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An Investigation on the Existence and Uniqueness Analysis of the Optimal Exercise Boundary of American Put Option

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Abstract. In this paper, we discuss the Banach fixed point theorem conditions on the optimal exercise boundary of American put option paying continuously dividend yield, to investigate whether its existence, uniqueness, and convergence are derived. In this respect, we consider the integral representation of the optimal exercise boundary which is extracted as a consequence of the Feynman-Kac formula. In order to prove the above features, we define a nonempty closed set in Banach space and prove that the proposed mapping is contractive and onto. At final, we illustrate the ratio convergence of the mapping on the optimal exercise boundary.

1. Introduction

The valuation of American option has a large literature in financial mathematics field ([6, 7, 10]). Since an American option can be exercised at any time prior to the expiration date, so the valuation of it needs to determine the optimal exercise boundary ([4, 12, 19, 21]). Therefore numerical and analytical representation of the optimal exercise boundary has attracted the attention of more researchers (see [23, 29, 32, 33, 35]). In this respect, some authors have introduced the integral representation for the optimal exercise boundary for American type option [28]. In [20], Goodman and et.al. developed an asymptotic expansion by using a boundary integral equation. Kim and Byun [22], presented the properties of the optimal boundary in an option pricing model and developed an efficient recursive valuation method. Also, Chen and Chadam[13] and Lauko [25] derived and rigorously proved high order asymptotic expansion for the early exercise boundary near expiry. As well, the differentiability of the optimal exercise boundary of American put option with the jump is considered in [8, 27]. Moreover, Chiarella and et.al. [16] and Zhu and et.al. [39] obtained the new integral equation for both the American put option and its optimal exercise, successfully, which is in advantages of one dimension and free of discontinuity and singularity at expiry date. In [38], Song-Ping Zhu introduced an analytical approximation formula for the optimal exercise boundary in the performance of the Laplace transform.
In particular, the existence and uniqueness of the optimal exercise boundary is of interest in some papers and references therein ([13, 29, 32]). It was realized that the existence and local uniqueness of a solution to

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nonlinear integral equations could be proved by applying the contraction principle (fixed point theorem) first for a small time interval and then extending it to any interval of time by induction [19]. Applying this method, Van Moerbeke [29] proved the existence and local uniqueness of a solution to the integral equations of a general optimal stopping problem. This paper revisits the classical topic of existence and uniqueness for the optimal early exercise boundary of an American put option when the underlying stock pays dividends continuously. Chen and Chadam [13] has explored this problem by using Schauder's fixed point theorem. In addition, Chen et al. [15] further proved that the free boundary is non-convex when the dividend rate is higher than the risk-free rate.

The new in this paper is that we reconsider the problem from the perspective of Banach fixed-point theorem. The presented approach clarifies this classical option valuation problem as a mathematical problem in functional analysis. We believe that the proposed lemmas and explored arguments in this paper would help not only for comprehension of the problem but also for the development of new analytical and numerical methods for its solution. So, we discuss the existence, uniqueness and convergence analysis of the optimal exercise boundary of American put option based on the Banach fixed point theorem [5], with local time. In order to achieve such a goal, we need to define a contractive and onto mapping on a nonempty closed set in Banach space. One of the technical difficulties in this context is that the derivative of the optimal boundary is not bounded at the initial point T. To avoid the singularity of the optimal exercise boundary near expiry we restrict our assumption on the case condition $q > r > 0$, where r and q denote the risk-free interest rate, and dividend yield respectively [34]. In the proceeding, the Lipschitz condition of the terms of the proposed integral equation is discussed. Moreover, the Hölder continuous with exponent $\frac{1}{2}$ of the optimal exercise boundary is also applied [9].

The outline of this paper is organized as follows. In section 2, we represented the dimensionless integral equation with respect to E of American put option, where E denotes the strike price. Some lemmas are discussed in section 3 and in the following, the Banach fixed point theorem is brought and its conditions are investigated on the proposed integral equation. In proceeding, two Remarks are developed, namely the first Remark discusses the existence and uniqueness of the optimal exercise boundary for American call option based on the put-call symmetry relationship, and the second Remark illustrates the convexity properties of the optimal exercise boundary near expiry based on different test cases for r and q. At final, in section 4, we illustrate the behavior of the optimal exercise boundary of American put option based on proposed mapping along with the ratio of its convergence.

2. Mathematical modelling

The Feynman-Kac formula named after Richard Feynman and Mark Kac establishes a link between parabolic partial differential equations (PDEs) and stochastic processes. It offers a method of solving certain partial differential equations by simulating random paths of a stochastic process. Conversely, an important class of expectations of random processes can be computed by deterministic methods. As in [38, 39], we consider the American put option, $P(S, t)$, which satisfies in the following partial differential equation

$$\frac{\partial P}{\partial t} + \frac{1}{2}S^2\sigma^2\frac{\partial^2 P}{\partial S^2} + (r-q)S\frac{\partial P}{\partial S} - rP = 0, \ S \in (S_f(t), \infty) \tag{1}$$

endowed with the initial conditions:

$$P(S,T) = \max(E - S, 0), \tag{2}$$

$$S_f^P(T) = E\min(1, \frac{r}{q}), \tag{3}$$

where $S_f^P(t)$, $t \in [0,T]$, denotes the optimal exercise boundary of American put option. Moreover the boundary condition for $P(S,t)$ when $S \to S_f^P(t)$, is writtern as follows:

$$P(S(t), t) = E - S_f^P(t), \tag{4}$$

along with the smooth pasting principle:

$$\frac{\partial P}{\partial S}(S_f^P(t), t) = -1, \tag{5}$$

Here $S > 0$ stands for the underlying stock price, $\sigma > 0$ is the volatility of the underlying stock process and T denotes the time of maturity.
Based on the Feynman-Kac formula, (see [24]) the solution $P(S, t)$ can be written by a conditional expectation as follows:

$$\begin{aligned} P(S,t) &= E_Q^t\left[\exp(-r(T-t))\max(E - S_T, 0)\right] \\ &\quad + \int_t^T \exp(-r(u-t))\, E_Q^u\left[(rE - qS_u)1_{S_u < S_f(u)}\right] du, \end{aligned} \tag{6}$$

under the probability measure Q such that stock price $S(t)$ driven by the equation

$$dS(t) = rS(t)\,dt + \sigma S(t)\,dW^Q(t),$$

where $W^Q(t)$ is a Wiener process defined on the probability space $\{\Omega, \mathcal{F}, Q\}$. Let us define $\tau = T - t$. The relation (6) can be evaluated to give the following representation of the American put price [10, 24, 25]:

$$\begin{aligned} P(S,\tau) &= E\exp(-r\tau)N(-d_2) - S\exp(-q\tau)N(-d_1) \\ &\quad + \int_0^\tau (Er\,\exp(-r\varepsilon)\,N(-d_{\varepsilon,2}) - qS\,\exp(-q\varepsilon)\,N(-d_{\varepsilon,1}))d\varepsilon, \end{aligned} \tag{7}$$

where

$$d_1 = \frac{\ln\frac{S}{E} + (r - q + \frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}}, \quad d_2 = d_1 - \sigma\sqrt{\tau}, \tag{8}$$

$$d_{\varepsilon,1} = \frac{\ln\frac{S}{S_f^P(\tau-\varepsilon)} + (r - q + \frac{\sigma^2}{2})\varepsilon}{\sigma\sqrt{\varepsilon}}, \quad d_{\varepsilon,2} = d_{\varepsilon,1} - \sigma\sqrt{\varepsilon}, \tag{9}$$

and $N(.)$ denotes the commulative normal distribution function.
The solution for $S_f^P(\tau)$ requires the knowledge of $S_f^P(\tau - \epsilon)$, $0 < \epsilon \le \tau$. If we apply the boundary condition (4) to the American put option price formula (7), we obtain the following integral equation for $S_f^P(\tau)$:

$$\begin{aligned} E - S_f^P(\tau) &= E\,\exp(-r\tau)\,N(-d_2') - S_f^P(\tau)\,\exp(-q\tau)\,N(-d_1') \\ &\quad + \int_0^\tau \left(Er\,\exp(-r\varepsilon)\,N(-d_{\varepsilon,2}') - qS_f^P(\tau)\,\exp(-q\varepsilon)\,N(-d_{\varepsilon,1}')\right) d\varepsilon, \end{aligned} \tag{10}$$

where

$$d_1' = \frac{\ln\frac{S_f^P}{E} + (r - q + \frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}}, \qquad d_2' = d_1' - \sigma\sqrt{\tau}, \tag{11}$$

$$d_{\varepsilon,1}' = \frac{\ln\frac{S_f^P(\tau)}{S_f^P(\tau-\varepsilon)} + (r - q + \frac{\sigma^2}{2})\varepsilon}{\sigma\sqrt{\varepsilon}}, \quad d_{\varepsilon,2}' = d_{\varepsilon,1}' - \sigma\sqrt{\varepsilon},\ 0 < \varepsilon \le \tau, \tag{12}$$

$$S_f^P(0) = \frac{r}{q}E. \tag{13}$$

Using the change of variable $y(\tau) = \frac{S_f^p(\tau)}{E}$, the relation (10) transforms to

$$1 - y(\tau) = \exp(-r\tau)\,N(-d_2^{'}) - y(\tau)\,\exp(-q\tau)\,N(-d_1^{'}) + \int_0^\tau \Big(r\,\exp(-r\varepsilon)\,N(-d_{\varepsilon,2}^{'}) - qy(\tau)\,\exp(-q\varepsilon)\,N(-d_{\varepsilon,1}^{'})\Big)\,d\varepsilon, \tag{14}$$

where

$$d_1^{'} = \frac{\ln y(\tau) + (r - q + \frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}}, \quad d_2^{'} = d_1^{'} - \sigma\sqrt{\tau}, \tag{15}$$

$$d_{\varepsilon,1}^{'} = \frac{\ln \frac{y(\tau)}{y(\tau-\varepsilon)} + (r - q + \frac{\sigma^2}{2})\varepsilon}{\sigma\sqrt{\varepsilon}}, \quad d_{\varepsilon,2}^{'} = d_{\varepsilon,1}^{'} - \sigma\sqrt{\varepsilon}, \tag{16}$$

$$y(0) = \frac{r}{q}. \tag{17}$$

3. Uniqueness and existence of the optimal exercise boundary

In this paper we investigate the conditions of the Banach fixed point theorem, to guarantee the uniqueness, existence, and convergence of the optimal exercise boundary of relation (14). First, we mention the following lemmas:

Lemma 3.1. *Let us define the functions h and g as folows:*

$$h(y(\tau), y(\tau-\varepsilon); \tau, \varepsilon) = r\,\exp(-r\varepsilon)\,N(-d_{\varepsilon,2}^{'}) - q\,y(\tau)\,\exp(-q\varepsilon)\,N(-d_{\varepsilon,1}^{'}),$$
$$g(y(\tau); \tau) = \exp(-r\tau)\,N(-d_2^{'}) - y(\tau)\,\exp(-q\tau)\,N(-d_1^{'}),$$

we claim that the above functions satisfy the Lipschitz conditions with respect to $y(\tau)$ and $y(\tau-\varepsilon)$.

Proof: First, we consider the function $h(y(\tau), y(\tau-\varepsilon); \tau, \varepsilon)$. It is sufficient to show that the function is differentiable with respect to $y(\tau)$. Similarly, the differentiability of h with respect to $y(\tau-\varepsilon)$ can be proved. So we have:

$$\frac{\partial h}{\partial y(\tau)} = r\,\exp(-r\varepsilon)\,\frac{\partial N(-d_{\varepsilon,2}^{'})}{\partial y(\tau)} - qy(\tau)\,\exp(-q\varepsilon)\,\frac{\partial N(-d_{\varepsilon,1}^{'})}{\partial y(\tau)} - q\,\exp(-q\varepsilon)N(-d_{\varepsilon,1}^{'}),$$

where

$$\frac{\partial N(-d_{\varepsilon,i}^{'})}{\partial y(\tau)} = \frac{\partial N(-d_{\varepsilon,i}^{'})}{\partial d_{\varepsilon,i}^{'}} \times \frac{\partial d_{\varepsilon,i}^{'}}{\partial y(\tau)} = -\frac{1}{\sqrt{2\pi}}\,\exp\Big(\frac{-d_{\varepsilon,i}^2}{2}\Big) \times \frac{y(\tau-\varepsilon) - y(\tau)}{y(\tau-\varepsilon) \times y(\tau)} \times \frac{1}{\sigma\sqrt{\varepsilon}}, \quad i = 1, 2. \tag{18}$$

Concerning the behavior of $\frac{\partial N(-d_{\varepsilon,i}^{'})}{\partial y(\tau)}$ for every $\tau \in [0, T]$, we have the following discussions:
1) Since the $y(\tau)$ is Hölder continuous with exponent $\frac{1}{2}$ for every time $\tau \in [0, T]$ [9], so there exists a constant $C \in \mathbb{R}$ such that

$$\|y(\tau-\varepsilon) - y(\tau)\|_\infty \leq C\sqrt{\varepsilon}. \tag{19}$$

2) As the functions $y(\tau)$ and $y(\tau-\varepsilon)$ are decreasing functions, so the given function $\frac{1}{y(\tau-\varepsilon)\times y(\tau)}$ in (18) is an increasing function and its supremum value is extracted as follows [24]:

$$\sup_{0\leq\varepsilon\leq\tau, 0\leq\tau\leq T} \frac{1}{y(\tau-\varepsilon)\times y(\tau)} = (\frac{\mu_- - 1}{\mu_-})^2, \tag{20}$$

where

$$\mu_- = \frac{-(r-q-\frac{\sigma^2}{2}) - \sqrt{(r-q-\frac{\sigma^2}{2})^2 + 2\sigma^2 r}}{\sigma^2} < 0. \tag{21}$$

Now based on points 1 and 2, we survey the upper bound for (18) in the following cases:
A) As $\tau \to 0^+$, from [24] for case $r < q$, we have:

$$y(T-\tau) \sim \frac{r}{q}(1 - \beta\sigma\sqrt{2\tau}), \tag{22}$$

where β is satisfying in the integral equation

$$-\beta^3 \exp(\beta^2) \int_\beta^\infty \exp(-u^2) du = \frac{1-2\beta^2}{4}, \tag{23}$$

which can be solved by the Newton iteration method ($\beta = 0.6438$). On the other hand, there exists a constant $0 < K < 1$ such that $\| \sqrt{\tau-\varepsilon} - \sqrt{\tau}\|_\infty \leq K\sqrt{\varepsilon}$. Therefore based on (17), (22) and (23) the simplified bounded for (18) will be

$$\begin{aligned} \|\frac{\partial N(-d'_{\varepsilon,i})}{\partial y(\tau)}\|_\infty &\leq \frac{1}{\sqrt{2\pi}} \|\exp(\frac{-d^2_{\varepsilon,i}}{2})\|_\infty \times \frac{\|y(\tau-\varepsilon) - y(\tau)\|_\infty}{\|y(\tau-\varepsilon)\times y(\tau)\|_\infty} \times \frac{1}{\sigma\sqrt{\varepsilon}}, \\ &\leq \frac{1}{\sqrt{2\pi}} \times 1 \times (\frac{q}{r})^2 \times \frac{r}{q}\beta\sigma K\sqrt{2\varepsilon} \times \frac{1}{\sigma\sqrt{\varepsilon}}, \\ &\leq \frac{1}{\sqrt{\pi}} \times \frac{q}{r} \times \beta \times K, \qquad i = 1,2, \end{aligned} \tag{24}$$

where the last inequality is less than one obviously, under the condition $r < q < \frac{r\sqrt{\pi}}{\beta K}$.
B) Secondly, for values of τ other than case A, containing $\tau \to \infty$, by using (16), (19) and (20) we have:

$$\begin{aligned} \|\frac{\partial N(-d'_{\varepsilon,i})}{\partial y(\tau)}\|_\infty &\leq \frac{1}{\sqrt{2\pi}} \|\exp(\frac{-d^2_{\varepsilon,i}}{2})\|_\infty \times \frac{\|y(\tau-\varepsilon) - y(\tau)\|_\infty}{\|y(\tau-\varepsilon)\times y(\tau)\|_\infty} \times \frac{1}{\sigma\sqrt{\varepsilon}} \\ &\leq \frac{1}{\sqrt{2\pi}} \exp(-a^2\varepsilon) \times C\sqrt{\varepsilon} \times (\frac{\mu_- - 1}{\mu_-})^2 \times \frac{1}{\sigma\sqrt{\varepsilon}}, \qquad i = 1,2, \end{aligned} \tag{25}$$

where a is a constant value derived from (16), and by choosing sufficiently large value $0 < \varepsilon \leq \tau$, the supremum value of (25) is less than one. Therefore, the $\frac{\partial N(-d'_{\varepsilon,i})}{\partial y(\tau)}$ and subsequently $\frac{\partial h}{\partial y(\tau)}$ are exist and satisfy the Lipschitz condition.

Now for the function $g(y(\tau);\tau)$ we can conclude that:

$$\begin{aligned} g(y(\tau),\tau) &= \exp(-r\tau) N(-d'_2) - y(\tau) \exp(-q\tau) N(-d'_1) \\ &= \frac{1}{\sqrt{2\pi}}\Big(\exp(-r\tau) \int_{-\infty}^{-d'_2} \exp(-\frac{x^2}{2}) dx \\ &\quad - y(\tau) \exp(-q\tau) \int_{-\infty}^{-d'_1} \exp(-\frac{x^2}{2}) dx\Big), \end{aligned}$$

where $d_1^{'}$ and $d_2^{'}$ are defined in (15).
So, the derivative of function $g(y(\tau), \tau)$ with respect to $y(\tau)$ is written:

$$\frac{\partial g}{\partial y(\tau)} = \frac{1}{\sqrt{2\pi}}\Big(\frac{\exp(-r\tau)\exp(-\frac{d_2'^2}{2})}{\sigma\sqrt{\tau}y(\tau)} - \exp(-q\tau)\int_{-\infty}^{-d_1'}\exp(-\frac{x^2}{2})dx - \frac{1}{\sigma\sqrt{\tau}}\exp(-q\tau)\exp(\frac{-d_1'^2}{2})\Big). \quad (26)$$

Concerning the first and third terms of (26), we compute:

$$\begin{aligned}&\frac{\exp(-r\tau)\exp(-\frac{-d_2'^2}{2})}{\sigma\sqrt{\tau}y(\tau)} - \exp(-q\tau)\frac{\exp(\frac{-d_1'^2}{2})}{\sigma\sqrt{\tau}}\\ &= \frac{1}{\sigma\sqrt{\tau}}\exp(\frac{-d_1'^2}{2})\Big(\frac{\exp(-r\tau)\exp(\sigma\sqrt{\tau}d_1' - \frac{\sigma^2\tau}{2})}{y(\tau)} - \exp(-q\tau)\Big)\\ &= \frac{1}{\sigma\sqrt{\tau}}\exp(\frac{-d_1'^2}{2})(\exp(-q\tau) - \exp(-q\tau)) = 0, \end{aligned} \quad (27)$$

where in the last line, we have used the relation (15).
So, relation (26) simplifies as follows:

$$\frac{\partial g}{\partial y(\tau)} = \exp(-q\tau)N(-d_1'). \quad (28)$$

Therefore, the function $g(y(\tau);\tau)$ is differentiable with respect to $y(\tau)$ for every time $\tau \in [0, T]$ and its derivative is smaller than one clearly, so it satisfies the Lipschitz condition.

Now we are moving to the Banach fixed point theorem to investigate the existence and uniqueness of the optimal exercise boundary for the American put option of relation (14).

Theorem 3.2. (Banach Fixed-Point Theorem) *Assume that K is a nonempty closed set in a Banach space V, and further, that is $H : K \longrightarrow K$ a contractive mapping with contractivity constant $\alpha, 0 \leq \alpha < 1$. Then the following results hold.*

1) *Existence and uniquenes: There exists a unique $u \in K$ such that $u = H(u)$.*
2) *Convergence and error estimates of the iteration: For any $u_0 \in K$, the sequence $\{u_n\} \subset K$ defined by $u_{n+1} = H(u_n), n = 0, 1, ...,$ converges to u:$\|u_n - u\|_V \to 0$ as $n \to \infty$.*
3) *For the error, the following bounds are valid:*

$$\begin{aligned}&\|u_n - u\|_V \leq \frac{\alpha^n}{1-\alpha}\|u_0 - u_1\|_V, \\ &\|u_n - u\|_V \leq \frac{\alpha}{1-\alpha}\|u_{n-1} - u_n\|_V, \\ &\|u_n - u\|_V \leq \|u_{n-1} - u\|_V. \end{aligned} \quad (29)$$

Based on the Theorem (3.2), let us consider $X = C[0, \delta]$ with the norm $\|.\|_\infty$, which the parameter δ can be a positive value. We define a mapping $H : X \to X$ as follows:

$$\begin{aligned}H(y)(\tau) &= 1 - \exp(-r\tau)N(-d_2^{'}) + y(\tau)\exp(-q\tau)N(-d_1^{'})\\ &- \int_0^\tau \Big(r\exp(-r\varepsilon)N(-d_{\varepsilon,2}^{'}) - qy(\tau)\exp(-q\varepsilon)N(-d_{\varepsilon,1}^{'})\Big)d\varepsilon. \end{aligned} \quad (30)$$

It is sufficient to prove that the defined mapping H in (30) is onto and contractive. First, we claim that H is an onto mapping.

Lemma 3.3. *Assume that Y is a nonempty closed ball in X, we show that $H : Y \to Y$ is onto mapping.*

proof: Let us define

$$Y = \{y(\tau) \in X \mid y(0) = \frac{r}{q}, \|y(\tau) - 1\|_\infty \leq 1\}, \tag{31}$$

it is sufficient to show that $\|H(y)(\tau) - 1\|_\infty < 1$. By considering the Lipschitz condition of function h in Lemma (3.1), we have:

$$\begin{aligned}\|H(y)(\tau) - 1\|_\infty &\leq \| - \exp(-r\tau)\,N(-d_2^{'}) + y(\tau)\,\exp(-q\tau)\,N(-d_1^{'}) \\ &- \int_0^\tau h(y(\tau), y(\tau-\epsilon); \tau, \epsilon)d\varepsilon\|_\infty \\ &\leq \| - \exp(-r\tau)\,N(-d_2^{'}) + y(\tau)\,\exp(-q\tau)\,N(-d_1^{'})\|_\infty \\ &+ \int_0^\tau \|h(y(\tau), y(\tau-\epsilon); \tau, \epsilon)\|_\infty d\varepsilon \\ &\leq L_1 + L_2\,\delta \leq 1,\end{aligned} \tag{32}$$

where the last inequality is derived due to the fact that

$$L_1 = \| - \exp(-r\tau)\,N(-d_2^{'}) + y(\tau)\,\exp(-q\tau)\,N(-d_1^{'})\|_\infty < 1, \tag{33}$$

since

$$0 < y(\tau) = \frac{S_f^p(\tau)}{E} < \frac{S_f^p(0)}{E} = \frac{r}{q} < 1, \tag{34}$$

and L_2 is extracted from the continuity of the function h which is given by Lemma (3.1) with respect to τ. So, by choosing $\delta \leq \frac{(1-L_1)}{L_2}$, we can conclude that the mapping H is onto.

Lemma 3.4. *By assumption the mapping H in relation* (30)*, and considering Lemma* (3.3)*, we show that H is a contractive mapping on Y.*

proof:

$$\begin{aligned}\|H(y_2)(\tau) - H(y_1)(\tau)\|_\infty &\leq \|g(y_2(\tau), \varepsilon) - \int_0^\tau h(y_2(\tau), y_2(\tau-\epsilon); \tau, \epsilon)d\varepsilon \\ &+ \int_0^\tau h(y_1(\tau), y_1(\tau-\epsilon); \tau, \epsilon)d\varepsilon - g(y_1(\tau), \varepsilon)\|_\infty \\ &\leq \|g(y_2(\tau), \varepsilon) - g(y_1(\tau), \varepsilon)\|_\infty \\ &+ \int_0^\tau \Big(\|h(y_1(\tau), y_1(\tau-\epsilon); \tau, \epsilon) \\ &- h(y_2(\tau), y_2(\tau-\epsilon); \tau, \epsilon)\|_\infty d\varepsilon\Big) \leq C_1\|y_2(\tau) - y_1(\tau)\|_\infty \\ &+ \int_0^\tau \Big(C_2\|y_2(\tau) - y_1(\tau)\|_\infty \\ &+ C_3\|y_2(\tau-\epsilon) - y_1(\tau-\epsilon)\|_\infty\Big)d\varepsilon \\ &\leq C_1\|y_2(\tau) - y_1(\tau)\|_\infty + (C_2 + C_3)\|y_2(\tau) - y_1(\tau)\|_\infty \tau \\ &\leq \|y_2(\tau) - y_1(\tau)\|_\infty (C_1 + (C_2 + C_3)\delta),\end{aligned} \tag{35}$$

where C_1 and C_2, C_3 are the Lipschitz constants of the functions g and h respectively. The equality $\|y_2(\tau) - y_1(\tau)\|_\infty = \|y_2(\tau-\epsilon) - y_1(\tau-\epsilon)\|_\infty$ is deduced by using the definition of $\|.\|_\infty$. Moreover, the last inequality

is obtained because of $\delta \leq \frac{(1-C_1)}{C_2+C_3}$, where $C_1 < 1$, which is discussed in Lemma (3.1). Therefore, we can conclude that the mapping H is a contractive mapping on Y, and the proof is complete.
At final, by taking proper value for δ such that $\delta \leq \min\left(\frac{(1-C_1)}{C_2+C_3}, \frac{(1-L_1)}{\sqrt{L_2}}\right)$, the conditions of the Banach fixed-point theorem are satisfied. So the existence and uniqueness of the optimal exercise boundary for the American put option in (10) are verified.

Remark 3.5. *Modification to American call option*
In [24, 30, 36], the authors show that in the standard model for stock price (geometric Brownian motion), the put-call symmetry relation for the American option with parameters S, E, r, q and T is as follows

$$C(S, E, r, q, T) = P(E, S, q, r, T), \tag{36}$$

where $C(S, E, r, q, T)$ denotes the American call option.
By setting $C(S_f^C(\tau), \tau) = S_f^C(\tau) - E$ for $S > S_f^C(\tau)$, and $S_f^C(\tau)$ denotes the optimal exercise boundary for American call option, the corresponding integral equation for the early exercise boundary $S_f^C(\tau)$ can be obtained as follows:

$$\begin{aligned} S_f^C(\tau) - E = {} & S_f^C(\tau)\, \exp(-q\tau)\, N(d_1^{'}) - E\, \exp(-r\tau)\, N(d_2^{'}) \\ & + \int_0^\tau \left(q S_f^C(\tau)\, \exp(-q\varepsilon)\, N(d_{\varepsilon,1}^{'}) - Er\, \exp(-r\varepsilon)\, N(d_{\varepsilon,2}^{'}) \right) d\varepsilon. \end{aligned} \tag{37}$$

Moreover, the corresponding put-call symmetry relation for the optimal exercise boundary is deduced to be (see [24]):

$$S_f^C(\tau, r, q) = \frac{E^2}{S_f^P(\tau, q, r)}. \tag{38}$$

Then, based on (3) *and* (38), *we can deduce $S_f^C(\tau) = E \max(1, \frac{r}{q})$ at $\tau \to 0^+$. The approximate solution of the optimal exercise boundary for an American call option on an asset with dividends for the case of $r > q$ as $\tau \to 0^+$ is given by (see [17] and [37]):*

$$S_f^C(T - \tau) \sim \frac{r}{q} E(1 + \beta\sigma \sqrt{2\tau}), \tag{39}$$

Similarly, we can conclude that the conditions of Banach fixed point theorem are satisfied for American call option in the case $r > q$.

Remark 3.6. *Nonconvexity of the American put option near expiry under the case $q > r$*
The convexity of the optimal exercise boundary are discussed and illustrated in the references [11, 14], whereas in [14, 18] the convexity property is proved only for the case $q = 0$. As refer to [15], the authors proved the non-convexity of the optimal exercise boundary for the American put option, under the case $q > r$. Moreover in [31], the author proved that the optimal exercise boundary for American put option is convex only for the case $q + \frac{\sigma^2}{2} \leq r$, but its convexity is an open problem under the case $q < r < q + \frac{\sigma^2}{2}$. Also, baesd on relation (10), *we illustrate the behavior of $S_f^P(\tau)$ for three different cases $q > r$, $q = r$ and $q < r$. As we see in Fig. 1, the optimal exercise boundary is convex for cases $q = r$ and $q < r$, but for the case $q > r$, we observe that the $S_f^P(\tau)$ is non-convex near expiry.*

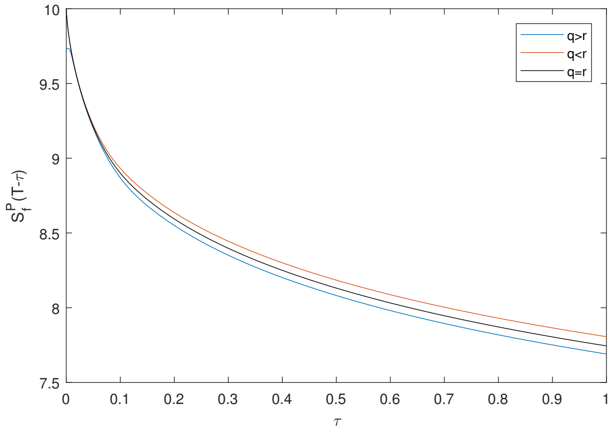

Figure 1: The behavior of the optimal exercise boundary with respect to τ with different cases $q > r$, $q < r$ and $q = r$

4. Numerical illustration

Based on Theorem (3.2), the parameter α adjusts the rate of convergence of mapping H defined in (30). The magnitude of value α relates to the Lipschitz constant C_1 which is derived in Lemma (3.1). Since the value C_1 is sufficiently smaller than 1, we can claim that the proposed mapping H defined in (30) converges to the exact solution in the interval $\tau \in [0, \delta]$. To give the reader an idea of the behavior of the optimal exercise boundary in (10), we provide in Figure (1), the numerical approximation of mapping H, with typical parameters. In Figure (1), we see the desired solution converges to the value $\frac{\mu_-}{\mu_- - 1}E$, as τ increases, where μ_- is obtained from (21).

As we observe for the behavior plot (blue line) near expiry in Figure (1), we can declare that the $S_f^P(\tau)$ based on (22) as $\tau \to 0^+$ follows (see [24]):

$$S_f^P(\tau) \sim \frac{rE}{q}(1 - \beta\sigma\sqrt{2\tau}), \tag{40}$$

so it is in negative infinite slope, but based on the decreasing property of the optimal exercise boundary for the American put option for the whole time domain, then the proposed plot in Figure (1) is in coincidence with our interpretation. However, for the cases $q < r$ and $q = r$, the optimal exersice boundary for the American put option is in positive infinite slope, as are illustraed in Figure (1) in red and black lines.

Let $S_{fAP}^P(\tau)$ denote the approximate solution using the numerical method on the fixed point mapping defined in (30). Moreover, in $[0, T]$ let us consider $N+1$ times $\tau_0, \tau_1, \ldots, \tau_N$, such that $\tau_i = i\Delta\tau, i = 0, 1, \ldots, N, \Delta\tau = \frac{T}{N_1}$. The error on $S_{fAP}^P(\tau)$ at time points $\tau_i = 0, 1, ..., N$, is computed as follows:

$$Error = \max_{i=0,1,\ldots,N} |S_{fAP}^P(\tau_i) - S_f^P(\tau_i)|. \tag{41}$$

Following a very common practice, in order to test the convergence of the mapping defined in (30), the number of N nodes is progressively doubled, and the convergence rate is empirically estimated by computing the ratios of successive values of error. Finally, as the true solution $S_f^P(\tau)$, needed in (41), is not available, a very accurate estimation of it is obtained by employing with a very large number of time steps ($N = 4096$).

Table 1: Model parameters

T	E	r	q	σ
1	10	0.1	0.12	0.15

Table 2: Convergence of the fixed point mapping (30), the *Error* measured according to (41) for data in Table 1

N	*Error*	*Ratio*
64	8.67×10^{-2}	
128	1.37×10^{-2}	6.33
256	5.80×10^{-3}	2.36
512	1.88×10^{-3}	3.08

The results obtained are shown in Table 2. We may observe that, as is reasonable to expect in part (iii) of Banach fixed point theorem, the ratio errors tend to be higher than 4 when N is doubled. In particular, a very accurate approximation (error of order 10^{-3} on the whole set of time nodes) with $N = 512$ is achieved.

5. Conclusion

In this paper, we are interested in investigating whether the existence and uniqueness of the optimal exercise boundary for the American put option can be derived based on the Banach fixed-point theorem. To deal with, we restrict our assumption on the case $q > r$, and define a mapping based on the fixed point theorem, which satisfies the onto and contractive conditions. In addition to them, the ratio convergence of the proposed mapping is investigated. Moreover, based on put-call symmetry, we conclude that the Banach fixed point theorem conditions are satisfied for the American call option too. As well, we discuss and illustrate the nonconvexity of the optimal exercise boundary for the American put option near expiry under the case $q > r$ whereas in the cases $q < r$ and $q = r$ the convexity holds.

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