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Quantitative Estimates for Szász Operators and its Hybrid Variant

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Abstract. The present article deals with the study on approximation properties of well known Szász-Mirakyan operators. We estimate the quantitative Voronovskaja type asymptotic formula for the Szász-Baskakov operators and difference between Szász-Mirakyan operators and the hybrid Szász operators having weights of Baskakov basis in terms of the weighted modulus of continuity.

1. Introduction

The Szász operators are defined as

$$S_n(f,x) = \sum_{k=0}^{\infty} s_{n,k}(x) f\left(\frac{k}{n}\right), \tag{1}$$

where

$$s_{n,k}(x) = e^{-nx}\frac{(nx)^k}{k!}.$$

The integral modification with the weights of Baskakov basis functions namely Szász-Baskakov type operators were first considered in [12] and later improved by Gupta in [3] are defined as

$$M_n(f;x) = (n-1)\sum_{k=0}^{\infty} s_{n,k}(x)\int_0^{\infty} v_{n,k}(t) f(t)dt, \tag{2}$$

where $s_{n,k}(x)$ is the Szász basis function defined in (1) and $v_{n,k}(t)$ is the Baskakov basis functions defined by

$$v_{n,k}(t) = \binom{n+k-1}{k}\frac{t^k}{(1+t)^{n+k}}.$$

Let us consider $B_2[0,\infty) := \{f : |f(x)| \leq M_f(1+x^2) \text{ with } M_f > 0\}$. Also let $C_2[0,\infty)$ denotes the subspace of all continuous functions in $B_2[0,\infty)$. Further $C_2^*[0,\infty)$ denotes the closed subspace of $C_2[0,\infty)$ for which

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$\lim_{x\to\infty} |f(x)|(1+x^2)^{-1} < C$ for some constant C, and $\|.\|_2 = \sup_{x\in[0,\infty)} |f(x)|(1+x^2)^{-1}$. In [10], Ispir considered for each $f \in C_2[0,\infty)$, the following weighted modulus of continuity:

$$W(f,\delta) = \sup_{x\geq 0, |h|<\delta} \frac{|f(x+h)-f(x)|}{(1+x^2)(1+h^2)}.$$

Motivated by the work of Acu and Rasa [1], Aral et al [2] considered two different general sequences of linear positive operators defined on an unbounded interval and obtained quantitative estimates for differences of these operators. The results considered in [2] dealt with the general estimate of the problem raised by well-known mathematician A. Lupaş in [11]. Some of the important results on the difference estimates can be found in [5], [6] and [7], very recently Gupta et al [9] compiled some of the results on difference of operators.

Very recently Aral et al [2] considered $F_k : D \to R$ be positive linear functional defined on a subspace D of $C[0,\infty)$, which contains $C_2[0,\infty)$ and the polynomials up to degree 6, such that $F_k(e_0) = 1, b^{F_k} := F_k(e_1), \mu_r^{F_k} = F_k(e_1 - b^F e_0)^r = \sum_{i=0}^{r} \binom{r}{i} (-1)^i F_k(e_{r-i})[F_k(e_1)]^i, r \in \mathbb{N}$ and established the approximation result on polynomial weighted spaces. They considered the operators having same basis function p_k given by

$$U(f,x) = \sum_{k=0}^{\infty} p_k(x) F_k(f), \quad V(f,x) = \sum_{k=0}^{\infty} p_k(x) G_k(f).$$

The main result given in [2] is the following:

Theorem A. Let $f \in C_2[0,\infty)$ with $f'' \in C_2^*[0,\infty)$ and $x \in [0,\infty)$, then for $n \in \mathbb{N}$, we have

$$\begin{aligned} |(U-V)(f,x)| \quad &\leq \quad \frac{\|f\|_2}{2}\beta(x) + 8W(f'',\delta_1(x))(1+\beta(x)) \\ &\quad +16W(f,\delta_2(x))(\gamma(x)+1), \end{aligned}$$

where

$$\beta(x) = \sum_{k=0}^{\infty} p_k(x)\left[(1+(b^{F_k})^2)\mu_2^{F_k} + (1+(b^{G_k})^2)\mu_2^{G_k}\right]$$

$$\gamma(x) = \sum_{k=0}^{\infty} p_k(x)(1+(b^{F_k})^2)$$

$$\delta_1(x) = \left[\sum_{k=0}^{\infty} p_k(x)\left((1+(b^{F_k})^2)\mu_6^{F_k} + (1+(b^{G_k})^2)\mu_6^{G_k}\right)\right]^{1/4}$$

and

$$\delta_2(x) = \left[\sum_{k=0}^{\infty} p_k(x)(1+(b^{F_k})^2)(b^{F_k} - b^{G_k})^4\right]^{1/4}.$$

In the next section, we estimate quantitative result for the difference of Szász operators and the Szász-Baskakov operators.

2. Auxiliary Results

Lemma 2.1. *[8] The following recurrence relation holds for moments*

$$S_n(e_{m+1},x) \quad = \quad \frac{x}{n} S_n'(e_m,x) + x S_n(e_m,x).$$

Some of the moments of Szász operators defined by (1) are given as:

$$S_n(e_0,x) = 1$$
$$S_n(e_1,x) = x$$
$$S_n(e_2,x) = x^2 + \frac{x}{n}$$
$$S_n(e_3,x) = x^3 + \frac{3x^2}{n} + \frac{x}{n^2}$$
$$S_n(e_4,x) = x^4 + \frac{6x^3}{n} + \frac{7x^2}{n^2} + \frac{x}{n^3}$$
$$S_n(e_5,x) = x^5 + \frac{10x^4}{n} + \frac{25x^3}{n^2} + \frac{15x^2}{n^3} + \frac{x}{n^4},$$
$$S_n(e_6,x) = x^6 + \frac{15x^5}{n} + \frac{65x^4}{n^2} + \frac{90x^3}{n^3} + \frac{31x^2}{n^4} + \frac{x}{n^5}$$
$$S_n(e_7,x) = x^7 + \frac{21x^6}{n} + \frac{140x^5}{n^2} + \frac{350x^4}{n^3} + \frac{301x^3}{n^4} + \frac{63x^2}{n^5} + \frac{x}{n^6}$$
$$S_n(e_8,x) = x^8 + \frac{28x^7}{n} + \frac{266x^6}{n^2} + \frac{1050x^5}{n^3} + \frac{1701x^4}{n^4} + \frac{966x^3}{n^5} + \frac{127x^2}{n^6} + \frac{x}{n^7}.$$

Lemma 2.2. *(see [8, pp. 36] and references therein) The m-th order ($m \in \mathbb{N}$) moment, with $e_m(x) = x^m, m \in \mathbb{N}$ are given by*

$$M_n(e_m,x) = \frac{(n-m-2)!m!}{(n-2)!}\, {}_1F_1\left(-m;1;-nx\right).$$

The central moments of the operators M_n can be obtained from Lemma 2.2, using the linearity property. In the following lemma which are obtained using recurrence relation.

Lemma 2.3. *[3] The central moments $\mu_{n,s}(x) = M_m((t-x)^s, x), s \in N$, satisfy the following recurrence relation:*

$$\begin{aligned} &(n-s-2)\mu_{n,s+1}(x) \\ = \quad &(s+1)(1+2x)\mu_{n,s}(x) + x\mu'_{n,s}(x) + sx(2+x)\mu_{n,s-1}(x), n > s+2. \end{aligned}$$

In particular

$$\mu_{n,1}(x) = \frac{1+2x}{n-2}, \mu_{n,2}(x) = \frac{(6+n)x^2 + 2x(n+3) + 2}{(n-2)(n-3)}.$$

If we denote

$$F_{k,n}(f) = f\left(\frac{k}{n}\right), G_{k,n}(f) = (n-1)\int_0^\infty v_{n,k}(t)f(t)dt,$$

then the operators (1) and (2) take the following forms:

$$S_n(f,x) = \sum_{k=0}^{\infty} s_{n,k}(x)F_{k,n}(f)$$

and

$$M_n(f,x) = \sum_{k=0}^{\infty} s_{n,k}(x)G_{k,n}(f).$$

3. Approximation

Theorem 3.1. *Let $f \in C[0,\infty)$ and $f'' \in C_2^*[0,\infty)$ then for $x \in [0,\infty)$ we have*

$$\begin{aligned} & \left| M_n(f,x) - f(x) - \left(\frac{1+2x}{n-2}\right) f'(x) - \left(\frac{(6+n)x^2 + 2x(n+3) + 2}{(n-2)(n-3)}\right) \frac{f''(x)}{2!} \right| \\ \leq \; & \frac{8(1+x^2)(1+2x)}{n-2} W(f'', \alpha^{1/4}). \end{aligned}$$

for every $x \in [0,\infty)$ and $\alpha = \frac{\mu_{n,6}(x)}{\mu_{n,2}(x)}$.

Proof. Using the Taylor series expansion of f, we have

$$f(t) = g(x) + (t-x)f'(x) + (t-x)^2 \frac{f''(x)}{2!} + \frac{(t-x)^2}{2!}(f''(\zeta) - f''(x)),$$

where ζ lies between t and x.
Applying the operator $M_n(f,x)$, we have

$$\begin{aligned} & M_n(f,x) - f(x) - f'(x)\mu_{n,1}(x) - \frac{f''(x)}{2!}\mu_{n,2}(x) \\ = \; & M_n\left(\frac{(t-x)^2}{2!}(f''(\zeta) - f''(x)), x\right), \end{aligned}$$

From Lemma 2.3, we obtain

$$\begin{aligned} & \left| M_n(f,x) - f(x) - \left(\frac{1+2x}{n-2}\right) f'(x) - \left(\frac{(6+n)x^2 + 2x(n+3) + 2}{(n-2)(n-3)}\right) \frac{f''(x)}{2!} \right| \\ \leq \; & M_n\left(\left|\frac{(t-x)^2}{2!}(f''(\zeta) - f''(x))\right|, x\right). \end{aligned}$$

Following [10], we have

$$\begin{aligned} & \left|\frac{(t-x)^2}{2!}(f''(\zeta) - f''(x))\right| \\ \leq \; & 8(1+x^2)\left((t-x)^2 + \frac{(t-x)^6}{\delta^4}\right) W(f'', x), \quad (0 < \delta < 1). \end{aligned}$$

Hence

$$\begin{aligned} & \left| M_n(f,x) - f(x) - \left(\frac{1+2x}{n-2}\right) f'(x) - \left(\frac{(6+n)x^2 + 2x(n+3) + 2}{(n-2)(n-3)}\right) \frac{f''(x)}{2!} \right| \\ \leq \; & \frac{8(1+x^2)(1+2x)}{n-2}\left(1 + \frac{\mu_{n,6}(x)}{\delta^4 \mu_{n,2}(x)}\right) W(f'', \delta). \end{aligned}$$

Taking $\delta = \left(\frac{\mu_{n,6}(x)}{\mu_{n,2}(x)}\right)^{1/4}$, we get the desired result. □

Below, we present the exact estimate for Szász-Baskakov operators and Szász operators.

Theorem 3.2. *Let $f \in C_2[0,\infty)$ with $f'' \in C_2^*[0,\infty)$ and $x \in [0,\infty)$, then for $n \in \mathbb{N}$, we have*

$$\begin{aligned}|(S_n - M_n)(f,x)| &\leq \frac{\|f\|_2}{2}\beta(x) + 8W(f'',\delta_1(x))(1+\beta(x)) \\ &+16W(f,\delta_2(x))\left(x^2 + \frac{x}{n} + 1\right),\end{aligned}$$

where

$$\begin{aligned}\beta(x) &= \frac{1}{(n-2)^4(n-3)}\Big(n^4x(x^3+x^2+x+1) + n^3(8x^3-3x+1) \\ &+n^2(17x^2+5x-5) + n(5x+9) - 5\Big),\end{aligned}$$

$$\begin{aligned}\delta_1(x) &= \frac{5^{1/4}}{[(n-7)_6(n-2)^7]^{1/4}}\Big[3n^{10}x^3(x+1)^3(x^2+1) + 2n^9x^2\Big(37x^6+156x^5+263x^4 \\ &+264x^3+191x^2+93x+22\Big) + n^8x\Big(-40x^7+2100x^6+5766x^5+6180x^4 \\ &+3884x^3+1780x^2+581x+123\Big) + n^7\Big(-1200x^7+20052x^6+38454x^5 \\ &+24258x^4+6380x^3+466x^2-189x+53\Big) + n^6\Big(-12128x^6+77484x^5 \\ &+104269x^4+37929x^3+3012x^2-354x-451\Big) + 2n^5\Big(-24768x^5+55519x^4 \\ &+51209x^3+12264x^2+1416x+833\Big) + n^4\Big(-74040x^4+32144x^3+13159x^2 \\ &-363x-3408\Big) - 3n^3(5440x^3+1330x^2+1145x-1401) + n^2\Big(16200x^2 \\ &+4626x-3221\Big) + 6n(60x+253) - 360\Big]^{1/4}\end{aligned}$$

and

$$\begin{aligned}\delta_2(x) &= \frac{1}{n^{5/4}(n-2)}\Big[16x + 16nx(2+31x) + 40n^2x(1+6x)^2 + 40n^3x(1+7x+20x^2+26x^3) \\ &+5n^4x(1+2x)^2(5+4x+12x^2) + n^5(1+2x)^4(1+x^2)\Big]^{1/4}.\end{aligned}$$

Proof. Following Theorem A, by simple computation, we have

$$b^{F_{k,n}} = F_{k,n}(e_1) = \frac{k}{n}, b^{G_{k,n}} = G_{k,n}(e_1) = \frac{k+1}{n-2}.$$

Also, we have

$$\mu_2^{F_{k,n}} := F_{k,n}(e_1 - b^{F_{k,n}}e_0)^2 = 0$$

and

$$\begin{aligned}\mu_2^{G_{k,n}} &:= G_{k,n}(e_1 - b^{G_{k,n}}e_0)^2 \\ &= G_{k,n}(e_2,x) + \left(\frac{k+1}{n-2}\right)^2 - 2G_{k,n}(e_1,x)\left(\frac{k+1}{n-2}\right) \\ &= \frac{(k+2)(k+1)}{(n-2)(n-3)} - \left(\frac{k+1}{n-2}\right)^2 \\ &= \frac{k^2+nk+n-1}{(n-2)^2(n-3)}.\end{aligned}$$

Next, using Lemma 2.1, we have

$$\begin{aligned}
\beta(x) &:= \sum_{k=0}^{\infty} s_{n,k}(x)\left[(1+(b^{F_{k,n}})^2)\mu_2^{F_{k,n}} + (1+(b^{G_{k,n}})^2)\mu_2^{G_{k,n}}\right] \\
&= \sum_{k=0}^{\infty} s_{n,k}(x)\left(1+\frac{(k+1)^2}{(n-2)^2}\right).\frac{k^2+nk+n-1}{(n-2)^2(n-3)} \\
&= \sum_{k=0}^{\infty} s_{n,k}(x)\left(\frac{k^2+nk+n-1}{(n-2)^2(n-3)} + \frac{(k^2+2k+1)(k^2+nk+n-1)}{(n-2)^4(n-3)}\right) \\
&= \sum_{k=0}^{\infty} s_{n,k}(x)\left(\frac{k^2+nk+n-1}{(n-2)^2(n-3)} + \frac{(k^4+(n+2)k^3+nk^2+(3n-2)k+n-1)}{(n-2)^4(n-3)}\right) \\
&= \frac{\left(n^2x^2+nx+n^2x+n-1\right)}{(n-2)^2(n-3)} \\
&\quad + \frac{1}{(n-2)^4(n-3)}\Big(n^4x^4+6n^3x^3+7n^2x^2+nx \\
&\quad + (n+2)[n^3x^3+3n^2x^2+nx] + n^3x^2+n^2x+(3n-2)nx+n-1\Big) \\
&= \frac{1}{(n-2)^4(n-3)}\Big(n^4x(x^3+x^2+x+1)+n^3(8x^3-3x+1) \\
&\quad + n^2(17x^2+5x-5)+n(5x+9)-5\Big)
\end{aligned}$$

Finally,

$$\mu_6^{F_{k,n}} := F_{k,n}(e_1-b^{F_{k,n}}e_0)^6 = 0$$

and

$$\begin{aligned}
\mu_6^{G_{k,n}} &:= G_{k,n}(e_1-b^{G_{k,n}}e_0)^6 \\
&= G_{k,n}(e_6,x) - 6G_{k,n}(e_5,x)\left(\frac{k+1}{n-2}\right) + 15G_{k,n}(e_4,x)\left(\frac{k+1}{n-2}\right)^2 \\
&\quad -20G_{k,n}(e_3,x)\left(\frac{k+1}{n-2}\right)^3 + 15G_{k,n}(e_2,x)\left(\frac{k+1}{n-2}\right)^4 \\
&\quad -6G_{k,n}(e_1,x)\left(\frac{k+1}{n-2}\right)^5 + \left(\frac{k+1}{n-2}\right)^6 \\
&= \frac{(k+1)_6}{(n-7)_6} - 6\frac{(k+1)_5}{(n-6)_5}\left(\frac{k+1}{n-2}\right) + 15\frac{(k+1)_4}{(n-5)_4}\left(\frac{k+1}{n-2}\right)^2 \\
&\quad -20\frac{(k+1)_3}{(n-4)_3}\left(\frac{k+1}{n-2}\right)^3 + 15\frac{(k+1)_2}{(n-3)_2}\left(\frac{k+1}{n-2}\right)^4 - 5\left(\frac{k+1}{n-2}\right)^6.
\end{aligned}$$

Then

$$
\begin{aligned}
\delta_1^4(x) &= \sum_{k=0}^{\infty} s_{n,k}(x)\left[(1+(b^{F_{n,k}})^2)\mu_6^{F_{k,n}} + (1+(b^{G_{n,k}})^2)\mu_6^{G_{k,n}}\right] \\
&= \sum_{k=0}^{\infty} s_{n,k}(x)(1+(b^{G_{n,k}})^2)\mu_6^{G_{k,n}} \\
&= \sum_{k=0}^{\infty} s_{n,k}(x)\left(1+\frac{(k+1)^2}{(n-2)^2}\right)\mu_6^{G_{k,n}} \\
&= \frac{5}{(n-7)_6(n-2)^7}\Big[3n^{10}x^3(x+1)^3(x^2+1) + 2n^9x^2\Big(37x^6+156x^5+263x^4 \\
&\quad +264x^3+191x^2+93x+22\Big) + n^8x\Big(-40x^7+2100x^6+5766x^5+6180x^4 \\
&\quad +3884x^3+1780x^2+581x+123\Big) + n^7\Big(-1200x^7+20052x^6+38454x^5 \\
&\quad +24258x^4+6380x^3+466x^2-189x+53\Big) + n^6\Big(-12128x^6+77484x^5 \\
&\quad +104269x^4+37929x^3+3012x^2-354x-451\Big) + 2n^5\Big(-24768x^5+55519x^4 \\
&\quad +51209x^3+12264x^2+1416x+833\Big) + n^4\Big(-74040x^4+32144x^3+13159x^2 \\
&\quad -363x-3408\Big) - 3n^3(5440x^3+1330x^2+1145x-1401) + n^2\Big(16200x^2 \\
&\quad +4626x-3221\Big) + 6n(60x+253) - 360\Big]
\end{aligned}
$$

and by using Lemma 2.1, we have

$$
\begin{aligned}
\delta_2^4(x) &= \sum_{k=0}^{\infty} s_{n,k}(x)(1+(b^{F_{n,k}})^2)(b^{F_{n,k}} - b^{G_{n,k}})^4 \\
&= \sum_{k=0}^{\infty} s_{n,k}(x)\left(1+\frac{k^2}{n^2}\right)\left(\frac{k}{n}-\frac{k+1}{n-2}\right)^4 \\
&= \sum_{k=0}^{\infty} s_{n,k}(x)\frac{(n+2k)^4(n^2+k^2)}{n^6(n-2)^4} \\
&= \frac{1}{(n-2)^4}\sum_{k=0}^{\infty} s_{n,k}(x)\frac{16k^6+32nk^5+40n^2k^4+40n^3k^3+25n^4k^2+8n^5k+n^6}{n^6} \\
&= \frac{1}{(n-2)^4}\Big[16S_n(e_6,x)+32S_n(e_5,x)+40S_n(e_4,x) \\
&\quad +40S_n(e_3,x)+25S_n(e_2,x)+8S_n(e_1,x)+1\Big] \\
&= \frac{1}{n^5(n-2)^4}\Big[16x+16nx(2+31x)+40n^2x(1+6x)^2+40n^3x(1+7x+20x^2+26x^3) \\
&\quad +5n^4x(1+2x)^2(5+4x+12x^2)+n^5(1+2x)^4(1+x^2)\Big]
\end{aligned}
$$

This completes the proof of the theorem. □

Remark 3.3. *One may find the quantitative difference estimates between the operators discussed here and Baskakov type operators [4], where different basis are concerned.*

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