Filomat 35:4 (2021), 1141–1154
https://doi.org/10.2298/FIL2104141W

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Existence of Positive Solutions for a Singular Nonlinear Semipositone Fractional Differential Equations With Parametric Dependence

Om Kalthoum Wanassi[a], Faten Toumi[b]

[a]*University of Monastir, Faculty of Sciences of Monastir, LR18ES17, 5019 Monastir, Tunisia.*
[b]*King Faisal University, College of Business Administration,*
P.O. 380 Ahsaa 31982, Saoudi Arabia.

Abstract.
We examine the existence and multiplicity of positive solutions for a class of nonlinear semipositone fractional differential equations involving integral boundary conditions. The results are obtained in terms of different intervals of the parameters by means of the Leray-Schauder and Guo-Krasnoselskii fixed point theorems. Examples are included to verify our main results.

1. Introduction

Fractional Calculus has been recently applied in various areas of engineering, science, finance, applied mathematics, and bio engineering. However, many researchers remain unaware of this field. Monographs [4, 8, 11, 17, 18] are excellent source for the theory and its applications.

Among all subjects, the existence of positive solutions of singular nonlinear semipositone fractional differential equations has been widely studied by many authors in recent years, see for example [2, 5–7, 9, 12–15, 19–24].
Namely, Luca and Tudorache [13] considered the following system

$$\begin{cases} D^{\alpha}u(t) + \mu f(t, u(t), v(t)) = 0 \quad in \ \ (0,1), n-1 \leq \alpha \leq n \\ D^{\beta}v(t) + \lambda g(t, u(t), v(t)) = 0 \quad in \ \ (0,1), m-1 \leq \beta \leq m \\ u^{(j)}(0) = 0, \ \ 0 \leq j \leq n-2, \ \ u(1) = \int_0^1 u(s)dH(s), \\ v^{(j)}(0) = 0, \ \ 0 \leq j \leq m-2, \ \ v(1) = \int_0^1 v(s)dK(s), \end{cases}$$

where $f, g : [0,1] \times [0,+\infty), \times[0,+\infty) \longrightarrow (-\infty,+\infty)$ are sign-changing and continuous. They presented two intervals for parameters μ and λ such that the above problem has at least one positive solution. But the existence of positive solutions is not treated when the nonlinearities f and g are singular at $t = 0$ or/and $t = 1$.

2010 *Mathematics Subject Classification*. Primary 34A08; Secondary 34B27, 34B18.
Keywords. Nonlinear fractional differential equation, semipositone, positive solution, Green's function.
Received: 26 March 2020; Revised: 20 April 2020; Accepted: 04 May 2020
Communicated by Maria Alessandra Ragusa
Email addresses: omkalthoum.wannassi@fsm.rnu.tn (Om Kalthoum Wanassi), ftowmi@kfu.edu.sa (Faten Toumi)

Then, in [16], Henderson and Luca investigated the existence of positive solutions for a system of semipositone coupled fractional boundary value problems

$$\begin{cases} D^{\alpha}u(t) + \mu f(t,u(t),v(t)) = 0 \quad in \ (0,1), n-1 \leq \alpha \leq n \\ D^{\beta}v(t) + \lambda g(t,u(t),v(t)) = 0 \quad in \ (0,1), m-1 \leq \beta \leq m \\ u^{(j)}(0) = 0, \ \ 0 \leq j \leq n-2, \ \ u(1) = \int_0^1 v(s)dH(s), \\ v^{(j)}(0) = 0, \ \ 0 \leq j \leq m-2, \ \ v(1) = \int_0^1 u(s)dK(s), \end{cases}$$

for $f, g : (0,1) \times [0,+\infty), \times [0,+\infty) \longrightarrow (-\infty,+\infty)$ sign-changing continuous functions satisfying $-p_1(t) \leq f(t,u,v) \leq \alpha_1(t)\beta_1(t,u,v)$ and $-p_2(t) \leq g(t,u,v) \leq \alpha_2(t)\beta_2(t,u,v)$ for all $t \in (0,1)$, $u, v \in [0,+\infty)$, with $p_i, \alpha_i \in C((0,1),[0,\infty))$ and $\beta_i \in C([0,1] \times [0,\infty) \times [0,\infty),[0,\infty))$, $0 < \int_0^1 p_i(s)d(s) < \infty$, $0 < \int_0^1 \alpha_i(s)d(s) < \infty$, $i = 1, 2$.

More recently, in [19], Toumi and Wanassi discussed, in the scalar case, the existence of positive solution for the following problem

$$\begin{cases} D^{\alpha}u(t) + \mu f(t,u(t)) = 0, \qquad t \in (0,1), \\ u^{(j)}(0) = 0, \ \ 0 \leq j \leq n-2, \ u(1) = \lambda \int_0^1 u(s)ds, \end{cases} \tag{1}$$

where $f : (0,1) \times [0,+\infty), \times [0,+\infty) \longrightarrow (-\infty,+\infty)$ is sing-changing continuous function which may be singular at $t = 0$ or/and $t = 1$ and satisfies $-p(t) \leq f(t,u) \leq q(t)g(t,u)$ with $p, q \in C((0,1),[0,\infty))$ and $g \in C([0,1] \times [0,\infty),[0,\infty))$. The authors derived a new condition on p and q such that the existence of positive solutions is proved. However, by developing asymptotic conditions on the nonlinearity f, they obtained sufficient conditions to confirm the existence of multiple solutions, solely for $\mu = 1$.

Motivated by the above cited works, the purpose of this paper focuses on the study of the existence of positive solutions for the following singular boundary value problem with fractional order involving semipositone nonlinearities

$$\begin{cases} D^{\alpha}u(t) + \mu_1 f(t,u(t),v(t)) = 0 \quad in \ (0,1), \ n-1 \leq \alpha \leq n \\ D^{\beta}v(t) + \mu_2 g(t,u(t),v(t)) = 0 \quad in \ (0,1), \ m-1 \leq \beta \leq m \\ u^{(j)}(0) = 0, \ \ 0 \leq j \leq n-2, \ \ u(1) = \lambda_1 \int_0^1 u(s)ds, \\ v^{(j)}(0) = 0, \ \ 0 \leq j \leq m-2, \ \ v(1) = \lambda_2 \int_0^1 v(s)ds, \end{cases} \tag{2}$$

depending on the real parameters $\mu_1, \mu_2 > 0$, where $n, m \in \mathbb{N}, n, m \geq 3, 0 < \lambda_1 < \alpha, 0 < \lambda_2 < \beta, D^{\delta}$ denotes the Riemann-Liouville derivative of order δ and $f, g \in C((0,1) \times [0,+\infty) \times [0,+\infty),(-\infty,+\infty))$ are sign-changing which may be singular at $t = 0$ or/and $t = 1$. In particular, we improve the existing results in the case when the nonlinear terms satisfy more general conditions than those given in [13, 16, 19]. Moreover, by using the positivity of the related Green's function, existence and multiplicity results are derived, through the well-known Leray-Schauder and Krasnoselskii fixed point theorems, from the construction of suitable cones on Banach spaces. Such a construction follows by using adequate properties of the associated Green's function.

The paper is organized as follows. In Section 2 we recall some properties of the Green's function and lemmas which are needed later. Section 3 is devoted to establish existence of one or two positive solutions for (2). In the last Section, some examples are given to illustrate our main results.

2. Preliminaries

In this section, we present the main tools that we will use throughout the paper.

Definition 2.1. *The Riemann-Liouville fractional integral of order $\alpha > 0$ for a measurable function $f : (0,+\infty) \to \mathbb{R}$ is defined as*

$$I^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)ds, \ t > 0,$$

where Γ is the Euler Gamma function, provided that the right-hand side is pointwise defined on $(0,+\infty)$.

Definition 2.2. *The Riemann-Liouville fractional derivative of order $\alpha > 0$ for a measurable function $f : (0,+\infty) \to \mathbb{R}$ is defined as*

$$D^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \Big(\frac{d}{dt}\Big)^n \int_0^t (t-s)^{n-\alpha-1} f(s)ds = \Big(\frac{d}{dt}\Big)^n I^{n-\alpha} f(t),$$

provided that the right-hand side is pointwise defined on $(0,+\infty)$. Here $n = [\alpha]+1$, $[\alpha]$ denotes the integer part of the real number α.

In [2, 3, 19], a careful study of the linear problem has done and, in particular, of Green's function. In the same way, we continue and generalize our study based on this function. To this end, we recall the explicit expression of the Green's function related to problem (2) and its positive properties which usually are the basic tool in the construction of the cone and the discussion of positive solutions of the considered problem.

Lemma 2.3. *([3]) Let $n \geq 3$, $n-1 < \alpha \leq n$ and $\lambda \in (0,\alpha)$. Let $y \in C([0,1])$. Then the boundary value problem*

$$\begin{cases} D^{\alpha} u(t) + y(t) = 0 \text{ in } (0,1), \\ u^{(j)}(0) = 0, \ \ 0 \leq j \leq n-2, \ u(1) = \lambda \int_0^1 u(s)ds, \end{cases} \tag{3}$$

has *a unique solution*

$$u(t) = \int_0^1 G_{\alpha,\lambda}(t,s) y(s)ds,$$

where $G_{\alpha,\lambda}(t,s)$ is the Green function given by

$$G_{\alpha,\lambda}(t,s) = \frac{t^{\alpha-1}(1-s)^{\alpha-1}(\alpha-\lambda+\lambda s) - (\alpha-\lambda)((t-s)^+)^{\alpha-1}}{(\alpha-\lambda)\Gamma(\alpha)}, \tag{4}$$

for all $t,s \in [0,1]$, with $(t-s)^+ = \max(t-s,0)$, $s,t \in [0,1]$.

The positive properties of the Green's function will be of fundamental importance in many of our arguments. Hence, we state the following proposition.

Proposition 2.4. *([3]) Let $n-1 < \alpha \leq n$, $n \geq 3$ and $\lambda \in (0,\alpha)$. Then $G_{\alpha,\lambda}$ defined by (4) satisfies the following assertions:*

i) *$G_{\alpha,\lambda}$ is nonnegative continuous function on $[0,1]\times[0,1]$ and $G_{\alpha,\lambda}(t,s) > 0$, for all $t,s \in (0,1)$.*

ii) *$G_{\alpha,\lambda}(t,s) \leq \eta_\alpha K_\alpha(s)$ for all $t,s \in [0,1]$, where $K_\alpha(s) = \frac{s(1-s)^{\alpha-1}}{\Gamma(\alpha)}$ and $\eta_\alpha = \frac{\alpha}{\alpha-\lambda}$.*

iii) *$G_{\alpha,\lambda}(t,s) \leq \eta_\alpha t^{\alpha-1} k_\alpha(s)$ for all $t,s \in [0,1]$, where $k_\alpha(s) = \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}$.*

iv) *$G_{\alpha,\lambda}(t,s) \geq \eta_\alpha \nu^*_{\alpha,\lambda} t^{\alpha-1} K_\alpha(s)$, $\forall t,s \in [0,1]$, where $\nu^*_{\alpha,\lambda} = \frac{\lambda}{\alpha}$.*

v) *Let $\theta \in (0,\frac{1}{2})$, $s \in [0,1]$, then $\min_{t\in[\theta,1-\theta]} G_{\alpha,\lambda}(t,s) \geq \gamma_\alpha K_\alpha(s)$, where*

$$\gamma_\alpha = \Big(\frac{\theta}{\alpha-1} + \frac{\lambda}{\alpha-\lambda}\Big)\theta^{\alpha-1}.$$

In the sequel, we state a key lemma in which it is improved the results given in Lemma 2.3. More precisely, a weaker condition on the linear term was assumed to prove the existence and uniqueness of solutions of the linear problem.

Lemma 2.5. *([2]) Let $n \geq 3$, $n-1 < \alpha \leq n$ and $\lambda \in (0, \alpha)$.*
Let $(1-t)^{\alpha-1}p(t) \in C(0,1) \cap L^1(0,1)$. Then the boundary value problem

$$\begin{cases} D^\alpha w(t) + p(t) = 0 \text{ in } (0,1), \\ w^{(j)}(0) = 0, \ \ 0 \leq j \leq n-2, \ w(1) = \lambda \int_0^1 w(s)ds, \end{cases} \tag{5}$$

has a unique solution $w(t) = \int_0^1 G_{\alpha,\lambda}(t,s)p(s)ds \in C([0,1])$ satisfying

$$w(t) \leq \eta_\alpha \frac{t^{\alpha-1}}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1}|p(s)|ds, \ \ \forall t \in [0,1].$$

The proofs of our results are based on the nonlinear alternative of Leray-Schauder type and the Krasnoselskii's fixed point theorem.

Theorem 2.6. *([1]) Let X be a Banach space and $\Omega \subset X$ closed and convex. Assume U is an open subset of Ω with $0 \in U$ and let $S : \overline{U} \to \Omega$ be a completely continuous operator. Then either*

(1) S has a fixed point in $\overline{U}$, or

(2) there exists $u \in \partial U$ and $\delta \in (0,1)$ such that $u = \delta S(u)$.

Theorem 2.7. *([10]) Let P be the cone of a real Banach space E and Ω_1, Ω_2 two bounded open balls of E centered at the origin with $\overline{\Omega}_1 \subset \Omega_2$. Suppose that $T : P \cap (\overline{\Omega}_2 \backslash \Omega_1) \longrightarrow P$ is completely continuous operator such that either*

(i) $\|Tx\| \geq \|x\|$, $x \in P \cap \partial\Omega_1$ and $\|Tx\| \leq \|x\|$, $x \in P \cap \partial\Omega_2$, or

(ii) $\|Tx\| \leq \|x\|$, $x \in P \cap \partial\Omega_1$ and $\|Tx\| \geq \|x\|$, $x \in P \cap \partial\Omega_2$.

holds. Then the operator T has at least one fixed point in $P \cap (\overline{\Omega}_2 \backslash \Omega_1)$.

3. Main results

This section is devoted to give existence results for the nonlinear boundary value problem (2). We shall prove existence and multiplicity results for some suitable values of positive real parameters μ_1 and μ_2.
In the sequel we need the following notations. For a function $b \in C(0,1)$ and a real $\alpha > 0$, we denote

$$\varphi_\alpha(b) = \int_0^1 k_\alpha(s)|b(s)|ds,$$

where k_α is given by Proposition 2.4 (*iii*) and for each $\theta \in [0, \frac{1}{2})$, we denote

$$\phi_\alpha^\theta(b) = \int_\theta^{1-\theta} K_\alpha(s)|b(s)|ds,$$

where K_α is given by Proposition 2.4 (*ii*).
Hereinafter, we adopt combinations of the following hypotheses:

(H$_1$) The functions $f, g \in C([0,1]\times[0+\infty)\times[0,\infty), (-\infty,+\infty))$ and there exist functions $p_1, p_2 \in C([0,1],[0,+\infty))$ such that $p_1(t), p_2(t) \neq 0$ on any subinterval of $(0,1)$ and satisfying

$$f(t,u,v) \geq -p_1(t) \text{ and } g(t,u,v) \geq -p_2(t) \ \text{ for any } \ t \in [0,1] \text{ and } u, v \in [0,+\infty).$$

(H$_2$) The functions $f, g \in C((0,1) \times [0 + \infty) \times [0, \infty), (-\infty, +\infty))$, and there exist functions $p_1, p_2, q_1, q_2 \in C((0,1), [0, +\infty))$ such that $p_1(t), p_2(t), q_1(t), q_2(t) \neq 0$ on any subinterval of $(0,1)$ and $h_1, h_2 \in C([0,1] \times [0, +\infty) \times [0, +\infty), [0, +\infty))$ such that

$$-p_1(t) \leq f(t, u, v) \leq q_1(t)h_1(t, u, v), \quad -p_2(t) \leq g(t, u, v) \leq q_2(t)h_2(t, u, v)$$

for all $t \in (0,1)$, $u, v \in [0, +\infty)$.

(H$_3$) $f(t, 0, 0) > 0,\ g(t, 0, 0) > 0$ for all $t \in [0,1]$.

(H$_4$) $\varphi_\alpha(p_1),\ \varphi_\beta(p_2),\ \phi^0_\alpha(q_1),\ \phi^0_\beta(q_2) \in (0, +\infty)$.

Remark 3.1. *It is clear that (H$_4$) implies that*
$0 < \phi^\theta_\alpha(p_1), \phi^\theta_\beta(p_2), \phi^\theta_\alpha(q_1), \phi^\theta_\beta(q_2) < \infty$ *for each* $\theta \in [0, \frac{1}{2})$.

In fact, for $\theta \in [0, \frac{1}{2})$, we have

$$0 < \phi^\theta_\alpha(p_1) = \int_\theta^{1-\theta} K_\alpha(s)p_1(s)ds \leq \int_0^1 K_\alpha(s)p_1(s)ds \leq \int_0^1 k_\alpha(s)p_1(s)ds = \varphi_\alpha(p_1) < \infty.$$

Similarly for $\phi^\theta_\beta(p_2), \phi^\theta_\alpha(q_1), \phi^\theta_\beta(q_2)$.

In this work, we intend to prove the existence of positive solution (u, v) of problem (2), that is $(u, v) \in C([0,1]) \times C([0,1])$ satisfying problem (2) and $u(t) > 0$ *or* $v(t) > 0$ $\forall t \in (0,1]$. To overcome the difficulty of positivity, we consider an auxiliary (intermediary) boundary value problem which will help us, combining with the assumptions imposed on f and g, to obtain positive solutions of the nonlinear problem.
Therefore, consider the following auxiliary problem

$$\begin{cases} D^\alpha x(t) + \mu_1(f(t, [x(t) - w_1(t)]^*, [y(t) - w_2(t)]^*) + p_1(t)) = 0,\ 0 < t < 1, \\ D^\beta y(t) + \mu_2(g(t, [x(t) - w_1(t)]^*, [y(t) - w_2(t)]^*) + p_2(t)) = 0,\ 0 < t < 1, \\ x^{(j)}(0) = 0,\ 0 \leq j \leq n-2,\ \ x(1) = \lambda_1 \int_0^1 x(s)ds, \\ y^{(j)}(0) = 0,\ 0 \leq j \leq m-2,\ \ y(1) = \lambda_2 \int_0^1 y(s)ds, \end{cases} \tag{6}$$

where

$$[x(t) - w(t)]^* = \begin{cases} x(t) - w(t), & \text{if } x(t) - w(t) \geq 0 \\ 0, & \text{if } x(t) - w(t) < 0, \end{cases}$$

and (w_1, w_2) is the unique solution of the boundary value problem

$$\begin{cases} D^\alpha w_1(t) + \mu_1 p_1(t) = 0,\ \ 0 < t < 1, \\ D^\beta w_2(t) + \mu_2 p_2(t) = 0,\ \ 0 < t < 1, \\ w_1^{(j)}(0) = 0,\ \ 0 \leq j \leq n-2,\ \ w_1(1) = \lambda_1 \int_0^1 w_1(s)ds, \\ w_2^{(j)}(0) = 0,\ \ 0 \leq j \leq m-2,\ \ w_2(1) = \lambda_2 \int_0^1 w_2(s)ds. \end{cases}$$

By Lemma 2.5, w_1 and w_2 satisfy

$$w_1(t) \leq \mu_1 \eta_\alpha t^{\alpha-1} \varphi_\alpha(p_1),\ \ \forall t \in [0,1], \tag{7}$$

$$w_2(t) \leq \mu_2 \eta_\beta t^{\beta-1} \varphi_\beta(p_2),\ \ \forall t \in [0,1], \tag{8}$$

We shall prove that there exists solution (x, y) for the boundary value problem (6) such that $x(t) > w_1(t)$ or $y(t) > w_2(t)$ for any $t \in (0,1]$. Then, it is easy to verify that $(x - w_1, y - w_2)$ represents a positive solution of boundary value problem (2).
So, we will concentrate our study on the boundary value problem (6). We consider the Banach space

$E = C([0,1]) \times C([0,1])$ endowed with standard norm $\left\|(x,y)\right\| = \|x\| + \left\|y\right\|$ where $\|x\| = \max_{0\leq t\leq 1} |x(t)|$, $x \in E$. We define the cone P by

$$P = \{(x,y) \in E : x(t) \geq 0,\ y(t) \geq 0, x(t) \geq \nu t^{\alpha-1} \|x\|,\ y(t) \geq \nu t^{\beta-1} \left\|y\right\|, \forall t \in [0,1]\},$$

where $\nu = \min(\nu^*_{\alpha,\lambda_1}, \nu^*_{\beta,\lambda_2})$ and ν^*_{α,λ_1}, ν^*_{β,λ_2} are given by Proposition 2.4 (*iv*).
For $r > 0$, let

$$\Omega_r = \{(x,y) \in P : \left\|(x,y)\right\| < r\}.$$

Next, we define the operator $T : E \to E$ as follows

$$T(x,y)(t) := (T_1(x,y)(t), T_2(x,y)(t)),\ \ \forall\ t \in [0,1],$$

where

$$T_1(x,y)(t) = \mu_1 \int_0^1 G_{\alpha,\lambda_1}(t,s)(f(s,[x(s)-w_1(s)]^*,[y(s)-w_2(s)]^*) + p_1(s))ds,$$

and

$$T_2(x,y)(t) = \mu_2 \int_0^1 G_{\beta,\lambda_2}(t,s)(g(s,[x(s)-w_1(s)]^*,[y(s)-w_2(s)]^*) + p_2(s))ds,$$

with G_{α,λ_1} and G_{β,λ_2} are defined by (4).

It is clear that if (x,y) is a fixed point of operator T, then (x,y) is a solution of problem (6).

Lemma 3.2. *If* (H_1) *and* (H_4) *or* (H_2) *and* (H_4) *hold. Then* $T : P \to P$ *is completely continuous.*

Proof. The operators T_1 and T_2 are well defined. To show this, let $(x,y) \in P$ with $\|(x,y)\| = M$. Then we obtain

$$[x(s)-w_1(s)]^* \leq x(s) \leq \|x\| \leq \|(x,y)\| = M,\ \ \forall\ s \in [0,1]$$
$$[y(s)-w_2(s)]^* \leq y(s) \leq \|y\| \leq \|(x,y)\| = M,\ \ \forall\ s \in [0,1].$$

If (H_1) and (H_4) hold, then we conclude that $T_1(x,y)(t) < \infty$ and $T_2(x,y)(t) < \infty$ for all $t \in [0,1]$. If (H_2) and (H_4) are satisfied, we obtain for all $t \in [0,1]$

$$\begin{aligned} T_1(x,y)(t) &\leq \mu_1\eta_\alpha \int_0^1 K_\alpha(s)(f(s,[x(s)-w_1(s)]^*,[y(s)-w_2(s)]^*) + p_1(s))ds \\ &\leq \mu_1\eta_\alpha \int_0^1 K_\alpha(s)(q_1(s)h_1(s,[x(s)-w_1(s)]^*,[y(s)-w_2(s)]^*) + p_1(s))ds \\ &\leq \mu_1\eta_\alpha L(\phi^0_\alpha(q_1) + \phi^0_\alpha(p_1)) < \infty, \end{aligned}$$

where

$$L = 1 + \max_{t\in[0,1],u,v\in[0,M]} h_1(t,u,v).$$

Similarly

$$T_2(x,y)(t) < \infty.$$

In addition, by Proposition 2.4 (iv) we deduce that

$$T_1(x,y)(t) \geq \nu t^{\alpha-1}\|T_1(x,y)\|,\ \ T_2(x,y)(t) \geq \nu t^{\beta-1}\|T_2(x,y)\|,\ \ t \in [0,1].$$

Then $T(\Omega) \subset \Omega$. By using standard arguments, we conclude that $T : P \to P$ is a compelety continuous operator. $\square$

Now, we prove the following existence results.

Theorem 3.3. *Assume that conditions* (H_1)*,* (H_3) *and* (H_4) *hold. Then there exist* $\mu_1^0, \mu_2^0 > 0$ *such that problem (2) has at least one positive solution for every* $0 < \mu_1 \leq \mu_1^0$ *and* $0 < \mu_2 \leq \mu_2^0$*.*

Proof. Let $\rho \in (0,1)$. Using (H_1) and (H_3), we deduce that there exists $R_0 \in (0,1]$ such that

$$f(t,u,v) \geq \rho f(t,0,0), \;\; g(t,u,v) \geq \rho g(t,0,0), \;\; \forall t \in [0,1], \; u,v \in [0,R_0]. \tag{9}$$

Define

$$M_1 = \max_{t\in[0,1],u,v\in[0,R_0]}\{f(t,u,v)+p_1(t)\} \geq \max_{t\in[0,1]}\{\rho f(t,0,0)+p_1(t)\} > 0,$$
$$M_2 = \max_{t\in[0,1],u,v\in[0,R_0]}\{g(t,u,v)+p_2(t)\} \geq \max_{t\in[0,1]}\{\rho g(t,0,0)+p_2(t)\} > 0,$$

and

$$\mu_1^0 = \frac{R_0}{4\eta_\alpha M_1 \phi_\alpha^0(1)}, \;\; \mu_2^0 = \frac{R_0}{4\eta_\beta M_2 \phi_\beta^0(1)}.$$

Let $\mu_1 \in (0,\mu_1^0]$ and $\mu_2 \in (0,\mu_2^0]$. Define the set $U = \{(x,y) \in P, \; \|(x,y)\| < R_0\}$. We suppose that there exist $(x,y) \in \partial U$ and $\delta \in (0,1)$ such that $(x,y) = \delta T(x,y)$, that is $x = \delta T_1(x,y)$ and $y = \delta T_2(x,y)$.
Then

$$[x(t)-w_1(t)]^* \leq x(t) \leq R_0,$$
$$[y(t)-w_2(t)]^* \leq y(t) \leq R_0.$$

Therefore by Proposition 2.4, for all $t \in [0,1]$, we have

$$\begin{aligned} x(t) = \delta T_1(x,y)(t) &\leq T_1(x,y)(t) = \mu_1 \int_0^1 G_\alpha(t,s)(f(s,[x(s)-w_1(s)]^*,[y(s)-w_2(s)]^*)+p_1(s))ds \\ &\leq \mu_1\eta_\alpha M_1 \int_0^1 K_\alpha(s)ds = \mu_1\eta_\alpha M_1\phi_\alpha^0(1) \leq \frac{R_0}{4}, \end{aligned}$$

$$\begin{aligned} y(t) = \delta T_2(x,y)(t) &\leq T_2(x,y)(t) = \mu_2 \int_0^1 G_\beta(t,s)(g(s,[x(s)-w_1(s)]^*,[y(s)-w_2(s)]^*)+p_2(s))ds \\ &\leq \mu_2\eta_\beta M_2 \int_0^1 K_\beta(s)ds = \mu_1\eta_\beta M_2\phi_\beta^0(1) \leq \frac{R_0}{4}. \end{aligned}$$

Thus, $\|x\| \leq \frac{R_0}{4}$ and $\|y\| \leq \frac{R_0}{4}$. Then $R_0 = \|(x,y)\| \leq \frac{R_0}{2}$, which is a contradiction. Hence, by Theorem 2.6, we deduce that T has a fixed point $(x_1,y_1) \in \overline{U} \subset P$.
Now, by (9), we have

$$\begin{aligned} x_1(t) &= T_1(x_1,y_1) \\ &\geq \mu_1 \int_0^1 G_\alpha(t,s)(\rho f(s,0,0)+p_1(s))ds \\ &> \mu_1 \int_0^1 G_\alpha(t,s)p_1(s)ds = w_1(t), \;\; \forall t \in (0,1], \end{aligned}$$

or

$$\begin{aligned} y_1(t) &= T_2(x_1,y_1) \\ &\geq \mu_2 \int_0^1 G_\alpha(t,s)(\rho g(s,0,0)+p_2(s))ds \\ &> \mu_2 \int_0^1 G_\alpha(t,s)p_2(s)ds = w_2(t), \;\; \forall t \in (0,1], \end{aligned}$$

Thus, $x_1(t) > w_1(t)$, $y_1(t) > w_2(t)$ for all $t \in (0,1]$. Let $u(t) = x_1(t) - w_1(t)$ and $v(t) = y_1(t) - w_2(t)$ for all $t \in [0,1]$. Then, $u(t) > 0$ or $v(t) > 0$ for all $t \in (0,1]$. So, $(u(t), v(t))$ is a positive solution of problem (2). □

Theorem 3.4. *Suppose that conditions (H_2) and (H_4) are satisfied. In addition, suppose that there exists $\theta \in (0, \frac{1}{2})$ such that*

$$f_\infty := \lim_{u+v\to+\infty} \min_{t\in[\theta,1-\theta]} \frac{f(t,u,v)}{u+v} = +\infty \;\; or \;\; g_\infty := \lim_{u+v\to+\infty} \min_{t\in[\theta,1-\theta]} \frac{g(t,u,v)}{u+v} = +\infty.$$

Then there exist $\mu_1^*, \mu_2^* > 0$ such that for any $0 < \mu_1 \le \mu_1^*$ and $0 < \mu_2 \le \mu_2^*$ problem (2) has at least one positive solution.

Proof. We choose $r > \max\{1, \frac{2\eta_\alpha\varphi_\alpha(p_1)}{\nu}, \frac{2\eta_\beta\varphi_\beta(p_2)}{\nu}\}$ and we put $\mu_1^* = \min\{1, \frac{r}{2\eta_\alpha M_1(\phi_\alpha^0(p_1)+\phi_\alpha^0(q_1))}\}$ and $\mu_2^* = \min\{1, \frac{r}{2\eta_\beta M_2(\phi_\beta^0(p_2)+\phi_\beta^0(q_2))}\}$ with $M_i = 1 + \max_{t\in[0,1],u,v\in[0,r]} h_i(t,u,v)$, $i = 1,2$.
Let $\mu_1 \in (0, \mu_1^*]$ and $\mu_2 \in (0, \mu_2^*]$. Then, for any $(x,y) \in \partial\Omega_r$ and $s \in [0,1]$, we have

$$[x(s) - w_1(s)]^* \le x(s) \le \|x\| \le r,$$

$$[y(s) - w_2(s)]^* \le y(s) \le \|y\| \le r.$$

Therefore, by Proposition 2.4 (*ii*) we obtain for any $(x,y) \in \partial\Omega_r$ and $t \in [0,1]$,

$$\begin{aligned} T_1(x,y)(t) &\le \mu_1\eta_\alpha \int_0^1 K_\alpha(s)(q_1(s)h_1(s,[x(s)-w_1(s)]^*,[y(s)-w_2(s)]^*) + p_1(s))ds \\ &\le \mu_1\eta_\alpha M_1 \int_0^1 K_\alpha(s)(q_1(s) + p_1(s))ds \\ &\le \mu_1^*\eta_\alpha M_1(\phi_\alpha^0(p_1) + \phi_\alpha^0(q_1)) \\ &\le \frac{r}{2}, \end{aligned}$$

and, similarly to the calculation of $T_1(x,y)(t)$, we get

$$\begin{aligned} T_2(x,y)(t) &\le \mu_2\eta_\beta \int_0^1 K_\beta(s)(q_2(s)h_2(s,[x(s)-w_1(s)]^*,[y(s)-w_2(s)]^*) + p_2(s))ds \\ &\le \mu_2^*\eta_\beta M_2(\phi_\beta^0(p_2) + \phi_\beta^0(q_2)) \\ &\le \frac{r}{2}. \end{aligned}$$

Thus

$$\left\|T(x,y)\right\| = \left\|T_1(x,y)\right\| + \left\|T_2(x,y)\right\| \le \left\|(x,y)\right\|, \;\text{ for all } (x,y) \in \partial\Omega_r. \tag{10}$$

On the other hand, by hypothesis, for $A = \max\left\{\frac{8}{\mu_1\nu\gamma_\alpha\phi_\alpha^\theta(1)\theta^{\alpha-1}}, \frac{8}{\mu_2\nu\gamma_\beta\phi_\beta^\theta(1)\theta^{\beta-1}}\right\}$ there exists $B > 0$ such that $f(t,u,v) \ge A(u+v)$ or $g(t,u,v) \ge A(u+v)$, $\forall t \in [\theta, 1-\theta]$, $u+v \ge B$.
Now, choose

$$R = \max\{2r, \frac{4B}{\nu\theta^{\alpha-1}}, \frac{4B}{\nu\theta^{\beta-1}}\}.$$

First, we suppose that $f_\infty = \infty$, that is, $f(t,u,v) \ge A(u+v)$ for all $t \in [\theta, 1-\theta]$, $u+v \ge B$. So, for any $(x,y) \in \partial\Omega_R$, we get $\|x\| + \left\|y\right\| = R$. Thus, we deduce that $\|x\| \ge \frac{R}{2}$ or $\left\|y\right\| \ge \frac{R}{2}$.

Assume that $\|x\| \geq \frac{R}{2}$. Then, by (7), for $(x,y) \in \partial\Omega_R$ and $t \in [0,1]$, we have

$$\begin{aligned} x(t) - w_1(t) &\geq x(t) - \mu_1 \eta_\alpha t^{\alpha-1} \varphi_\alpha(p_1) \\ &\geq x(t) - \frac{x(t)}{\nu \|x\|} \eta_\alpha \varphi_\alpha(p_1) \\ &\geq x(t)(1 - \frac{2\eta_\alpha \varphi_\alpha(p_1)}{\nu R}) \\ &\geq x(t)(1 - \frac{\eta_\alpha \varphi_\alpha(p_1)}{\nu r}) \\ &\geq \frac{x(t)}{2} \geq 0. \end{aligned}$$

Then, for $t \in [\theta, 1-\theta]$, we obtain

$$\begin{aligned} [x(t) - w_1(t)]^* &= x(t) - w_1(t) \geq \frac{x(t)}{2} \geq \frac{\nu}{2} t^{\alpha-1} \|x\| \\ &\geq \frac{\nu}{4} \theta^{\alpha-1} R \geq B. \end{aligned}$$

So

$$[x(t) - w_1(t)]^* + [y(t) - w_2(t)]^* \geq [x(t) - w_1(t)]^* = x(t) - w_1(t) \geq B.$$

Therefore, for any $(x,y) \in \partial\Omega_R$, $t \in [\theta, 1-\theta]$, we deduce

$$f(t, [x(t) - w_1(t)]^*, [y(t) - w_2(t)]^*) \geq A([x(t) - w_1(t)]^* + [y(t) - w_2(t)]^*) \geq A[x(t) - w_1(t)]^* \geq \frac{A}{2} x(t). \tag{11}$$

Using (11) and Proposition 2.4 (v), we obtain for any $(x,y) \in \partial\Omega_R$,
and $t \in [\theta, 1-\theta]$,

$$\begin{aligned} T_1(x,y)(t) &\geq \mu_1 \gamma_\alpha \int_\theta^{1-\theta} K_\alpha(s)(f(s, [x(s) - w_1(s)]^*, [y(s) - w_2(s)]^*) + p_1(s)) ds \\ &\geq \mu_1 \gamma_\alpha \frac{A}{2} \int_\theta^{1-\theta} K_\alpha(s)[x(s) - w_1(s)]^* ds \\ &\geq \mu_1 \gamma_\alpha \frac{\nu}{8} \phi_\alpha^\theta(1) \theta^{\alpha-1} A R = R. \end{aligned}$$

Thus,

$$\left\| T_1(x,y) \right\| \geq \left\| (x,y) \right\|, \text{ for all } (x,y) \in \partial\Omega_R.$$

Then

$$\left\| T(x,y) \right\| \geq \left\| (x,y) \right\|, \text{ for all } (x,y) \in \partial\Omega_R. \tag{12}$$

If $\left\| y \right\| \geq \frac{R}{2}$, then by the same manner, we prove again relation (12).
Now, we suppose that $g_\infty = \infty$, that is, $g(t,u,v) \geq A(u+v)$ for all $t \in [\theta, 1-\theta]$, $u + v \geq B$. So, for any $(x,y) \in \partial\Omega_R$, we get $\|x\| + \left\| y \right\| = R$. Thus, we deduce that $\|x\| \geq \frac{R}{2}$ or $\left\| y \right\| \geq \frac{R}{2}$.
If $\|x\| \geq \frac{R}{2}$, then for any $(x,y) \in \partial\Omega_R$ we obtain in a similar manner that $x(t) - w_1(t) \geq \frac{x(t)}{2}$ for all $t \in [0,1]$ and

$$\begin{aligned} T_2(x,y)(t) &\geq \mu_2 \gamma_\beta \int_\theta^{1-\theta} K_\beta(s)(g(s, [x(s) - w_1(s)]^*, [y(s) - w_2(s)]^*) + p_2(s)) ds \\ &\geq \mu_2 \gamma_\beta \frac{A}{2} \int_\theta^{1-\theta} K_\beta(s)[x(s) - w_1(s)]^* ds \\ &\geq \mu_2 \gamma_\beta \frac{\nu}{8} \theta^{\beta-1} \phi_\beta^\theta(1) A R = R. \end{aligned}$$

Therefore

$$\|T_2(x,y)\| \geq \|(x,y)\|, \text{ for all } (x,y) \in \partial\Omega_R.$$

Hence

$$\|T(x,y)\| \geq \|(x,y)\|, \text{ for all } (x,y) \in \partial\Omega_R. \tag{13}$$

If $\|y\| \geq \frac{R}{2}$, then by the same manner, we prove again relation (13). Therefore, by Theorem 2.7 and inequalities (10) and (12) or (10) and (13), we conclude that T has a fixed point $(x,y) \in \overline{\Omega_R}\backslash\Omega_r$, that is

$$r \leq \|(x,y)\| \leq R. \tag{14}$$

Now, since $\|(x,y)\| \geq r$, then $\|x\| \geq \frac{r}{2}$ or $\|y\| \geq \frac{r}{2}$.
First, if $\|x\| \geq \frac{r}{2}$, then, by (7), we obtain for $t \in [0,1]$

$$x(t) - w_1(t) \quad \geq \quad x(t) - \mu_1\eta_\alpha t^{\alpha-1}\varphi_\alpha(p_1) \geq t^{\alpha-1}[\nu\frac{r}{2} - \eta_\alpha\varphi_\alpha(p_1)] \geq 0.$$

By the same manner if $\|y\| \geq \frac{r}{2}$ we get, by (8),

$$y(t) - w_2(t) \geq t^{\beta-1}[\nu\frac{r}{2} - \eta_\beta\varphi_\beta(p_2)] \geq 0, \quad t \in [0,1].$$

Let $u(t) = x_1(t) - w_1(t)$ and $v(t) = y_1(t) - w_2(t)$ for all $t \in [0,1]$. Then $(u(t), v(t))$ is a positive solution of problem (2) with $u(t) \geq t^{\alpha-1}[\nu\frac{r}{2} - \eta_\alpha\varphi_\alpha(p_1)]$ and $v(t) \geq t^{\beta-1}[\nu\frac{r}{2} - \eta_\beta\varphi_\beta(p_2)]$ for all $t \in [0,1]$. □

Theorem 3.5. *Suppose that conditions (H_2) and (H_4) hold. In addition, if we have*

(A_1) there exists $\theta \in (0, \frac{1}{2})$ such that

$$f_\infty^* := \lim_{u+v\to+\infty} \min_{t\in[\theta,1-\theta]} f(t,u,v) = \infty \text{ or } g_\infty^* := \lim_{u+v\to+\infty} \min_{t\in[\theta,1-\theta]} g(t,u,v) = \infty$$

(A_2) $h_i^\infty := \lim_{u+v\to+\infty} \max_{t\in[0,1]} \frac{h_i(t,u,v)}{u+v} = 0, \ i = 1,2.$

Then there exist $\mu_1^*, \mu_2^* > 0$ such that for any $\mu_1 \geq \mu_1^*$ and $\mu_2 \geq \mu_2^*$ problem (2) has at least one positive solution.

Proof. Suppose that (A_1) holds. Then for $A = \max\left\{\frac{4\eta_\alpha\varphi_\alpha(p_1)}{\nu\gamma_\alpha\phi_\alpha^\theta(1)}, \frac{4\eta_\beta\varphi_\beta(p_2)}{\nu\gamma_\beta\phi_\beta^\theta(1)}\right\}$ there exists $L > 0$ such that

$$f(t,u,v) \geq A, \quad \forall t \in [\theta, 1-\theta], \ u+v \geq L, \tag{15}$$

or

$$g(t,u,v) \geq A, \quad \forall t \in [\theta, 1-\theta], \ u+v \geq L. \tag{16}$$

We define $\mu_1^* = \frac{L}{\eta_\alpha\varphi_\alpha(p_1)\theta^{\alpha-1}}$ and $\mu_2^* = \frac{L}{\eta_\beta\varphi_\beta(p_2)\theta^{\beta-1}}$. Let $\mu_1 \geq \mu_1^*$ and $\mu_2 \geq \mu_2^*$. Choose $R = \max\{\frac{4\mu_1\eta_\alpha\varphi_\alpha(p_1)}{\nu}, \frac{4\mu_2\eta_\beta\varphi_\beta(p_2)}{\nu}\}$.
First, if $f_\infty^* = \infty$, then (15) holds. Let $(x,y) \in \partial\Omega_R$. Then $\|x\| + \|y\| = R$, hence $\|x\| \geq \frac{R}{2}$ or $\|y\| \geq \frac{R}{2}$. We suppose that $\|x\| \geq \frac{R}{2}$. Then for any $t \in [0,1]$ we have

$$\begin{aligned} x(t) - w_1(t) \quad &\geq \quad \nu t^{\alpha-1}\|x\| - \mu_1 t^{\alpha-1}\eta_\alpha\varphi_\alpha(p_1) \geq t^{\alpha-1}[\nu\frac{R}{2} - \mu_1\eta_\alpha\varphi_\alpha(p_1)] \\ &\geq \quad \mu_1\eta_\alpha\varphi_\alpha(p_1)t^{\alpha-1} \geq \mu_1^*\eta_\alpha\varphi_\alpha(p_1)t^{\alpha-1} \geq \frac{L}{\theta^{\alpha-1}}t^{\alpha-1} \geq 0. \end{aligned}$$

Thus, for any $(x,y)\in\partial\Omega_R$ and $t\in[\theta,1-\theta]$, we have

$$[x(t)-w_1(t)]^*+[y(t)-w_2(t)]^*\geq[x(t)-w_1(t)]^*=x(t)-w_1(t)\geq\frac{L}{\theta^{\alpha-1}}t^{\alpha-1}\geq L,$$

and so, for any $(x,y)\in\partial\Omega_R$ and $t\in[\theta,1-\theta]$, we deduce

$$f(t,[x(t)-w_1(t)]^*,[y(t)-w_2(t)]^*)\geq A.$$

Then, for any $(x,y)\in\partial\Omega_R$ and $t\in[\theta,1-\theta]$ we obtain

$$\begin{aligned}T_1(x,y)(t) &\geq \mu_1\gamma_\alpha\int_\theta^{1-\theta}K_\alpha(s)(f(s,[x(s)-w_1(s)]^*,[y(s)-w_2(s)]^*)+p_1(s))ds\\ &\geq \mu_1\gamma_\alpha A\int_\theta^{1-\theta}K_\alpha(s)ds\\ &\geq \mu_1\gamma_\alpha\phi_\alpha^\theta(1)A\geq R.\end{aligned}$$

Hence, $\left\|T_1(x,y)\right\|\geq R,$ for all $(x,y)\in\partial\Omega_R$. Therefore

$$\left\|T(x,y)\right\|\geq\left\|(x,y)\right\|=R,\ \ \forall(x,y)\in\partial\Omega_R. \tag{17}$$

If $\left\|y\right\|\geq\frac{R}{2}$, then by the same manner, we prove again inequality (17).
Now, suppose that $g_\infty^*=\infty$, then (16) holds. Similarly we prove (17). On the other hand, by (A_2), for $\varepsilon=\min\{\frac{1}{4\mu_1\eta_\alpha\phi_\alpha^0(q_1)},\frac{1}{4\mu_2\eta_\beta\phi_\beta^0(q_2)}\}$ there exists $M>0$ such that

$$h_i(t,u,v)\leq\varepsilon(u+v),\ \ \forall(u+v)\geq M,\ \ \forall t\in[0,1],\ \ i=1,2.$$

So

$$h_i(t,u,v)\leq l_i+\varepsilon(u+v),\ \ \forall t\in[0,1],u,v\geq0,\ \ i=1,2,$$

where $l_i=\max\limits_{t\in[0,1],u,v\geq0,u+v\leq M}h_i(t,u,v)$. Put $l=\max(l_1,l_2)$ and fix a positive real R_1 such that

$$R_1>\max\left\{2R,\mu_1\eta_\alpha\Big(l\phi_\alpha^0(q_1)+\phi_\alpha^0(p_1)\Big)\Big(\frac{1}{2}-\mu_1\eta_\alpha\varepsilon\phi_\alpha^0(q_1)\Big)^{-1},\mu_2\eta_\beta\Big(l\phi_\beta^0(q_2)+\phi_\beta^0(p_2)\Big)\Big(\frac{1}{2}-\mu_2\eta_\beta\varepsilon\phi_\beta^0(q_2)\Big)^{-1}\right\}.$$

Therefore, for any $(x,y)\in\partial\Omega_{R_1}$ and $t\in[0,1]$, we have

$$\begin{aligned}T_1(x,y)(t) &\leq \mu_1\eta_\alpha\int_0^1K_\alpha(s)(f(s,[x(s)-w_1(s)]^*,[y(s)-w_2(s)]^*)+p_1(s))ds\\ &\leq \mu_1\eta_\alpha\int_0^1K_\alpha(s)(q_1(s)h_1(s,[x(s)-w_1(s)]^*,[y(s)-w_2(s)]^*)+p_1(s))ds\\ &\leq \mu_1\eta_\alpha\int_0^1K_\alpha(s)(q_1(s)(l+\varepsilon([x(s)-w_1(s)]^*+[y(s)-w_2(s)]^*))+p_1(s))ds\\ &\leq \mu_1l\,\eta_\alpha\phi_\alpha^0(q_1)+\mu_1\eta_\alpha\varepsilon\phi_\alpha^0(q_1)R_1+\mu_1\eta_\alpha\phi_\alpha^0(p_1)\\ &\leq R_1(\frac{1}{2}-\mu_1\eta_\alpha\varepsilon\phi_\alpha^0(q_1))+\mu_1\eta_\alpha\varepsilon\phi_\alpha^0(q_1)R_1\\ &\leq \frac{R_1}{2}.\end{aligned}$$

Thus

$$\left\|T_1(x,y)\right\|\leq\frac{1}{2}\left\|(x,y)\right\|,\ \ \forall(x,y)\in\partial\Omega_{R_1}.$$

Similary, we prove

$$\left\|T_2(x,y)\right\| \le \frac{1}{2}\left\|(x,y)\right\|, \ \forall (x,y) \in \partial\Omega_{R_1}.$$

Hence, we obtain

$$\left\|T(x,y)\right\| \le \left\|(x,y)\right\|, \ \forall (x,y) \in \partial\Omega_{R_1}. \tag{18}$$

Therefore, by Theorem 2.7 and inequalities (17) and (18), we conclude that T has a fixed point $(x_1, y_1) \in \overline{\Omega_{R_1}}\backslash\Omega_R$, that is

$$R \le \left\|(x_1,y_1)\right\| \le R_1.$$

Since $\left\|(x_1,y_1)\right\| \ge R$, then $\|x_1\| \ge \frac{R}{2}$ and $\|y_1\| \ge \frac{R}{2}$.
Assume that $\|x_1\| \ge \frac{R}{2}$, then

$$x_1(t) - w_1(t) \ \ \ge \ \ t^{\alpha-1}(\nu\frac{R}{2} - \mu_1\eta_\alpha\varphi_\alpha(p_1)) \ge \mu_1\eta_\alpha\varphi_\alpha(p_1)t^{\alpha-1} \ge \frac{L}{\theta^{\alpha-1}}t^{\alpha-1} \ge 0, \ \text{for all} \ \ t \in [0,1].$$

Similarly, if $\|y_1\| \ge \frac{R}{2}$, then we conclude again that $y_1(t) - w_2(t) \ge \frac{L}{\theta^{\beta-1}}t^{\beta-1} \ge 0$, for all $t \in [0,1]$. Let $u(t) = x_1(t) - w_1(t)$ and $v(t) = y_1(t) - w_2(t)$ for all $t \in [0,1]$. Then $(u(t), v(t))$ is a positive solution of problem (2) with $u(t) \ge \frac{L}{\theta^{\alpha-1}}t^{\alpha-1}$, $v(t) \ge \frac{L}{\theta^{\beta-1}}t^{\beta-1}$ for all $t \in [0,1]$. □

Now, we give the multiplicity result.

Theorem 3.6. *Suppose that* (H_3) *and* (H_4) *hold. In addition suppose that*

$$f_\infty := \lim_{u+v\to+\infty}\ \min_{t\in[\theta,1-\theta]} \frac{f(t,u,v)}{u+v} = +\infty \ \ or \ \ g_\infty := \lim_{u+v\to+\infty}\ \min_{t\in[\theta,1-\theta]} \frac{g(t,u,v)}{u+v} = +\infty,$$

$(H_1^{'})$ *The functions* $f, g \in C([0,1]\times[0+\infty)\times[0,\infty),(-\infty,+\infty))$ *and there exist functions* $p_1, p_2, q_1, q_2 \in C([0,1],[0,+\infty))$ *and* $h_1, h_2 \in C([0,1]\times[0,+\infty)\times[0,+\infty),[0,+\infty))$ *such that*

$$-p_1(t) \le f(t,u,v) \le q_1(t)h_1(t,u,v), \quad -p_2(t) \le g(t,u,v) \le q_2(t)h_2(t,u,v)$$

for all $t \in [0,1]$, $u, v \in [0,+\infty)$. *hold.*
Then the problem (2) has at least two positive solutions for $\mu_1 > 0$ *and* $\mu_2 > 0$ *sufficiently small.*

Proof. Applying Theorem 3.3 and Theorem 3.4, we conclude that, for $0 < \mu_1 \le \min\{\mu_1^0, \mu_1^*\}$ and $0 < \mu_2 \le \min\{\mu_2^0, \mu_2^*\}$, problem (2) has at least two positive solutions satisfying $0 \le \|(u_1 + w_1, v_1 + w_2)\| \le 1$ and $\|(u_2 + w_1, v_2 + w_2)\| > 1$. □

4. Examples

In this section, we consider some examples which illustrate our main results.

Example 4.1. *We consider the system of fractional differential equations*

$$\begin{cases} D^{\frac{7}{2}}u(t) + \mu_1((u(t)+v(t))^2 - \frac{1}{(1-t)^2}) = 0 \ \ in \ \ (0,1), \\ D^{\frac{7}{2}}v(t) + \mu_2(\exp(u(t)+v(t)) - \frac{1}{1-t}) = 0 \ \ in \ \ (0,1), \\ u(0) = u'(0) = 0, \ \ u(1) = \int_0^1 u(s)ds, \\ v(0) = v'(0) = 0, \ \ v(1) = \frac{1}{2}\int_0^1 v(s)ds, \end{cases} \tag{19}$$

Let $\alpha = \beta = \frac{7}{2}$, $\lambda_1 = 1$, $\lambda_2 = \frac{1}{2}$, $p_1(t) = \frac{1}{(1-t)^2}$, $p_2(t) = \frac{1}{1-t}$, $q_1(t) = \frac{1}{t}$, $q_2(t) = \frac{1}{\sqrt{t}}$ *for all* $t \in (0,1)$. *Let* $h_1(t,u,v) = (u+v)^2 t$, $h_2(t,u,v) = \exp(u+v)\sqrt{t}$ *for all* $t \in [0,1]$, $f(t,u,v) = (u+v)^2 - \frac{1}{(1-t)^2}$ *and* $g(t,u,v) = \exp(u+v) - \frac{1}{1-t}$. *For* $\theta = \frac{1}{4}$, *we verify that* $f_\infty = g_\infty = +\infty$. *By direct calculs, we obtain* $\varphi_\alpha(p_1) \simeq 0.2006$, $\varphi_\beta(p_2) \simeq 0.12036$, $\phi^0_\alpha(p_1) \simeq 0.08024$, $\phi^0_\alpha(q_1) \simeq 0.08597$, $\phi^0_\beta(p_2) \simeq 0.03438$ *and* $\phi^0_\beta(q_2) \simeq 0.03692$. *Therefore, using notation of proof of Theorem 3.4, we choose* $r = 4$ *and* $R = 100$. *Then a simple calculs yields to* $\mu_1^* = 0.13223$ *and* $\mu_2^* \simeq 0.008638$. *Hence, Theorem 3.4 ensures the existence of positive solution of problem (19) for every* $\mu_1 \le 0.13223$ *and* $\mu_2 \le 0.008638$.

Example 4.2. *We consider the following nonlinear fractional differential equation*

$$\begin{cases} D^{\frac{5}{2}}u(t) + \mu_1(\sqrt{u(t)+v(t)} - \frac{1}{\sqrt{(1-t)^3}}) = 0 & in\ \ (0,1), \\ D^{\frac{7}{2}}v(t) + \mu_2(ln(1+u(t)+v(t)) - \frac{1}{\sqrt{(1-t)^3}}) = 0 & in\ \ (0,1), \\ u(0) = u'(0) = 0,\ \ u(1) = 2\int_0^1 u(s)ds, \\ v(0) = v'(0) = 0,\ \ v(1) = 2\int_0^1 v(s)ds, \end{cases} \tag{20}$$

Set $\alpha = \frac{5}{2}$, $\beta = \frac{7}{2}$, $p_1(t) = p_2(t) = \frac{1}{\sqrt{(1-t)^3}}$, $q_1(t) = \frac{1}{t}$, $q_2(t) = \frac{1}{\sqrt{t^2(1-t)}}$, $h_1(t,u,v) = (\sqrt{u+v})t$, $h_2(t,u,v) = ln(1+u+v)\sqrt{t^2(1-t)}$, $f(t,u,v) = \sqrt{u+v} - \frac{1}{\sqrt{(1-t)^3}}$ *and* $g(t,u,v) = ln(1+u+v) - \frac{1}{\sqrt{(1-t)^3}}$. *We verify that* $h_i^\infty = 0$, $i = 1,2$ *and for* $\theta = \frac{1}{3}$, $f^*_\infty = g^*_\infty = +\infty$. *We get also* $\varphi_\alpha(p_1) \simeq 0.75225$, $\varphi_\beta(p_2) \simeq 0.15045$, $\phi^0_\alpha(p_1) \simeq 0.37613$, $\phi^0_\alpha(q_1) \simeq 0.3009$, $\phi^0_\beta(p_2) \simeq 0.05015$ *and* $\phi^0_\beta(q_2) \simeq 0.1003$. *A simple calculation yields to* $\mu_1^* \simeq 8.289$ *and* $\mu_2^* \simeq 266.43$. *Thus, by Theorem 3.5, we conclude that problem (20) has at least one positive solution for every* $\mu_1 \ge 8.289$ *and* $\mu_2 \ge 266.43$.

Example 4.3. *We consider the following system*

$$\begin{cases} D^{\frac{7}{2}}u(t) + \mu_1((2+u(t)+v(t))^{\frac{3}{2}} + t\cos u(t)) = 0, & in\ \ (0,1), \\ D^{\frac{7}{2}}v(t) + \mu_2(\exp(u(t)+v(t)) + t\cos v(t)) = 0, & in\ \ (0,1), \\ u(0) = u'(0) = 0,\ \ u(1) = \frac{3}{2}\int_0^1 u(s)ds, \\ v(0) = v'(0) = 0,\ \ v(1) = 2\int_0^1 v(s)ds, \end{cases} \tag{21}$$

Let $\alpha = \frac{7}{2}$, $\beta = \frac{7}{2}$, $f(t,u,v) = (2+u+v)^{\frac{3}{2}} + t\cos u$ *and* $g(t,u,v) = \exp(u+v) + t\cos v$, $p_1(t) = p_2(t) = t$, *for all* $t \in [0,1]$, *and then hypthesis* (H_1) *is verified. Also, assumption* (H_3) *is satisfied, because* $f(t,0,0) = 1+t$ *and* $g(t,0,0) = 1+t$, *for all* $t \in [0,1]$.
Let $\rho = \frac{1}{3}$ *and* $R_0 = 1$. *Then*

$$f(t,u,v) \ge \rho f(t,0,0) = \frac{1}{3}(1+t),\ \ g(t,u,v) \ge \rho f(t,0,0) = \frac{1}{3}(1+t),\ \ for\ all\ t \in [0,1]\ u,v\ \in [0,1],$$

and

$$M_1 = \max_{t\in[0,1],u,v\in[0,1]}\{f(t,u,v) + p_1(t)\} \simeq 7.1861,$$

$$M_2 = \max_{t\in[0,1],u,v\in[0,1]}\{g(t,u,v) + p_2(t)\} \simeq 8.3890.$$

In addition, we have $\phi^0_\alpha(p_1) = \phi^0_\beta(p_2) \simeq 0.076728$, $\phi^0_\alpha(q_1) = \phi^0_\beta(q_2) \simeq 0.038687$. *A simple calculation yeilds to* $\mu_1^0 = \frac{R_0}{4\eta_\alpha M_1 \phi^0_\alpha(1)} \simeq 0.5138$ *and* $\mu_2^0 = \frac{R_0}{4\eta_\beta M_2 \phi^0_\beta(1)} \simeq 0.3306$. *Thus, by Theorem 3.3, for* $\mu_1 \le \mu_1^0$ *and* $\mu_2 \le \mu_2^0$, *we conclude that problem (21) has at least one positive solution.*
On the other hand, hypothesis $(H_1^{'})$ *is verified for* $q_1(t) = q_2(t) = 1$ *and* $h_1(t,u,v) = (2+u+v)^{\frac{3}{2}} + t$, $h_2(t,u,v) = \exp(u+v) + t$ *for all* $t \in [0,1]$. *Also, for* $\theta = \frac{1}{3}$, *we verify that* $f_\infty = g_\infty = +\infty$. *Then, by direct calculs, we get* $\mu_1^* \simeq 0.31442$ *and* $\mu_2^* = 0.01377$. *Thus, by Theorem 3.6, we deduce that (21) has at least two positive solutions for* $\mu_1 \le \min\{\mu_1^0, \mu_1^*\} \simeq 0.31442$ *and* $\mu_2 \le \min\{\mu_2^0, \mu_2^*\} \simeq 0.01377$.

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