Filomat 35:4 (2021), 1087–1093
https://doi.org/10.2298/FIL2104087Z

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Generalized Inverses and Solutions to Related Equations

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Abstract. In this paper, some new characterizations of partial isometries and strongly EP elements are investigated. Especially, we discuss the existence of the solutions of certain equations in a given set to characterize partial isometries, strongly EP elements and so on.

1. Introduction

Let R be an associative ring with 1. An element $a \in R$ is said to be group invertible if there exists $a^\# \in R$ such that

$$aa^\# a = a, \quad a^\# aa^\# = a^\#, \quad aa^\# = a^\# a.$$

The element $a^\#$ is called the group inverse of a, which is uniquely determined by the above equations [1]. The set of all group invertible elements of R will be denoted by $R^\#$.
An involution $* : a \longmapsto a^*$ in a ring R is an anti-isomorphism of degree 2, that is,

$$(a^*)^* = a, \quad (a+b)^* = a^* + b^*, \quad (ab)^* = b^* a^*.$$

An element a^+ in R is called the Moore-Penrose inverse (MP-inverse) of a [6] when satisfying the following conditions.

$$aa^+ a = a, \quad a^+ aa^+ = a^+, \quad (aa^+)^* = aa^+, \quad (a^+ a)^* = a^+ a.$$

If such a^+ exists, then it is unique [6]. We write R^+ for the set of all MP-invertible elements of R. a is said to be EP if $a \in R^\# \cap R^+$ and satisfies $a^\# = a^+$ [2, 9]. We then denote by R^{EP} the set of all EP elements of R. If $a \in R^+$ and $a^+ = a^*$, the element a is called partial isometry. Furthermore, a is called strongly EP element if $a \in R^{EP}$ is a partial isometry. Denote by R^{PI} and R^{SEP} the set of all partial isometry elements and strongly EP elements [7] of R.

In [3], D. Mosić and D. S. Djordjević presented some equivalent conditions for the element a in a ring with involution to be a partial isometry. In addition, some characterizations of EP elements in ring with involution were given. Recently, some studies on partial isometries and EP elements have come to some

2010 *Mathematics Subject Classification*. 15A09; 16U99; 16W10
Keywords. strongly EP element, partial isometry, solutions of equation.
Received: 22 March 2020; Revised: 18 November 2020; Accepted: 23 November 2020
Communicated by Dijana Mosić
Corresponding author: Yue Sui
Research supported by the National Natural Science Foundation of China (11471282) and Innovation and Entrepreneurship Training Program for College Students in Jiangsu Province (202011117097Y)
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meaningful conclusions in [5, 8, 10, 11]. Moreover, the description of *EP* elements by using solutions of equations has been explored in [8, 11, 12].

Inspired by the above articles, we consider the characterization of partial isometries and strongly *EP* elements from the perspective of the solutions of the certain equations in this paper. We give some new equivalent conditions for elements in a ring with involution to be partial isometries and strongly *EP* elements. Let $\chi_a = \{a, a^\#, a^+, a^*, (a^\#)^*, (a^+)^*\}$. It will be proved that the equation $a^*x = a^+xa^\#a$ has at least one solution in χ_a if and only if $a \in R^{PI}$. Also, we show that $a \in R^{SEP}$ if and only if the equation $a^*x = xa^+a^\#a$ has at least one solution in χ_a. By constantly revising the above equation, we get similar results in the following equations $a^*xa = xa^+a$, $a^*xa = xa^+a$ and $a^*xy = xa^+y$.

2. Results

Lemma 2.1. *Let $a \in R^\# \cap R^+$ and $x \in R$. If $a^+a^*x = 0$, then $a^*x = 0$.*

Proof. Pre-multiply $a^+a^*x = 0$ by $a^*(a^\#)^*a$, we arrive at the conclusion. □

The proof of [3, Thoerem 2.3] shows that: Let $a \in R^\# \cap R^+$. Then $a \in R^{SEP}$ if and only if $a^*a^+ = a^+a^\#$. Noting that $a^\# = a^\#a^+a$. Hence $a \in R^{SEP}$ if and only if $a^*a^+ = a^+a^\#a^+a$. This elicits the following equation.

$$a^*x = a^+a^\#xa. \tag{1}$$

Theorem 2.2. *Let $a \in R^\# \cap R^+$. Then $a \in R^{PI}$ if and only if the equation (1) has at least one solution in $\{a, a^\#, a^+, (a^\#)^*\}$.*

Proof. $\Rightarrow$ Assume that $a \in R^{PI}$, then $a^+ = a^*$, this infers $x = a$ is a solution.

$\Leftarrow$ (1) If $x = a$ is a solution, then $a^*a = a^+a^\#a^2 = a^+a$. Hence $a \in R^{PI}$ by [3, Theorem 2.1].

(2) If $x = a^\#$ is a solution, then $a^*a^\# = a^+a^\#a^\#a = a^+a^\#$. We can soon deduce that $a \in R^{PI}$ by post-multiplying it by a^2.

(3) If $x = a^+$ is a solution, then $a^*a^+ = a^+a^\#a^+a = a^+a^\#$. By [3, Theorem 2.3], $a \in R^{PI}$.

(4) If $x = (a^\#)^*$ is a solution, then $a^*(a^\#)^* = a^+a^\#(a^\#)^*a$. Post-multiply the equality by aa^+, we have

$$a^+a^\#(a^\#)^*a = a^+a^\#(a^\#)^*a^2a^+.$$

Pre-multiply the last equality by a^3, we get

$$a(a^\#)^*a = a(a^\#)^*a^2a^+.$$

Again, pre-multiply the above mentioned equality by $a^*a^*a^+$, one has $a^*a = a^*a^2a^+$, so

$$a = (a^+)^*a^*a = (a^+)^*a^*a^2a^+ = a^2a^+.$$

It follows that $a \in R^{EP}$ by [12, Corollary 2.14].

Now, we observe that

$$a^+a = (a^+a)^* = a^*(a^+)^* = a^*(a^\#)^* = a^+a^\#(a^\#)^*a.$$

and then

$$a = aa^+a = aa^+a^\#(a^\#)^*a = a^\#(a^\#)^*a.$$

Thus $a^+a = aa^+ = a^\#(a^\#)^*aa^+ = a^\#(a^\#)^* = a^+(a^+)^*$. It is immediate that $a^* = a^+aa^* = a^+(a^+)^*a^* = a^+aa^+ = a^+$, which leads to $a \in R^{PI}$. □

Question 2.3. *If $x = a^*$ or $x = (a^+)^*$ is a solution of the equation* (1), *does $a \in R^{PI}$?*

Modifying the equation (1), we have the equation as follows.

$$a^*x = a^+xa^\#a. \tag{2}$$

Theorem 2.4. *Let $a \in R^{\#} \cap R^{+}$. Then $a \in R^{PI}$ if and only if the equation (2) has at least one solution in χ_a.*

Proof. $\Rightarrow$ Obviously, $x = a$ is a solution of the above equation.

$\Leftarrow$ (1) If $x = a$ is a solution, then $a^*a = a^+aa^{\#}a = a^+a$. This means $a \in R^{PI}$ by [3, Theorem 2.1].

(2) If $x = a^{\#}$ is a solution, then $a^*a^{\#} = a^+a^{\#}a^{\#}a = a^+a^{\#}$. It evident that $a \in R^{PI}$.

(3) If $x = a^+$ is a solution, then $a^*a^+ = a^+a^+a^{\#}a$. Multiply the equality on the right by a, we have $a^*a^+a = a^+a^+a$. Taking involution of the above equality, we deduce that $a^+a^2 = a^+a(a^+)^*$. Pre-multiply the last equality by a, we get $a^2 = a(a^+)^*$. Multiply the equation from the right by a^*, we get $a^2a^* = a^2a^+$. Per-multiply the equation by $a^+a^{\#}$, one has $a^* = a^+$. Hence $a \in R^{PI}$.

(4) If $x = a^*$ is a solution, then $a^*a^* = a^+a^*a^{\#}a$, this gives $a^*a^*(1 - a^+a) = 0$. Notice that

$$a^*(1 - a^+a) = (a^{\#})^*a^*a^*(1 - a^+a) = 0,$$

which means $a^* = a^*a^+a = (a^+a^2)^*$. Apply involution to the equation, one has $a = a^+a^2$. It follows from [8, Corollary 2.12] that $a \in R^{EP}$. We further obtain that

$$a^*a^* = a^+a^*a^{\#}a = a^+a^*a^a\# = a^+a^*aa^+ = a^+a^*,$$

it follows that $a^2 = a(a^+)^*$. Hence $a \in R^{PI}$.

(5) If $x = (a^{\#})^*$ is a solution, then $a^*(a^{\#})^* = a^+(a^{\#})^*a^{\#}a$. We soon get that $a^*(a^{\#})^*(1 - a^+a) = 0$. This gives

$$a^*(1 - a^+a) = a^*a^*(a^{\#})^*(1 - a^+a) = 0.$$

It is easy to see $a \in R^{EP}$. On the other hand,

$$a^+a = a^*(a^+)^* = a^*(a^{\#})^* = a^+(a^{\#})^*a^{\#}a = a^+(a^+)^*a^+a = a^+(a^+)^*.$$

Consequently, $a \in R^{PI}$.

(6) If $x = (a^+)^*$ is a solution, then $a^*(a^+)^* = a^+(a^+)^*a^{\#}a$, which means that $a^+a = a^+(a^+)^*a^{\#}a$. It is straightforward that

$$a = aa^+a = aa^+(a^+)^*a^{\#}a = (a^+)^*a^{\#}a.$$

Post-multiply the last equation by a, we have $a^2 = (a^+)^*a$. Hence $a \in R^{PI}$. □

Observing the proof of Theorem 2.4, we have the following corollary.

Corollary 2.5. *Let $a \in R^{\#} \cap R^{+}$. Then $a \in R^{PI}$ if and only if $a^*a^+a = a^+a^+a$.*

Next, we revise the equation (2) as follows.

$$a^*x = xa^+a^{\#}a. \tag{3}$$

Theorem 2.6. *Let $a \in R^{\#} \cap R^{+}$. Then $a \in R^{SEP}$ if and only if the equation (3) has at least one solution in χ_a.*

Proof. $\Rightarrow$ Since $a \in R^{SEP}$, $x = a$ is a solution by [4, Theorem 2.2(iv)].

$\Leftarrow$ (1) If $x = a$ is a solution, then $a^*a = aa^+a^{\#}a = aa^{\#}$. By [3, Theorem 2.3(v)], we know $a \in R^{SEP}$.

(2) If $x = a^{\#}$ is a solution, then $a^*a^{\#} = a^{\#}a^+a^{\#}a = a^{\#}a^{\#}$. It is immediate that $a \in R^{SEP}$ by [3, Theorem 2.3(xiv)].

(3) If $x = a^+$ is a solution, then $a^*a^+ = a^+a^+a^{\#}a$. Post-multiply it by a, we have

$$a^*a^+a = a^+a^+a.$$

Hence $a \in R^{PI}$ by Corollary 2.5. In addition, we know

$$a^*a^* = a^*a^+ = a^+a^+a^{\#}a = a^*a^*a^{\#}a.$$

Pre-multiply it by $(a^{\#})^*$, one has $a^* = a^*a^{\#}a$, which implies $a \in R^{EP}$. Consequently, $a \in R^{SEP}$.

(4) If $x = a^*$ is a solution, then $a^*a^* = a^*a^+a^\# a$. Pre-multiply it by $a^+(a^+)^*$, we get

$$a^+a^* = a^+a^+a^\# a = a^+a^*(a^+)^*a^\#.$$

By Lemma 2.1, we have $a^* = a^*(a^+)^*a^\# = a^+aa^\#$. Then we multiply the equality on the right by a and obtain that $a^*a = a^+a$, which shows $a \in R^{PI}$. Now, we have $a^*a^* = a^*a^+a^\# a = a^*a^*a^\# a$, it follows from the proof of (3) that $a \in R^{SEP}$

(5) If $x = (a^\#)^*$ is a solution, then $a^*(a^\#)^* = (a^\#)^*a^+a^\# a$. Pre-multiply it by a^*a^*, one has

$$a^*a^* = a^*a^+a^\# a.$$

It follows from (4) that $a \in R^{SEP}$.

(6) If $x = (a^+)^*$ is a solution, then $a^*(a^+)^* = (a^+)^*a^+a^\# a$, that is $a^+a = (a^+)^*a^+a^\# a$. Multiply it from the right by a, one has $a^+a^2 = (a^+)^*$. Hence $a \in R^{SEP}$ by [3, Theorem 2.3(xviii)]. □

Lemma 2.7. *Let $a \in R^\# \cap R^+$ and $x \in R$. If $a^+a^+x = 0$, then $a^+x = 0$.*

Proof. Since $a^+a^+x = 0$, $a^*a^+x = a^*aa^+a^+x = 0$. Pre-multiply the equality by $(a^\#)^*$, one has $(a^\#)^*a^*a^+x = 0$. Noting that $(a^\#)^*a^*a^+ = (a^\#)^*a^*a^+aa^+ = (a^+a^2a^\#)^*a^+ = a^+$. Then $a^+x = 0$. □

Further, we get the following equation by post-multiplying the equation (3) by a.

$$a^*xa = xa^+a. \tag{4}$$

Theorem 2.8. *Let $a \in R^\# \cap R^+$. Then $a \in R^{PI}$ if and only if the equation (4) has at least one solution in χ_a.*

Proof. ⇒ $x = a^+$ is a solution because $a^* = a^+$.

⇐ (1) If $x = a$ is a solution, then $a^*a^2 = aa^+a = a$. Thus, we deduce that $a \in R^{PI}$ by [3, Theorem 2.3(xix)].

(2) If $x = a^\#$ is a solution, then $a^*a^\# a = a^\# a^+a = a^\#$. Hence $a \in R^{PI}$ from [3, Theorem 2.3].

(3) If $x = a^+$ is a solution, then $a^*a^+a = a^+a^+a$. It follows from Corollary 2.5 that $a \in R^{PI}$.

(4) If $x = a^*$ is a solution, then $a^*a^*a = a^*a^+a$. Pre-multiply the equation by $a^+(a^+)^*$, one has $a^+a^*a = a^+a^+a$, it follows

$$a^+a^* = a^+a^*aa^+ = a^+a^+aa^+ = a^+a^+.$$

Post-multiply the last equality by $(a^+)^*$, one has $a^+a^+a = a^+a^+(a^+)^*$. By Lemma 2.7, we have $a^+a = a^+(a^+)^*$, it follows that $a^* = a^+aa^* = a^+(a^+)^*a^* = a^+$. Thus $a \in R^{PI}$.

(5) If $x = (a^\#)^*$ is a solution, then $a^*(a^\#)^*a = (a^\#)^*a^+a$. Multiply it by a^*a^* from the left, we obtain

$$a^*a^*a = a^*a^+a.$$

From the proof of (4), $a \in R^{PI}$.

(6) If $x = (a^+)^*$ is a solution, then $a^*(a^+)^*a = (a^+)^*a^+a$. That is $a^+a^2 = (a^+)^*$, this gives $a^2 = aa^+a^2 = a(a^+)^*$, which implies $a \in R^{PI}$. □

From the proof of (4) in Theorem 2.8, we have the following corollary.

Corollary 2.9. *Let $a \in R^\# \cap R^+$. Then the following conditions are equivalent:*

1) $a \in R^{PI}$;

2) $a^+a^* = a^+a^+$;

3) $a^*a^+ = a^+a^+$.

Naturally, by observing the equation (1), equation (2) and equation (3), we come up with the following equation.

$$a^*x = a^+a^\# ax. \tag{5}$$

Considering whether a is a partial isometry in relation to the solution of the equation (5) leads to the following problem.

Question 2.10. *Let $a \in R^{\#} \cap R^{+}$. If $a^*a^+a = a^+a^{\#}a$, does $a \in R^{PI}$?*

Lemma 2.11. *Let $a \in R^{\#} \cap R^{+}$. If $a^*a^+a^+ = a^+a^+a^+$, then $a \in R^{PI}$.*

Proof. Since $a^*a^+a^+ = a^+a^+a^+$, we get $a^*a^+a^+a = a^+a^+a^+a$. Applying the involution on the equality, we have

$$a^+a(a^+)^*a = a^+a(a^+)^*(a^+)^*.$$

Pre-multiply the last equality by a, we have

$$a(a^+)^*(a - (a^+)^*) = 0.$$

Noting that $(a^+)^* = aa^+(a^+)^*$, then $a^2a^+(a^+)^*(a - (a^+)^*) = 0$. Multiply it by $a^{\#}$ on the left, one has

$$(a^+)^*(a - (a^+)^*) = 0.$$

Hence $a^*a^+ = a^+a^+$. By Corollary 2.9, $a \in R^{PI}$. □

The proof of Lemma 2.11 infers the following corollary.

Corollary 2.12. *Let $a \in R^{\#} \cap R^{+}$. If $a(a^+)^*x = 0$, then $(a^+)^*x = 0$.*

Lemma 2.13. *Let $a \in R^{\#} \cap R^{+}$. If $a^*a^*a^+ = a^*a^+a^+$, then $a \in R^{PI}$.*

Proof. Pre-multiply the equality $a^*a^*a^+ = a^*a^+a^+$ by $(a^+)^*$, we have

$$aa^+a^*a^+ = aa^+a^+a^+,$$

it follows that $a^+a^*a^+ = a^+a^+a^+$. So $a^+a^*a^+a = a^+a^+a^+a$. Applying the involution on the last equality, one has

$$a^+a^2(a^+)^* = a^+a(a^+)^*(a^+)^* = a^+a(a^+)^*a^+a(a^+)^*.$$

Multiply it on the right by $a^*a^{\#}a$, we have $a^+a^2 = a^+a(a^+)^*$. Therefore $a^2 = a(a^+)^*$, which implies $a \in R^{PI}$. □

The proof of Lemma 2.13 implies the following corollary.

Corollary 2.14. *Let $a \in R^{\#} \cap R^{+}$. If $xa(a^+)^* = 0$, then $xa = 0$.*

Lemma 2.15. *Let $a \in R^{\#} \cap R^{+}$. If $a^3 = a(a^+)^*a$, then $a \in R^{PI}$.*

Proof. Since

$$a^3 = a(a^+)^*a = a(aa^+(a^+)^*a^+a)a = a^2a^+(a^+)^*a^+a^2,$$

we know that

$$a = a^{\#}a^3a^{\#} = a^{\#}a^2a^+(a^+)^*a^+a^2a^{\#} = aa^+(a^+)^*a^+a = (a^+)^*.$$

Then it is obvious that $a \in R^{PI}$. □

Theorem 2.16. *Let $a \in R^{\#} \cap R^{+}$. Then $a \in R^{PI}$ if and only if the equation $a^*xy = xa^+y$ has at least one solution in $\chi_a^2 = \{(x, y) | x, y \in \chi_a\}$.*

Proof. ⇒ If $a \in R^{PI}$, then $a^* = a^+$, this implies $\begin{cases} x = a^* \\ y = a \end{cases}$ is a solution.

⇐ (1) If $y = a$, then $a^*xa = xa^+a$. We know from Theorem 2.8 that $a \in R^{PI}$.

(2) If $y = a^{\#}$, then $a^*xa^{\#} = xa^+a^{\#}$. Post-multiply the equation by a^2, one gets $a^*xa = xa^+a$. It is immediate from Theorem 2.8 that $a \in R^{PI}$.

(3) If $y = a^+$, then we have the following equation

$$a^*xa^+ = xa^+a^+. \tag{6}$$

(i) If $x = a$, then $a^*aa^+ = aa^+a^+$, that is $a^* = aa^+a^+$. This clearly forces

$$(1 - aa^+)a^* = (1 - aa^+)aa^+a^+ = 0,$$

it follows $a = a^2a^+$ which yields $a \in R^{EP}$. Thus $a^* = aa^+a^+ = a^+$. Consequently, $a \in R^{PI}$.

(ii) If $x = a^\#$, then $a^*a^\#a^+ = a^\#a^+a^+$. It is clear that

$$(1 - aa^+)a^*a^\#a^+ = (1 - aa^+)a^\#a^+a^+ = 0.$$

Multiply the equality on the right by a^3a^+, we get $(1 - aa^+)a^* = 0$. This gives $a \in R^{EP}$. Hence, we obtain that

$$a^* = a^*aa^+ = a^*a^+a = a^*a^\#a^+a^2 = a^\#a^+a^+a^2 = a^\#a^+a = a^\# = a^+,$$

which proves $a \in R^{PI}$.

(iii) If $x = a^+$, then $a^*a^+a^+ = a^+a^+a^+$. By Lemma 2.11, $a \in R^{PI}$.

(iv) If $x = a^*$, then $a^*a^*a^+ = a^*a^+a^+$. By Lemma 2.13, we know that $a \in R^{PI}$.

(v) If $x = (a^\#)^*$, then $a^*(a^\#)^*a^+ = (a^\#)^*a^+a^+$. Post-multiply the equality by a and applying the involution, one has

$$a^+aa^\#a = a^+a(a^+)^*a^\#,$$

that is, $a^+a = a^+a(a^+)^*a^\#$. We find out that

$$a = aa^+a = aa^+a(a^+)^*a^\# = a(a^+)^*a^\#.$$

Then we know $a(a^+)^* = a(a^+)^*a^\#(a^+)^*$. It is immediate from Corollary 2.12 that

$$(a^+)^* = (a^+)^*a^\#(a^+)^*.$$

Taking the involution of the equality, we get $a^+ = a^+(a^\#)^*a^+$. Hence $a = aa^+a = aa^+(a^\#)^*a^+a$. Applying the involution of the equality, one has

$$a^* = a^+aa^\#aa^+ = a^+aa^+ = a^+.$$

Consequently, $a \in R^{PI}$.

(vi) If $x = (a^+)^*$, then $a^*(a^+)^*a^+ = (a^+)^*a^+a^+$, that is $a^+ = (a^+)^*a^+a^+$. This forces that $a^+a = (a^+)^*a^+a^+a$. Applying the involution of the equality, we obtain that

$$a^+a = a^+a(a^+)^*a^+.$$

Pre-multiply it by a and then we know $a = a(a^+)^*a^+$. As a result, $a^2 = a(a^+)^*a^+a = a(a^+)^*$, which indicates $a \in R^{PI}$.

(4) If $y = a^*$, then we have the following equation.

$$a^*xa^* = xa^+a^*. \tag{7}$$

(i) If $x = a$, then $a^*aa^* = aa^+a^*$. Applying the involution on the equality, we have $aa^*a = a^2a^+$. Observe that $aa^*a(1 - aa^+) = a^2a^+(1 - aa^+) = 0$. Then accordingly we know $a^*a(1 - aa^+) = 0$, which gives $a \in R^{EP}$. Moreover, $aa^*a = a^2a^+ = a$. This means $a \in R^{PI}$.

(ii) If $x = a^\#$, then $a^*a^\#a^* = a^\#a^+a^*$. It is evident that $(1 - aa^+)a^*a^\#a^* = (1 - aa^+)a^\#a^+a^* = 0$. Then we obtain that

$$(1 - aa^+)a^*a^\# = (1 - aa^+)^*a^*a^\#a^*(a^+)^* = 0.$$

Accordingly, we get

$$(1 - aa^+)a^* = (1 - aa^+)a^*aa^+ = (1 - aa^+)a^*a^\#a^2a^+ = 0.$$

This gives $a \in R^{EP}$. On the other hand,

$$a^*a^\# = a^*a^\#a^*(a^+)^* = a^\#a^+a^*(a^+)^* = a^\#a^+a^+a = a^\#a^+ = a^+a^\#.$$

By [3, Theorem 2.3], we know $a \in R^{PI}$.

(iii) If $x = a^+$, then $a^*a^+a^* = a^+a^+a^*$. Post-multiply it by $(a^+)^*$, one has

$$a^*a^+a^+a = a^+a^+a^+a.$$

This gives $a^*a^+a^+ = a^+a^+a^+$. It follows from Lemma 2.11 that $a \in R^{PI}$.

(iv) If $x = a^*$, then $a^*a^*a^* = a^*a^+a^*$. Applying the involution on the equation, we have $a^3 = a(a^+)^*a$. Hence $a \in R^{PI}$ by Lemma 2.15.

(v) If $x = (a^\#)^*$, then $a^*(a^\#)^*a^* = (a^\#)^*a^+a^*$. Taking involution of the equality, we get

$$a = a(a^+)^*a^\#.$$

Post-multiply it by a^2 and we thus obtain $a^3 = a(a^+)^*a$. It is immediate from Lemma 2.15 that $a \in R^{PI}$.

(vi) If $x = (a^+)^*$, then $a^*(a^+)^*a^* = (a^+)^*a^+a^*$. That is $a^* = (a^+)^*a^+a^*$. Applying the involution of it, we get $a = a(a^+)^*a^+$. Thus $a^2 = a(a^+)^*$ which gives $a \in R^{PI}$.

(5) If $y = (a^\#)^*$, then we obtain the following equation.

$$a^*x(a^\#)^* = xa^+(a^\#)^*. \tag{8}$$

Post-multiply the equation (8) by a^*a^*, we have

$$a^*xa^* = xa^+a^*.$$

By (4), we know $a \in R^{PI}$.

(6) If $y = (a^+)^*$, then we get the following equation.

$$a^*x(a^+)^* = xa^+(a^+)^*. \tag{9}$$

Post-multiply the equation (9) by a^*a, one obtains the equation (2.4) as follows

$$a^*xa = xa^+a.$$

By Theorem 2.8, $a \in R^{PI}$. □

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