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The Continuity and the Simplest Possible Expression of Inner Inverses of Linear Operators in Banach Space

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Abstract. The main topic of this paper is the relationship between the continuity and the simplest possible expression of inner inverses. We first provide some new characterizations for the simplest possible expression to be an inner inverse of the perturbed operator. Then we obtain the equivalence conditions on the continuity of the inner inverse. Furthermore, we prove that if $T_n \to T$ and the sequence of inner inverses $\{T_n^-\}$ is convergent, then T is inner invertible and we can find a succinct expression of the inner inverse of T_n, which converge to any given inner inverse T^-. This is very useful and convenient in applications.

1. Introduction and Preliminaries

Let X, Y be Banach spaces and $B(X, Y)$ denote all bounded linear operators from X into Y. For any operator $T \in B(X, Y)$, $N(T)$ denotes the null space and $R(T)$ denotes the range space respectively. I denotes the identity operator in Banach space.

Let $T \in B(X, Y)$ be invertible and $T^{-1} \in B(Y, X)$ be its inverse. It is well known that if $I + \delta T T^{-1}$ or $I + T^{-1}\delta T$ is invertible with $\delta T \in B(X, Y)$, then $I + \delta T T^{-1}$ and $I + T^{-1}\delta T$ are invertible (See the Proposition 2.1 in [5]) . Further, the perturbed operator $\overline{T} = T + \delta T$ is also invertible with

$$\overline{T}^{-1} = T^{-1}(I + \delta T T^{-1})^{-1} = (I + T^{-1}\delta T)^{-1}T^{-1}.$$

If T is not invertible, we can consider its inverse in the generalized sense. Recall that an operator $S \in B(Y, X)$ is said to be *an inner inverse* of T if $TST = T$ and *an outer inverse* if $STS = S$. If S is both an inner inverse and outer inverse of T, then S is called *a generalized inverse* of T and T is said to be generalized invertible. The inner inverse, outer inverse and generalized inverse of T is always denoted by T^-, $T^{\{2\}}$ and T^+, respectively. As is well known, the nonzero outer inverse of any nonzero bounded linear operator always exists, while the inner inverse or generalized inverse may not exist and it is not unique even if it exists [12].

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Let T have an inner inverse T^-. If $I+\delta TT^-$ or $I+T^-\delta T$ is invertible, then $I+\delta TT^-$ and $I+T^-\delta T$ are invertible [5], and it follows from $(I+T^-\delta T)T^- = T^-(I+\delta TT^-)$ that the operator

$$B = T^-(I+\delta TT^-)^{-1} = (I+T^-\delta T)^{-1}T^-$$

is well defined. It is natural to ask whether B is an inner inverse of $\overline{T} = T+\delta T$. Such expression is called the simplest possible expression and the similar problem has been studied for the outer inverse in [9, 14, 19], generalized inverse in [2, 3, 6, 7, 11, 13, 16], Moore-Penrose inverse in [4, 7, 18, 20], group inverse in [1, 10, 15, 17, 20] and Drazin inverse in [1, 15]. For example, in 1993, Nashed and Chen [14] showed that $T^+(I+\delta TT^+)^{-1} = (I+T^+\delta T)^{-1}T^+$ is a generalized inverse of $\overline{T}$ if the small perturbation δT satisfies the "Nashed condition", i.e.,

$$(I+\delta TT^+)^{-1}\overline{T} \quad \text{maps} \quad N(T) \quad \text{into} \quad R(T).$$

It is noteworthy that their proof is strongly dependent on the following fact.

Theorem 1.1. [14] *Let $T \in B(X,Y)$ be with an outer inverse $T^{\{2\}} \in B(Y,X)$. Assume that $\delta T \in B(X,Y)$ satisfies $\|\delta TT^{\{2\}}\| < 1$, then*

$$B = T^{\{2\}}(I+\delta TT^{\{2\}})^{-1} = (I+T^{\{2\}}\delta T)^{-1}T^{\{2\}}$$

and B is an outer inverse of $\overline{T} = T+\delta T$ with $R(B) = R(T^{\{2\}})$ and $N(B) = N(T^{\{2\}})$.

Theorem 1.1 says that the outer inverse is stable and continuous, i.e., the simplest possible expression $T^{\{2\}}(I+\delta TT^{\{2\}})^{-1} = (I+T^{\{2\}}\delta T)^{-1}T^{\{2\}}$ is an outer inverse of the perturbed operator $\overline{T}$ and obviously, $\overline{T}^{\{2\}} \to T^{\{2\}}$ as $\delta T \to 0$. Unlike the outer inverse, the inner inverse is not stable or continuous, that is, the perturbed operator $\overline{T}$ may not be inner invertible and even if $\overline{T}$ is inner invertible, the inner inverse $\overline{T}^-$ may not be convergent to T^- as $\delta T \to 0$ (see [12] and Example 2.1). Therefore, the simplest possible expression

$$T^-(I+\delta TT^-)^{-1} = (I+T^-\delta T)^{-1}T^-$$

may not necessarily be an inner inverse of $\overline{T} = T+\delta T$. In [8], the authors gave some necessary and sufficient conditions for the simplest possible expression to be an inner inverse of $\overline{T}$. For convenience, we list the theorem.

Theorem 1.2. [8] *Let $T^- \in B(Y,X)$ be an inner inverse of $T \in B(X,Y)$. If $I+\delta TT^-$ is invertible with $\delta T \in B(X,Y)$, then the following statements are equivalent:*
(1) *$B = T^-(I+\delta TT^-)^{-1} = (I+T^-\delta T)^{-1}T^-$ is an inner inverse of $\overline{T}$;*
(2) *$R(\overline{T}) \cap N(T^-) = \{0\}$ and $(I+\delta TT^-)^{-1}\overline{T}N(T) \subseteq N(T^-TT^- - T^-)$;*
(3) *$R(\overline{T}) \cap N(TT^-) = \{0\}$;*
(4) *$(I+\delta TT^-)^{-1}\overline{T}N(T) \subseteq R(T)$;*
(5) *$(I+\delta TT^-)^{-1}R(\overline{T}) = R(T)$;*
(6) *$(I+T^-\delta T)^{-1}N(T) = N(\overline{T})$.*

As we know, it remains a problem whether the continuity of inner inverse can imply that the simplest possible expression $B = T^-(I+\delta TT^-)^{-1} = (I+T^-\delta T)^{-1}T^-$ is an inner inverse of $\overline{T} = T+\delta T$. We shall give a complete solution to this problem in this paper. First, from the point of view of the topological direct sum decompositions, we derive some new and useful characteristics for the simplest possible expression to be an inner inverse of the perturbed operator. Based on them, we prove that the continuity of inner inverse is equivalent to the statement that the simplest possible expression is an inner inverse of the perturbed operator. Thus we can conclude that if there exists a convergent sequence of inner inverses $\{T_n^-\}$, then T is inner invertible and for any inner inverse T^- of T, the operator $B_n = T^-[I+(T_n-T)T^-]^{-1} = [I+T^-(T_n-T)]^{-1}T^-$ is an inner inverse of T_n for all sufficiently large n. This is very useful and convenient in applications.

2. Main Results

The following example shows that the simplest possible expression may not be the inner inverse of T_n and even if T_n is inner invertible, maybe there is no convergent sequence of inner inverse $\{T_n^-\}$.

Example 2.1. Let

$$T = \begin{bmatrix} 1 & 0 & 2 \\ -1 & -1 & 1 \\ 3 & 2 & 0 \end{bmatrix} \quad \text{and} \quad T_n = \begin{bmatrix} \frac{n+2}{n+1} & 0 & 2 \\ -1 & -1 & 1 \\ 3 & 2 & 0 \end{bmatrix},$$

then T_n is invertible and T_n has a unique inner inverse

$$T_n^- = \begin{bmatrix} n & -2n & -n \\ -\frac{3n}{2} & 3n & \frac{3n+1}{2} \\ -\frac{n}{2} & n+1 & \frac{n+1}{2} \end{bmatrix},$$

which is divergent. Hence for any inner inverse T^- of T, the simplest possible expression

$$B_n = T^-[I + (T_n - T)T^-]^{-1} = [I + T^-(T_n - T)]^{-1}T^-$$

is not an inner inverse of T_n.

To prove our main results, we need the following lemmas.

Lemma 2.2. *Let X, Y be linear spaces and X_1, X_2 be two linear subspaces of X. Let T be an invertible linear operator from X onto Y. Then*

$$X = X_1 \oplus X_2$$

if and only if

$$Y = TX_1 \oplus TX_2$$

Furthermore, if X, Y are Banach spaces, X_1, X_2 are linear closed subspaces of X, then above two direct sums are topological direct ones.

Proof. The proof is straightforward, we omit it. □

Lemma 2.3. *Let X and Y be Banach spaces and $T^- \in B(Y, X)$ be an inner inverse of $T \in B(X, Y)$. Let $I + \delta TT^-$ or $I + T^-\delta T$ be invertible with $\delta T \in B(X, Y)$ and*

$$B = T^-(I + \delta TT^-)^{-1} = (I + T^-\delta T)^{-1}T^-,$$

then

$$B\overline{T}T^-T = T^-T \quad \text{and} \quad TT^-\overline{T}B = TT^-.$$

Further,

$$R(T^-T) \subseteq R(B\overline{T}) \quad \text{and} \quad N(\overline{T}B) \subseteq N(TT^-).$$

Proof. It is obvious to see that

$$\begin{aligned} B\overline{T}T^-T &= T^-(I + \delta TT^-)^{-1}(T + \delta T)T^-T. \\ &= T^-(I + \delta TT^-)^{-1}(T + \delta TT^-T) \\ &= T^-(I + \delta TT^-)^{-1}(I + \delta TT^-)T \\ &= T^-T \end{aligned}$$

and

$$\begin{aligned} TT^{-}\overline{T}B &= TT^{-}(T+\delta T)(I+T^{-}\delta T)^{-1}T^{-} \\ &= (T+TT^{-}\delta T)(I+T^{-}\delta T)^{-1}T^{-} \\ &= T(I+T^{-}\delta T)(I+T^{-}\delta T)^{-1}T^{-} \\ &= TT^{-}. \end{aligned}$$

Then

$$R(T^{-}T) = R(B\overline{T}T^{-}T) \subseteq R(B\overline{T})$$

and

$$N(\overline{T}B) \subseteq N(TT^{-}\overline{T}B) = N(TT^{-}).$$

□

From the point of view of the topological direct sums decompositions, we shall give some new characteristics for the simplest possible expression is an inner inverse of $T+\delta T$ in our main theorem.

Theorem 2.4. *Let X and Y be Banach spaces and $T^{-} \in B(Y,X)$ be an inner inverse of $T \in B(X,Y)$. If $I+\delta TT^{-}$ or $I+T^{-}\delta T$ is invertible with $\delta T \in B(X,Y)$, then the following statements are equivalent:*
(1) $B = T^{-}(I+\delta TT^{-})^{-1} = (I+T^{-}\delta T)^{-1}T^{-}$ *is an inner inverse of* $\overline{T} = T+\delta T$;
(2) $X = N(\overline{T}) \oplus R(BT)$;
(3) $X = N(\overline{T}) + R(BT)$;
(4) $R(\overline{T}) = R(\overline{T}T^{-})$ *and* $N(TT^{-}) = N(\overline{T}B)$;
(5) $R(\overline{T}) \subseteq R(\overline{T}T^{-})$ *and* $N(TT^{-}) \subseteq N(\overline{T}B)$;
(6) $N(T^{-}\overline{T}) = N(\overline{T})$ *and* $R(B\overline{T}) = R(T^{-}T)$;
(7) $N(T^{-}\overline{T}) \subseteq N(\overline{T})$ *and* $R(B\overline{T}) \subseteq R(T^{-}T)$;
(8) $X = N(\overline{T}) \oplus R(T^{-}T)$;
(9) $X = N(\overline{T}) + R(T^{-}T)$;
(10) $Y = R(\overline{T}) \oplus N(TT^{-})$.

Proof. (1) ⇒ (2). If (1) holds, then by Theorem 1.2, $(I+T^{-}\delta T)^{-1}N(T) = N(\overline{T})$. It follows from Lemma 2.2 and $X = N(T) \oplus R(T^{-}T)$ that we can get

$$(I+T^{-}\delta T)^{-1}X = (I+T^{-}\delta T)^{-1}N(T) \oplus (I+T^{-}\delta T)^{-1}R(T^{-}T),$$

i.e.,

$$\begin{aligned} X &= (I+T^{-}\delta T)^{-1}N(T) \oplus (I+T^{-}\delta T)^{-1}R(T^{-}T) \\ &= (I+T^{-}\delta T)^{-1}N(T) \oplus R[(I+T^{-}\delta T)^{-1}T^{-}T] \\ &= (I+T^{-}\delta T)^{-1}N(T) \oplus R(BT) \end{aligned}$$

Then $X = N(\overline{T}) \oplus R(BT)$.

(2) ⇒ (3). Obviously.

(3) ⇒ (1). From Theorem 1.2, we only need to show $(I+T^{-}\delta T)^{-1}N(T) = N(\overline{T})$. For any $x \in N(\overline{T})$, we have

$$(I+T^{-}\delta T)x = x + T^{-}\overline{T}x - T^{-}Tx = x - T^{-}Tx.$$

Then $T(I+T^{-}\delta T)x = 0$, i.e., $(I+T^{-}\delta T)x \in N(T)$. Hence $(I+T^{-}\delta T)N(\overline{T}) \subseteq N(T)$. By $X = N(T) \oplus R(T^{-}T)$, we can get $N(T) \cap R(T^{-}T) = \{0\}$ and so

$$(I+T^{-}\delta T)N(\overline{T}) \cap R(T^{-}T) = \{0\}$$

Since $X = N(\overline{T}) + R(BT)$, we can obtain

$$\begin{aligned} X &= (I + T^{-}\delta T)N(\overline{T}) + (I + T^{-}\delta T)R(BT) \\ &= (I + T^{-}\delta T)N(\overline{T}) + R[(I + T^{-}\delta T)BT] \\ &= (I + T^{-}\delta T)N(\overline{T}) + R[(I + T^{-}\delta T)(I + T^{-}\delta T)^{-1}T^{-}T] \\ &= (I + T^{-}\delta T)N(\overline{T}) + R(T^{-}T) \end{aligned}$$

and

$$X = (I + T^{-}\delta T)N(\overline{T}) \oplus R(T^{-}T).$$

Thus, by $X = N(T) \oplus R(T^{-}T)$ and Lemma 2.1 in [8], we have

$$(I + T^{-}\delta T)N(\overline{T}) = N(T).$$

(1) ⇒ (4). If B is an inner inverse of $\overline{T}$, then

$$R(\overline{T}) = R(\overline{T}B) = R(\overline{T}T^{-}(I + \delta TT^{-})^{-1}) = R(\overline{T}T^{-}).$$

On the other hand, for any $x \in N(TT^{-})$, then $T^{-}x \in N(T)$, and by (1) ⇔ (6) in Theorem 1.2, we have

$$Bx = (I + T^{-}\delta T)^{-1}T^{-}x \in (I + T^{-}\delta T)^{-1}N(T) = N(\overline{T}),$$

i.e. $x \in N(\overline{T}B)$. This implies $N(TT^{-}) \subseteq N(\overline{T}B)$. By Lemma 2.3, we can get

$$N(TT^{-}) = N(\overline{T}B).$$

(4) ⇒ (5). Obviously.

(5) ⇒ (1). We first prove that $\overline{T}B$ is an idempotent operator. By Lemma 2.3, $TT^{-}\overline{T}B = TT^{-}$. then $TT^{-}(I - \overline{T}B) = 0$ and $R(I - \overline{T}B) \subseteq N(TT^{-})$. Since $N(TT^{-}) \subseteq N(\overline{T}B)$, we can get

$$\overline{T}B(I - \overline{T}B) = 0,$$

i.e., $\overline{T}B$ is idempotent and so $R(\overline{T}B) = N(I - \overline{T}B)$. Hence

$$R(\overline{T}) \subseteq R(\overline{T}T^{-}) = \overline{T}R(T^{-}) = \overline{T}R(B) = R(\overline{T}B) = N(I - \overline{T}B).$$

Thus $(I - \overline{T}B)\overline{T} = 0$, i.e., B is an inner inverse of $\overline{T}$.

(1) ⇒ (6). If B is an inner inverse of $\overline{T}$, then

$$N(\overline{T}) = N(B\overline{T}) = N[(I + T^{-}\delta T)^{-1}T^{-}\overline{T}] = N(T^{-}\overline{T}).$$

By Theorem 1.2, we get $(I + \delta TT^{-})^{-1}R(\overline{T}) = R(T)$, and

$$\begin{aligned} R(B\overline{T}) &= BR(\overline{T}) \\ &= B(I + \delta TT^{-})R(T) \\ &= T^{-}(I + \delta TT^{-})^{-1}(I + \delta TT^{-})R(T) \\ &= T^{-}R(T) \\ &= R(T^{-}T) \end{aligned}$$

(6) ⇒ (7). Obviously.

(7) ⇒ (1). We first claim that $B\overline{T}$ is idempotent. In fact, it follows from Lemma 2.3 that $(I - B\overline{T})T^{-}T = 0$. If $R(B\overline{T}) \subseteq R(T^{-}T)$, then $R(B\overline{T}) \subseteq N(I - B\overline{T})$ and

$$(I - B\overline{T})B\overline{T} = 0.$$

i.e., $B\overline{T}$ is idempotent and so $R(I - B\overline{T}) = N(B\overline{T})$. If $N(T^-\overline{T}) \subseteq N(\overline{T})$, then

$$N(B\overline{T}) = N[(I + T^-\delta T)^{-1}T^-\overline{T}] = N(T^-\overline{T}) \subseteq N(\overline{T}).$$

Hence

$$\overline{T}(I - B\overline{T}) = 0,$$

i.e. B is an inner inverse of $\overline{T}$.

(1) ⇒ (8). If B is an inner inverse of $\overline{T}$, then by Theorem 1.2, $(I + \delta TT^-)^{-1}R(\overline{T}) = R(T)$. So $R(\overline{T}) = (I + \delta TT^-)R(T)$ and

$$R(B\overline{T}) = BR(\overline{T}) = B(I + \delta TT^-)R(T) = T^-R(T) = R(T^-T).$$

Hence

$$X = N(\overline{T}) \oplus R(B\overline{T}) = N(\overline{T}) \oplus R(T^-T).$$

(8) ⇒ (9). Obviously.

(9) ⇒ (1). If $X = N(\overline{T}) + R(T^-T)$, then by Lemma 2.3,

$$R(B\overline{T}) = B\overline{T}X = B\overline{T}[N(\overline{T}) + R(T^-T)] = B\overline{T}R(T^-T) = R(B\overline{T}T^-T) = R(T^-T),$$

i.e., $R(B\overline{T}) = R(T^-T)$. Similar to (7) ⇒ (1), we can prove that $B\overline{T}$ is idempotent. Then for any $x \in X$, we can find $x_1 \in N(\overline{T})$ and $x_2 \in R(T^-T) = R(B\overline{T})$ such that

$$(I - B\overline{T})x = x_1 + x_2.$$

Hence $x_2 = B\overline{T}x_2 = B\overline{T}[(I - B\overline{T})x - x_1] = 0$ and $\overline{T}(I - B\overline{T})x = \overline{T}x_1 = 0$. This indicates that B is an inner inverse of $\overline{T}$.

(1) ⇒ (10). If B is an inner inverse of $\overline{T}$, then

$$Y = R(\overline{T}) \oplus N(\overline{T}B)$$

and by (1) ⇒ (4), $N(TT^-) = N(\overline{T}B)$. Hence

$$Y = R(\overline{T}) \oplus N(TT^-).$$

(10) ⇒ (1). If $Y = R(\overline{T}) \oplus N(TT^-)$, then

$$R(\overline{T}) \cap N(TT^-) = \{0\}.$$

Using Theorem 1.2, we can conclude that B is an inner inverse of $\overline{T}$. □

Remark 2.5. *Perhaps the main difference between the inner inverse and the outer inverse is that* $B = T^{\{2\}}(I+\delta TT^{\{2\}})^{-1}$ *is always an outer inverse of* $\overline{T}$ *and both* $B\overline{T}$ *and* $\overline{T}B$ *are idempotent, whereas* $T^-(I + \delta TT^-)^{-1}$ *may not be an inner inverse of* $\overline{T}$ *and neither* $T^-(I + \delta TT^-)^{-1}\overline{T}$ *nor* $\overline{T}T^-(I + \delta TT^-)^{-1}$ *is necessarily idempotent.*

Now we can use Theorem 1.1 and Theorem 2.4 to consider the generalized inverse.

Corollary 2.6. *Let X and Y be Banach spaces and $T^+ \in B(Y, X)$ be a generalized inverse $T \in B(X, Y)$. If $I + \delta TT^+$ or $I + T^+\delta T$ is invertible with $\delta T \in B(X, Y)$, then the following statements are equivalent:*
(1) $B = T^+(I + \delta TT^+)^{-1} = (I + T^+\delta T)^{-1}T^+$ *is a generalized inverse of* $\overline{T}$;
(2) $X = N(\overline{T}) \oplus R(T^+)$;
(3) $X = N(\overline{T}) + R(T^+)$;
(4) $R(\overline{T}) = R(\overline{T}T^+)$;
(5) $R(\overline{T}) \subseteq R(\overline{T}T^+)$;
(6) $N(T^+\overline{T}) = N(\overline{T})$;
(7) $N(T^+\overline{T}) \subseteq N(\overline{T})$;
(8) $Y = R(\overline{T}) \oplus N(T^+)$.

Proof. If T^+ is a generalized inverse of T, then by Theorem 1.1, B is an outer inverse of $\overline{T}$. Hence

$$N(TT^+) = N(T^+) = N(B) = N(\overline{T}B)$$

and

$$R(T^+T) = R(T^+) = R(B) = R(B\overline{T}).$$

In addition,

$$\begin{aligned} R(BT) &= R[(I + T^+\delta T)^{-1}T^+T] \\ &= (I + T^+\delta T)^{-1}R(T^+T) \\ &= (I + T^+\delta T)^{-1}R(T^+) \\ &= R(B) = R(T^+). \end{aligned}$$

Hence $R(B\overline{T}) = R(BT) = R(T^+)$. By Theorem 2.4, we can get what we wanted. □

Remark 2.7. *From Corollary 2.6, we can recover some results in* [6, 8, 9, 11, 19].

Next, we show that the two equations in (4) and (6) of Theorem 2.4 are independent.

Example 2.8. Let

$$T = \begin{bmatrix} 0 & -1 & -2 \\ -1 & 2 & 3 \\ -2 & 3 & 4 \end{bmatrix} \quad \text{and} \quad \delta T = \frac{1}{60}\begin{bmatrix} 30 & 20 & 0 \\ 10 & 15 & -30 \\ 40 & 12 & 20 \end{bmatrix},$$

then we can get

$$\overline{T} = T + \delta T = \frac{1}{60}\begin{bmatrix} 30 & -40 & -120 \\ -50 & 135 & 150 \\ -80 & 192 & 260 \end{bmatrix} \text{ and } T^- = \frac{1}{6}\begin{bmatrix} 0 & 11 & -8 \\ 11 & 20 & -9 \\ -8 & -9 & 4 \end{bmatrix}$$

is an inner inverse of T. Noting that $\overline{T}$ and T^- are invertible, we know

$$\mathrm{Rank}(\overline{T}T^-) = \mathrm{Rank}(T^-\overline{T}) = 3$$

and hence

$$R(\overline{T}) = R(\overline{T}T^-), \quad N(T^-\overline{T}) = N(\overline{T}).$$

However,

$$\mathrm{Rank}(TT^-) = \mathrm{Rank}(T^-T) = \mathrm{Rank}(T) = 2, \quad \mathrm{Rank}(\overline{T}B) = \mathrm{Rank}(B\overline{T}) = 3.$$

Then

$$N(TT^-) \neq N(\overline{T}B) \quad \text{and} \quad R(B\overline{T}) \neq R(T^-T).$$

Example 2.9. Let

$$T = \begin{bmatrix} 1 & -2 & 0 \\ -3 & 1 & 2 \\ 0 & -5 & 2 \end{bmatrix}, \; T^- = \frac{1}{3}\begin{bmatrix} 8 & 3 & -1 \\ 16 & 6 & -5 \\ 40 & 15 & -11 \end{bmatrix} \text{ and } \delta T = \frac{1}{12}\begin{bmatrix} 12 & 6 & 0 \\ 0 & 3 & -6 \\ 4 & 6 & 0 \end{bmatrix}.$$

Then we can get

$$\overline{T} = \frac{1}{12}\begin{bmatrix} 24 & -18 & 0 \\ -36 & 15 & 18 \\ 4 & -54 & 24 \end{bmatrix} \text{ and } I + T^-\delta T = \frac{1}{36}\begin{bmatrix} 128 & 51 & -18 \\ 172 & 120 & -36 \\ 436 & 219 & -54 \end{bmatrix}.$$

Hence, $\overline{T}$ and $I + T^-\delta T$ are invertible and

$$\mathrm{Rank}(\overline{T}B) = \mathrm{Rank}(B\overline{T}) = \mathrm{Rank}(T^-) = 2,$$

and

$$\mathrm{Rank}(TT^-) = \mathrm{Rank}(T^-T) = \mathrm{Rank}(T) = 2.$$

Therefore

$$N(TT^-) = N(\overline{T}B) \quad \text{and} \quad R(B\overline{T}) = R(T^-T).$$

However, $\mathrm{Rank}(\overline{T}T^-) = \mathrm{Rank}(T^-\overline{T}) = \mathrm{Rank}(T^-) = 2$,

$$R(\overline{T}) \neq R(\overline{T}T^-) \quad \text{and} \quad N(T^-\overline{T}) \neq N(\overline{T}).$$

For the convenience of applications, we shall give some equivalent conditions on (4) and (6) in Theorem 2.4.

Theorem 2.10. *Let X and Y be Banach spaces and $T^- \in B(Y, X)$ be an inner inverse of $T \in B(X, Y)$. Assume that $I + \delta TT^-$ or $I + T^-\delta T$ is invertible with $\delta T \in B(X, Y)$ and $B = T^-(I + \delta TT^-)^{-1} = (I + T^-\delta T)^{-1}T^-$. Then the following statements hold:*
(1) $R(\overline{T}) = R(\overline{T}T^-) \Leftrightarrow \overline{T}N(T) \subset R(\overline{T}T^-)$;
(2) $N(TT^-) = N(\overline{T}B) \Leftrightarrow R(T^-TT^- - T^-) \subset (I + T^-\delta T)N(\overline{T})$;
(3) $N(\overline{T}) = N(T^-\overline{T}) \Leftrightarrow R(\overline{T}) \cap N(T^-) = \{0\}$;
(4) $R(T^-T) = R(B\overline{T}) \Leftrightarrow (I + \delta TT^-)^{-1}R(\overline{T}) \subset N(T^-TT^- - T^-)$.

Proof. (1) If $R(\overline{T}) = R(\overline{T}T^-)$, then

$$\overline{T}N(T) \subset R(\overline{T}) = R(\overline{T}T^-).$$

Conversely, if $\overline{T}N(T) \subset R(\overline{T}T^-)$, then

$$R(\overline{T}) = \overline{T}X = \overline{T}[N(T) + R(T^-T)] \subset R(\overline{T}T^-) + R(\overline{T}T^-T) = R(\overline{T}T^-).$$

Hence

$$R(\overline{T}) = R(\overline{T}T^-).$$

(2) Since TT^- is an idempotent operator and Lemma 2.3, we get

$$N(\overline{T}B) \subset N(TT^-) = R(I - TT^-).$$

Therefore, $N(TT^-) = N(\overline{T}B)$

$$\begin{aligned}
&\Leftrightarrow\ R(I - TT^-) \subset N(\overline{T}B)\\
&\Leftrightarrow\ \overline{T}B = \overline{T}BTT^-\\
&\Leftrightarrow\ \overline{T}(I + T^-\delta T)^{-1}T^- = \overline{T}(I + T^-\delta T)^{-1}T^-TT^-\\
&\Leftrightarrow\ \overline{T}(I + T^-\delta T)^{-1}(T^-TT^- - T^-) = 0\\
&\Leftrightarrow\ (I + T^-\delta T)^{-1}R(T^-TT^- - T^-) \subset N(\overline{T})\\
&\Leftrightarrow\ R(T^-TT^- - T^-) \subset (I + T^-\delta T)N(\overline{T}).
\end{aligned}$$

(3) Assume $N(T^-\overline{T}) = N(\overline{T})$, then for any $y \in R(\overline{T}) \cap N(T^-)$, there exists $x \in X$ such that $y = \overline{T}x$ and $T^-y = 0$, i.e. $T^-\overline{T}x = 0$. If $N(\overline{T}) = N(T^-\overline{T})$, then $x \in N(\overline{T})$ and $y = \overline{T}x = 0$. This implies

$$R(\overline{T}) \cap N(T^-) = \{0\}.$$

Conversely, if $R(\overline{T}) \cap N(T^-) = \{0\}$, then for any $x \in N(T^-\overline{T})$, i.e. $T^-\overline{T}x = 0$, hence $\overline{T}x \in N(T^-) \cap R(\overline{T}) = \{0\}$. This indicates $N(T^-\overline{T}) \subseteq N(\overline{T})$. Obviously, $N(\overline{T}) \subseteq N(T^-\overline{T})$. Thus

$$N(T^-\overline{T}) = N(\overline{T}).$$

(4) $R(T^-T) = R(B\overline{T})$

$\Leftrightarrow$ $R(B\overline{T}) \subset R(T^-T) = N(I - T^-T)$

$\Leftrightarrow$ $(I - T^-T)B\overline{T} = 0$

$\Leftrightarrow$ $B\overline{T} = T^-TB\overline{T}$

$\Leftrightarrow$ $T^-(I + \delta TT^-)^{-1}\overline{T} = T^-TT^-(I + \delta TT^-)^{-1}\overline{T}$

$\Leftrightarrow$ $(T^-TT^- - T^-)(I + \delta TT^-)^{-1}\overline{T} = 0$

$\Leftrightarrow$ $(I + \delta TT^-)^{-1}R(\overline{T}) \subset N(T^-TT^- - T^-)$.

□

Next we shall consider the continuity of the inner inverse and prove that the continuity implies that the simplest possible expression $T^-[I + (T_n - T)T^-]^{-1}$ is an inner inverse of T_n.

Theorem 2.11. *Let X and Y be Banach spaces and $T^- \in B(Y, X)$ be an inner inverse of $T \in B(X, Y)$. Let $T_n \in B(X, Y)$ satisfy $T_n \to T$, then the following statements are equivalent:*
(1) *for all sufficiently large n, T_n has an inner inverse T_n^- with*

$$T_n^- \to T^-;$$

(2) *there exists $N \in \mathbf{N}$, such that for all $n \geq N$,*

$$B_n = T^-[I + (T_n - T)T^-]^{-1} = [I + T^-(T_n - T)]^{-1}T^-$$

is an inner inverse of T_n;
(3) *there exists $N \in \mathbf{N}$, such that for all $n \geq N$,*

$$R(T_n) \subseteq R(T_nT^-) \quad and \quad N(TT^-) \subseteq N(T_nB_n).$$

Proof. From Theorem 2.4, we can obtain (2) $\Leftrightarrow$ (3). It is obvious to see (2) $\Rightarrow$ (1). To complete the proof, we shall show (1) $\Rightarrow$ (3). In fact, if T_n has an inner inverse T_n^- with $T_n^- \to T^-$, then $T_nT_n^- \to TT^-$ and $T_n^-T_n \to T^-T$. Hence there exists $N \in \mathbf{N}$, such that for all $n \geq N$, both $I - T_nT_n^- + TT^-$, $I - T_n^-T_n + T^-T$ and $I + (T_n - T)T^-$ are invertible. Thus, we can get

$$R(T_n) = R[T_n(I - T_n^-T_n + T^-T)] = R(T_nT^-T) \subseteq R(T_nT^-)$$

and by Lemma 2.3,

$$(I - T_nT_n^- + TT^-)T_nB_n = TT^-T_nB_n = TT^-,$$

which means $N(TT^-) \subseteq N(T_nB_n)$. □

Example 2.12. *Let X be the Banach space l_1 and $T_n, T \in B(X)$ be defined by*

$$\begin{aligned} T_n(x_1, x_2, x_3, ...) &= (0, \frac{n+1}{n}x_2, \frac{n+1}{n}x_3, ...), \\ T(x_1, x_2, x_3, ...) &= (0, x_2, x_3, ...), \qquad (x_1, x_2, x_3, ...) \in l_1. \end{aligned}$$

Obviously, $T_n \to T$, T_n and T are inner invertible with

$$\begin{aligned} T_n^-(x_1, x_2, x_3, ...) &= \frac{n}{n+1}(2x_1, x_2, x_3, ...), \\ T^-(x_1, x_2, x_3, ...) &= (2x_1, x_2, x_3, ...), \qquad (x_1, x_2, x_3, ...) \in l_1. \end{aligned}$$

Noting $T_n^- \to T^-$, by Theorem 2.11, we can conclude that

$$B_n = T^-[I + (T_n - T)T^-]^{-1}$$

is an inner inverse of T_n.

In the following theorem, we should note that for any inner inverse T^- of T, the simplest possible expression $B_n = T^-[I + (T_n - T)T^-]^{-1} = [I + T^-(T_n - T)]^{-1}T^-$ is an inner inverse of T_n.

Theorem 2.13. *Let X and Y be Banach spaces and $T_n, T \in B(X,Y)$ satisfy $T_n \to T$. If T_n has an inner inverse $T_n^- \in B(Y,X)$ and the sequence $\{T_n^-\}$ is convergent, then T is inner invertible and for any inner inverse T^- of T, there exists $N \in \mathbf{N}$, such that for all $n \geq N$,*

$$B_n = T^-[I + (T_n - T)T^-]^{-1} = [I + T^-(T_n - T)]^{-1}T^-$$

is an inner inverse of T_n.

Proof. Let $T_n^- \to S \in B(Y,X)$, then we can take the limit in $T_nT_n^-T_n = T_n$ and get $TST = T$. This means that T is inner invertible. For any inner inverse T^- of T, there exists $N \in \mathbf{N}$, such that for all $n \geq N$, both $I - T_n^-T_n + T_n^-T_nT^-T$, $I - T_nT_n^- + TT^-T_nT_n^-$ and $I + (T_n - T)T^-$ are invertible. Thus, we can get

$$R(T_n) = R[T_n(I - T_n^-T_n + T_n^-T_nT^-T)] = R(T_nT^-T) \subseteq R(T_nT^-)$$

and by Lemma 2.3,

$$(I - T_nT_n^- + TT^-T_nT_n^-)T_nB_n = TT^-T_nB_n = TT^-.$$

This implies $N(TT^-) \subseteq N(T_nB_n)$ and it follows from Theorem 2.11 that B_n is an inner inverse of T_n. □

Since one of inner inverses of the idempotent operator is itself, we can obtain the following interesting result.

Corollary 2.14. *Let X be a Banach space and $P_n, P \in B(X)$ be idempotent operators on X. If $P_n \to P$, then for any inner inverse P^-, there exists $N \in \mathbf{N}$, such that for all $n \geq N$,*

$$B_n = P^-[I + (P_n - P)P^-]^{-1} = [I + P^-(P_n - P)]^{-1}P^-$$

is an inner inverse of P_n.

By Theorems 1.1, 2.11 and 2.13, we can obtain the following corollaries.

Corollary 2.15. *Let X and Y be Banach spaces and $T^+ \in B(Y,X)$ be a generalized inverse of $T \in B(X,Y)$. Let $T_n \in B(X,Y)$ satisfy $T_n \to T$, then the following statements are equivalent:*
(1) *for all sufficiently large n, T_n has a generalized inverse T_n^+ with*

$$T_n^+ \to T^+;$$

(2) *there exists $N \in \mathbf{N}$, such that for all $n \geq N$,*

$$B_n = T^+[I + (T_n - T)T^+]^{-1} = [I + T^+(T_n - T)]^{-1}T^+$$

is a generalized inverse of T_n.

Corollary 2.16. *Let X and Y be Banach spaces and $T_n, T \in B(X,Y)$ satisfy $T_n \to T$. If T_n has a generalized inverse $T_n^+ \in B(Y,X)$ and the sequence $\{T_n^+\}$ is convergent, then T is generalized invertible and for any generalized inverse T^+ of T, there exists $N \in \mathbf{N}$, such that for all $n \geq N$,*

$$B_n = T^+[I + (T_n - T)T^+]^{-1} = [I + T^+(T_n - T)]^{-1}T^+$$

is a generalized inverse of T_n.

3. Conclusions

As a continuation of our previous work [8], we have provided some new characterizations for the simplest possible expression $B = T^-(I + \delta TT^-)^{-1} = (I + T^-\delta T)^{-1}T^-$ to be an inner inverse of $\overline{T} = T + \delta T$. We also prove that the continuity of inner inverse is equivalent to that the simplest possible expression is an inner inverse of $\overline{T}$. It turns out that the same property is also enjoyed by the generalized inverse. Thus we can conclude that the instability of the generalized inverse focuses on the inner inverse.

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