the topological condition in stratified L-neighborhood topological groups, Ahsanullah etal [4] investigated the notion of stratified L-neighborhood groups. Quite recently, the author observed that the topological condition for stratified L-neighborhood space is not used in the characterized theorem of stratified L-neighborhood topological groups [3, Theorem 2.21]. We also observed a similar result from the characterized theorem of topological groups [1, Theorem 1.3.12]. Inspired by these two observations, we showed in [18, Theorem 3.15] that the stratified L-neighborhood space associated with a stratified L-neighborhood group is naturally topological. Hence the topological condition is redundant, and stratified L-neighborhood groups are equivalent to stratified L-neighborhood topological groups. In this sense, we say that when defining (lattice-valued) topological groups, (lattice-valued) neighborhood spaces are enough. Therefore, the study on lattice-valued neighborhood spaces is important to the theory of lattice-valued topological groups.

In this paper, consider L being a complete residuated lattice, we shall study the categorical properties of $\top$-neighborhood spaces and their categorical relationships to neighborhood spaces and stratified L-neighborhood spaces.

The contents are arranged as follows. Section 2 recalls some basic notions as preliminary. Section 3 focuses on the categorical properties of $\top$-neighborhood spaces and their categorical relationships to neighborhood spaces. The main results are: (1) the category of $\top$-neighborhood spaces is a topological category; (2) neighborhood spaces can be embedded in $\top$-neighborhood spaces as a reflective subcategory; and if L is a continuous complete residuated lattice, then neighborhood spaces can embed in $\top$-neighborhood spaces as a simultaneously reflective and coreflective subcategory. Section 4 presents the categorical properties of stratified L-neighborhood spaces and their categorical relationships to $\top$-neighborhood spaces. The main results are: (1) the category of stratified L-neighborhood spaces is a topological category; (2) if L is a meet-continuous complete residuated lattice, then $\top$-neighborhood spaces can be embedded in stratified L-neighborhood spaces as a reflective subcategory; and if L is a continuous complete residuated lattice, then

$\top$-neighborhood spaces can embed in stratified L-neighborhood spaces as a simultaneously reflective and coreflective subcategory. Section 5 makes a conclusion.

2. Preliminaries

In this section, we recall briefly some basic notions of neighborhood spaces, complete residuated lattice, lattice-valued fuzzy sets and lattice-valued filters.

2.1. Neighborhood Spaces

In this subsection, we recall some concepts about neighborhood spaces from [15].

Let $(X, \mathscr{U})$ and $(Y, \mathscr{V})$ be neighborhood spaces. Then a function $f : (X, \mathscr{U}) \longrightarrow (Y, \mathscr{V})$ is said to be *continuous* at x if $B \in \mathscr{V}^{f(x)}$ implies $f^{-1}(B) \in \mathscr{U}^x$, and f is said to be *continuous* if it is continuous at each $x \in X$. Let **NS** denote the category of neighborhood spaces and continuous functions.

For a source $(X \xrightarrow{f_j} (X_j, \mathscr{U}_j))_{j\in J}$ in **NS**, we mean that $(X \xrightarrow{f_j} X_j)_{j\in J}$ is a family of functions and $(X_j, \mathscr{U}_j)_{j\in J}$ is a family of neighborhood spaces. By the *initial structure* w.r.t. the source $(X \xrightarrow{f_j} (X_j, \mathscr{U}_j))_{j\in J}$ is meant the neighborhood system $\mathscr{U}$ on X such that the conditions of an initial lift in the category of **NS** are fulfilled, that is,

(1) all functions $f_j : (X, \mathscr{U}) \longrightarrow (X_j, \mathscr{U}_j)$ are continuous,

(2) for any neighborhood space $(Y, \mathscr{V})$ and any function $f : Y \longrightarrow X$, it holds that $f : (Y, \mathscr{V}) \longrightarrow (X, \mathscr{U})$ is continuous if and only if for all $j \in J$, the functions $f_j \circ f : (Y, \mathscr{V}) \longrightarrow (X_j, \mathscr{U}_j)$ are continuous.

Each source $(X \xrightarrow{f_j} (X_j, \mathscr{U}_j))_{j\in J}$ in **NS** has an initial structure $\mathscr{U}$ defined by $\forall x \in X$,

$$\mathscr{U}^x = \{A \supseteq \bigcap_{j\in F} f_j^{-1}(B_j) | F \in 2^{(J)}, \forall j \in F, B_j \in \mathscr{U}_j^{f_j(x)}\},$$

where $2^{(J)}$ denote all the non-empty finite subsets of J.

For a sink $((X_j, \mathscr{U}_j) \xrightarrow{f_j} X)_{j\in J}$ in **NS**, we mean that $(X_j \xrightarrow{f_j} X)_{j\in J}$ is a family of functions and $(X_j, \mathscr{U}_j)_{j\in J}$ is a family of neighborhood spaces. By the *final structure* w.r.t. the sink $((X_j, \mathscr{U}_j) \xrightarrow{f_j} X)_{j\in J}$ is meant the neighborhood system $\mathscr{U}$ on X such that the conditions of a final lift in the category of **NS** are fulfilled, that is,

(1) all functions $f_j : (X_j, \mathscr{U}_i) \longrightarrow (X, \mathscr{U})$ are continuous.

(2) for any neighborhood space $(Y, \mathscr{V})$ and any function $f : X \longrightarrow Y$, it holds that $f : (X, \mathscr{U}) \longrightarrow (Y, \mathscr{V})$ is continuous if and only if for all $j \in J$, the functions $f \circ f_j : (X_j, \mathscr{U}_j) \longrightarrow (Y, \mathscr{V})$ are continuous.

Each sink $((X_j, \mathscr{U}_j) \xrightarrow{f_j} X)_{j\in J}$ in **NS** has a final structure $\mathscr{U}$ defined by $\forall x \in X$,

$$\mathscr{U}^x = \begin{cases} \dot{x}, & x \notin \bigcup_{j\in J} f_j(X_j); \\ \bigcap_{j\in J, f_j(x_j)=x} f_j(\mathscr{U}_j^{x_j}), & \text{otherwise}. \end{cases}$$

A category over **Set** is called *topological* iff it has initial structures iff it has final structures.

The reader is referred to [2] for other notions and results in category theory.

A filter on a set X is an upper set of $(2^X, \subseteq)$ (2^X denotes the power set of X) that is closed under finite meets and does not contain the empty set $\emptyset$. For any set X, let $\mathscr{F}(X)$ denote the set of filters on X. Let $f : X \longrightarrow Y$ be a function and $\mathscr{F} \in \mathscr{F}(X), \mathscr{G} \in \mathscr{G}(Y)$. Then we denote $f(\mathscr{F})$ the filter on Y generated by the family $\{f(A)|A \in \mathscr{F}\}$ as filter base, and call it the image of $\mathscr{F}$ under f. In addition, if for all $B \in \mathscr{G}$ it holds that $f^{-1}(B) \neq \emptyset$, then the family $\{f^{-1}(B)|B \in \mathscr{G}\}$ forms a filter base on X, and we denote the filter generated by it as $f^{-1}(\mathscr{G})$ and call it the inverse image of $\mathscr{G}$ under f.

Let $(X, \mathscr{U}_i)$ $(i = 1, 2)$ be neighborhood spaces. Then we say $\mathscr{U}_1$ is finer than $\mathscr{U}_2$ or $\mathscr{U}_2$ is coarser than $\mathscr{U}_1$, and denoted as $\mathscr{U}_2 \leq \mathscr{U}_1$ if the function $id_X : (X, \mathscr{U}_1) \longrightarrow (X, \mathscr{U}_2)$ is continuous, that is $\mathscr{U}_2^x \subseteq \mathscr{U}_1^x$ for all $x \in X$.

Let $N(X)$ denote the set of neighborhood systems on X. Then $(N(X), \leq)$ forms a complete lattice. For $\{\mathscr{U}_j\}_{j\in J} \subseteq N(X)$, it is easily seen that:

(1) the supremum of $\{\mathscr{U}_j\}_{j\in J}$ is defined as: $\forall x \in X$, $(\bigvee_{j\in J} \mathscr{U}_j)^x$ is the filter generated by $\{\bigcap\limits_{j\in F} A_j | F \in 2^{(J)}, \forall j \in F, A_j \in \mathscr{U}_j^x\}$ as a base;

(2) the infimum of $\{\mathscr{U}_j\}_{j\in J}$ is defined as: $\forall x \in X$, $(\bigwedge_{j\in J} \mathscr{U}_j)^x = \bigcap_{j\in J} \mathscr{U}_j^x$.

2.2. *Complete Residuated Lattice and Lattice-Valued Fuzzy Sets*

A *complete residuated lattice* is a pair $(L, *)$, where L is a complete lattice with respect to a partial order $\leq$ on it, with the top (resp., bottom) element $\top$ (resp., $\perp$), and

(i) $*$ is a commutative semigroup operation on L such that $a * \bigvee\limits_{j\in J} b_j = \bigvee\limits_{j\in J}(a * b_j)$ for all $a \in L$ and $\{b_j\}_{j\in J} \subseteq L$,

(ii) $\top$ is the unique unit in the sense of $\top * a = a$ for all $a \in L$.

For a given complete residuated lattice, the binary operation $\to$ on L is defined by $a \to b = \bigvee\{c \in L | a*c \leq b\}$ for all $a, b \in L$. The binary operation $\to$ is called the *implication operation* with respect to $*$. The basic properties of $*$ and $\to$ are collected as below [7, 14, 16, 19, 38].

(I1) $a \to b = \top \Leftrightarrow a \leq b$;

(I2) $a * (a \to b) \leq b$;

(I3) $a \to (b \to c) = (a * b) \to c = b \to (a \to c)$;

(I4) $(\bigvee\limits_{j\in J} a_j) \to b = \bigwedge\limits_{j\in J}(a_j \to b)$, (I4) implies that (I4′) $a \leq b \Rightarrow a \to c \geq b \to c$;

(I5) $a \to (\bigwedge\limits_{j\in J} b_j) = \bigwedge\limits_{j\in J}(a \to b_j)$, (I5) implies that (I5′) $b \leq c \Rightarrow a \to b \leq a \to c$;

(I6) $a * b \leq c \Leftrightarrow a \leq b \to c$.

In this paper, if not otherwise specified, $L = (L, *)$ is always assumed to be a complete residuated lattice.

$L = (L, *)$ is called a *complete Heyting algebra or a frame* when $* = \wedge$.

$L = (L, *)$ is called a *complete MV-algebra* if L satisfies the condition:

(MV) $a \vee b = (a \to b) \to b$ for any $a, b \in L$.

$L = (L, *)$ is said to be *meet-continuous* if the underlying lattice $(L, \leq)$ is a meet-continuous lattice [10], that is, L satisfies the condition:

(MC) $a \wedge \bigvee\limits_{j\in J} a_j = \bigvee\limits_{j\in J}(a \wedge a_j)$ for any $a \in L$ and any non-empty directed subset $\{a_j | j \in J\} \subseteq L$.

$L = (L, *)$ is said to be *continuous* if $(L, \leq)$ is a continuous lattice [10], i.e., for any non-empty subfamily $\{a_{j,k} | j \in J, k \in K(j)\} \subseteq L$ with $\{a_{j,k} | k \in K(j)\}$ is directed for all $j \in J$, the following identity holds,

$$(\mathbf{CC}) \qquad \bigwedge_{j\in J} \bigvee_{k\in K(j)} a_{j,k} = \bigvee_{h\in N} \bigwedge_{j\in J} a_{j,h(j)},$$

where N is the set of all choice functions $h : J \longrightarrow \bigcup\limits_{j\in J} K(j)$ with $h(j) \in K(j)$ for all $j \in J$.

Remark 2.1. For a complete MV-algebra, it is pointed out in [9, Lemma 2.2] that $a \wedge \bigvee\limits_{j\in J} a_j = \bigvee\limits_{j\in J}(a \wedge a_j)$ for any $a \in L$ and $\{a_j | j \in J\} \subseteq L$. Hence, a complete MV-algebra is meet-continuous.

In a residuated lattice $(L, *)$, for all $a, b \in L$, $a * b \leq a, b$ and hence $a * b \leq a \wedge b$.

We call a function $\mu : X \to L$ an L-fuzzy set in X, and use L^X to denote the set of all L-fuzzy sets in X. The operations $\bigvee, \bigwedge, *, \to$ on L can be translated onto L^X pointwise. That is, for any $\mu, \nu, \mu_j (j \in J) \in L^X$,

$$\mu \leq \nu \text{ iff } \mu(x) \leq \nu(x) \text{ for any } x \in X,$$
$$(\bigvee_{j\in J} \mu_j)(x) = \bigvee_{j\in J} \mu_j(x), (\bigwedge_{j\in J} \mu_j)(x) = \bigwedge_{j\in J} \mu_j(x),$$
$$(\mu * \nu)(x) = \mu(x) * \nu(x), (\mu \to \nu)(x) = \mu(x) \to \nu(x).$$

For any $\alpha \in L$, we use $\underline{\alpha}$ to denote the constant L-fuzzy set with value α. For any $A \subseteq X$, we use $\top_A$ to denote the characteristic function of A.

Let $f : X \longrightarrow Y$ be a function. We define $f^{\rightarrow} : L^X \longrightarrow L^Y$ and $f^{\leftarrow} : L^Y \longrightarrow L^X$ [16] by $f^{\rightarrow}(\mu)(y) = \bigvee_{f(x)=y} \mu(x)$ for $\mu \in L^X$ and $y \in Y$, and $f^{\leftarrow}(\nu)(x) = \nu(f(x))$ for $\nu \in L^Y$ and $x \in X$.

Let μ, ν be L-fuzzy sets in X. The subsethood degree [6] of μ, ν, denoted by $S_X(\mu, \nu)$, is defined by

$$S_X(\mu, \nu) = \bigwedge_{x \in X} (\mu(x) \rightarrow \nu(x)).$$

The following theorem collects some basic properties of subsethood degree, they can be found in many place such as [6, 8, 9, 20, 24, 28, 41, 42, 44].

Lemma 2.2. *Let $f : X \longrightarrow Y$ be a function and $\mu_i, \nu_i \in L^X$, $w_i \in L^Y$ $(i = 1, 2)$. Then*

(1) $\mu_1 \leq \mu_2 \Leftrightarrow S_X(\mu_1, \mu_2) = \top$;

(2) $S_X(\mu_1, \mu_2) * S_X(\mu_2, \nu_1) \leq S_X(\mu_1, \nu_1)$;

(3) $S_X(\mu_1, \nu_1) \wedge S_X(\mu_2, \nu_2) \leq S_X(\mu_1 \wedge \mu_2, \nu_1 \wedge \nu_2)$;

(4) $\nu_1 \leq \nu_2 \Rightarrow S_X(\mu_1, \nu_1) \leq S_X(\mu_1, \nu_2)$, $S_X(\nu_1, \mu_1) \geq S_X(\nu_2, \mu_1)$;

(5) $S_X(\mu_1, \nu_1) \leq S_Y(f^{\rightarrow}(\mu_1), f^{\rightarrow}(\nu_1)), S_Y(w_1, w_2) \leq S_X(f^{\leftarrow}(w_1), f^{\leftarrow}(w_2))$.

2.3. $\top$-Filters.

The notion of filter has been extended to the fuzzy setting in two approaches: prefilters (or $\top$-filters more general) and L-filters. Both prefilters ($\top$-filters) and L-filters play important roles in (lattice-valued) fuzzy topology, see e.g. Lowen [30–32], Höhel [16, 17], Gutiérrez García [11],Fang [9], Jäger [22], Lai [23], Li [27] and Pang [34].

Definition 2.3. ([16]) A non-empty subset $\mathbb{F} \subseteq L^X$ is called a $\top$-filter on the set X whenever:

(TF1) $\bigvee\limits_{x \in X} \lambda(x) = \top$ for all $\lambda \in \mathbb{F}$,

(TF2) $\lambda \wedge \mu \in \mathbb{F}$ for all $\lambda, \mu \in \mathbb{F}$,

(TF3) if $\lambda \in L^X$ such that $\bigvee\limits_{\mu \in \mathbb{F}} S_X(\mu, \lambda) = \top$, then $\lambda \in \mathbb{F}$.

The set of all $\top$-filters on X is denoted by $\mathbb{F}_L^{\top}(X)$.

Definition 2.4. ([16]) A non-empty subset $\mathbb{B} \subseteq L^X$ is called a $\top$-filter base on the set X provided:

(TB1) $\bigvee\limits_{x \in X} \lambda(x) = \top$ for all $\lambda \in \mathbb{B}$.

(TB2) if $\lambda, \mu \in \mathbb{B}$, then $\bigvee\limits_{\nu \in \mathbb{B}} S_X(\nu, \lambda \wedge \mu) = \top$.

Each $\top$-filter base generates a $\top$-filter $\mathbb{F}_{\mathbb{B}}$ defined by $\mathbb{F}_{\mathbb{B}} := \{\lambda \in L^X | \bigvee\limits_{\mu \in \mathbb{B}} S_X(\mu, \lambda) = \top\}$.

Lemma 2.5. ([36, Lemma 3.1]) *Let $\mathbb{F} \in \mathbb{F}_L^{\top}(X)$ and $\mathbb{B}$ be a $\top$-filter base of $\mathbb{F}$. Then for any $\lambda \in L^X$, $\bigvee\limits_{\mu \in \mathbb{B}} S_X(\mu, \lambda) = \bigvee\limits_{\mu \in \mathbb{F}} S_X(\mu, \lambda)$.*

Example 2.6. ([16]) Let $f : X \longrightarrow Y$ be a function, $\mathbb{F} \in \mathbb{F}_L^{\top}(X)$ and $\mathbb{G} \in \mathbb{F}_L^{\top}(Y)$. Then:

(1) The family $\{f^{\rightarrow}(\lambda) | \lambda \in \mathbb{F}\}$ forms a $\top$-filter base on Y, and the $\top$-filter $f^{\Rightarrow}(\mathbb{F})$ generated by it is called the image of $\mathbb{F}$ under f. It is easy to check that $\mu \in f^{\Rightarrow}(\mathbb{F})$ iff $f^{\leftarrow}(\mu) \in \mathbb{F}$.

(2) The family $\{f^{\leftarrow}(\mu) | \mu \in \mathbb{G}\}$ forms a $\top$-filter base on Y iff $\bigvee\limits_{y \in f(X)} \mu(y) = \top$ holds for all $\mu \in \mathbb{G}$, and the $\top$-filter $f^{\Leftarrow}(\mathbb{G})$ (if it exists) generated by it is called the inverse image of $\mathbb{G}$ under f.

(3) For each $x \in X$, $[x]_{\top} = \{\lambda \in L^X | \lambda(x) = \top\}$ is a $\top$-filter on X.

Lemma 2.7. ([9, Lemma 2.11]) *Let $f : X \longrightarrow Y$ be a function, $\mathbb{F} \in \mathbb{F}_L^{\top}(X)$ and and $\mathbb{G} \in \mathbb{F}_L^{\top}(Y)$.*

(1) If $\mathbb{B}$ is a $\top$-filter base of $\mathbb{F}$, then $\{f^{\rightarrow}(\lambda) | \lambda \in \mathbb{B}\}$ is a $\top$-filter base of $f^{\Rightarrow}(\mathbb{F})$. Thus, $f^{\Rightarrow}(\mathbb{F})$ can be calculated by $f^{\Rightarrow}(\mathbb{F}) = \{\lambda \in L^Y | \bigvee_{\mu \in \mathbb{B}} S_Y(f^{\rightarrow}(\mu), \lambda) = \top\}$.

(2) If $\mathbb{B}$ is a $\top$-filter base of $\mathbb{G}$ and $f^{\Leftarrow}(\mathbb{G})$ exists, then $\{f^{\leftarrow}(\mu) | \mu \in \mathbb{B}\}$ is a $\top$-filter base of $f^{\Leftarrow}(\mathbb{G})$. Thus, $f^{\Leftarrow}(\mathbb{G})$ can be calculated by $f^{\Leftarrow}(\mathbb{G}) = \{\lambda \in L^X | \bigvee_{\mu \in \mathbb{B}} S_Y(f^{\leftarrow}(\mu), \lambda) = \top\}$.

Remark 2.8. Although the results in this subsection are stated for complete MV-algebra or complete Heyting algebra in [9, 16, 36], it is not difficult to check that the results still hold for general complete residuated lattice since for any $\{a_i|i \in I\}, \{b_j|j \in J\} \subseteq L$ with $\bigvee_{i\in I} a_i = \top, \bigvee_{j\in J} b_j = \top$, it holds that

$$\top = \bigvee_{i\in I} a_i * \bigvee_{j\in J} b_j = \bigvee_{i\in I, j\in J} (a_i * b_j) \leq \bigvee_{i\in I, j\in J} (a_i \wedge b_j).$$

2.4. Stratified L-Filters

Stratified L-filter is also a lattice-valued extension of crisp filter.

Definition 2.9. ([16]) A stratified L-filter on a set X is a function $\mathcal{F} : L^X \longrightarrow L$ such that:
(LF1) $\mathcal{F}(\underline{\bot}) = \bot, \mathcal{F}(\underline{\top}) = \top$;
(LF2) $\forall \lambda, \mu \in L^X, \mathcal{F}(\lambda) \wedge \mathcal{F}(\mu) = \mathcal{F}(\lambda \wedge \mu)$;
(LFs) $\forall \lambda \in L^X, \forall \alpha \in L, \mathcal{F}(\underline{\alpha} * \lambda) \geq \alpha * \mathcal{F}(\lambda)$.
The set of all stratified L-filters on X is denoted by $\mathcal{F}_L^s(X)$.

Example 2.10. ([16]) Let $f : X \longrightarrow Y$ be a function and $\mathcal{F} \in \mathcal{F}_L^s(X), \mathcal{G} \in \mathcal{F}_L^s(Y)$.
(1) The function $f^{\Rightarrow}(\mathcal{F}) : L^Y \longrightarrow L$ defined by $\mu \mapsto \mathcal{F}(\mu \circ f)$ is a stratified L-filter on Y.
(2) If L is meet-continuous, then the function $f^{\Leftarrow}(\mathcal{G}) : L^Y \longrightarrow L$ defined by $f^{\Leftarrow}(\mathcal{G})(\lambda) = \bigvee\{\mathcal{G}(\mu)|f^{\leftarrow}(\mu) \leq \lambda\}$ is a stratified L-filter on Y whenever $f^{\leftarrow}(\mu) = \underline{\bot}$ implies $\mathcal{G}(\mu) = \bot$[when proving that $f^{\Leftarrow}(\mathcal{G})$ fulfills (LF2), the meet-continuity is used].
(3) For any $x \in X$, the function $[x] : L^X \longrightarrow L, [x](\lambda) = \lambda(x)$ is a stratified L-filter on X.

3. ⊤-Neighborhood Spaces and their Relationships to Neighborhood Spaces

In this section, we shall study the categorical properties of ⊤-neighborhood spaces and their categorical relationships to neighborhood spaces. The main results are two: (1) the category of ⊤-neighborhood spaces is a topological category; (2) neighborhood spaces can be embedded in ⊤-neighborhood spaces as a reflective subcategory, and if L is continuous, then neighborhood spaces can embed in ⊤-neighborhood spaces as a simultaneously reflective and coreflective subcategory.

3.1. ⊤-Neighborhood Spaces

Definition 3.1. ([16]) A ⊤-neighborhood space is a pair $(X, \mathbb{U} := \{\mathbb{U}^x\}_{x\in X})$ satisfies
(TN1) $\forall x \in X, \mathbb{U}^x \subseteq L^X$ is a ⊤-filter, and
(TN2) $\forall x \in X, \mathbb{U}^x \subseteq [x]_\top$.
Then $\mathbb{U}$ is called a ⊤-neighborhood system on X.

Remark 3.2. Let $L = ([0, 1], *)$ be the Łukasiewicz t-norm. It is mentioned in [23, Remark 3.11] that a ⊤-filter (i.e., L-saturated prefilter consisting of inhabited fuzzy sets) is precisely a saturated prefilter in the sense of Lowen [32]. It follows easily that a ⊤-neighborhood space is precisely a fuzzy neighborhood space dropping the topological condition (N3) in [32, Definition 2.1]. Hence, ⊤-neighborhood space is an extension of Lowen's fuzzy neighborhood space.

A function $f : (X, \mathbb{U}) \longrightarrow (Y, \mathbb{V})$ between ⊤-neighborhood spaces is said to be *continuous* at x if $\nu \in \mathbb{V}^{f(x)}$ implies $f^{\leftarrow}(\nu) \in \mathbb{U}^x$, and f is said to be *continuous* if it is continuous at each $x \in X$.
The following lemma about continuity is obviously.

Lemma 3.3. *Let $f : (X, \mathbb{U}) \longrightarrow (Y, \mathbb{V})$ be a function between ⊤-neighborhood spaces and $\mathbb{B}$ be a ⊤-neighborhood base of $\mathbb{V}^{f(x)}$ for $x \in X$. Then f is continuous at x iff for any $\lambda \in \mathbb{B}$, $f^{\leftarrow}(\lambda) \in \mathbb{U}^x$.*

Let **TNS** denote the category of ⊤-neighborhood spaces and continuous functions. The notions of source, sink, initial structure and final structure in **TNS** can be defined similarly as for **NS**.

Theorem 3.4. *Each source* $(X \xrightarrow{f_j} (X_j, \mathbb{U}_j))_{j\in J}$ *in* **TNS** *has an initial structure.*

Proof. We consider the pair $(X, \mathbb{U})$ defined by

$$\forall x \in X, \mathbb{U}^x = \Big\{\mu \in L^X \Big| \bigvee_{F\in 2^{(J)}, \forall j\in F, \mu_j \in \mathbb{U}_j^{f_j(x)}} S_X(\bigwedge_{j\in F} f_j^{\leftarrow}(\mu_j), \mu) = \top\Big\}.$$

Firstly, we prove that $(X, \mathbb{U})$ is a $\top$-neighborhood space.

(TF1): Note that (TN2) implies (TF1), so we need only check (TN2). Let $x \in X$ and $\mu \in \mathbb{U}^x$. Then

$$\begin{aligned}
\top &= \bigvee_{F\in 2^{(J)}, \forall j\in F, \mu_j \in \mathbb{U}_j^{f_j(x)}} S_X(\bigwedge_{j\in F} f_j^{\leftarrow}(\mu_j), \mu) \\
&\le \bigvee_{F\in 2^{(J)}, \forall j\in F, \mu_j \in \mathbb{U}_j^{f_j(x)}} \Big(\big(\bigwedge_{j\in F} f_j^{\leftarrow}(\mu_j)(x)\big) \to \mu(x)\Big), \text{ by } \mu_j(f_j(x)) = \top \\
&= \bigvee_{F\in 2^{(J)}, \forall j\in F, \mu_j \in \mathbb{U}_j^{f_j(x)}} (\top \to \mu(x)) = \mu(x).
\end{aligned}$$

Thus $\mu(x) = \top$.

(TF2): Let $\mu, \nu \in \mathbb{U}^x$. Then

$$\bigvee_{F\in 2^{(J)}, \forall j\in F, \mu_j \in \mathbb{U}_j^{f_j(x)}} S_X(\bigwedge_{j\in F} f_j^{\leftarrow}(\mu_j), \mu) = \top, \quad \bigvee_{G\in 2^{(J)}, \forall k\in G, \nu_k \in \mathbb{U}_k^{f_k(x)}} S_X(\bigwedge_{k\in G} f_k^{\leftarrow}(\nu_k), \nu) = \top.$$

It follows that

$$\begin{aligned}
\top &= \Big[\bigvee_{F\in 2^{(J)}, \forall j\in F, \mu_j \in \mathbb{U}_j^{f_j(x)}} S_X(\bigwedge_{j\in F} f_j^{\leftarrow}(\mu_j), \mu)\Big] * \Big[\bigvee_{G\in 2^{(J)}, \forall k\in G, \nu_k \in \mathbb{U}_k^{f_k(x)}} S_X(\bigwedge_{k\in G} f_k^{\leftarrow}(\nu_k), \nu)\Big] \\
&\le \bigvee_{F\in 2^{(J)}, \forall j\in F, \mu_j \in \mathbb{U}_j^{f_j(x)}} \bigvee_{G\in 2^{(J)}, \forall k\in G, \nu_k \in \mathbb{U}_k^{f_k(x)}} \Big(S_X(\bigwedge_{j\in F} f_j^{\leftarrow}(\mu_j), \mu) \wedge S_X(\bigwedge_{k\in G} f_k^{\leftarrow}(\nu_k), \nu)\Big) \\
&\le \bigvee_{F\in 2^{(J)}, \forall j\in F, \mu_j \in \mathbb{U}_j^{f_j(x)}} \bigvee_{G\in 2^{(J)}, \forall k\in G, \nu_k \in \mathbb{U}_k^{f_k(x)}} S_X\Big(\bigwedge_{j\in F} f_j^{\leftarrow}(\mu_j) \wedge \bigwedge_{k\in G} f_k^{\leftarrow}(\nu_k), \mu \wedge \nu\Big) \\
\text{by } H := F \cup G \in 2^{(J)} &\le \bigvee_{H\in 2^{(J)}, \forall j\in H, w_j \in \mathbb{U}_j^{f_j(x)}} S_X(\bigwedge_{j\in H} f_j^{\leftarrow}(w_j), \mu \wedge \nu),
\end{aligned}$$

where the first inequality holds since $*$ distributes over arbitrary joins and $* \le \wedge$. Thus $\mu \wedge \nu \in \mathbb{U}^x$.

(TF3): Let $\mu \in L^X$ satisfy the condition $\bigvee_{\nu \in \mathbb{U}^x} S_X(\nu, \mu) = \top$. Note that for any $\nu \in \mathbb{U}^x$ we have

$$\bigvee_{F\in 2^{(J)}, \forall j\in F, \mu_j \in \mathbb{U}_j^{f_j(x)}} \Big(S_X(\bigwedge_{j\in F} f_j^{\leftarrow}(\mu_j), \nu)\Big) = \top,$$

then

$$\begin{aligned}\top &= \bigvee_{\nu\in\mathbb{U}^x}\Big(S_X(\nu,\mu)*\bigvee_{F\in 2^{(J)},\forall j\in F,\mu_j\in\mathbb{U}_j^{f_j(x)}}\Big(S_X(\bigwedge_{j\in F}f_j^{\leftarrow}(\mu_j),\nu)\Big)\Big)\\ &= \bigvee_{\nu\in\mathbb{U}^x}\ \bigvee_{F\in 2^{(J)},\forall j\in F,\mu_j\in\mathbb{U}_j^{f_j(x)}}\Big(S_X(\bigwedge_{j\in F}f_j^{\leftarrow}(\mu_j),\nu)*S_X(\nu,\mu)\Big)\\ &\le \bigvee_{F\in 2^{(J)},\forall j\in F,\mu_j\in\mathbb{U}_j^{f_j(x)}} S_X(\bigwedge_{j\in F}f_j^{\leftarrow}(\mu_j),\mu).\end{aligned}$$

Thus $\mu\in\mathbb{U}^x$.

Now, we have proved that $\mathbb{U}$ satisfies (TN1) and (TN2), so $(X,\mathbb{U})$ is a $\top$-neighborhood space.

Secondly, we prove the initial conditions.

(1) Each $f_j:(X,\mathbb{U})\longrightarrow(X_j,\mathbb{U}_j)$ is continuous. Indeed, let $\mu_j\in\mathbb{U}_j^{f_j(x)}$ then

$$\top = S_X(f_j^{\leftarrow}(\mu_j),f_j^{\leftarrow}(\mu_j))\le\bigvee_{F\in 2^{(J)},\forall i\in F,\mu_i\in\mathbb{U}_i^{f_i(x)}}S_X(\bigwedge_{i\in F}f_i^{\leftarrow}(\mu_i),f_j^{\leftarrow}(\mu_j)).$$

Thus $f_j^{\leftarrow}(\mu_j)\in\mathbb{U}^x$.

(2) Let $f:(Y,\mathbb{V})\longrightarrow(X,\mathbb{U})$ be a function between $\top$-neighborhood spaces. Then f is continuous iff each $f_j\circ f$ is continuous. Obviously, if f is continuous, then each $f_j\circ f$ is continuous. Conversely, let $y\in Y$ and $\mu\in\mathbb{U}^{f(y)}$, i.e.,

$$\begin{aligned}\top &= \bigvee_{F\in 2^{(J)},\forall j\in F,\mu_j\in\mathbb{U}_j^{f_j(f(y))}}S_X(\bigwedge_{j\in F}f_j^{\leftarrow}(\mu_j),\mu)\\ &\le \bigvee_{F\in 2^{(J)},\forall j\in F,\mu_j\in\mathbb{U}_j^{f_j(f(y))}}S_Y(f^{\leftarrow}(\bigwedge_{j\in F}f_j^{\leftarrow}(\mu_j)),f^{\leftarrow}(\mu)),\text{continuity of } f_j\circ f \text{ implies } \nu:=f^{\leftarrow}(\bigwedge_{j\in F}f_j^{\leftarrow}(\mu_j))\in\mathbb{V}^y\\ &\le \bigvee_{\nu\in\mathbb{V}^y}S_Y(\nu,f^{\leftarrow}(\mu)).\end{aligned}$$

Hence $f^{\leftarrow}(\mu)\in\mathbb{V}^y$, and so f is continuous at y. □

From the above theorem we get the following corollary.

Corollary 3.5. *The category* **TNS** *is topological over the category* **Set***.*

Remark 3.6. (1) For a source $(X\xrightarrow{f_j}(X_j,\mathbb{U}_j))_{j\in J}$ in **TNS**, the initial $\top$-neighborhood system $\mathbb{U}^x$ at x, is indeed the $\top$-filter generated by

$$\{\bigwedge_{j\in F}f_j^{\leftarrow}(\mu_j)|F\in 2^{(J)},\forall j\in F,\mu_j\in\mathbb{U}_j^{f_j(x)}\}$$

as a base.

(2) Corollary 3.5 shows that each sink $((X_j,\mathbb{U}_j)\xrightarrow{f_j}X)_{j\in J}$ in **TNS** has a final structure. Here we give the construction of the final structure $(X,\mathbb{U})$ for later use.

$$\forall x\in X,\mathbb{U}^x=\begin{cases}[x]_\top, & x\notin\bigcup_{j\in J}f_j(X_j);\\ \bigcap_{j\in J,f_j(x_j)=x}f_j^{\Rightarrow}(\mathbb{U}_j^{x_j}), & \text{otherwise.}\end{cases}$$

Let $(X, \mathbb{U}_i)$ $(i = 1, 2)$ be $\top$-neighborhood spaces. Then we say $\mathbb{U}_1$ is finer than $\mathbb{U}_2$ or $\mathbb{U}_2$ is coarser than $\mathbb{U}_1$, and denoted as $\mathbb{U}_2 \leq \mathbb{U}_1$ if the function $id_X : (X, \mathbb{U}_1) \longrightarrow (X, \mathbb{U}_2)$ is continuous, that is $\forall x \in X$, $\mathbb{U}_2^x \subseteq \mathbb{U}_1^x$.

Let $TN(X)$ denote the set of $\top$-neighborhood systems on X.

Corollary 3.7. $(TN(X), \leq)$ *forms a complete lattice.*

Proof. Obviously, $(TN(X), \leq)$ is a partially ordered set. For any collection $\{\mathbb{U}_j\}_{j\in J} \subseteq TN(X)$, let $\mathbb{U}$ be the initial structure on X w.r.t. the source $(X \xrightarrow{f_j} (X_j, \mathbb{U}_j))_{j\in J}$ with each $f_j = id_X$. It is easily seen from Theorem 3.4 that $\mathbb{U}$ is the coarsest $\top$-neighborhood systems on X finer than all $\mathbb{U}_j$. Thus $\mathbb{U}$ is the supremum (so denote $\mathbb{U}$ as $\bigvee_{j\in J} \mathbb{U}_j$) of $\{\mathbb{U}_j\}_{j\in J}$. Hence $(TN(X), \leq)$ forms a complete lattice.

In addition, it is easily seen from Remark 3.6 (2) that the final structure on X w.r.t. the sink $((X_j, \mathbb{U}_j) \xrightarrow{f_j} X)_{j\in J}$ with all $f_j = id_X$, defined as $\bigwedge_{j\in J} \mathbb{U}_j$, is the infimum of $\{\mathbb{U}_j\}_{j\in J}$. □

3.2. Embedding of Neighborhood Spaces in ⊤-Neighborhood Spaces as a Reflective and Coreflective Subcategory

For a given $\mathscr{F} \in \mathscr{F}(X)$, it is easily seen that the family $\{\top_A | A \in \mathscr{F}\}$ forms a $\top$-filter base, the $\top$-filter generated by it is denoted as $\omega(\mathscr{F})$. Indeed,

$$\omega(\mathscr{F}) = \{\lambda \in L^X | \bigvee_{A\in\mathscr{F}} S_X(\top_A, \lambda) = \top\}.$$

Particularly, for any $x \in X$, $\omega(\dot{x}) = [x]_\top$.

Lemma 3.8. *Let $f : X \longrightarrow Y$ be a function and $\mathscr{F} \in \mathscr{F}(X)$. Then $\omega(f(\mathscr{F})) = f^{\Rightarrow}(\omega(\mathscr{F}))$.*

Proof. Note that $\omega(f(\mathscr{F}))$ has a $\top$-filter base $\{\top_{f(A)} | A \in \mathscr{F}\}$, and $f^{\Rightarrow}(\omega(\mathscr{F}))$ has a $\top$-filter base $\{f^{\rightarrow}(\top_A) | A \in \mathscr{F}\}$. It is easy to check that $\top_{f(A)} = f^{\rightarrow}(\top_A)$. Hence, $\omega(f(\mathscr{F}))$ and $f^{\Rightarrow}(\omega(\mathscr{F}))$ have a common $\top$-filter base, and so they are equal. □

Definition 3.9. Let $(X, \mathscr{U})$ be a neighborhood space. Then it is not difficult to prove that the pair $(X, \omega(\mathscr{U}))$ is a $\top$-neighborhood space, where $\forall x \in X, \omega(\mathscr{U})^x = \omega(\mathscr{U}^x)$. A $\top$-neighborhood space $(X, \mathbb{U})$ is called generated by a neighborhood space $(X, \mathscr{U})$ if $\mathbb{U} = \omega(\mathscr{U})$.

Lemma 3.10. *Let $f : X \longrightarrow Y$ be a function and $(X, \mathscr{U})$ and $(Y, \mathscr{V})$ be neighborhood spaces. Then $f : (X, \mathscr{U}) \longrightarrow (Y, \mathscr{V})$ is continuous if and only if $f : (X, \omega(\mathscr{U})) \longrightarrow (Y, \omega(\mathscr{V}))$ is continuous.*

Proof. Assume that $f : (X, \mathscr{U}) \longrightarrow (Y, \mathscr{V})$ is continuous at $x \in X$. Then for any $A \in \mathscr{V}^{f(x)}$, we have $f^{-1}(A) \in \mathscr{U}^x$, and so $f^{\leftarrow}(\top_A) = \top_{f^{-1}(A)} \in \omega(\mathscr{U})^x$. It follows by Lemma 3.3 that $f : (X, \omega(\mathscr{U})) \longrightarrow (Y, \omega(\mathscr{V}))$ is continuous at x. Conversely, let $f : (X, \omega(\mathscr{U})) \longrightarrow (Y, \omega(\mathscr{V}))$ be continuous at x. Then for any $B \in \mathscr{V}^{f(x)}$, we have $\top_B \in \omega(\mathscr{V})^{f(x)}$ and so $\top_{f^{-1}(B)} = f^{\leftarrow}(\top_B) \in \omega(\mathscr{U})^x$. That means $\bigvee_{A\in\mathscr{U}^x} S_X(\top_A, \top_{f^{-1}(B)}) = \top$, i.e., there exists an $A \in \mathscr{U}^x$ such that $A \subseteq f^{-1}(B)$. Hence $f^{-1}(B) \in \mathscr{U}^x$ and so $f : (X, \mathscr{U}) \longrightarrow (Y, \mathscr{V})$ is continuous at $x \in X$. □

It is easily seen that the correspondence $(X, \mathscr{U}) \mapsto (X, \omega(\mathscr{U}))$ defines an embedding functor ω : **NS** $\longrightarrow$ **TNS**. The following theorem shows that ω preserves the initial structures.

Theorem 3.11. *Let $\mathscr{U}$ be the initial structure w.r.t. the source $(X \xrightarrow{f_j} (X_j, \mathscr{U}_j))_{j\in J}$ in* **NS**. *Then $\omega(\mathscr{U})$ is the initial structure w.r.t. the source $(X \xrightarrow{f_j} (X_j, \omega(\mathbb{U}_j)))_{j\in J}$ in* **TNS**.

Proof. Since it has been known that $\omega(\mathscr{U})$ is a $\top$-neighborhood system on X, we need only check the initial conditions.

(1) Each $f_j : (X, \omega(\mathscr{U})) \longrightarrow (X_j, \omega(\mathscr{U}_j))$ is continuous. Indeed, it follow immediately by Lemma 3.10 and the continuity of $f_j : (X, \mathscr{U}) \longrightarrow (X_j, \mathscr{U}_j)$.

(2) Let $f : (Y, \mathbb{V}) \longrightarrow (X, \omega(\mathscr{U}))$ be a function between $\top$-neighborhood spaces. Then f is continuous iff each $f_j \circ f$ is continuous.

Obviously, if f is continuous, then each $f_j \circ f$ is continuous.

Conversely, let each $f_j \circ f$ is continuous. For any $y \in Y$, note that

$$\{\bigcap_{j\in F} f_j^{-1}(A_j) | F \in 2^{(J)}, \forall j \in F, A_j \in \mathscr{U}_j^{f_j(f(y))}\}$$

forms a base of $\mathscr{U}^{f(y)}$. Then for any $\lambda \in \omega(\mathscr{U})^{f(y)}$, it follows by Lemma 2.5 that

$$\begin{aligned}
\top &= \bigvee_{F\in 2^{(J)}, \forall j\in F, A_j \in \mathscr{U}_j^{f_j(f(y))}} S_X(\top_{\bigcap_{j\in F} f_j^{-1}(A_j)}, \lambda) \\
&= \bigvee_{F\in 2^{(J)}, \forall j\in F, A_j \in \mathscr{U}_j^{f_j(f(y))}} S_X(\bigwedge_{j\in F} f_j^{\leftarrow}(\top_{A_j}), \lambda), \text{ by } \top_{A_j} \in \omega(\mathscr{U}_j)^{f_j(f(y))} \\
&\leq \bigvee_{F\in 2^{(J)}, \forall j\in F, \mu_j \in \omega(\mathscr{U}_j)^{f_j(f(y))}} S_X(\bigwedge_{j\in F} f_j^{\leftarrow}(\mu_j), \lambda), \text{ by Lemma2.2(5)} \\
&\leq \bigvee_{F\in 2^{(J)}, \forall j\in F, \mu_j \in \omega(\mathscr{U}_j)^{f_j(f(y))}} S_Y\Big(f^{\leftarrow}(\bigwedge_{j\in F} f_j^{\leftarrow}(\mu_j)), f^{\leftarrow}(\lambda)\Big) \\
&= \bigvee_{F\in 2^{(J)}, \forall j\in F, \mu_j \in \omega(\mathscr{U}_j)^{f_j(f(y))}} S_Y\Big(\bigwedge_{j\in F} (f_j \circ f)^{\leftarrow}(\mu_j), f^{\leftarrow}(\lambda)\Big), \text{by continuity of } f_j \circ f \text{ in } \mathbf{NS} \\
&\leq \bigvee_{F\in 2^{(J)}, \forall j\in F, \nu_j \in \mathbb{V}^y} S_Y\Big(\bigwedge_{j\in F} \nu_j, f^{\leftarrow}(\lambda)\Big), \text{by (TF2)} \\
&\leq \bigvee_{\nu\in\mathbb{V}^y} S_Y\Big(\nu, f^{\leftarrow}(\lambda)\Big),
\end{aligned}$$

i.e., $f^{\leftarrow}(\lambda) \in \mathbb{V}^y$. Hence f is continuous at y. □

Definition 3.12. Let $f : X \longrightarrow Y$ be a function and $(Y, \mathbb{V})$ (resp., $(Y, \mathscr{V})$) be a $\top$-neighborhood space (resp., neighborhood space). Then $f : X \longrightarrow (Y, \mathbb{V})$ (resp., $f : X \longrightarrow (Y, \mathscr{V})$) can be regarded as a source in **TNS** (resp., **NS**) consisting of one function. We denote the corresponding initial structure as $f^{\Leftarrow}(\mathbb{V})$ (resp., $f^{-1}(\mathscr{V})$), i.e.,

$$\forall x \in X, f^{\Leftarrow}(\mathbb{V})^x := f^{\Leftarrow}(\mathbb{V}^{f(x)}) \text{ (resp., } f^{-1}(\mathscr{V})^x = f^{-1}(\mathscr{V}^{f(x)})).$$

Note that $f^{\Leftarrow}(\mathbb{V}^{f(x)})$ exists since $\lambda(f(x)) = \top$ for any $\lambda \in \mathbb{V}^{f(x)} \subseteq [f(x)]_{\top}$.

Corollary 3.13. (1) *For any $\{\mathscr{U}_j\}_{j\in J} \subseteq N(X)$, we have $\omega(\bigvee_{j\in J} \mathscr{U}_j) = \bigvee_{j\in J} \omega(\mathscr{U}_j)$.*

(2) *For a function $f : X \longrightarrow Y$ and a neighborhood space $(Y, \mathscr{V})$, we have $\omega(f^{-1}(\mathscr{V})) = f^{\Leftarrow}(\omega(\mathscr{V}))$.*

(3) *For a continuous function $f : (X, \mathbb{U}) \longrightarrow (Y, \mathbb{V})$ in* **TNS**, *we have $f^{\Leftarrow}(\mathbb{V}) \leq \mathbb{U}$.*

Proof. (1) It follows from Theorem 3.11 by considering the source $(X \xrightarrow{id_X} (X, \mathscr{U}_j))_{j\in J}$ in **NS**.

(2) It follows from Theorem 3.11 by considering the source $X \xrightarrow{f} (Y, \mathscr{V})$ in **NS**.

(3) Note that $f = f \circ id_X : (X, \mathbb{U}) \longrightarrow (X, f^{\Leftarrow}(\mathbb{V})) \longrightarrow (Y, \mathbb{V})$ is continuous, then by the initial condition (2), we get that $id_X : (X, \mathbb{U}) \longrightarrow (X, f^{\Leftarrow}(\mathbb{V}))$ is continuous, so $f^{\Leftarrow}(\mathbb{V}) \leq \mathbb{U}$. □

Lemma 3.14. *Let $f : X \longrightarrow Y$ be a function and $\{\mathbb{U}_j\}_{j\in J}$ be a family of $\top$-neighborhood systems on Y. Then $f^{\Leftarrow}(\bigvee_{j\in J} \mathbb{U}_j) = \bigvee_{j\in J} f^{\Leftarrow}(\mathbb{U}_j)$. A similar result for neighborhood spaces also holds.*

Proof. Obviously, $f^{\Leftarrow}(\bigvee_{j\in J}\mathbb{U}_j) \geq \bigvee_{j\in J} f^{\Leftarrow}(\mathbb{U}_j)$. On the other hand, for any $x \in X$, let $\lambda \in f^{\Leftarrow}(\bigvee_{j\in J}\mathbb{U}_j)^x = f^{\Leftarrow}(\bigvee_{j\in J}\mathbb{U}_j^{f(x)})$. Then by Remark 3.6 (1) and Lemma 2.7 (2)

$$\begin{aligned}\top &= \bigvee_{F\in 2^{(J)},\forall j\in F,\mu_j\in\mathbb{U}_j^{f(x)}} S_X(\bigwedge_{j\in F} f^{\leftarrow}(\mu_j),\lambda),\ \text{by } f^{\leftarrow}(\mu_j)\in f^{\Leftarrow}(\mathbb{U}_j^{f(x)}) = f^{\Leftarrow}(\mathbb{U}_j)^x\\ &\leq \bigvee_{F\in 2^{(J)},\forall j\in F,\nu_j\in f^{\Leftarrow}(\mathbb{U}_j)^x} S_X(\bigwedge_{j\in F}\nu_j,\lambda),\end{aligned}$$

i.e., $\lambda \in \bigvee_{j\in J} f^{\Leftarrow}(\mathbb{U}_j)^x$. Hence, $f^{\Leftarrow}(\bigvee_{j\in J}\mathbb{U}_j) \leq \bigvee_{j\in J} f^{\Leftarrow}(\mathbb{U}_j)$. □

Given a $\top$-neighborhood space $(X,\mathbb{U})$, define $\rho(\mathbb{U}) = \bigvee\{\mathscr{U} \in N(X)|\omega(\mathscr{U}) \leq \mathbb{U}\}$. Note that $(N(X), \leq)$ forms a complete lattice. Hence $\rho(\mathbb{U})$ is a neighborhood system on X, i,e., $(X,\rho(\mathbb{U}))$ is a neighborhood space.

Proposition 3.15. *Let $f : X \longrightarrow Y$ be a continuous function between $\top$-neighborhood spaces $(X,\mathbb{U})$ and $(Y,\mathbb{V})$. Then f is also continuous between $(X,\rho(\mathbb{U}))$ and $(Y,\rho(\mathbb{V}))$.*

Proof. Since $f : (X,\mathbb{U}) \longrightarrow (Y,\mathbb{V})$ is continuous, then $f^{\Leftarrow}(\mathbb{V}) \leq \mathbb{U}$ by Corollary 3.13 (3). Hence,

$$\begin{aligned}f^{-1}(\rho(\mathbb{V})) &= f^{-1}(\vee\{\mathscr{V}|\mathscr{V}\in N(Y),\omega(\mathscr{V})\leq\mathbb{V}\}),\ \text{by Lemma3.14}\\ &= \vee\{f^{-1}(\mathscr{V})|\mathscr{V}\in N(Y),\omega(\mathscr{V})\leq\mathbb{V}\},\ \text{by Corollary3.13(2)}\\ &\leq \vee\{f^{-1}(\mathscr{V})|\mathscr{V}\in N(Y),\omega(f^{-1}(\mathscr{V})) = f^{\Leftarrow}(\omega(\mathscr{V}))\leq f^{\Leftarrow}(\mathbb{V})\leq\mathbb{U}\}\\ &\leq \vee\{f^{-1}(\mathscr{V})|\mathscr{V}\in N(Y),\omega(f^{-1}(\mathscr{V}))\leq\mathbb{U}\}\\ &\leq \vee\{\mathscr{U}|\mathscr{U}\in N(X),\omega(\mathscr{U})\leq\mathbb{U}\}\\ &= \rho(\mathbb{U}).\end{aligned}$$

Therefore, $f : (X,\rho(\mathbb{U})) \longrightarrow (Y,\rho(\mathbb{V}))$ is continuous. □

It is easily seen that the correspondence $(X,\mathbb{U}) \mapsto (X,\rho(\mathbb{U}))$ defines a concrete functor $\rho : \mathbf{TNS} \longrightarrow \mathbf{NS}$.

Theorem 3.16. *The pair (ρ,ω) is a Galois correspondence and ρ is a left inverse of ω. Hence the category* **NS** *can be embedded in the category* **TNS** *as a reflective subcategory.*

Proof. Given $(X,\mathscr{U}) \in \mathbf{NS}$ and $(X,\mathbb{U}) \in \mathbf{TNS}$.

(1) Since ω is an embedding, then $\omega(\mathscr{V}) \leq \omega(\mathscr{U}) \Leftrightarrow \mathscr{V} \leq \mathscr{U}$. Hence, $\rho\circ\omega(\mathscr{U}) = \bigvee\{\mathscr{V} \in N(X)|\omega(\mathscr{V}) \leq \omega(\mathscr{U})\} = \bigvee\{\mathscr{V} \in N(X)|\mathscr{V} \leq \mathscr{U}\} = \mathscr{U}$.

(2) $\omega\circ\rho(\mathbb{U}) \leq \mathbb{U}$. Indeed,

$$\begin{aligned}\omega\circ\rho(\mathbb{U}) &= \omega(\vee\{\mathscr{V}|\mathscr{V}\in N(X),\omega(\mathscr{V})\leq\mathbb{U}\}),\ \text{by Corollary3.13(1)}\\ &= \vee\{\omega(\mathscr{V})|\mathscr{V}\in N(X),\omega(\mathscr{V})\leq\mathbb{U}\}\leq\mathbb{U}.\end{aligned}$$

□

The following Theorem 3.18 shows that ω preserves the final structures. But first we start with the following:

Lemma 3.17. *Let $(X,\{\mathscr{U}_j\})_{j\in J}$ be a family of neighborhood spaces. If L is continuous, then $\omega(\bigwedge_{j\in J}\mathscr{U}_j) = \bigwedge_{j\in J}\omega(\mathscr{U}_j)$.*

Proof. Obviously, $\omega(\bigwedge_{j\in J}\mathscr{U}_j) \leq \bigwedge_{j\in J}\omega(\mathscr{U}_j)$.

Let $x \in X$ and $\lambda \in \bigcap_{j\in J} \omega(\mathscr{U}_j)^x$. Then for any $j \in J$, $\bigvee_{A_j \in \mathscr{U}_j^x} S_X(\top_{A_j}, \lambda) = \top$, and $\{S_X(\top_{A_j}, \lambda) | A_j \in \mathscr{U}_j^x\}$ is a directed subset in L. Hence

$$\begin{aligned} \top &= \bigwedge_{j\in J} \bigvee_{A_j\in\mathscr{U}_j^x} S_X(\top_{A_j}, \lambda), \text{ by continuous condition } (\mathbf{CC}) \\ &= \bigvee_{h\in N} \bigwedge_{j\in J} S_X(\top_{h(j)}, \lambda), \text{where } N = \{f : J \longrightarrow \bigcup_{j\in J} \mathscr{U}_j^x | \forall j \in J, f(j) \in \mathscr{U}_j^x\} \\ &= \bigvee_{h\in N} S_X(\bigvee_{j\in J} \top_{h(j)}, \lambda), \text{ by } \bigvee_{j\in J} h(j) \in \bigcap_{j\in J} \mathscr{U}_j^x \\ &\le \bigvee_{B\in\bigcap_{j\in J}\mathscr{U}_j^x} S_X(\top_B, \lambda), \end{aligned}$$

i.e., $\lambda \in \omega(\bigwedge_{j\in J} \mathscr{U}_j)^x$. Hence $\omega(\bigwedge_{j\in J} \mathscr{U}_j) \ge \bigwedge_{j\in J} \omega(\mathscr{U}_j)$. □

Theorem 3.18. *Let $\mathscr{U}$ be the final structure w.r.t. the sink $((X_j, \mathscr{U}_j) \xrightarrow{f_j} X)_{j\in J}$ in* **NS**. *If L is continuous, then $(X, \omega(\mathscr{U}))$ is the final structure w.r.t. the sink $((X_j, \omega(\mathscr{U}_j)) \xrightarrow{f_j} X)_{j\in J}$ in* **TNS**.

Proof. Since it has been known that $\omega(\mathscr{U})$ is a $\top$-neighborhood system on X, we need only check the final conditions.

(1) Each $f_j : (X_j, \omega(\mathscr{U}_j)) \longrightarrow (X, \omega(\mathscr{U}))$ is continuous. Indeed, it follow immediately by Lemma 3.10 and the continuity of $f_j : (X_j, \mathscr{U}_j) \longrightarrow (X, \mathscr{U})$.

(2) Let $f : (X, \omega(\mathscr{U})) \longrightarrow (Y, \mathbb{V})$ be a function between $\top$-neighborhood spaces. Then f is continuous iff each $f \circ f_j$ is continuous.

Obviously, if f is continuous, then each $f \circ f_j$ is continuous.

Conversely, let each $f \circ f_j$ is continuous. For any $x \in X$, it has two cases.

Case 1: $x \notin \bigcup_{j\in J} f_j(\mathscr{U}_j)$, then $\omega(\mathscr{U})^x = \omega(\dot{x}) = [x]_\top$. For any $\lambda \in \mathbb{V}^{f(x)}$, since $\mathbb{V}^{f(x)} \subseteq [f(x)]_\top$ we have $f^{\leftarrow}(\lambda)(x) = \lambda(f(x)) = \top$, so $f^{\leftarrow}(\lambda) \in \omega(\mathscr{U})^x$. Hence f is continuous at x.

Case 2: $x \in \bigcup_{j\in J} f_j(\mathscr{U}_j)$, then

$$\omega(\mathscr{U})^x = \omega(\mathscr{U}^x) = \omega\Big(\bigcap_{j\in J, f_j(x_j)=x} f_j(\mathscr{U}_j^{x_j})\Big) = (\text{by Lemma3.17}) \bigcap_{j\in J, f_j(x_j)=x} \omega(f_j(\mathscr{U}_j^{x_j})).$$

For any $\lambda \in \mathbb{V}^{f(x)}$, we have $\lambda \in \mathbb{V}^{f(f_j(x_j))}$ for any $x_j \in X_j$ with $f_j(x_j) = x$. By $f \circ f_j$ is continuous at x_j we get

$$f_j^{\leftarrow}(f^{\leftarrow}(\lambda)) = (f \circ f_j)^{\leftarrow}(\lambda) \in \omega(\mathscr{U}_j^{x_j}),$$

so $f^{\leftarrow}(\lambda) \in f_j^{\Rightarrow}(\omega(\mathscr{U}_j^{x_j})) = (\text{by Lemma3.8})\omega(f_j(\mathscr{U}_j^{x_j}))$, and then $f^{\leftarrow}(\lambda) \in \bigcap_{j\in J, f_j(x_j)=x} \omega(f_j(\mathscr{U}_j^{x_j})) = \omega(\mathscr{U})^x$. Hence f is continuous at x. □

Definition 3.19. Let $f : X \longrightarrow Y$ be a function and $(X, \mathscr{U})$ be a neighborhood space (resp., $(X, \mathbb{U})$ be a $\top$-neighborhood space). Then $f : (X, \mathscr{U}) \longrightarrow Y$ (resp., $f : (X, \mathbb{U}) \longrightarrow Y$) can be regarded as a sink in **NS** (resp., **TNS**) consisting of one function. We denote the corresponding final structure as $f(\mathscr{U})$ (resp., $f^{\Rightarrow}(\mathbb{U})$), i.e.,

$$\forall y \in Y, f(\mathscr{U})^y = \begin{cases} \dot{y}, & y \notin f(X); \\ \bigcap_{f(x)=y} f(\mathscr{U}^x), & \text{otherwise.} \end{cases} \quad (\text{resp., } f^{\Rightarrow}(\mathbb{U})^y = \begin{cases} [y]_\top, & y \notin f(X); \\ \bigcap_{f(x)=y} f^{\Rightarrow}(\mathbb{U}^x), & \text{otherwise.} \end{cases}).$$

Corollary 3.20. *Let $f : X \longrightarrow Y$ be a function and L be continuous.*
(1) *For any $(X, \mathscr{U}) \in$ **NS**, we have $\omega(f(\mathscr{U})) = f^{\Rightarrow}(\omega(\mathscr{U}))$.*
(2) *For a continuous function $f : (X, \mathbb{U}) \longrightarrow (Y, \mathbb{V})$ in **TNS**, we have $f^{\Rightarrow}(\mathbb{U}) \geq \mathbb{V}$.*

Proof. (1) It follows from Theorem 3.18 by considering the sink $(X, \mathscr{U}) \xrightarrow{f} Y$ in **NS**.
(2) Note that $f = id_Y \circ f : (X, \mathbb{U}) \longrightarrow (Y, f^{\Rightarrow}(\mathbb{U})) \longrightarrow (Y, \mathbb{V})$ is continuous, then by the final condition (2), we get that $id_Y : (Y, f^{\Rightarrow}(\mathbb{U})) \longrightarrow (Y, \mathbb{V})$ is continuous, so $f^{\Rightarrow}(\mathbb{U}) \geq \mathbb{V}$. □

Let $(X, \mathbb{U})$ be a $\top$-neighborhood space, define

$$\iota(\mathbb{U}) = \wedge\{\mathscr{U} \in N(X) | \omega(\mathscr{U}) \geq \mathbb{U}\}.$$

Note that $(N(X), \leq)$ forms a complete lattice. Hence $\iota(\mathbb{U})$ is a neighborhood system on X, i,e., $(X, \iota(\mathbb{U}))$ is a neighborhood space.

Lemma 3.21. *Let $f : X \longrightarrow Y$ be a function and $\{\mathscr{U}_j\}_{j\in J} \subseteq N(X)$. Then $f(\bigwedge_{j\in J} \mathscr{U}_j) = \bigwedge_{j\in J} f(\mathscr{U}_j)$.*

Proof. Let $y \notin f(X)$. Then $f(\bigwedge_{j\in J} \mathscr{U}_j)^y = \dot{y} = \bigwedge_{j\in J} f(\mathscr{U}_j)^y$. Let $y \in f(X)$. Then

$$f(\bigwedge_{j\in J} \mathscr{U}_j)^y = \bigcap_{f(x)=y} f(\bigcap_{j\in J} \mathscr{U}_j^x) = \bigcap_{f(x)=y} \bigcap_{j\in J} f(\mathscr{U}_j^x) = \bigcap_{j\in J} \bigcap_{f(x)=y} f(\mathscr{U}_j^x) = \bigwedge_{j\in J} f(\mathscr{U}_j)^y.$$

□

Proposition 3.22. *If L is continuous and $f : X \longrightarrow Y$ is a continuous function between $\top$-neighborhood spaces $(X, \mathbb{U})$ and $(Y, \mathbb{V})$, then f is also continuous between $(X, \iota(\mathbb{U}))$ and $(Y, \iota(\mathbb{V}))$.*

Proof. Note that

$$\begin{aligned}
f(\iota(\mathbb{U})) &= f(\wedge\{\mathscr{V} \in N(X) | \mathscr{V} \in N(X), \omega(\mathscr{V}) \geq \mathbb{U}\}), \text{ by Lemma3.21} \\
&= \wedge\{f(\mathscr{V}) | \mathscr{V} \in N(X), \omega(\mathscr{V}) \geq \mathbb{U}\} \\
&\geq \wedge\{f(\mathscr{V}) | \mathscr{V} \in N(X), f^{\Rightarrow}(\omega(\mathscr{V})) \geq f^{\Rightarrow}(\mathbb{U})\}, \text{ by Corollary3.20(1)} \\
&= \wedge\{f(\mathscr{V}) | \mathscr{V} \in N(X), \omega(f(\mathscr{V})) \geq f^{\Rightarrow}(\mathbb{U})\}, \text{ by Corollary3.20(2)} \\
&= \wedge\{f(\mathscr{V}) | \mathscr{V} \in N(X), \omega(f(\mathscr{V})) \geq f^{\Rightarrow}(\mathbb{U}) \geq \mathbb{V}\} \\
&\geq \wedge\{f(\mathscr{V}) | \mathscr{V} \in N(X), \omega(f(\mathscr{V})) \geq \mathbb{V}\} \\
&\geq \wedge\{\mathscr{W} | \mathscr{W} \in N(Y), \omega(\mathscr{W}) \geq \mathbb{V}\} = \iota(\mathbb{V}).
\end{aligned}$$

Hence, for any $x \in X$, it holds that

$$\iota(\mathbb{V})^{f(x)} \subseteq f(\iota(\mathbb{U}))^{f(x)} = \bigcap_{f(z)=f(x)} f^{\Rightarrow}(\iota(\mathbb{U})^z) \subseteq f^{\Rightarrow}(\iota(\mathbb{U})^x).$$

Therefore, $f : (X, \iota(\mathbb{U})) \longrightarrow (Y, \iota(\mathbb{V}))$ is continuous at x. □

It is easily seen that the correspondence $(X, \mathbb{U}) \mapsto (X, \iota(\mathbb{U}))$ defines a concrete functor $\iota :$ **TNS** $\longrightarrow$ **NS**.

Theorem 3.23. *If L is continuous, then the pair (ω, ι) is a Galois correspondence and ι is a left inverse of ω. Thus the category **NS** can be embedded in the category **TNS** as a coreflective subcategory.*

Proof. Given $(X, \mathscr{U}) \in$ **NS** and $(X, \mathbb{U}) \in$ **TNS**.
(1) Since ω is an embedding, then $\omega(\mathscr{V}) \geq \omega(\mathscr{U}) \Leftrightarrow \mathscr{V} \geq \mathscr{U}$. Hence, $\iota \circ \omega(\mathscr{U}) = \wedge\{\mathscr{V} \in N(X) | \omega(\mathscr{V}) \geq \omega(\mathscr{U})\} = \wedge\{\mathscr{V} \in N(X) | \mathscr{V} \geq \mathscr{U}\} = \mathscr{U}$.
(2) By ω preserving $\wedge$, it holds that

$$\omega \circ \iota(\mathbb{U}) = \omega(\wedge\{\mathscr{V} | \mathscr{V} \in N(X), \omega(\mathscr{V}) \geq \mathbb{U}\} = \wedge\{\omega(\mathscr{V}) | \mathscr{V} \in N(X), \omega(\mathscr{V}) \geq \mathbb{U}\} \geq \mathbb{U}.$$

□

From Theorem 3.16 and Theorem 3.23 we conclude the following corollary.

Corollary 3.24. *If L is continuous, then the category* **NS** *can be embedded in the category* **TNS** *as a simultaneously reflective and coreflective subcategory.*

4. Stratified L-Neighborhood Spaces and their Relationships to $\top$-Neighborhood Spaces

In this section, we shall focus on the categorical properties of stratified L-neighborhood spaces and their categorical relationships to $\top$-neighborhood spaces. The main results are: (1) the category of stratified L-neighborhood spaces is a topological category; (2) if L is meet-continuous, then $\top$-neighborhood spaces is embedded in stratified L-neighborhood spaces as a reflective subcategory, and if L is continuous, then $\top$-neighborhood spaces is embedded in stratified L-neighborhood spaces as a simultaneously reflective and coreflective subcategory.

4.1. Stratified L-Neighborhood Spaces

Definition 4.1. ([16]) A stratified L-neighborhood space is a pair $(X, \mathcal{U} := \{\mathcal{U}^x\}_{x\in X})$, where

(LN1) $\forall x \in X$, $\mathcal{U}^x$ is a stratified L-filter; and

(LN2) $\forall x \in X$, $\mathcal{U}^x \leq [x]$.

Then $\mathcal{U}$ is called a stratified L-neighborhood system on X.

Let $(X, \mathcal{U})$ and $(Y, \mathcal{V})$ be stratified L-neighborhood spaces. Then a function $f : (X, \mathcal{U}) \longrightarrow (Y, \mathcal{V})$ is said to be *continuous* at x if $\mathcal{V}^{f(x)}(\nu) \leq \mathcal{U}^x(f^{\leftarrow}(\nu))$ for any $\nu \in L^Y$, and f is said to be *continuous* if it is continuous at each $x \in X$.

Let **SLNS** denote the category of stratified L-neighborhood spaces and continuous functions. The notions of source, sink, initial structure and final structure in **SLNS** can be defined similarly as for **NS**.

Theorem 4.2. *Each sink* $((X_j, \mathcal{U}_j) \xrightarrow{f_j} X)_{j\in J}$ *in* **SLNS** *has a final structure.*

Proof. We consider the pair $(X, \mathcal{U})$ defined by $\forall x \in X$:

$$\mathcal{U}^x = \begin{cases} [x], & x \notin \bigcup\limits_{j\in J} f_j(X_j); \\ \bigwedge\limits_{j\in J, f_j(x_j)=x} f_j^{\Rightarrow}(\mathcal{U}_j^{x_j}), & \text{otherwise.} \end{cases}$$

Obviously, $(X, \mathcal{U})$ is a stratified L-neighborhood space. Next, we prove the final conditions.

(1) Each $f_j : (X_j, \mathcal{U}_j) \longrightarrow (X, \mathcal{U})$ is continuous. Indeed, for any $x_j \in X_j$ and any $\mu \in L^X$, it holds that

$$\mathcal{U}^{f_j(x_j)}(\mu) = \bigwedge_{j\in J, f_j(z_j)=f(x_j)} f_j^{\Rightarrow}(\mathcal{U}_j^{z_j}) \leq f_j^{\Rightarrow}(\mathcal{U}_j^{x_j})(\mu) = \mathcal{U}_j^{x_j}(f^{\leftarrow}(\mu)).$$

Thus f_j is continuous at x.

(2) Let $f : (X, \mathcal{U}) \longrightarrow (Y, \mathcal{V})$ be a function between stratified L-neighborhood spaces. Then f is continuous iff each $f \circ f_j$ is continuous. Obviously, if f is continuous, then each $f \circ f_j$ is continuous. Conversely, let $x \in X$ and $\mu \in L^Y$. If $x \notin \bigcup_{j\in J} f_j(X_j)$, then

$$\mathcal{V}^{f(x)}(\mu) \leq [f(x)](\mu) = \mu(f(x)) = f^{\leftarrow}(\mu)(x) = [x](f^{\leftarrow}(\mu)) = \mathcal{U}^x(f^{\leftarrow}(\mu)).$$

If $x \in \bigcup_{j\in J} f_j(X_j)$, then for any $x_j \in X_j$ with $f_j(x_j) = x$, from the continuity of $f \circ f_j$, we have

$$\mathcal{V}^{f(x)}(\mu) = \mathcal{V}^{f\circ f_j(x_j)}(\mu) \leq \mathcal{U}^{x_j}((f \circ f_j)^{\leftarrow}(\mu)) = f_j^{\Rightarrow}(\mathcal{U}^{x_j})(f^{\leftarrow}(\mu)),$$

it follows that

$$\mathcal{V}^{f(x)}(\mu) \leq \bigwedge_{f_j(x_j)=x} f_j^{\Rightarrow}(\mathcal{U}^{x_j})(f^{\leftarrow}(\mu)) = \mathcal{U}^x(f^{\leftarrow}(\mu)).$$

Hence f is continuous at x. □

From the above theorem we get the following corollary.

Corollary 4.3. *The category* **SLNS** *is topological over* **Set**.

Remark 4.4. Corollary 4.3 shows that each source $(X \xrightarrow{f_j} (X_j, \mathcal{U}_j))_{j\in J}$ in **SLNS** has an initial structure. When $* = \wedge$, Mufarrij and Ahsanullah [3, Theorem 2.17] proved that **SLNS** is topological over **Set** by constructing the initial structure of source in **SLNS**. But their approach seems to be hard to generalize to the case of $* \neq \wedge$. Therefore, Corollary 4.3 extends the result in [3] from the lattice-context.

Let $\mathcal{U}_i$ $(i = 1, 2)$ be two stratified L-neighborhood systems on X. Then we say $\mathcal{U}_1$ is finer than $\mathcal{U}_2$ or $\mathcal{U}_2$ is coarser than $\mathcal{U}_1$, and write $\mathcal{U}_2 \leq \mathcal{U}_1$ if the function $id_X : (X, \mathcal{U}_1) \longrightarrow (X, \mathcal{U}_2)$ is continuous, that is $\mathcal{U}_2^x \leq \mathcal{U}_1^x$ for all $x \in X$.

Let $LN(X)$ denote the set of stratified L-neighborhood systems on X.

Corollary 4.5. $(LN(X), \leq)$ *forms a complete lattice.*

Proof. Obviously, $(LN(X), \leq)$ is a partial ordered set. For any collection $\{\mathcal{U}_j\}_{j\in J} \subseteq LN(X)$, it is easily seen from Theorem 4.2 that the final structure on X w.r.t. the sink $((X_j, \mathcal{U}_j) \xrightarrow{f_j} X)_{j\in J}$ with all $f_j = id_X$, defined by $\bigwedge_{j\in J} \mathcal{U}_j$, is the infimum of $\{\mathcal{U}_j\}_{j\in J}$. Thus $(LN(X), \leq)$ forms a complete lattice. Similarly, let $\mathcal{U}$ be the initial structure on X w.r.t. the source $(X \xrightarrow{f_j} (X_j, \mathcal{U}_j))_{j\in J}$ with each $f_j = id_X$. Then $\mathcal{U}$ is the coarsest stratified L-neighborhood systems on X finer than all $\mathcal{U}_j$, and then $\mathcal{U}$ is the supremum (so denote $\mathcal{U}$ as $\bigvee_{j\in J} \mathcal{U}_j$) of $\{\mathcal{U}_j\}_{j\in J}$. □

4.2. *Embedding of $\top$-Neighborhood Spaces in Stratified L-Neighborhood Spaces as a Reflective and Coreflective Subcategory*

Let L be meet-continuous and $\mathbb{F} \in \mathbb{F}_L^\top(X)$. Then it is proved in [23] that the function $\omega^\top : L^X \longrightarrow L$ defined by

$$\forall \lambda \in L^X, \omega^\top(\mathbb{F})(\lambda) = \bigvee_{\mu\in\mathbb{F}} S_X(\mu, \lambda)$$

is a stratified L-filter on X [when we prove that $\omega^\top(\mathbb{F})$ satisfies (LF2), the meet-continuity is used]. Particularly, $\omega^\top([x]_\top) = [x]$ for any $x \in X$.

In this subsection, we always assume that L to be meet-continuous if not otherwise statement.

Lemma 4.6. *Let $f : X \longrightarrow Y$ be a function and $\mathbb{F} \in \mathbb{F}_L^\top(X)$. Then $\omega^\top(f^\Rightarrow(\mathbb{F})) = f^\Rightarrow(\omega^\top(\mathbb{F}))$.*

Proof. It is similar to Lemma 3.8. □

Definition 4.7. Let $(X, \mathbb{U})$ be a $\top$-neighborhood space. A stratified L-neighborhood space $(X, \mathcal{U})$ is said to be generated by a $\top$-neighborhood space if $\mathcal{U} = \omega^\top(\mathbb{U})$.

In [27], the author and co-author proved that $(X, \omega^\top(\mathbb{U}))$ is a stratified L-neighborhood space, where $\forall x \in X, \omega^\top(\mathbb{U})^x = \omega^\top(\mathbb{U}^x)$.

Lemma 4.8. *Let $f : X \longrightarrow Y$ be a function and $(X, \mathbb{U})$ and $(Y, \mathbb{V})$ be $\top$-neighborhood spaces. Then $f : (X, \mathbb{U}) \longrightarrow (Y, \mathbb{V})$ is continuous iff $f : (X, \omega^\top(\mathbb{U})) \longrightarrow (Y, \omega^\top(\mathbb{V}))$ is continuous.*

Proof. Assume that $f : (X, \mathbb{U}) \longrightarrow (Y, \mathbb{V})$ is continuous at $x \in X$. It follows from [27] that $(X, \omega^\top(\mathbb{U}))$ is a stratified L-neighborhood space, where $\forall x \in X, \omega^\top(\mathbb{U})^x = \omega^\top(\mathbb{U}^x)$. Then for any $\lambda \in L^Y$,

$$\begin{aligned} \omega^\top(\mathbb{V})^{f(x)}(\lambda) &= \bigvee_{\mu\in\mathbb{V}^{f(x)}} S_Y(\mu, \lambda) \\ &\leq \bigvee_{\mu\in\mathbb{V}^{f(x)}} S_X(f^\leftarrow(\mu), f^\leftarrow(\lambda)), \text{ by } f^\leftarrow(\mu) \in \mathbb{U}^x \\ &\leq \bigvee_{\nu\in\mathbb{U}^x} S_X(\nu, f^\leftarrow(\lambda)) = \omega^\top(\mathbb{U})^x(f^\leftarrow(\lambda)). \end{aligned}$$

Hence, $f : (X, \omega^\top(\mathbb{U})) \longrightarrow (Y, \omega^\top(\mathbb{V}))$ is continuous at x. Conversely, let $f : (X, \omega^\top(\mathbb{U})) \longrightarrow (Y, \omega^\top(\mathbb{V}))$ be continuous at x. Then for any $\lambda \in \mathbb{V}^{f(x)}$, we have $\top = \omega^\top(\mathbb{V})^{f(x)}(\lambda) \leq \omega^\top(\mathbb{U})^x(f^\leftarrow(\lambda))$, which means $f^\leftarrow(\lambda) \in \mathbb{U}^x$. Hence, $f : (X, \mathbb{U}) \longrightarrow (Y, \mathbb{V})$ is continuous at $x \in X$. □

It is easily seen that the correspondence $(X, \mathbb{U}) \mapsto (X, \omega^\top(\mathbb{U}))$ defines an embedding functor $\omega^\top$: **TNS** $\longrightarrow$ **SLNS**.

The following theorem shows that $\omega^\top$ preserves the initial structures.

Theorem 4.9. *Let $\mathbb{U}$ be the initial structure of the source $(X \xrightarrow{f_j} (X_j, \mathbb{U}_j))_{j\in J}$ in* **TNS**. *Then $\omega^\top(\mathbb{U})$ is the initial structure of the source $(X \xrightarrow{f_j} (X_j, \omega^\top(\mathbb{U}_j)))_{j\in J}$ in* **SLNS**.

Proof. Since it is known that $(X, \omega^\top(\mathbb{U}))$ is a stratified L-neighborhood space, we need only check the initial conditions.

(1) Each $f_j : (X, \omega^\top(\mathbb{U})) \longrightarrow (X_j, \omega^\top(\mathbb{U}_j))$ is continuous. Indeed, it follow immediately by Lemma 4.8 and the continuity of $f_j : (X, \mathbb{U}) \longrightarrow (X_j, \mathbb{U}_j)$.

(2) Let $f : (Y, \mathcal{V}) \longrightarrow (X, \omega^\top(\mathbb{U}))$ be a function between stratified L-neighborhood spaces. Then f is continuous iff each $f_j \circ f$ is continuous. Obviously, if f is continuous, then each $f_j \circ f$ is continuous. Conversely, let $\lambda \in L^X$ and $y \in Y$. Remark 3.6 (1) shows that

$$\forall x \in X, \{\bigwedge_{j\in F} f_j^\leftarrow(\mu_j) | F \in 2^{(J)}, \forall j \in F, \mu_j \in \mathbb{U}_j^{f_j(x)}\}$$

is a base of $\mathbb{U}^x$. Then

$$\begin{aligned}
\omega^\top(\mathbb{U}^{f(y)})(\lambda) &= \bigvee_{\mu\in\mathbb{U}^{f(y)}} S_X(\mu, \lambda), \text{ by Lemma2.5} \\
&= \bigvee_{F\in 2^{(J)}, \forall j\in F, \mu_j\in\mathbb{U}_j^{f_j(f(y))}} S_X(\bigwedge_{j\in F} f_j^\leftarrow(\mu_j), \lambda), \text{by } \omega^\top(\mathbb{U}_j^{f_j(f(y))})(\mu_j) = \top \\
&= \bigvee_{F\in 2^{(J)}, \forall j\in F, \mu_j\in\mathbb{U}_j^{f_j(f(y))}} \Big([\bigwedge_{j\in F} \omega^\top(\mathbb{U}_j^{f_j(f(y))})(\mu_j)] * S_X(\bigwedge_{j\in F} f_j^\leftarrow(\mu_j), \lambda)\Big) \\
&\leq \bigvee_{F\in 2^{(J)}, \forall j\in F, \mu_j\in L^{X_j}} \Big([\bigwedge_{j\in F} \omega^\top(\mathbb{U}_j^{f_j(f(y))})(\mu_j)] * S_X(\bigwedge_{j\in F} f_j^\leftarrow(\mu_j), \lambda)\Big), \text{ by continuity of } f_j \circ f \\
&\leq \bigvee_{F\in 2^{(J)}, \forall j\in F, \mu_j\in L^{X_j}} \Big([\bigwedge_{j\in F} \mathcal{V}^y\big((f_j \circ f)^\leftarrow(\mu_j)\big)] * S_X(\bigwedge_{j\in F} f_j^\leftarrow(\mu_j), \lambda)\Big), \text{ by Lemma2.2(5)} \\
&\leq \bigvee_{F\in 2^{(J)}, \forall j\in F, \mu_j\in L^{X_j}} \Big([\bigwedge_{j\in F} \mathcal{V}^y\big((f_j \circ f)^\leftarrow(\mu_j)\big)] * S_Y\big(f^\leftarrow[\bigwedge_{j\in F} f_j^\leftarrow(\mu_j)], f^\leftarrow(\lambda)\big)\Big) \\
&= \bigvee_{F\in 2^{(J)}, \forall j\in F, \mu_j\in L^{X_j}} \Big([\bigwedge_{j\in F} \mathcal{V}^y\big((f_j \circ f)^\leftarrow(\mu_j)\big)] * S_Y\big(\bigwedge_{j\in F}(f_j \circ f)^\leftarrow(\mu_j), f^\leftarrow(\lambda)\big)\Big), \text{ by (LFs)} \\
&\leq \mathcal{V}^y(f^\leftarrow(\lambda)).
\end{aligned}$$

Hence $f : (Y, \mathcal{V}) \longrightarrow (X, \omega^\top(\mathbb{U}))$ is continuous at y. □

Definition 4.10. Let $f : X \longrightarrow Y$ be a function and $(Y, \mathcal{V})$ be a stratified L-neighborhood space. Then $f : X \longrightarrow (Y, \mathcal{V})$ can be regarded as a source in **SLNS** consisting of one function. We denote the corresponding initial structure as $f^\Leftarrow(\mathcal{V})$, i,e.,

$$\forall x \in X, f^\Leftarrow(\mathcal{V})^x = f^\Leftarrow(\mathcal{V}^{f(x)}).$$

Note that $f^\Leftarrow(\mathcal{V}^{f(x)})$ exists since for any $\mu \in L^Y$ with $f^\leftarrow(\mu) = \underline{\perp}$, we have $\mathcal{V}^{f(x)}(\mu) \leq [f(x)](\mu) = \mu(f(x)) = f^\leftarrow(\mu)(x) = \perp$.

Corollary 4.11. (1) *For any* $\{\mathscr{U}_j\}_{j\in J} \subseteq TN(X)$, *we have* $\omega^\top(\bigvee\limits_{j\in J} \mathbb{U}_j) = \bigvee\limits_{j\in J} \omega^\top(\mathbb{U}_j)$.

(2) *For a function* $f : X \longrightarrow Y$ *and a* $\top$*-neighborhood space* $(Y, \mathbb{V})$, *we have* $\omega^\top(f^\Leftarrow(\mathbb{V})) = f^\Leftarrow(\omega^\top(\mathbb{V}))$.

(3) *For a continuous function* $f : (X, \mathcal{U}) \longrightarrow (Y, \mathcal{V})$ *in* **TNS**, *we have* $f^\Leftarrow(\mathcal{V}) \leq \mathcal{U}$.

Proof. It is similar to Corollary 3.13. □

Given a stratified L-neighborhood space $(X, \mathcal{U})$, define $\rho^\top(\mathcal{U}) = \bigvee\{\mathbb{U} \in TN(X) | \omega^\top(\mathbb{U}) \leq \mathcal{U}\}$. Note that $(TN(X), \leq)$ forms a complete lattice. Hence $\rho^\top(\mathcal{U})$ is a $\top$-neighborhood system on X, i,e., $(X, \rho^\top(\mathcal{U}))$ is a $\top$-neighborhood space.

Proposition 4.12. *Let* $f : X \longrightarrow Y$ *be a continuous function between* $\top$*-neighborhood spaces* $(X, \mathcal{U})$ *and* $(Y, \mathcal{V})$. *Then* f *is also continuous between* $(X, \rho^\top(\mathcal{U}))$ *and* $(Y, \rho^\top(\mathcal{V}))$.

Proof. Assume that $f : (X, \mathcal{U}) \longrightarrow (Y, \mathcal{V})$ is continuous. Then

$$\begin{aligned}
f^\Leftarrow(\rho^\top(\mathcal{V})) &= f^\Leftarrow(\vee\{\mathbb{V} | \mathbb{V} \in TN(Y), \omega^\top(\mathbb{V}) \leq \mathcal{V}\}), \text{ by Lemma3.14} \\
&= \vee\{f^\Leftarrow(\mathbb{V}) | \mathbb{V} \in TN(Y), \omega^\top(\mathbb{V}) \leq \mathcal{V}\}, \text{ by Corollary4.11(2), (3)} \\
&\leq \vee\{f^\Leftarrow(\mathbb{V}) | \mathbb{V} \in TN(Y), \omega^\top(f^\Leftarrow(\mathbb{V})) = f^\Leftarrow(\omega^\top(\mathbb{V})) \leq f^\Leftarrow(\mathcal{V}) \leq \mathcal{U}\} \\
&\leq \vee\{f^\Leftarrow(\mathbb{V}) | \mathbb{V} \in TN(Y), \omega^\top(f^\Leftarrow(\mathbb{V})) \leq \mathcal{U}\} \\
&\leq \vee\{\mathbb{U} | \mathbb{U} \in TN(X), \omega^\top(\mathbb{U}) \leq \mathcal{U}\} \\
&= \rho^\top(\mathcal{U}).
\end{aligned}$$

It follows that for any $\lambda \in \rho^\top(\mathcal{V})^{f(x)}$, we have $f^\leftarrow(\lambda) \in \rho^\top(\mathcal{U})^x$. Therefore, $f : (X, \rho^\top(\mathcal{U})) \longrightarrow (Y, \rho^\top(\mathcal{V}))$ is continuous. □

It is easily seen that the correspondence $(X, \mathcal{U}) \mapsto (X, \rho^\top(\mathcal{U}))$ defines a concrete functor $\rho^\top$: **SLNS** $\longrightarrow$ **TNS**.

Theorem 4.13. *The pair* $(\rho^\top, \omega^\top)$ *is a Galois correspondence and* $\rho^\top$ *is a left inverse of* $\omega^\top$. *Thus the category* **TNS** *is embedded in the category* **SLNS** *as a reflective subcategory.*

Proof. Given $(X, \mathbb{U}) \in$ **TNS** and $(X, \mathcal{U}) \in$ **SLNS**.

(1) Since $\omega^\top$ is an embedding, then $\omega^\top(\mathbb{V}) \leq \omega^\top(\mathbb{U}) \Leftrightarrow \mathbb{V} \leq \mathbb{U}$. Hence, $\rho^\top \circ \omega^\top(\mathbb{U}) = \bigvee\{\mathbb{V} \in TN(X) | \omega^\top(\mathbb{V}) \leq \omega^\top(\mathbb{U})\} = \bigvee\{\mathbb{V} \in TN(X) | \mathbb{V} \leq \mathbb{U}\} = \mathbb{U}$.

(2) $\omega^\top \circ \rho^\top(\mathcal{U}) \leq \mathcal{U}$. Indeed,

$$\begin{aligned}
\omega^\top \circ \rho^\top(\mathcal{U}) &= \omega^\top(\vee\{\mathbb{V} | \mathbb{V} \in TN(X), \omega^\top(\mathbb{V}) \leq \mathcal{U}\}, \text{ by } \omega^\top \text{ preserving } \vee \\
&= \vee\{\omega^\top(\mathbb{V}) | \mathbb{V} \in TN(X), \omega^\top(\mathbb{V}) \leq \mathcal{U}\} \leq \mathcal{U}.
\end{aligned}$$

□

The following Theorem 4.15 shows that $\omega^\top$ preserves the final structures. But first we begin with the following:

Lemma 4.14. *Let* $(X, \{\mathbb{U}_j\})_{j\in J}$ *be a family of* $\top$*-neighborhood spaces. If* L *is continuous, then* $\omega^\top(\bigwedge\limits_{j\in J} \mathbb{U}_j) = \bigwedge\limits_{j\in J} \omega^\top(\mathbb{U}_j)$.

Proof. Obviously, $\omega^\top(\bigwedge_{j\in J}\mathbb{U}_j) \leq \bigwedge_{j\in J}\omega^\top(\mathbb{U}_j)$. On the other hand, let $x \in X$ and $\lambda \in L^X$. Note that the set $\{S_X(\mu_j, \lambda)| j \in J, \mu_j \in \mathbb{U}_j^x\}$ is a directed subset in L. Hence,

$$\begin{aligned}
\bigwedge_{j\in J}\omega^\top(\mathbb{U}_j)^x(\lambda) &= \bigwedge_{j\in J}\bigvee_{\mu_j\in\mathbb{U}_j^x} S_X(\mu_j,\lambda), \text{ by continuous condition } (\mathbf{CC}) \\
&= \bigvee_{h\in N}\bigwedge_{j\in J} S_X(h(j),\lambda), \text{ where } N = \{f : J \longrightarrow \bigcup_{j\in J}\mathbb{U}_j^x | \forall j \in J, f(j) \in \mathbb{U}_j^x\} \\
&= \bigvee_{h\in N} S_X(\bigvee_{j\in J} h(j),\lambda), \text{ by } \bigvee_{j\in J} h(j) \in \bigcap_{j\in J}\mathbb{U}_j^x \\
&\leq \bigvee_{\nu\in\bigcap_{j\in J}\mathbb{U}_j^x} S_X(\nu,\lambda) = \omega^\top(\bigwedge_{j\in J}\mathbb{U}_j)^x(\lambda).
\end{aligned}$$

Therefore, $\omega^\top(\bigwedge_{j\in J}\mathbb{U}_j) \geq \bigwedge_{j\in J}\omega^\top(\mathbb{U}_j)$. □

Theorem 4.15. *Let $\mathbb{U}$ be the final structure w.r.t. the sink $((X_j, \mathbb{U}_j) \xrightarrow{f_j} X)_{j\in J}$ in* **TNS**. *If L is continuous, then $(X, \omega^\top(\mathbb{U}))$ is the final structure w.r.t. the sink $((X_j, \omega^\top(\mathbb{U}_j)) \xrightarrow{f_j} X)_{j\in J}$ in* **SLNS**.

Proof. Since it is well-known that $\omega^\top(\mathbb{U})$ is a stratified L-neighborhood system on X, we need only check the final conditions.

(1) Each $f_j : (X_j, \omega^\top(\mathbb{U}_j)) \longrightarrow (X, \omega^\top(\mathbb{U}))$ is continuous. Indeed, it follow immediately by Lemma 4.8 and the continuity of $f_j : (X_j, \mathbb{U}_j) \longrightarrow (X, \mathbb{U})$.

(2) Let $f : (X, \omega^\top(\mathbb{U})) \longrightarrow (Y, \mathcal{V})$ be a function between $\top$-neighborhood spaces. Then f is continuous iff each $f \circ f_j$ is continuous.

Obviously, if f is continuous, then each $f \circ f_j$ is continuous.

Conversely, let each $f \circ f_j$ is continuous. For any $x \in X$, it has two cases.

Case 1: $x \notin \bigcup_{j\in J} f_j(\mathbb{U}_j)$, then $\omega^\top(\mathbb{U})^x = \omega^\top([x]_\top) = [x]$. For any $\lambda \in L^Y$, we have $\mathcal{V}^{f(x)}(\lambda) \leq [f(x)](\lambda) = f^\leftarrow(\lambda)(x) = [x](f^\leftarrow(\lambda)) = \omega^\top(\mathbb{U})^x(f^\leftarrow(\lambda))$. Hence f is continuous at x.

Case 2: $x \in \bigcup_{j\in J} f_j(\mathbb{U}_j)$, then

$$\omega^\top(\mathbb{U})^x = \omega^\top(\mathbb{U}^x) = \omega^\top\Big(\bigwedge_{j\in J, f_j(x_j)=x} f_j^\Rightarrow(\mathbb{U}_j^{x_j})\Big) = (\text{by Lemma4.14}) \bigwedge_{j\in J, f_j(x_j)=x}\omega^\top(f_j^\Rightarrow(\mathbb{U}_j^{x_j})).$$

For any $\lambda \in L^Y$ and any $x_j \in X_j$ with $f_j(x_j) = x$, by $f \circ f_j$ is continuous at x_j we get

$$\mathcal{V}^{f(x)}(\lambda) = \mathcal{V}^{f(f_j(x_j))}(\lambda) \leq \omega^\top(\mathbb{U}_j^{x_j})(f_j^\leftarrow(f^\leftarrow(\lambda))) = f_j^\Rightarrow(\omega^\top(\mathbb{U}_j^{x_j}))(f^\leftarrow(\lambda)) = (\text{by Lemma4.6})\omega^\top(f_j^\Rightarrow(\mathbb{U}_j^{x_j}))(f^\leftarrow(\lambda)),$$

so

$$\mathcal{V}^{f(x)}(\lambda) \leq \bigwedge_{j\in J, f_j(x_j)=x}\omega^\top(f_j^\Rightarrow(\mathbb{U}_j^{x_j}))(f^\leftarrow(\lambda)) = \omega^\top(\mathbb{U})^x(f^\leftarrow(\lambda)).$$

Hence f is continuous at x. □

Definition 4.16. Let $f : X \longrightarrow Y$ be a function and $(X, \mathcal{U})$ be a stratified L-neighborhood space. Then $f : (X, \mathcal{U}) \longrightarrow Y$ can be regarded as a sink in **SLNS** consisting of one function. We denote the corresponding final structure as $f(\mathcal{U})$, i.e.,

$$\forall y \in Y, f(\mathcal{U})^y = \begin{cases} [y], & y \notin f(X); \\ \bigwedge_{f(x)=y} f^\Rightarrow(\mathcal{U}^x), & \text{otherwise.} \end{cases}$$

Corollary 4.17. *Let $f : X \longrightarrow Y$ be a function and L be continuous.*
(1) *For any $(X, \mathbb{U}) \in$ **TNS**, we have $\omega^\top(f^\Rightarrow(\mathbb{U})) = f^\Rightarrow(\omega^\top(\mathbb{U}))$.*
(2) *For a continuous function $f : (X, \mathcal{U}) \longrightarrow (Y, \mathcal{V})$ in **SLNS**, we have $f^\Rightarrow(\mathcal{U}) \geq \mathcal{V}$.*

Proof. It is similar to Corollary 3.20. □

Let $(X, \mathcal{U})$ be a stratified L-neighborhood space, define

$$\iota^\top(\mathcal{U}) = \wedge\{\mathbb{U} \in TN(X) | \omega^\top(\mathbb{U}) \geq \mathcal{U}\}.$$

Note that $(TN(X), \leq)$ forms a complete lattice. Hence $\iota^\top(\mathcal{U})$ is a $\top$-neighborhood system on X, i,e., $(X, \iota^\top(\mathcal{U}))$ is a $\top$-neighborhood space.

Lemma 4.18. *Let $f : X \longrightarrow Y$ be a function and $\{\mathbb{U}_j\}_{j\in J} \subseteq TN(X)$. Then $f^\Rightarrow(\bigwedge_{j\in J} \mathbb{U}_j) = \bigwedge_{j\in J} f^\Rightarrow(\mathbb{U}_j)$.*

Proof. Let $y \notin f(X)$. Then $f^\Rightarrow(\bigwedge_{j\in J} \mathbb{U}_j)^y = [y]_\top = \bigwedge_{j\in J} f^\Rightarrow(\mathbb{U}_j)^y$. Let $y \in f(X)$. Then

$$f^\Rightarrow(\bigwedge_{j\in J} \mathbb{U}_j)^y = \bigcap_{f(x)=y} f^\Rightarrow(\bigcap_{j\in J} \mathbb{U}_j^x) = \bigcap_{f(x)=y} \bigcap_{j\in J} f^\Rightarrow(\mathbb{U}_j^x) = \bigcap_{j\in J} \bigcap_{f(x)=y} f^\Rightarrow(\mathbb{U}_j^x) = \bigwedge_{j\in J} f^\Rightarrow(\mathbb{U}_j)^y.$$

□

Proposition 4.19. *Let $f : X \longrightarrow Y$ be a continuous function between stratified L-neighborhood spaces $(X, \mathcal{U})$ and $(Y, \mathcal{V})$. Then f is also continuous between $(X, \iota^\top(\mathcal{U}))$ and $(Y, \iota^\top(\mathcal{V}))$.*

Proof. Note that

$$\begin{aligned}
f^\Rightarrow(\iota^\top(\mathcal{U})) &= f^\Rightarrow(\wedge\{\mathbb{V} | \mathbb{V} \in TN(X), \omega^\top(\mathbb{V}) \geq \mathcal{U}\}), \text{ by Lemma4.18} \\
&= \wedge\{f^\Rightarrow(\mathbb{V}) | \mathbb{V} \in TN(X), \omega^\top(\mathbb{V}) \geq \mathcal{U}\} \\
&\geq \wedge\{f^\Rightarrow(\mathbb{V}) | \mathbb{V} \in TN(X), f^\Rightarrow(\omega^\top(\mathbb{V})) \geq f^\Rightarrow(\mathcal{U})\}, \text{ by Corollar4.17(1)} \\
&= \wedge\{f^\Rightarrow(\mathbb{V}) | \mathbb{V} \in TN(X), \omega^\top(f^\Rightarrow(\mathbb{V})) \geq f^\Rightarrow(\mathcal{U})\}, \text{ by Corollary4.17(2)} \\
&\geq \wedge\{f^\Rightarrow(\mathbb{V}) | \mathbb{V} \in TN(X), \omega^\top(f^\Rightarrow(\mathbb{V})) \geq \mathcal{V}\} \\
&\geq \wedge\{\mathbb{W} | \mathbb{W} \in TN(Y), \omega^\top(\mathbb{W}) \geq \mathcal{V}\} = \iota^\top(\mathcal{V}).
\end{aligned}$$

Hence, for any $x \in X$,

$$\iota^\top(\mathcal{V})^{f(x)} \leq f^\Rightarrow(\iota^\top(\mathcal{U}))^{f(x)} = \bigwedge_{f(z)=f(x)} f^\Rightarrow(\iota^\top(\mathcal{U})^z) \leq f^\Rightarrow(\iota^\top(\mathcal{U})^x).$$

Therefore, $f : (X, \iota^\top(\mathcal{U})) \longrightarrow (Y, \iota^\top(\mathcal{V}))$ is continuous at x. □

It is easily seen that the correspondence $(X, \mathcal{U}) \mapsto (X, \iota^\top(\mathcal{U}))$ defines a concrete functor $\iota^\top$: **SLNS** $\longrightarrow$ **TNS**.

Theorem 4.20. *If L is continuous, then the pair $(\omega^\top, \iota^\top)$ is a Galois correspondence and $\iota^\top$ is a left inverse of $\omega^\top$. Thus the category **TNS** is embedded in the category **SLNS** as a coreflective subcategory.*

Proof. Given $(X, \mathbb{U}) \in$ **TNS** and $(X, \mathcal{U}) \in$ **SLNS**.
(1) Since $\omega^\top$ is an embedding, then $\omega^\top(\mathbb{V}) \geq \omega^\top(\mathbb{U}) \Leftrightarrow \mathbb{V} \geq \mathbb{U}$. Hence, $\iota^\top \circ \omega^\top(\mathbb{U}) = \wedge\{\mathbb{V} \in TN(X) | \omega^\top(\mathbb{V}) \geq \omega^\top(\mathbb{U})\} = \wedge\{\mathbb{V} \in TN(X) | \mathbb{V} \geq \mathbb{U}\} = \mathbb{U}$.
(2) By $\omega^\top$ preserving $\wedge$,

$$\omega^\top \circ \iota^\top(\mathcal{U}) = \omega^\top(\wedge\{\mathbb{V} | \mathbb{V} \in TN(X), \omega^\top(\mathbb{V}) \geq \mathcal{U}\} = \wedge\{\omega^\top(\mathbb{V}) | \mathbb{V} \in TN(X), \omega^\top(\mathbb{V}) \geq \mathcal{U}\} \geq \mathcal{U}.$$

□

From Theorem 4.13 and Theorem 4.20 we conclude the following:

Corollary 4.21. *If L is continuous, then the category* **TNS** *is embedded in the category* **SLNS** *as a simultaneously reflective and coreflective subcategory.*

Remark 4.22. For $L = ([0,1], \wedge)$, the well-known Lowen functors ω_L, ι_L [30] play an important role in the study of the connection between topological spaces and stratified L-topological spaces. Precisely, for a topological space (X, τ), a function $\lambda : X \longrightarrow L$ is called lower semicontinuous if

$$\forall x \in X, \lambda(x) = \bigvee_{A \in \mathscr{U}_\tau^x} \bigwedge_{y \in A} \lambda(y), \text{ where } \mathscr{U}_\tau^x \text{ is the neighborhood system associated with } \tau.$$

Then the family $\omega_L(\tau) := \{\lambda \in L^X | \lambda \text{ is a lower semicontinuous function}\}$ forms a stratified L-topology on X, and the correspondence $(X, \tau) \mapsto (X, \omega_L(\tau))$ defines a concrete functor from the category **Top** of topological spaces to the category **SLTop** of stratified L-topological spaces. Conversely, for a stratified L-topological space $(X, \mathcal{T})$, define $\iota_L(\mathcal{T})$ as the finest topology on X such that all $\lambda \in \mathcal{T}$ are lower semicontinuous. Then the correspondence $(X, \mathcal{T}) \mapsto (X, \iota_L(\mathcal{T}))$ defines a concrete functor from the category **SLTop** to the category **Top**. It has been known that the pair (ω_L, ι_L) is a Galois correspondence and ι_L is a left inverse of ω_L. Nowadays, these two functors have been extended by many scholars from different aspects [11, 17, 25, 40, 47]. In the following, we will show that the functor $\omega^\top \circ \omega$ considered in this paper can be regarded as an extension of the Lowen functor ω_L. Indeed, for a topological space (X, τ), note that

$$\forall \lambda \in L^X, [\omega_L \circ \omega](\mathscr{U}_\tau)^x(\lambda) = [\omega_L \circ \omega](\mathscr{U}_\tau^x)(\lambda) = \bigvee_{\mu \in \omega(\mathscr{U}_\tau^x)} S_X(\mu, \lambda) = (\text{by Lemma2.5}) \bigvee_{A \in \mathscr{U}_\tau^x} S_X(\top_A, \lambda) = \bigvee_{A \in \mathscr{U}_\tau^x} \bigwedge_{y \in A} \lambda(y).$$

In addition, it has been known in [23, Proposition 4.6] that there is a bijection between stratified L-topologies and stratified L-neighborhood systems on a set X. Furthermore, the stratified L-topology $\mathcal{T}$ associated with the stratified L-neighborhood systems $[\omega^\top \circ \omega](\mathscr{U}_\tau)$ is given by

$$\lambda \in \mathcal{T} \Longleftrightarrow \forall x \in X, [\omega^\top \circ \omega](\mathscr{U}_\tau)^x(\lambda) = \lambda(x) \Longleftrightarrow \forall x \in X, \bigvee_{A \in \mathscr{U}_\tau^x} \bigwedge_{y \in A} \lambda(y) = \lambda(x) \Longleftrightarrow \lambda \in \omega_L(\tau).$$

This means that if we distinguish no (stratified L-)topological spaces and (stratified L-)topological neighborhood spaces, then $[\omega^\top \circ \omega]|_{\mathbf{Top}} = \omega_L$. Because $L = [0,1]$ is continuous, it follows by Theorem 3.23 and Theorem 4.20 that $(\omega^\top \circ \omega, \iota \circ \iota^\top)$ is a Galois correspondence and $\iota \circ \iota^\top$ is a left inverse of $\omega^\top \circ \omega$, and so $\iota \circ \iota^\top|_{\mathbf{SLTop}} = \iota_L$. Hence, we say that the functors $\omega^\top \circ \omega$ and $\iota \circ \iota^\top$ can be regarded as extensions of Lowen functors ω_L and ι_L.

5. Conclusion

In this paper, under a more general lattice-context, we discussed the categorical properties and categorical relationships about two types of lattice-valued neighborhood spaces. The main results can be summarized as below:

TNS and **SLNS** are all topological category over **Set**;

NS $\overset{r}{\hookrightarrow}$ **TNS**; and when L is meet-continuous **TNS** $\overset{r}{\hookrightarrow}$ **SLNS**;

NS $\overset{r,c}{\hookrightarrow}$ **TNS** $\overset{r,c}{\hookrightarrow}$ **SLNS**, when L is continuous;

where r, c mean, respectively, reflective and coreflective.

In the future, we shall introduce a category of $\top$-neighborhood groups, which is expected to characterize precisely strong L-topological groups. The results in this paper will help us to establish the relationships between new category and related (lattice-valued) topological group categories.

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