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On a Quaternionic Sequence with Vietoris' Numbers

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Abstract. Special integers sequences have been the center of attention for many researchers, as well as the sequences of quaternions where its components are the elements of these sequences. Motivated by a rational sequence, we consider the quaternions with components Vietoris' numbers and investigate some of its properties. For this sequence a two and three term recurrence relation is established, as well as a Binet's type formula. Moreover the generating function for this sequence is introduced and also the determinant of some tridiagonal matrices are used in order to find elements of this sequence.

1. Introduction

One of the main motivation about this work comes from the study of Vietoris' sequence included in the paper [6]. As the authors have mentioned, this sequence of rational numbers "is a sequence that can be considered on the crossroad of positivity of trigonometric sums, stable behavior of some classes of holomorphic functions and a set of Appell polynomials in several hypercomplex variables"[6, p. 77]. Regarding the class of Appell polynomials and their generalizations, one can find in the literature some research of these polynomials from several points of view, see for example [4, 5, 7] and [25–27]. Recalling some combinatorial properties from Leopold Vietoris (1891-2002) [33], some interesting generalizations can be seen in [24]. These and other properties led us to further study this sequence (see [9]) and yet another sequence formed from it (see [10]).

Also nowadays, several studies involving quaternion sequences, in which the respective components are elements of some number sequences, have been a research topic for many authors (see, [1–3], [11–13], [17, 18], [20–22], [28–31], [34], among others). Motivated by these works, we started to investigate a quaternion sequence whose components are Vietoris' numbers. As a results of this study, we present in this paper a two terms recurrence formula, a way to find the general term of this sequence by the use of the determinant of a special tridiagonal matrices, a generating function and also a Binet's type formula satisfied by this sequence. In a additions to a series of properties presented in ([6], [9], [10]) for the Vietoris's number sequence, we also introduce new results that are stated in the next Section.

The structure of this paper is as follows: as we mentioned before, in the Section 2 we recall the definition of Vietoris' number sequence and present some new results that will be useful in the study of the quaternion

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sequence whose components are Vietoris' numbers; in the Section 3 we introduce a new quaternion sequence where Vietoris' numbers are involved. A two terms recurrence formula, as well as some of its properties are stated; the determinant of some special tridiagonal matrices with quaternion entries is used to find the general term of this new sequence, as can be seen in Section 4; the generating function of this quaternions sequence as well as its Binet type formula is presented in the last section.

2. Vietoris' number sequence

As shown in [15, Theorem 3.9], the elements of the Vietoris number sequence $\{v_n\}_{n\geq 0}$ can be written using the generalized central binomial coefficient

$$\binom{n}{\lfloor \frac{n}{2} \rfloor}$$

in the form

$$v_n = \frac{1}{2^n}\binom{n}{\lfloor \frac{n}{2} \rfloor}, \qquad n \geq 0.$$

The first Vietoris' numbers for $n = 0, 1, 2, 3, \cdots$ are

$$1, \frac{1}{2}, \frac{1}{2}, \frac{3}{8}, \frac{3}{8}, \frac{5}{16}, \frac{5}{16}, \frac{35}{128}, \frac{35}{128}, \frac{63}{256}, \frac{63}{256}, \frac{231}{1024}, \frac{231}{1024}, \cdots, \tag{1}$$

sequence related with the sequence A283208 in the OEIS in [23].

Recall that in [32], Vietoris stated that the even members of the Vietoris number sequence $\{v_n\}_{n\geq 0}$ are given by

$$v_{2n} = \frac{1}{2^{2n}}\binom{2n}{n}, \qquad n \geq 0, \tag{2}$$

where $v_{2n} = v_{2n-1}$, and $\binom{2n}{n}$ are the central binomial coefficient.

It is also known that the recurrence relation for $\{v_{2n}\}_{n\geq 0}$ is the following identity:

$$v_{2n+2} = d(2n)v_{2n}, \qquad n \geq 0, \tag{3}$$

where

$$d(k) = \frac{k+1}{k+2}, \quad k \geq 0. \tag{4}$$

Notice that we can write v_{2n} in terms of v_{2k}, using (4)

$$v_{2n} = \prod_{l=1}^{n-k} d(2n-2l)v_{2k}, \qquad n > k. \tag{5}$$

In order to present a Binet's like formula for the Vietoris numbers, we recall the following recurrence of order two of the Vietoris numbers.

Lemma 2.1. *Let $\{v_n\}_{n\geq 0}$ be the Vietoris' number sequence, then*

$$v_{2n+2} = \frac{1}{2}v_{2n+1} + \frac{1}{2}d(2n)v_{2n}.$$

Proof. Due to tha fact that $v_{2n+2} = v_{2n+1}$ and the recurence relation (3), we have

$$v_{2n+2} = \frac{1}{2}v_{2n+1} + \frac{1}{2}v_{2n+1} = \frac{1}{2}v_{2n+1} + \frac{1}{2}d(2n)v_{2n}.$$

□

Furthermore, we obtain a new recurrence of order two for the even index Vietoris' number.

Lemma 2.2. *Let $\{v_n\}_{n\geq 0}$ be the Vietoris' number sequence, then*

$$v_{2n+2} = \frac{1}{2}d(2n)v_{2n} + \frac{1}{2}d(2n)d(2n-2)v_{2n-2}.$$

Proof. Using Lemma 2.1, due to $v_{2n+1} = v_{2n+2}$ and the recurrence relation (3) we have

$$v_{2n+2} = \frac{1}{2}v_{2n+1} + \frac{1}{2}d(2n)v_{2n} = \frac{1}{2}d(2n)v_{2n} + \frac{1}{2}d(2n)d(2n-2)v_{2n-2}.$$

□

Using Lemma 2.1 and the fact that $v_{2n-1} = v_{2n}$, we obtain the following Binet's like formula for the Vietoris number sequence.

Theorem 2.3. *Let $\{v_n\}_{n\geq 0}$ be the Vietoris' number sequence, then*

$$v_{2n} = c_1(2n)r_1^{2n}(2n) + c_2(2n)r_2^{2n}(2n),$$

where

$$r_1(2n) = \frac{1}{4}\left(1 - \sqrt{1+8d(2n)}\right), \qquad r_2(2n) = \frac{1}{4}\left(1 + \sqrt{1+8d(2n)}\right),$$

and

$$\begin{cases} c_1(2n) = \dfrac{r_2^{2n}(2n) - v_2}{r_2^{2n}(2n) - r_1^{2n}(2n)} \displaystyle\prod_{k=1}^{n-1}(2r_1(2k)-1)r_1(2k) \\ c_2(2n) = \dfrac{v_2 - r_1^{2n}(2n)}{r_2^{2n}(2n) - r_1^{2n}(2n)} \displaystyle\prod_{k=1}^{n-1}(2r_2(2k)-1)r_2(2k) \end{cases} \tag{6}$$

Proof. Using Lemma 2.1, we have

$$v_{2n+2} = \frac{1}{2}v_{2n+1} + \frac{1}{2}d(2n)v_{2n}.$$

The Binet formula for Vietoris' numbers explicitly gives v_{2n} as a function of the index n and the roots $r_1(2n)$ and $r_2(2n)$ of the characteristic equation

$$x^2 - \frac{1}{2}x - \frac{1}{2}d(2n) = 0,$$

where

$$r_1(2n) = \frac{1}{4}\left(1 - \sqrt{1+8d(2n)}\right), \qquad r_2(2n) = \frac{1}{4}\left(1 + \sqrt{1+8d(2n)}\right).$$

Substituting (6), we obtain

$$c_1(2n)r_1^{2n}(2n)+c_2(2n)r_2^{2n}(2n) = \frac{(r_2^{2n}(2n)-v_2)\prod_{k=1}^{n-1}(2r_1(2k)-1)r_1(2k)r_1^{2n}(2n)+(v_2-r_1^{2n}(2n))\prod_{k=1}^{n-1}(2r_2(2k)-1)r_2(2k)r_2^{2n}(2n)}{r_2^{2n}(2n)-r_1^{2n}(2n)}$$

$$= \frac{r_2^{2n}(2n)r_1^{2n}(2n)\prod_{k=1}^{n-1}\left(2r_1^2(2k)-r_1(2k))-(2r_2^2(2k)-r_2(2k))\right)}{r_2^{2n}(2n)-r_1^{2n}(2n)}$$

$$+v_2\frac{-r_1^{2n}(2n)\prod_{k=1}^{n-1}2r_1^2(2k)-r_1(2k)+r_2^{2n}(2n)\prod_{k=1}^{n-1}2r_2^2(2k)-r_2(2k)}{r_2^{2n}(2n)-r_1^{2n}(2n)}.$$

Since $r_1(2n)$ and $r_2(2n)$ satisfies $x^2-\frac{1}{2}x-\frac{1}{2}d(2n)=0$, then

$$2r_1^2(2k)-r_1(2k)=d(2k), \qquad 2r_2^2(2k)-r_2(2k)=d(2k)$$

and using the fact $v_{2n}=\prod_{k=1}^{n-1}d(2k)v_2$, we have

$$c_1(2n)r_1^{2n}(2n)+c_2(2n)r_2^{2n}(2n) = \frac{v_2\left(-r_1^{2n}(2n)\prod_{k=1}^{n-1}d(2k)+r_2^{2n}(2n)\prod_{k=1}^{n-1}d(2k)\right)}{r_2^{2n}(2n)-r_1^{2n}(2n)} = v_2\prod_{k=1}^{n-1}d(2k)=v_{2n}.$$

□

Remark 2.4. *Some basic properties can be noticed for r_1 and r_2 defined in Theorem 2.3:*

(i) The value of $r_1(0)=\frac{1+\sqrt{5}}{4}$, *which is half of golden ratio.*

(ii) $r_1(2n)+r_2(2n)=\frac{1}{2}$.

(iii) $r_1(2n)r_2(2n)=-\frac{d(2n)}{2}$.

In the next theorem, using Lemma 2.2, a new Binet's like formula is obtained for the Vietoris number sequence.

Theorem 2.5. *Let* $\{v_n\}_{n\geq 0}$ *be the Vietoris number sequence, then*

$$v_{2n}=c_1(2n)r_1^{2n}(2n)+c_2(2n)r_2^{2n}(2n),$$

where

$$r_1(2n)=\frac{d(2n)}{4}\left(1-\sqrt{1+8\frac{d(2n-2)}{d(2n)}}\right),\ r_2(2n)=\frac{d(2n)}{4}\left(1+\sqrt{1+8\frac{d(2n-2)}{d(2n)}}\right)$$

and

$$\begin{cases} c_1(2n)=\dfrac{(2n-1)!!(-d(2n)r_2^{2n}(2n)+r_2^{2n+2}(2n+2))}{n!\left(2^n r_1^{2n}(2n)r_2^{2n+2}(2n+2)-2^n r_2^{2n}(2n)r_1^{2n+2}(2n+2)\right)} \\[2ex] c_2(2n)=\dfrac{(2n-1)!!(d(2n)r_1^{2n}(2n)-r_1^{2n+2}(2n+2))}{n!(2^n r_1^{2n}(2n)r_2^{2n+2}(2n+2)-2^n r_2^{2n}(2n)r_1^{2n+2}(2n+2))} \end{cases} \tag{7}$$

Proof. Using Lemma 2.2, the Binet formula for Vietoris' numbers explicitly gives v_{2n} as a function of the index n and the roots $r_1(2n)$ and $r_2(2n)$ of the characteristic equation

$$x^2 - \frac{1}{2}d(2n)x - \frac{1}{2}d(2n)d(2n-2) = 0,$$

where

$$r_1(2n) = \frac{d(2n)}{4}\left(1 - \sqrt{1 + 8\frac{d(2n-2)}{d(2n)}}\right),\ r_2(2n) = \frac{d(2n)}{4}\left(1 + \sqrt{1 + 8\frac{d(2n-2)}{d(2n)}}\right).$$

Substituting $c_1(2n)$ and $c_2(2n)$, defined in (7), in

$$c_1(2n)r_1^{2n}(2n) + c_2(2n)r_2^{2n}(2n),$$

we obtain

$$\begin{aligned}
c_1(2n)r_1^{2n}(2n) + c_2(2n)r_2^{2n}(2n) &= \frac{(2n-1)!!(-d(2n)r_2^{2n}(2n) + r_2^{2n+2}(2n+2))r_1^{2n}(2n)}{n!\left(2^n r_1^{2n}(2n)r_2^{2n+2}(2n+2) - 2^n r_2^{2n}(2n)r_1^{2n+2}(2n+2)\right)} \\
&= \frac{(2n-1)!!(d(2n)r_1^{2n}(2n) - r_1^{2n+2}(2n+2))r_2^{2n}(2n)}{n!(2^n r_1^{2n}(2n)r_2^{2n+2}(2n+2) - 2^n r_2^{2n}(2n)r_1^{2n+2}(2n+2))} \\
&= \frac{(2n-1)!!}{n!2^n} \\
&= v_{2n}.
\end{aligned}$$

□

Remark 2.6. *For r_1 and r_2 defined in Theorem 2.5, one has:*

$$r_1(2n) + r_2(2n) = \frac{d(2n)}{2}, \qquad r_1(2n)r_2(2n) = -\frac{d(2n)d(2n-2)}{2}.$$

3. Quaternion sequence involving Vietoris' numbers $\{V_s\}_{s\geq 0}$

In order to introduce a new quaternion sequence involving Vietoris' numbers, we begin this section by presenting some basic notions and properties of quaternion algebra, for more detail see for example [14] and [16].

The set of quaternions form a four-dimensional associative and noncommutative algebra over the set of real numbers. Let $\{1, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$ standard basis in $\mathbb{R}^4$ satisfying the following multiplication rules:

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1, \quad \mathbf{ij} = -\mathbf{ji} = \mathbf{k}, \quad \mathbf{jk} = -\mathbf{kj} = \mathbf{i}, \quad \mathbf{ki} = -\mathbf{ik} = \mathbf{j}.$$

Let $\mathbb{H}$ be the skew field of quaternions defined by

$$\mathbb{H} = \{q = q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k}, \quad q_s \in \mathbb{R},\ s = 0, 1, 2, 3\},$$

and for $q \in \mathbb{H}$ given by $q = q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k}$, where $\mathrm{Sc}(q) = q_0$ is called the scalar part of the quaternion and $\mathrm{Vec}(q) = q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k}$ vector part.

We recall that the prime involution in $\mathbb{H}$ is the mapping $q \to q'$ defined by

$$q' = q_0 - q_1\mathbf{i} - q_2\mathbf{j} + q_3\mathbf{k}$$

and the reversion in $\mathbb{H}$ is the mapping $q \to q*$ defined by

$$q^* = q_0 + q_1\mathbf{i} + q_2\mathbf{j} - q_3\mathbf{k}.$$

The conjugation in $\mathbb{H}$ is the mapping $q \to \overline{q}$ defined by

$$\overline{q} = (q')^* = (q^*)' = q_0 - q_1\mathbf{i} - q_2\mathbf{j} - q_3\mathbf{k}.$$

These involutions satisfy the following product rules

$$(q\,p)' = q'\,p', \quad (q\,p)^* = p^*\,q^*, \quad \overline{q\,p} = \overline{p}\,\overline{q}, \quad \forall q, p \in \mathbb{H}.$$

Furthermore, one has

$$q\mathbf{k} = \mathbf{k}q', \quad \forall q \in \mathbb{H}.$$

One also has the relation between the scalar part and as well as

$$q + \overline{q} = 2\mathrm{Sc}(q), \qquad q - \overline{q} = 2\mathrm{Vec}(q). \tag{8}$$

The modulus or absolute value of the quaternion, is given by

$$\|q\|^2 = q\overline{q} = \overline{q}q, \qquad \forall q \in \mathbb{H}. \tag{9}$$

3.1. A quaternion sequence $\{V_n\}_{n\geq 0}$ and basic properties

We define the sth element of the quaternionic sequence with Vietoris' number by

$$V_s = v_s + v_{s+1}\mathbf{i} + v_{s+2}\mathbf{j} + v_{s+3}\mathbf{k}, \quad s \in \mathbb{N}_0. \tag{10}$$

If we denote the complex sequence with Vietoris' numbers by

$$V_s^c = v_s + v_{s+1}\mathbf{i}, \tag{11}$$

one can rewrite the quaternionic sequence with Vietoris' number by

$$V_s = v_s + v_{s+1}\mathbf{i} + (v_{s+2} + v_{s+3}\mathbf{i})\,\mathbf{j} = V_s^c + V_{s+2}^c\mathbf{j}.$$

In the case, $s = 2n$, and due to the fact that $v_{2n-1} = v_{2n}$, we have

$$\begin{aligned} V_{2n} &= v_{2n} + v_{2n+1}\mathbf{i} + v_{2n+2}\mathbf{j} + v_{2n+3}\mathbf{k} \\ &= v_{2n} + v_{2n+2}(\mathbf{i} + \mathbf{j}) + v_{2n+4}\mathbf{k}. \end{aligned}$$

Using the recurrence formula (3), one obtains

$$V_{2n} = v_{2n}\left(1 + d(2n)(\mathbf{i} + \mathbf{j}) + d(2n)d(2n+2)\mathbf{k}\right), \tag{12}$$

which can be rewritten as

$$V_{2n} = v_{2n}\Big((1 + d(2n)\mathbf{i}) + d(2n)\,(1 + d(2n+2)\mathbf{i})\,\mathbf{j}\Big) \tag{13}$$

Let $\widehat{a}(2n) = 1 + d(2n)(\mathbf{i} + \mathbf{j}) + d(2n)d(2n+2)\mathbf{k}$, we can rewritten V_{2n} as

$$V_{2n} = v_{2n}\,\widehat{a}(2n). \tag{14}$$

In the case, $s = 2n + 1$, and due to the fact that $v_{2n-1} = v_{2n}$ and the recurrence formula of the Vietoris' numbers (3), one gets

$$\begin{aligned} V_{2n+1} &= v_{2n+1} + v_{2n+2}\mathbf{i} + v_{2n+3}\mathbf{j} + v_{2n+4}\mathbf{k} \\ &= v_{2n+2}(1 + \mathbf{i}) + v_{2n+4}(\mathbf{j} + \mathbf{k}) \\ &= v_{2n+2}\,(1 + \mathbf{i} + d(2n+2)(\mathbf{j} + \mathbf{k})) \\ &= v_{2n+2}(1 + \mathbf{i})\,(1 + d(2n+2)\mathbf{j}). \end{aligned} \tag{15}$$

Let $\widehat{b}(2n+2) = 1 + \mathbf{i} + d(2n+2)(\mathbf{j} + \mathbf{k})$, we can rewritten V_{2n+1} as

$$V_{2n+1} = v_{2n+2}\,\widehat{b}(2n+2). \tag{16}$$

Remark 3.1. *If we consider the expression (11) we obtain*

(i) $V_{2n} = V^c_{2n} + V^c_{2n+2}\mathbf{j}$;
(ii) $V_{2n+1} = V^c_{2n+1} + V^c_{2n+3}\mathbf{j} = 2v_{2n+2}V^c_1 + 2v_{2n+4}V^c_1\mathbf{j} = 2V^c_1\left(v_{2n+2} + v_{2n+4}\mathbf{j}\right).$

We also can rewrite

$$\widehat{a}(2n) = (1 + d(2n)\mathbf{i}) + d(2n)\left(1 + d(2n+2)\mathbf{i}\right)\mathbf{j}$$

while

$$\widehat{b}(2n+2) = 1 + \mathbf{i} + d(2n+2)(\mathbf{j} + \mathbf{k}) = 2V^c_1 + 2V^c_1 d(2n+2)\mathbf{j}.$$

In the next result one presents some relations between this quaternion sequence and its norm.

Proposition 3.2. *Let V_s defined in (10). Then*

(a) $\|V_{2n}\|^2 - \|V_{2n+1}\|^2 = v^2_{2n} - v^2_{2n+4}$;

(b) $\|V_{2n+1}\| < \|V_{2n}\|$;

(c) $V^2_s + \|V_s\|^2 = 2v_sV_s$.

Proof. One starts by proving (a), using (14) and (16) , we get

$$V_{2n}\overline{V_{2n}} = v^2_{2n}\left(1 + 2d^2(2n) + d^2(2n)d^2(2n+2)\right)$$

and

$$V_{2n+1}\overline{V_{2n+1}} = v^2_{2n+2}\left(2 + 2d^2(2n+2)\right),$$

therefore,

$$\|V_{2n}\|^2 - \|V_{2n+1}\|^2 = \frac{v^2_{2n+2}}{d^2(2n)} - v^2_{2n+2}d^2(2n+2) = v^2_{2n} - v^2_{2n+4}.$$

Using (a), we have

$$v^2_{2n} - v^2_{2n+4} = (v_{2n} - v_{2n+4})(v_{2n} + v_{2n+4}),$$

where

$$v_{2n} - v_{2n+4} = 4n + 5 > 0, \qquad v_{2n} + v_{2n+4} > 0.$$

Therefore

$$\|V_{2n}\|^2 - \|V_{2n+1}\|^2 > 0$$

proving (b). To prove (c) we use (9) and (8)

$$V^2_s + \|V_s\|^2 = V_sV_s + \overline{V_s}V_s = \left(V_s + \overline{V_s}\right)V_s = 2v_sV_s.$$

□

Further relations can be encountered using the prime involution.

Proposition 3.3. *Let V_s be defined as in (10). Then*

(a) $V_{2n+1}\mathbf{i} = (-V_{2n+1})'$;

(b) $V_{2n} + V'_{2n} = 2\left(v_{2n} + v_{2n+4}\mathbf{k}\right)$;

(c) $V_{2n} - V'_{2n} = 2v_{2n+2}\left(\mathbf{i} + \mathbf{j}\right)$;

(d) $V_{2n+1} + V'_{2n+1} = 2\left(v_{2n+2} + v_{2n+4}\mathbf{k}\right)$;

(e) $V_{2n+1} - V'_{2n+1} = 2\left(v_{2n+2}\mathbf{i} + v_{2n+4}\mathbf{j}\right);$

(f) $V_{2n} - V'_{2n+1} = (v_{2n} - v_{2n+2}) + 2v_{2n+2}\mathbf{i} + 2v_{2n+4}\mathbf{j};$

(g) $V_{2n} + V'_{2n+1} = (v_{2n} + v_{2n+2}) + 2v_{2n+4}\mathbf{k}.$

Proof. Let start by proving (a), using (16) and the fact that $v_{2s-1} = v_{2s}$ for all $s \in \mathbb{N}$, we have

$$\begin{aligned} V_{2n+1}\mathbf{i} &= v_{2n+1}\mathbf{i} + v_{2n+2}\mathbf{i}^2 + v_{2n+3}\mathbf{ji} + v_{2n+4}\mathbf{ki} \\ &= v_{2n+1}\mathbf{i} - v_{2n+2} - v_{2n+3}\mathbf{k} + v_{2n+4}\mathbf{j} \\ &= v_{2n+2}\mathbf{i} - v_{2n+1} - v_{2n+4}\mathbf{k} + v_{2n+3}\mathbf{j} \\ &= \left(-V_{2n+1}\right)'. \end{aligned}$$

The equalities $(b), (c)$ and $(f), (g)$ are a direct result of the prime involution property. In the proof of (d) and (e) one also uses the fact that $v_{2s-1} = v_{2s}$ for all $s \in \mathbb{N}$. □

A relation between a combination of three consecutive terms of $\{V_s\}_{s\geq 0}$, starting with an odd index term of this sequence, is presented in the following proposition.

Proposition 3.4. *Let V_s be defined as in (10). Then*

$$\sum_{k=1}^{3}(-1)^k V_{2n+k} = (V_{2n+2}\ \mathbf{i})' = -(V_{2n+2})'\mathbf{i}$$

Proof. Using the definition of V_n defined in (10), we obtain:

$$\begin{aligned} \sum_{k=1}^{3}(-1)^k V_{2n+k} &= -\Big(v_{2n+1} + v_{2n+2}\mathbf{i} + v_{2n+3}\mathbf{j} + v_{2n+4}\mathbf{k}\Big) + v_{2n+2} + v_{2n+3}\mathbf{i} \\ &\quad + v_{2n+4}\mathbf{j} + v_{2n+5}\mathbf{k} - v_{2n+3} - v_{2n+4}\mathbf{i} - v_{2n+5}\mathbf{j} - v_{2n+6}\mathbf{k} \\ &= -v_{2n+2}\mathbf{i} + v_{2n+4}(-\mathbf{k} - 1) - v_{2n+6}\mathbf{j} \\ &= \Big(v_{2n+2} + v_{2n+4}(-\mathbf{i} - \mathbf{j}) + v_{2n+6}\mathbf{k}\Big)(-\mathbf{i}) \\ &= \Big(V_{2n+2}\ \mathbf{i}\Big)'. \end{aligned}$$

□

Proposition 3.5. *Let V_s be defined as in (10). Then*

(a) $V_{2n+1}V_{2n-1} = 2(v_{2n+2}\mathbf{i} + v_{2n+4}\mathbf{j})(v_{2n} + v_{2n+2}\mathbf{j});$

(b) $V_{2n-1}V_{2n+1} = 2(v_{2n}\mathbf{i} + v_{2n+2}\mathbf{j})(v_{2n+2} + v_{2n+4}\mathbf{j});$

(c) $V_{2n} = V_{2n+1} + v_{2n}\Big((1 - d(2n)) + d(2n)(1 - d(2n+2))\mathbf{j}\Big);$

(d) $V_{2n+1} = V_{2n+2} + d(2n)v_{2n}((1 - d(2n+2))\mathbf{i} + d(2n+2)(1 - d(2n+4))\mathbf{k});$

(e) $V^2_{2n+1} = 2d^2(2n)v^2_{2n}(\mathbf{i} + d(2n+2)\mathbf{j})(1 + d(2n+2)\mathbf{j});$

(f) $V_{2n+1}V_{2n} = V^2_{2n+1} + d(2n)v^2_{2n}((1 + \mathbf{i}) + d(2n+2)(\mathbf{j} + \mathbf{k}))((1 - d(2n)) + d(2n)(1 - d(2n+2))\mathbf{j});$

(g) $V_{2n}V_{2n+1} = V^2_{2n+1} + d(2n)v^2_{2n}((1 - d(2n)) + d(2n)(1 - d(2n+2))\mathbf{j})((1 + \mathbf{i}) + d(2n+2)(\mathbf{j} + \mathbf{k}));$

(h) $V_{2n+2}V_{2n} = V_{2n+1}V_{2n} - d(2n)v_{2n}((1 - d(2n+2)\mathbf{i} + d(2n+2)(1 - d(2n+4))\mathbf{k})V_{2n};$

(i) $V_{2n}V_{2n+2} = V_{2n}V_{2n+1} - V_{2n}d(2n)v_{2n}((1 - d(2n+2))\mathbf{i} + d(2n+2)(1 - d(2n+4))\mathbf{k}).$

Proof. By a straightforward calculation, and the fact that

$$(1+\mathbf{i})^2 = 2\mathbf{i}, \quad (1+\mathbf{i})(\mathbf{j}+\mathbf{k}) = 2\mathbf{k}, \quad (\mathbf{j}+\mathbf{k})(1+\mathbf{i}) = 2\mathbf{j}, \quad (\mathbf{j}+\mathbf{k})^2 = -2 \tag{17}$$

we obtain the equality (*a*)

$$\begin{aligned} V_{2n+1}V_{2n-1} &= (v_{2n+1} + v_{2n+2}\mathbf{i} + v_{2n+3}\mathbf{j} + v_{2n+4}\mathbf{k})(v_{2n-1} + v_{2n}\mathbf{i} + v_{2n+1}\mathbf{j} + v_{2n+2}\mathbf{k}) \\ &= 2\left(v_{2n+2}v_{2n}\mathbf{i} + v_{2n+2}^2\mathbf{k} + v_{2n}v_{2n+4}\mathbf{j} - v_{2n+2}v_{2n+4}\right) \\ &= 2\left((v_{2n+2}\mathbf{i} + v_{2n+4}\mathbf{j})v_{2n} + v_{2n+2}(v_{2n+2}\mathbf{k} - v_{2n+4})\right) \\ &= 2\left((v_{2n+2}\mathbf{i} + v_{2n+4}\mathbf{j})v_{2n} + v_{2n+2}(v_{2n+2}\mathbf{i} + v_{2n+4}\mathbf{j})\mathbf{j}\right) \\ &= 2(v_{2n+2}\mathbf{i} + v_{2n+4}\mathbf{j})(v_{2n} + v_{2n+2}\mathbf{j}). \end{aligned}$$

Furthermore

$$\begin{aligned} V_{2n-1}V_{2n+1} &= (v_{2n-1} + v_{2n}\mathbf{i} + v_{2n+1}\mathbf{j} + v_{2n+2}\mathbf{k})(v_{2n+1} + v_{2n+2}\mathbf{i} + v_{2n+3}\mathbf{j} + v_{2n+4}\mathbf{k}) \\ &= 2\left(v_{2n+2}v_{2n}\mathbf{i} + v_{2n+2}^2\mathbf{j} + v_{2n}v_{2n+4}\mathbf{k} - v_{2n+2}v_{2n+4}\right) \\ &= 2\left((v_{2n+2}\mathbf{i} + v_{2n+4}\mathbf{k})v_{2n} + v_{2n+2}(v_{2n+2}\mathbf{j} - v_{2n+4})\right) \\ &= 2\left(\mathbf{i}(v_{2n+2} + v_{2n+4}\mathbf{j})v_{2n} + v_{2n+2}\mathbf{j}(v_{2n+2} + v_{2n+4}\mathbf{j})\right) \\ &= 2(v_{2n}\mathbf{i} + v_{2n+2}\mathbf{j})(v_{2n+2} + v_{2n+4}\mathbf{j}), \end{aligned}$$

proving (*b*). In order to prove (*c*) we use (16) and the fact that $v_{2n-1} = v_{2n}$, hence

$$\begin{aligned} V_{2n} - V_{2n+1} &= v_{2n}\Big((1 + d(2n)\mathbf{i}) + d(2n)(1 + d(2n+2)\mathbf{i})\mathbf{j}\Big) - v_{2n+2}(1 + \mathbf{i} + d(2n+2)(\mathbf{j}+\mathbf{k})) \\ &= v_{2n}\Big((1 + d(2n)\mathbf{i}) + d(2n)(1 + d(2n+2)\mathbf{i})\mathbf{j}\Big) - v_{2n}(d(2n) + d(2n)\mathbf{i} + d(2n)d(2n+2)(\mathbf{j}+\mathbf{k})) \\ &= v_{2n}\Big((1 - d(2n)) + d(2n)(1 - d(2n+2))\mathbf{j}\Big). \end{aligned}$$

Now, for the proof of (*d*) we use (10) and the fact that $v_{2n-1} = v_{2n}$, hence

$$\begin{aligned} V_{2n+1} &= V_{2n+2} + (v_{2n+2} - v_{2n+4})\,\mathbf{i} + (v_{2n+4} - v_{2n+6})\,\mathbf{k} \\ &= V_{2n+2} + d(2n)v_{2n}\Big((1 - d(2n+2))\,\mathbf{i} + d(2n+2)(1 - d(2n+4))\,\mathbf{k}\Big). \end{aligned}$$

By the use of (10), the fact that $v_{2n-1} = v_{2n}$ and, once more, by (17) we get the identity (*e*),

$$\begin{aligned} V_{2n+1}^2 &= v_{2n+2}^2(1+\mathbf{i})^2 + v_{2n+2}v_{2n+4}(1+\mathbf{i})(\mathbf{j}+\mathbf{k}) + v_{2n+2}v_{2n+4}(\mathbf{j}+\mathbf{k})(1+\mathbf{i}) + v_{2n+4}^2(\mathbf{j}+\mathbf{k})^2 \\ &= -2v_{2n+4}^2 + 2v_{2n+2}^2\mathbf{i} + 2v_{2n+2}v_{2n+4}\mathbf{j} + 2v_{2n+2}v_{2n+4}\mathbf{k} \\ &= 2(v_{2n+2}\mathbf{i} + v_{2n+4}\mathbf{j})(v_{2n+2} + v_{2n+4}\mathbf{j}). \end{aligned}$$

In order to prove (*f*) we use the equality (*c*) and also (10), giving the following

$$\begin{aligned} V_{2n+1}V_{2n} &= V_{2n+1}\Big(V_{2n+1} + v_{2n}\left((1 - d(2n)) + d(2n)(1 - d(2n+2))\,\mathbf{j}\right)\Big) \\ &= V_{2n+1}^2 + V_{2n+1}v_{2n}\left((1 - d(2n)) + d(2n)(1 - d(2n+2))\,\mathbf{j}\right) \\ &= V_{2n+1}^2 + v_{2n}\left(v_{2n+2}(1+\mathbf{i}) + v_{2n+4}(\mathbf{j}+\mathbf{k})\right)\left((1 - d(2n)) + d(2n)(1 - d(2n+2))\,\mathbf{j}\right) \\ &= V_{2n+1}^2 + v_{2n}^2 d(2n)\left((1+\mathbf{i}) + d(2n+2)(\mathbf{j}+\mathbf{k})\right)\left(1 - d(2n) + d(2n)(1 - d(2n+2))\mathbf{j}\right). \end{aligned}$$

To prove equality (*g*) just apply the same technique used in proof of the equality (*f*), hence we omit the respective proof here. For the last two equalities we use the identity (*d*) and we immediately get the results. □

3.2. Recurrence relations for $\{V_n\}_{n\geq 0}$

Several two and three terms recurrence relations for $\{V_n\}_{n\geq 0}$ are introduced. Due to the structure of the V_{2n} versus V_{2n+1} presented in (14) and (16) respectively, the following four two term recurrence relations are deduced.

Proposition 3.6. *Consider V_s defined in (10) and $d(s)$ defined in (4). Then*

$$V_{2n+2} = V_{2n+1}\phi_R(2n+2), \qquad V_{2n+2} = \phi_L(2n+2)V_{2n+1},$$

where

$$\phi_R(2n+2) = \frac{\phi_0(2n+2) + \phi_1(2n+2)\mathbf{i} + \phi_2(2n+2)\mathbf{j} + \phi_3(2n+2)\mathbf{k}}{2 + 2d^2(2n+2)}$$

and

$$\phi_L(2n+2) = \frac{\phi_0(2n+2) + \phi_4(2n+2)\mathbf{i} - \phi_2(2n+2)\mathbf{j} + \phi_2(2n+2)\mathbf{k}}{2 + 2d^2(2n+2)}$$

with

$$\begin{aligned}
\phi_0(2n+2) &= 1 + d(2n+2) + d^2(2n+2)[1 + d(2n+4)];\\
\phi_1(2n+2) &= d(2n+2) - 1 + d^2(2n+2)[1 - d(2n+4)];\\
\phi_2(2n+2) &= d(2n+2)[d(2n+4) - d(2n+2)];\\
\phi_3(2n+2) &= d(2n+2)[d(2n+4) + d(2n+2)] - 2d(2n+4);\\
\phi_4(2n+2) &= d(2n+2) - 1 + d^2(2n+2)[d(2n+4) - 1].
\end{aligned}$$

Proof. In order to prove $V_{2n+2} = V_{2n+1}\phi_R(2n+2)$ we use (14) and (16), getting

$$\begin{aligned}
V_{2n+2} &= v_{2n+2}\widehat{a}(2n+2)\\
&= v_{2n+2}\widehat{b}(2n+2)\tfrac{\overline{\widehat{b}(2n+2)}}{\|\widehat{b}(2n+2)\|^2}\widehat{a}(2n+2)\\
&= V_{2n+1}\tfrac{\phi_0(2n+2)+\phi_1(2n+2)\mathbf{i}+\phi_2(2n+2)\mathbf{j}+\phi_3(2n+2)\mathbf{k}}{2+2d^2(2n+2)}\\
&= V_{2n+1}\phi_R(2n+2),
\end{aligned}$$

due to $\overline{\widehat{b}(2n+2)}\widehat{a}(2n+2) = \phi_0(2n+2) + \phi_1(2n+2)\mathbf{i} + \phi_2(2n+2)\mathbf{j} + \phi_3(2n+2)\mathbf{k}$.
In order to prove $V_{2n+2} = \phi_L(2n+2)V_{2n+1}$ we use (14) and (16), getting

$$\begin{aligned}
V_{2n+2} &= v_{2n+2}\widehat{a}(2n+2)\\
&= \widehat{a}(2n+2)\tfrac{\overline{\widehat{b}(2n+2)}}{\|\widehat{b}(2n+2)\|^2}\widehat{b}(2n+2)v_{2n+2}\\
&= \tfrac{\phi_0(2n+2)+\phi_4(2n+2)\mathbf{i}-\phi_2(2n+2)\mathbf{j}+\phi_2(2n+2)\mathbf{k}}{2+2d^2(2n+2)}V_{2n+1}\\
&= \phi_L(2n+2)V_{2n+1},
\end{aligned}$$

due to $\widehat{a}(2n+2)\overline{\widehat{b}(2n+2)} = \phi_0(2n+2) + \phi_4(2n+2)\mathbf{i} - \phi_2(2n+2)\mathbf{j} + \phi_2(2n+2)\mathbf{k}$. □

Proposition 3.7. *Consider V_s defined in (10), and $d(s)$ defined in (4). Then*

$$V_{2n+1} = V_{2n}\psi_R(2n), \qquad V_{2n+1} = \psi_L(2n)V_{2n},$$

where

$$\psi_R(2n+2) = \frac{d(2n)\left(\psi_0(2n) + \psi_1(2n)\mathbf{i} + \psi_2(2n)\mathbf{j} + \psi_3(2n)\mathbf{k}\right)}{1 + d^2(2n)(2 + d^2(2n+2))}$$

and

$$\psi_L(2n+2) = \frac{d(2n)\left(\psi_0(2n) + \psi_4(2n)\mathbf{i} + \psi_2(2n)\mathbf{j} + \psi_2(2n)\mathbf{k}\right)}{1 + d^2(2n)(2 + d^2(2n+2))}$$

with

$$\begin{aligned}
\psi_0(2n) &= 1 + d(2n) + d(2n)d(2n+2)[1 + d(2n+2)];\\
\psi_1(2n) &= 1 - d(2n) + d(2n)d(2n+2)[d(2n+2) - 1];\\
\psi_2(2n) &= d(2n+2) - d(2n);\\
\psi_3(2n) &= d(2n+2) + d(2n) - 2d(2n+2)d(2n);\\
\psi_4(2n) &= 1 - d(2n) + d(2n)d(2n+2)[1 - d(2n+2)].
\end{aligned}$$

Proof. In order to prove $V_{2n+1} = V_{2n}\psi_R(2n)$, one uses (14) and (16), hence

$$\begin{aligned} V_{2n+1} &= v_{2n+2}\widehat{b}(2n+2) \\ &= v_{2n}\widehat{a}(2n)d(2n)\frac{\overline{\widehat{a}(2n)}}{\|\widehat{a}(2n)\|^2}\widehat{b}(2n+2) \\ &= V_{2n}\psi_R(2n), \end{aligned}$$

where $\overline{\widehat{a}(2n)\widehat{b}}(2n+2) = \psi_0(2n) + \psi_1(2n)\mathbf{i} + \psi_2(2n)\mathbf{j} + \psi_3(2n)\mathbf{k}$.

Analogously, one proves

$$\begin{aligned} V_{2n+1} &= v_{2n+2}\widehat{b}(2n+2) \\ &= d(2n)\widehat{b}(2n+2)\frac{\overline{\widehat{a}(2n)}}{\|\widehat{a}(2n)\|^2}\widehat{a}(2n)v_{2n} \\ &= \psi_L(2n)V_{2n}, \end{aligned}$$

where $\widehat{b}(2n+2)\overline{\widehat{a}(2n)} = \psi_0(2n) + \psi_4(2n)\mathbf{i} + \psi_2(2n)\mathbf{j} + \psi_2(2n)\mathbf{k}$. □

Proposition 3.8. *Consider V_s defined in (10) and $d(s)$ defined in (4). Then*

$$V_{2n+2} = V_{2n}\theta_R(2n), \qquad V_{2n+2} = \theta_L(2n)V_{2n},$$

where

$$\theta_R(2n) = \frac{d(2n)\,(\theta_0(2n) + \theta_1(2n)\mathbf{i} + \theta_2(2n)\mathbf{j} + \theta_3(2n)\mathbf{k})}{1 + d^2(2n)(2 + d^2(2n+2))}$$

and

$$\theta_L(2n) = \frac{d(2n)\,(\theta_0(2n) + \theta_2(2n)\mathbf{i} + \theta_1(2n)\mathbf{j} + \theta_3(2n)\mathbf{k})}{1 + d^2(2n)(2 + d^2(2n+2))}$$

with

$$\begin{aligned} \theta_0(2n) &= 1 + d(2n)d(2n+2)(2 + d(2n+4)d(2n+2)); \\ \theta_1(2n) &= d(2n+2) - d(2n) + d(2n)d(2n+2)(d(2n+2) - d(2n+4)); \\ \theta_2(2n) &= d(2n+2) - d(2n) + d(2n)d(2n+2)(d(2n+4) - d(2n+2)); \\ \theta_3(2n) &= d(2n+2)(d(2n+4) - d(2n)). \end{aligned}$$

Proof. To prove these results we rely on Proposition 3.6 and Proposition 3.7. Using the equality (14) , in order to prove $V_{2n+2} = V_{2n}\theta_R(2n)$, one gets

$$V_{2n+2} = V_{2n+1}\phi_R(2n+2) = V_{2n}\psi_R(2n)\phi_R(2n+2).$$

Hence, due to (14) and (16) we have

$$\psi_R(2n)\phi_R(2n+2) = \frac{d(2n)}{\|\widehat{a}(2n)\|^2}\overline{\widehat{a}(2n)}\frac{\widehat{b}(2n+2)}{\|\widehat{b}(2n+2)\|^2}\overline{\widehat{b}(2n+2)}\widehat{a}(2n+2)$$

and then

$$V_{2n+2} = V_{2n}\frac{d(2n)}{\|\widehat{a}(2n)\|^2}\overline{\widehat{a}(2n)}\widehat{a}(2n+2) = V_{2n}\theta_R(2n),$$

where $\overline{\widehat{a}(2n)}\widehat{a}(2n+2) = \theta_0(2n) + \theta_1(2n)\mathbf{i} + \theta_2(2n)\mathbf{j} + \theta_3(2n)\mathbf{k}$.

Furthermore, one has

$$V_{2n+2} = \phi_L(2n+2)V_{2n+1} = \phi_L(2n+2)\psi_L(2n)V_{2n}$$

due to

$$\phi_L(2n+2)\psi_L(2n) = \widehat{a}(2n+2)\frac{\overline{\widehat{b}(2n+2)}}{\|\widehat{b}(2n+2)\|^2}\widehat{b}(2n+2)\frac{d(2n)}{\|\widehat{a}(2n)\|^2}\overline{\widehat{a}(2n)}$$

one gets

$$V_{2n+2} = \widehat{a}(2n+2)\frac{d(2n)}{\|\widehat{a}(2n)\|^2}\overline{\widehat{a}(2n)}V_{2n} = \theta_L(2n)V_{2n},$$

where $\widehat{a}(2n+2)\overline{\widehat{a}(2n)} = \theta_0(2n) + \theta_2(2n)\mathbf{i} + \theta_1(2n)\mathbf{j} + \theta_3(2n)\mathbf{k}$. □

Proposition 3.9. *Consider V_s defined in (10) and $d(s)$ defined in (4). Then*

$$V_{2n+1} = V_{2n-1}\kappa_R(2n), \qquad V_{2n+1} = \kappa_L(2n)V_{2n-1},$$

where

$$\kappa_R(2n) = \frac{d(2n)\,(\kappa_0(2n) + \kappa_1(2n)\mathbf{j})}{1 + d^2(2n)}, \quad \kappa_L(2n) = \frac{d(2n)\,(\kappa_0(2n) + \kappa_1(2n)\mathbf{k})}{1 + d^2(2n)}$$

with

$$\kappa_0(2n) = 1 + d(2n)d(2n+2); \qquad \kappa_1(2n) = d(2n+2) - d(2n).$$

Proof. Using (16) we have

$$\begin{aligned} V_{2n+1} &= v_{2n+2}\widehat{b}(2n+2) \\ &= d(2n)v_{2n}\widehat{b}(2n+2) \\ &= v_{2n}\widehat{b}(2n)\overline{\widehat{b}(2n)}\tfrac{d(2n)}{\|\widehat{b}(2n)\|^2}\widehat{b}(2n+2) \\ &= V_{2n-1}\kappa_R(2n), \end{aligned}$$

the last equality is due to the fact $\overline{\widehat{b}(2n)}\widehat{b}(2n+2) = \kappa_0(2n) + \kappa_1(2n)\mathbf{j}$.

In a analogous way we prove

$$\begin{aligned} V_{2n+1} &= v_{2n+2}\widehat{b}(2n+2) \\ &= d(2n)v_{2n}\widehat{b}(2n+2) \\ &= \widehat{b}(2n+2)\overline{\widehat{b}(2n)}\tfrac{d(2n)}{\|\widehat{b}(2n)\|^2}v_{2n}\widehat{b}(2n) \\ &= \kappa_L(2n)V_{2n-1}, \end{aligned}$$

since $\widehat{b}(2n+2)\overline{\widehat{b}(2n)} = \kappa_0(2n) + \kappa_1(2n)\mathbf{k}$. □

In the next two results, a three term recurrence relation are presented. In Theorem 3.10 the recurrence relation is given by a three consecutive terms, while in Theorem 3.11 the recurrence relation is given by a three consecutive terms with index odd or even.

Theorem 3.10. *Let V_s be defined as in (10). Then*

$$V_{s+1} = V_sP_1(s) + V_{s-1}P_0(s-1)$$

where

$$P_1(s) = \begin{cases} \frac{1}{2}\psi_R(s), & s = 2n \\ \frac{1}{2}\phi_R(s+1), & s = 2n+1 \end{cases}, \quad P_0(s-1) = \begin{cases} \frac{1}{2}\kappa_R(s), & s = 2n \\ \frac{1}{2}\theta_R(s-1), & s = 2n+1 \end{cases}$$

ϕ_R, ψ_R, θ_R and κ_R are defined in Proposition 3.6 - Proposition 3.9.

Proof. Let $s = 2n + 1$, then using Proposition 3.6 and Proposition 3.8, one has

$$\begin{aligned} V_{2n+2} &= \tfrac{1}{2}V_{2n+2} + \tfrac{1}{2}V_{2n+2} \\ &= \tfrac{1}{2}V_{2n+1}\phi_R(2n+2) + \tfrac{1}{2}V_{2n}\theta_R(2n) \\ &= V_{2n+1}P_1(2n+1) + V_{2n}P_0(2n). \end{aligned}$$

For the case, $s = 2n$, then using Proposition 3.7 and Proposition 3.9, one gets

$$\begin{aligned} V_{2n+1} &= \tfrac{1}{2}V_{2n+1} + \tfrac{1}{2}V_{2n+1} \\ &= \tfrac{1}{2}V_{2n}\psi_R(2n) + \tfrac{1}{2}V_{2n-1}\kappa_R(2n) \\ &= V_{2n}P_1(2n) + V_{2n-1}P_0(2n-1). \end{aligned}$$

□

The next result one obtains a three term recurrence relation, where the terms are consecutive for V_n with index odd or even.

Theorem 3.11. *Let V_s be defined as in (10). Then*

$$V_{s+2} = V_sF_1(s) + V_{s-2}F_0(s-2),$$

where

$$F_1(s) = \begin{cases} \frac{\theta_R(s)}{2}, & s = 2n \\ \frac{\kappa_R(s+1)}{2}, & s = 2n+1 \end{cases}, \quad F_0(s-2) = \begin{cases} \theta_R(s-2)F_1(s), & s = 2n \\ \kappa_R(s-1)F_1(s), & s = 2n+1 \end{cases}$$

with θ_R is defined in Proposition 3.8 and κ_R defined in Proposition 3.9.

Proof. We start by proving the case $s = 2n$,

$$V_{2n+2} = V_{2n}F_1(2n) + V_{2n-2}F_0(2n-2).$$

Consider V_{2n+2} be defined as in (14), and using Lemma 2.2, we get

$$\begin{aligned} V_{2n+2} &= v_{2n+2}\widehat{a}(2n+2) \\ &= \left(\tfrac{1}{2}d(2n)v_{2n} + \tfrac{1}{2}d(2n)d(2n-2)v_{2n-2}\right)\widehat{a}(2n+2) \\ &= v_{2n}\widehat{a}(2n)\tfrac{1}{2}d(2n)\frac{\overline{\widehat{a}(2n)}}{\|\widehat{a}(2n)\|}\widehat{a}(2n+2) + v_{2n-2}\widehat{a}(2n-2)\tfrac{1}{2}d(2n)d(2n-2)\frac{\overline{\widehat{a}(2n-2)}}{\|\widehat{a}(2n-2)\|}\widehat{a}(2n+2) \\ &= V_{2n}\frac{\theta_R(2n)}{2} + V_{2n-2}\tfrac{1}{2}d(2n)d(2n-2)\frac{\overline{\widehat{a}(2n-2)}}{\|\widehat{a}(2n-2)\|}\widehat{a}(2n)\frac{\overline{\widehat{a}(2n)}}{\|\widehat{a}(2n)\|}\widehat{a}(2n+2) \\ &= V_{2n}\frac{\theta_R(2n)}{2} + V_{2n-2}\theta_R(2n-2)\frac{\theta_R(2n)}{2} \\ &= V_{2n}F_1(2n) + V_{2n-2}F_0(2n-2). \end{aligned}$$

For the case $s = 2n + 1$,

$$V_{2n+3} = V_{2n+1}F_1(2n+1) + V_{2n-1}F_0(2n-1).$$

Consider V_{2n+3} be defined as in (16), and using Lemma 2.2, we obtain

$$\begin{aligned} V_{2n+3} &= v_{2n+4}\,\widehat{b}(2n+4) \\ &= \left(\tfrac{1}{2}d(2n+2)v_{2n+2} + \tfrac{1}{2}d(2n+2)d(2n)v_{2n}\right)\widehat{b}(2n+4) \\ &= v_{2n+2}\widehat{b}(2n+2)\tfrac{1}{2}d(2n+2)\frac{\overline{\widehat{b}(2n+2)}}{\|\widehat{b}(2n+2)\|^2}\widehat{b}(2n+4) + v_{2n}\widehat{b}(2n)\frac{\overline{\widehat{b}(2n)}}{\|\widehat{b}(2n)\|^2}\tfrac{1}{2}d(2n+2)d(2n)\widehat{b}(2n+4) \\ &= V_{2n+1}\frac{\kappa(2n+2)}{2} + V_{2n-1}d(2n)\frac{\overline{\widehat{b}(2n)}}{\|\widehat{b}(2n)\|^2}\widehat{b}(2n+2)\frac{d(2n+2)}{2}\frac{\overline{\widehat{b}(2n)}}{\|\widehat{b}(2n)\|^2}\widehat{b}(2n+4) \\ &= V_{2n+1}\frac{\kappa(2n+2)}{2} + V_{2n-1}\kappa(2n)\frac{\kappa(2n+2)}{2} \\ &= V_{2n+1}F_1(2n+1) + V_{2n-1}F_0(2n-1). \end{aligned}$$

□

A type of Catalan's identity for the sequence $\{V_s\}_{s\geq 0}$ is presented next.

Theorem 3.12. *Let V_s be defined as in (10), $d(l)$ defined as in (4) and $\widehat{a},\widehat{b}$ be defined as in (14), (16) respectively. Then*

$$V_s^2 - V_{s-p}V_{s+p} = V_p^2 T(s,p), \qquad s > p$$

with $T(s,p) = L(s,p)K(s,p)$ having

$$L(s,p) = \begin{cases} \left(\frac{\overline{\widehat{a}(p)}}{\|\widehat{a}(p)\|^2} \prod_{l=1}^{\lfloor \frac{s+1-p}{2} \rfloor} d\left(2\lfloor \frac{s+1}{2} \rfloor - 2l\right)\right)^2, & p \quad even \\ \left(\frac{\overline{\widehat{b}(p+1)}}{\|\widehat{b}(p+1)\|^2} \prod_{l=1}^{\lfloor \frac{s-p-1}{2} \rfloor} d(\lfloor \frac{s+1}{2} \rfloor - 2l)\right)^2, & p \quad odd \end{cases}$$

and

$$K(s,p) = \begin{cases} \widehat{a}^2(s) - t(s,p)\widehat{a}(s-p)\widehat{a}(s+p), & s = 2n \quad p = 2k \\ \widehat{b}^2(s+1) - t(s,p)\widehat{b}(s+1-p)\widehat{b}(s+1+p), & s = 2n+1 \quad p = 2k \\ \widehat{a}^2(s) - \frac{t(s,p)}{d(s)}\widehat{b}(s-p+1)\widehat{b}(s+1+p), & s = 2n \quad p = 2k+1 \\ \widehat{b}^2(s+1) - t(s,p)d(s)\widehat{a}(s-p)\widehat{a}(s+p), & s = 2n+1 \quad p = 2k+1 \end{cases}$$

where

$$t(s,p) = \begin{cases} \prod_{l=1}^{\lfloor \frac{p}{2} \rfloor} \frac{d(s+p-2l)}{d(s-p-2+2l)}, & s+p \quad even \\ \prod_{l=1}^{\lfloor \frac{p}{2} \rfloor} \frac{d(s+p+1-2l)}{d(s-p-1+2l)}, & s+p \quad odd \end{cases}$$

Proof. The proof done for the following four cases:

(i) $s = 2n$ and $p = 2k$, i.e,

$$V_{2n}^2 - V_{2n-2k}V_{2n+2k} = V_{2k}^2 T(2n,2k), \qquad n > k.$$

Using (14) and (5) one has

$$V_{2n+2k} = v_{2n+2k}\widehat{a}(2n+2k) = \prod_{l=1}^{k} d(2n+2k-2l)v_{2n}\widehat{a}(2n+2k), \tag{18}$$

and

$$V_{2n-2k} = v_{2n-2k}\widehat{a}(2n-2k) = \prod_{l=1}^{k} \frac{1}{d(2n-2k+2l-2)} v_{2n}\widehat{a}(2n-2k) \tag{19}$$

Substituting (18) and (19) we have

$$\begin{aligned} V_{2n}^2 - V_{2n-2k}V_{2n+2k} &= v_{2n}^2\widehat{a}^2(2n) - v_{2n-2k}v_{2n+2k}\widehat{a}(2n-2k)\widehat{a}(2n+2k) \\ &= v_{2n}^2\widehat{a}^2(2n) - v_{2n}^2 \prod_{l=1}^{k} \frac{d(2n+2k-2l)}{d(2n-2k+2l-2)}\widehat{a}(2n-2k)\widehat{a}(2n+2k) \\ &= v_{2k}^2 \prod_{l=1}^{n-k} d^2(2n-2l)\left(\widehat{a}^2(2n) - t(2n,2k)\widehat{a}(2n-2k)\widehat{a}(2n+2k)\right) \\ &= v_{2k}^2\widehat{a}^2(2k)L(2n,2k)K(2n,2k) \\ &= V_{2k}^2 T(2n,2k). \end{aligned}$$

(ii) $s = 2n$ and $p = 2k + 1$

$$V_{2n}^2 - V_{2n-2k-1}V_{2n+2k+1} = V_{2k+1}^2 T(2n, 2k+1), \qquad n > k$$

Using (16) and (5) one has

$$\begin{aligned} V_{2n+2k+1} &= v_{2n+2k+2}\widehat{b}(2n+2k+2) \\ &= \prod_{l=1}^{k} d(2n+2k+2-2l)v_{2n+2}\widehat{b}(2n+2k+2), \end{aligned} \tag{20}$$

and

$$V_{2n-2k-1} = v_{2n-2k}\widehat{b}(2n-2k) = \prod_{l=1}^{k} \tfrac{1}{d(2n-2k+2l-2)} v_{2n}\widehat{b}(2n-2k) \tag{21}$$

Substituting (20) and (21) we have

$$\begin{aligned} V_{2n}^2 - V_{2n-2k-1}V_{2n+2k+1} &= v_{2n}^2\widehat{a}^2(2n) - v_{2n-2k}v_{2n+2k+2}\widehat{b}(2n-2k)\widehat{b}(2n+2k+2) \\ &= v_{2n}^2\widehat{a}^2(2n) - v_{2n+2}^2 \prod_{l=1}^{k} \tfrac{d(2n+2k+2-2l)}{d(2n-2k+2l-2)d(2n)}\widehat{b}(2n-2k)\widehat{b}(2n+2k+2) \\ &= v_{2k+2}^2 \prod_{l=1}^{n-k} d^2(2n+2-2l)\left(\tfrac{\widehat{a}^2(2n)}{d^2(2n)} - t(2n, 2k+1)\tfrac{\widehat{b}(2n-2k)\widehat{b}(2n+2k+2)}{d(2n)}\right) \\ &= v_{2k+2}^2\widehat{b}^2(2k+2)L(2n, 2k+1)K(2n, 2k+1) \\ &= V_{2k+1}^2 T(2n, 2k+1). \end{aligned}$$

(iii) $s = 2n + 1$ and $p = 2k$

$$V_{2n+1}^2 - V_{2n+1-2k}V_{2n+1+2k} = V_{2k}^2 T(2n+1, 2k), \qquad n > k.$$

Using (16) and (5) one has

$$V_{2n-2k+1} = v_{2n-2k+2}\widehat{b}(2n-2k+2) = \prod_{l=1}^{k} \frac{1}{d(2n-2k+2l)} v_{2n+2}\widehat{b}(2n-2k+2). \tag{22}$$

Substituting (20), (22) and (16), one gets

$$\begin{aligned} V_{2n+1}^2 - V_{2(n-k)+1}V_{2(n+k)+1} &= v_{2n+2}^2\widehat{b}^2(2n+2) - v_{2n-2k+2}v_{2n+2k+2}\widehat{b}(2n-2k+2)\widehat{b}(2n+2k+2) \\ &= v_{2n+2}^2\widehat{b}^2(2n+2)- \\ &\quad -v_{2n+2}^2 \prod_{l=1}^{k} \tfrac{d(2n+2k+2-2l)}{d(2n-2k+2l)}\widehat{b}(2n+2-2k)\widehat{b}(2n+2k+2) \\ &= v_{2k}^2 \prod_{l=1}^{n-k} d^2(2n-2l)d^2(2n)\times \\ &\quad \times\left(\widehat{b}^2(2n+2) - t(2n+1, 2k)\widehat{b}(2n-2k+2)\widehat{b}(2n+2k+2)\right) \\ &= v_{2k}^2\widehat{a}^2(2k)L(2n+1, 2k)K(2n+1, 2k) \\ &= V_{2k}^2 T(2n+1, 2k). \end{aligned}$$

(iv) $s = 2n + 1$ and $p = 2k + 1$

$$V_{2n+1}^2 - V_{2n-2k}V_{2n+2k+2} = V_{2k+1}^2 T(2n+1, 2k+1), \qquad n > k.$$

Using (14) and (5) one has

$$\begin{aligned} V_{2n+2k+2} &= v_{2n+2k+2}\widehat{a}(2n+2k+2) \\ &= \prod_{l=1}^{k} d(2n+2k+2-2l)v_{2n+2}\widehat{a}(2n+2k+2). \end{aligned} \tag{23}$$

Substituting (19) and (23) we have

$$\begin{aligned}
V_{2n+1}^2 - V_{2n-2k}V_{2n+2k+2} &= v_{2n+2}^2\widehat{b^2}(2n+2) - v_{2n-2k}v_{2n+2k+2}\widehat{a}(2n-2k)\widehat{a}(2n+2k+2)\\
&= v_{2n+2}^2\widehat{b^2}(2n+2)v_{2n+2}^2\prod_{l=1}^{k}\frac{d(2n+2k+2-2l)}{d(2n-2k+2l-2)}\widehat{a}(2n+2)\widehat{a}(2n+2k+2)\\
&= v_{2k+2}^2\prod_{l=1}^{n-k}d^2(2n+2-2l)\times\\
&\quad\times\left(\widehat{b^2}(2n+2) - t(2n+1,2k+1)\widehat{a}(2n+2)\widehat{a}(2n+2k+2)\right)\\
&= v_{2k+2}^2\widehat{b^2}(2k+2)L(2n+1,2k+1)K(2n+1,2k+1)\\
&= V_{2k+1}^2 T(2n+1,2k+1).
\end{aligned}$$

□

3.3. Generating function and Binet's type formula

Here we start by presenting the generating function for the sequence of $\{V_s\}_{s\geq 0}$, and finish with a Binet's type formula for this sequence.
Using the generating function of the Vietoris' sequence, (see [6])

$$g(t) = \frac{\sqrt{1+t} - \sqrt{1-t}}{t\sqrt{1-t}} = \sum_{n=0}^{+\infty} v_n t^n, \qquad 0 < |t| < 1$$

we obtain for the sequence $\{V_s\}_{s\geq 0}$ the following result:

Theorem 3.13. *The generating function for $\{V_s\}_{s\geq 0}$ is*

$$G(t) = \frac{1}{t^3}\left(g(t)(t^3 + t^2\mathbf{i} + t\mathbf{j} + \mathbf{k}) - H(t)\right), \qquad 0 < |t| < 1$$

where $H(t) = \frac{1}{2}\left(t^2(2\mathbf{i} + \mathbf{j} + \mathbf{k}) + t(2\mathbf{j} + \mathbf{k}) + 2\mathbf{k}\right)$.

Proof. Let

$$G(t) = \sum_{n=0}^{+\infty} V_n t^n$$

be the generating function for $\{V_s\}_{s\geq 0}$. Then

$$\begin{aligned}
t^3G(t) &= \sum_{n=0}^{+\infty} V_n t^{n+3} = \sum_{n=0}^{+\infty}(v_n + v_{n+1}\mathbf{i} + v_{n+2}\mathbf{j} + v_{n+3}\mathbf{k})t^{n+3}\\
&= \sum_{n=0}^{+\infty} v_n t^{n+3} + \mathbf{i}\sum_{n=0}^{+\infty} v_{n+1}t^{n+3} + \mathbf{j}\sum_{n=0}^{+\infty} v_{n+2}t^{n+3} + \mathbf{k}\sum_{n=0}^{+\infty} v_{n+3}t^{n+3}\\
&= t^3\sum_{n=0}^{+\infty} v_n t^n + \mathbf{i}t^2\sum_{n=0}^{+\infty} v_{n+1}t^{n+1} + \mathbf{j}t\sum_{n=0}^{+\infty} v_{n+2}t^{n+2} + \mathbf{k}\sum_{n=0}^{+\infty} v_{n+3}t^{n+3}\\
&= t^3g(t) + \mathbf{i}t^2\left(-v_0 + v_0 + \sum_{n=0}^{+\infty} v_{n+1}t^{n+1}\right)\\
&+ \mathbf{j}t\left(-v_0 - v_1t + v_0 + v_1t + \sum_{n=0}^{+\infty} v_{n+2}t^{n+2}\right)\\
&+ \mathbf{k}\left(-v_0 - v_1t - v_2t^2 + v_0 + v_1t + v_2t^2 + \sum_{n=0}^{+\infty} v_{n+3}t^{n+3}\right)\\
&= t^3g(t) + \mathbf{i}t^2(-v_0 + g(t)) + \mathbf{j}t(-v_0 - v_1t + g(t))\\
&+ \mathbf{k}\left(-v_0 - v_1t - v_2t^2 + g(t)\right)\\
&= g(t)(t^3 + \mathbf{i}t^2 + \mathbf{j}t + \mathbf{k}) - \left(\mathbf{i}t^2v_0 + \mathbf{j}t(v_0 + v_1t) + \mathbf{k}(v_0 + v_1t + v_2t^2)\right)\\
&= g(t)(t^3 + \mathbf{i}t^2 + \mathbf{j}t + \mathbf{k}) - H(t),
\end{aligned}$$

where, substituting the first elements of Vietoris' number sequence

$$H(t) = \frac{1}{2}\left(\mathbf{i}2t^2 + \mathbf{j}t(2+t) + \mathbf{k}(2+t+t^2)\right).$$

□

Using the Binet like formula presented in Theorem 2.3 for the Vietoris number sequence, one presents in the next result a Binet-like formula for $\{V_s\}_{s\geq 0}$.

Theorem 3.14. *If V_s is defined in (10). Then a Binet-like formulas is*

$$V_s = \widehat{\alpha_1}(s) r_1^{2\lfloor\frac{s+1}{2}\rfloor}\left(2\left\lfloor\frac{s+1}{2}\right\rfloor\right) + \widehat{\alpha_2}(s) r_2^{2\lfloor\frac{s+1}{2}\rfloor}\left(2\left\lfloor\frac{s+1}{2}\right\rfloor\right),$$

where, for $i = 1, 2$,

$$\widehat{\alpha_i}(s) = \begin{cases} c_i(s)\widehat{a}(s), & s = 2n \\ \\ c_i(s+1)\widehat{b}(s+1), & s = 2n+1 \end{cases},$$

with $r_i(\cdot)$, $c_i(\cdot)$ is defined in Theorem 2.3, and $\widehat{a}$, $\widehat{b}$ defined in (14) and (16) respectively.

Proof. One start be proving the theorem, for $s = 2n$. Using (14) and Theorem 2.3, one gets

$$\begin{aligned} V_{2n} &= v_{2n}\widehat{a}(2n) \\ &= \left(c_1(2n) r_1^{2n}(2n) + c_2(2n) r_2^{2n}(2n)\right)\widehat{a}(2n) \\ &= c_1(2n)\widehat{a}(2n) r_1^{2n}(2n) + c_2(2n)\widehat{a}(2n) r_2^{2n}(2n) \\ &= \widehat{\alpha_1}(2n) r_1^{2n}(2n) + \widehat{\alpha_2}(2n) r_2^{2n}(2n). \end{aligned}$$

For the case $s = 2n+1$, one uses (16) and Theorem 2.3

$$\begin{aligned} V_{2n+1} &= v_{2n+2}\widehat{b}(2n+2) \\ &= \left(c_1(2n+2) r_1^{2n+2}(2n+2) + c_2(2n+2) r_2^{2n+2}(2n+2)\right)\widehat{b}(2n+2) \\ &= c_1(2n+2)\widehat{b}(2n+2) r_1^{2n+2}(2n+2) + c_2(2n+2)\widehat{b}(2n+2) r_2^{2n+2}(2n+2) \\ &= \widehat{\alpha_1}(2n+2) r_1^{2n+2}(2n+2) + \widehat{\alpha_2}(2n+2) r_2^{2n+2}(2n+2). \end{aligned}$$

□

Remark 3.15. *Using the Binet type formula for the Vietoris number sequence presented in Theorem 2.5, one also can obtain a Binet type formula for the quaternion sequence $\{V_n\}_{n\geq 0}$.*

4. The determinant of a special kind of tridiagonal matrices that generates $\{V_n\}_{n\geq 0}$

We consider two different tridiagonal matrices and we stated that, for each type of matrices and considering its dimension, the determinant of these matrices generates the quaternion number sequence presented in (10).

Due to the noncommutativity of the quaternion elements, we calculate the determinant of a matrix with quaternion number using the Laplace expansion starting always with all entries of the last column, i.e., for any squart matrix $X = \left[x_{ij}\right]_{n\times n}$ the determinant is given by

$$\det X = \sum_{i=1}^{n} c_{in} x_{in},$$

with

$$c_{in} = (-1)^{i+n} \det Y_{in},$$

where $\det Y_{in}$ is the i, n minor of X.

We start by using a similar technique as given in [8] and [19]. Let us consider the tridiagonal matrix $U_n = \left[u_{ij}\right]_{n+1 \times n+1}$ defined by

$$U_{n+1} = \begin{bmatrix} V_1 & -V_0 & 0 & 0 & 0 & \dots & 0 \\ P_0(0) & P_1(1) & -1 & 0 & 0 & \dots & 0 \\ 0 & P_0(1) & P_1(2) & -1 & 0 & \dots & 0 \\ 0 & 0 & P_0(2) & P_1(3) & -1 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \\ 0 & 0 & 0 & 0 & P_0(n-1) & P_1(n) & -1 \\ 0 & 0 & 0 & 0 & 0 & P_0(n-1) & P_1(n) \end{bmatrix}_{(n+1)\times(n+1)} \tag{24}$$

where $P_0(\cdot)$ and $P_1(\cdot)$ are defined in Theorem 3.10.

Theorem 4.1. *Considere the matrix U_{n+1} defined in (24). Then*

$$\det U_{n+1} = V_{n+1}.$$

Proof. For de case $n = 0$, one gets $\det U_1 = V_1$ and for $n = 1$, using Theorem 3.10, one obtains

$$\det U_2 = \begin{bmatrix} V_1 & -V_0 \\ P_0(0) & P_1(1) \end{bmatrix} = V_1 P_1(1) + V_0 P_0(0) = V_2.$$

Furthermore, for the case of $n = 2$, $U_3 = \det \begin{bmatrix} V_1 & -V_0 & 0 \\ P_0(0) & P_1(1) & -1 \\ 0 & P_0(1) & P_1(2) \end{bmatrix}$ one obtains

$$\det U_3 = (-1)^6 \det U_2 P_1(2) - (-1)^5 \det U_1 P_0(1) = V_2 P_1(2) + V_1 P_0(1) = V_3.$$

Supposing that $\det U_n = V_n$ and using Theorem 3.10, one gets

$$\begin{aligned} \det U_{n+1} &= \det U_n P_1(n) + \det U_{n-1} P_0(n-1) \\ &= V_n P_1(n) + V_{n-1} P_0(n-1) = Vn + 1. \end{aligned}$$

□

Furthermore, for a different tridiagonal matrix, with quaternionic entries, we obtain the following result.

Theorem 4.2. *Considere the matrix*

$$Z_{n+1} = \begin{bmatrix} V_0 & 0 & 0 & 0 & 0 & \dots & 0 \\ -1 & 2P_1(0) & P_0(0) & 0 & 0 & \dots & 0 \\ 0 & -1 & P_1(1) & P_0(1) & 0 & \dots & 0 \\ 0 & 0 & -1 & P_1(2) & P_0(2) & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \\ 0 & 0 & 0 & 0 & -1 & P_1(n-2) & P_0(n-2) \\ 0 & 0 & 0 & 0 & 0 & -1 & P_1(n-1) \end{bmatrix}_{(n+1)\times(n+1)} \tag{25}$$

where $P_0(\cdot)$ and $P_1(\cdot)$ are defined in Theorem 3.10. Then

$$\det Z_{n+1} = V_n.$$

Proof. For de case $n = 0$, one gets $\det Z_1 = V_o$ and for $n = 1$, using Theorem 3.10, one obtains

$$\det Z_2 = \begin{bmatrix} V_0 & 0 \\ -1 & 2P_1(0) \end{bmatrix} = 2V_0P_1(0) = V_1.$$

Furthermore, for the case of $n = 2$, $Z_3 = \begin{bmatrix} V_0 & 0 & 0 \\ -1 & 2P_1(0) & P_0(0) \\ 0 & -1 & P_1(1) \end{bmatrix}$ one obtains

$$\det Z_3 = \det Z_2 P_1(1) + \det Z_1 P_0(0) = V_1P_1(1) + V_0P_0(0) = V_2.$$

Supposing that $\det Z_n = V_{n-1}$ and using Theorem 3.10, one gets

$$\begin{aligned} \det Z_{n+1} &= \det Z_n P_1(n-1) + \det Z_{n-1} P_0(n-2) \\ &= V_{n-1}P_1(n-1) + V_{n-2}P_0(n-2) = V_n. \end{aligned}$$

□

In the next result, one uses a tridiagonal matrix $A^k_{n+1} = \left[a_{ij}\right]_{n+1\times n+1}$ with $k = 0, 1$, defined by

$$A^k_{n+1} = \begin{bmatrix} V_k & 0 & 0 & 0 & \dots & 0 \\ -1 & 2F_1(k) & F_0(k) & 0 & \dots & 0 \\ 0 & -1 & F_1(2+k) & F_0(2+k) & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \\ 0 & 0 & 0 & -1 & F_1(2n-4+k) & F_0(2n-4+k) \\ 0 & 0 & 0 & 0 & -1 & F_1(2n-2+k) \end{bmatrix}_{(n+1)\times(n+1)} \tag{26}$$

where $F_1(\cdot)$, and $F_0(\cdot)$ are defined in the three term relation presented in Theorem 3.11.

For this type of matrices, considering its dimension, the determinant of these matrices generates the quaternion number sequence presented in (10), as can be seen in the next theorem, using a similar proof to the one given in Theorem 4.2. In particular for $k = 0$ one can prove that $\{\det A^0_{n+1}\}_{n\geq 0}$ generated a the subsequence $\{V_{2n}\}_{n\geq 0}$, while for the case $k = 1$ one gets the subsequence with odd index $\det A^1_{n+1} = V_{2n+1}$.

Theorem 4.3. *Considere the matrix A^k_{n+1} defined in (26). Then*

$$\det A^0_{n+1} = V_{2n}, \qquad \det A^1_{n+1} = V_{2n+1}.$$

Proof. One starts with $k = 0$. For $n = 0$, we have $\det A^0_1 = V_0$. For $n = 1$, we have

$$\det A^0_2 = \det \begin{bmatrix} V_0 & 0 \\ -1 & 2F_1(0) \end{bmatrix} = 2V_0F_1(0) = V_2.$$

In the case of $n = 2$ and using Theorem 3.11 in the last equation, $\det A^0_3 = \det \begin{bmatrix} V_0 & 0 & 0 \\ -1 & 2F_1(0) & F_0(0) \\ 0 & -1 & F_1(2) \end{bmatrix}$ where

$$\det A^0_3 = \det A^0_2 F_1(2) + \det A^0_1 F_0(0) = V_2F_1(2) + V_0F_0(0) = V_4.$$

Suppose that $\det A^0_n = V_{2n-2}$. Let us prove that $\det A^0_{n+1} = V_{2n}$,

$$A^0_{n+1} = \begin{bmatrix} V_0 & 0 & 0 & 0 & 0 & \dots & 0 \\ -1 & 2F_1(0) & F_0(0) & 0 & 0 & \dots & 0 \\ 0 & -1 & F_1(2) & F_0(2) & 0 & \dots & 0 \\ 0 & 0 & -1 & F_1(4) & F_0(4) & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \\ 0 & 0 & 0 & 0 & -1 & F_1(2n-4) & F_0(2n-4) \\ 0 & 0 & 0 & 0 & 0 & -1 & F_1(2n-2) \end{bmatrix}_{(n+1)\times(n+1)},$$

calculating the determinant we have

$$\begin{aligned} \det A^0_{n+1} &= \det A^0_n F_1(2n-2) + \det A^0_{n-2} F_0(2n-4) \\ &= V_{2n-2}F_1(2n-2) + V_{2n-4}F_2(2n-4) \\ &= V_{2n} \end{aligned}$$

For $k = 1$ and $n = 0$, we have $\det A^1_1 = V_1$. For $n = 1$, we have

$$\det A^1_2 = \det \begin{bmatrix} V_1 & 0 \\ -1 & 2F_1(1) \end{bmatrix} = 2V_1F_1(1) = V_3.$$

In the case of $n = 2$ and using Theorem 3.11 in the last equation, we have

$$\det A_3^1 = \det \begin{bmatrix} V_1 & 0 & 0 \\ -1 & 2F_1(1) & F_0(1) \\ 0 & -1 & F_1(3) \end{bmatrix}$$

where

$$\det A_3^1 = \det A_2^1 F_1(3) + \det A_1^1 F_0(1) = V_3 F_1(3) + V_1 F_0(1) = V_5$$

Suppose that $\det A_n^1 = V_{2n+1}$. Let us prove that $\det A_{n+1}^1 = V_{2n+3}$,

$$A_{n+1}^1 = \begin{bmatrix} V_1 & 0 & 0 & 0 & 0 & \dots & 0 \\ -1 & 2F_1(1) & F_0(1) & 0 & 0 & \dots & 0 \\ 0 & -1 & F_1(3) & F_0(3) & 0 & \dots & 0 \\ 0 & 0 & -1 & F_1(5) & F_0(5) & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \\ 0 & 0 & 0 & 0 & -1 & F_1(2n-1) & F_0(2n-1) \\ 0 & 0 & 0 & 0 & 0 & -1 & F_1(2n+1) \end{bmatrix}_{(n+1)\times(n+1)},$$

calculating the determinant we have

$$\begin{aligned} \det A_{n+1}^1 &= \det A_n^1 F_1(2n+1) + \det A_{n-2}^1 F_0(2n-1) \\ &= V_{2n+1} F_1(2n+1) + V_{2n-1} F_2(2n-1) \\ &= V_{2n+3} \end{aligned}$$

□

Theorem 4.4. *Considere the matrix $M_n^k = \left[m_{ij}\right]_{n\times n}$ with $k = 0, 1$, defined by*

$$M_n^k = \begin{bmatrix} V_k & -V_{k+2} & 0 & 0 & \dots & 0 \\ F_0(k) & F_1(2+k) & -1 & 0 & \dots & 0 \\ 0 & F_0(2+k) & F_1(4+k) & -1 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \\ 0 & 0 & 0 & F_0(2n-4+k) & F_1(2n-2+k) & -1 \\ 0 & 0 & 0 & 0 & F_0(2n-2+k) & F_1(2n+k) \end{bmatrix}_{n\times n} \tag{27}$$

where $F_1(\cdot)$, and $F_0(\cdot)$ are defined in the three term relation presented in Theorem 3.11. Then

$$\det M_n^0 = V_{2n}, \qquad \det M_n^1 = V_{2n+1}.$$

Using these type of matrices we obtain a pattern of recurrence found in $\{V_n\}_{n\geq 0}$, where the proof is similar to the one given in Theorem 4.1.

References

[1] M. Akyiğit, H. H. K¨sal, M. Tosun, Split Fibonacci quaternions, Advances in Applied Clifford Algebras 23 (2013) 535–545.
[2] M. Akyiğit, H. H. K¨sal, M. Tosun, Fibonacci generalized quaternions, Advances in Applied Clifford Algebras 24 (2014) 631–641.
[3] G. Bilgici, U. Tokeşer, Z. Ünal, *k*-Fibonacci and *k*-Lucas generalized quaternions, Konuralp Journal of Mathematics 5 (2017) 102–113.
[4] I. Cação, M. I. Falcão, M. Malonek, Matrix Representations of a Special Polynomial Sequence in Arbitrary Dimension, Computational Methods and Function Theory 12(2) (2012) 371–391.
[5] I. Cação, M. I. Falcão, M. Malonek, Hypercomplex polynomials, Vietoris' rational numbers and a related integer numbers sequence, Complex Analysis and Operator Theory 11(5) (2017) 1059–1076.
[6] I. Cação, M. I. Falcão, M. Malonek, On generalized Vietoris' number sequences, Discrete Applied Mathematics 269 (2019) 77–85.
[7] I. Cação, H.R. Malonek, G. Tomaz, Shifted generalized pascal matrices in the context of clifford algebra-valued polynomial sequences, in: B. Murgante,et al. (Eds.), Lecture Notes in Computer Science, vol. 10405, Springer-Verlag, Berlin, Heidelberg, 2017, 409–421.
[8] N. D. Cahill, J. R. D'Errico, D. A. Narayan, J. Y. Narayan, Fibonacci Determinants, The College Mathematics Journal 33 (3) (2002) 221–225.
[9] P. Catarino, R. De Almeida, A note on Vietoris' number sequence, submitted.
[10] P. Catarino, R. De Almeida, On Vietoris' Difference sequences, submitted.
[11] P. Catarino, The modified Pell and the modified *k*-Pell quaternions and octonions, Advances in Applied Clifford Algebras 26 (2016) 577–590.
[12] Y. Choo, A generalized quaternion with generalized Fibonacci number components, Applied Mathematical Sciences 14(1) (2020) 31–38.
[13] C. B. Çimen, A. İpek, On Pell quaternions and Pell-Lucas quaternions, Advances in Applied Clifford Algebras 26 (2016) 39–51.
[14] J. H. Conway, D. A. Smith, On Quaternions and Octonions: their Geometry, Arithmetic and Symmetry, A. K. Peters, 2003.
[15] M. I. Falcão, M. Malonek, A note on a one-parameter family of nonsymmetric number triangles, Opuscula Mathematica 32(4) (2012) 661–673.
[16] K. Gürlebeck, K. Habetha, W. Sprößig, Holomorphic Functions in the Plane and n-dimensional Space, Birkhäuser, Basel, 2008.
[17] S. Halici, On Fibonacci quaternions, Advances in Applied Clifford Algebras 22 (2012) 321–327.
[18] S. Halici, A. Karataş, On a generalization for Fibonacci quaternions, Chaos, Solitons and Fractals 98 (2017) 178–182.
[19] E. Kilic, D. Tasci, P. Haukkanen, On the generalized Lucas sequences by Hessenberg matrices, Ars Combinatoria 95 (2010) 383–395.
[20] B. K. Patel, P. K. Ray, On the properties of (p, q)-Fibonacci and (p, q)-Lucas quaternions, Mathematical Reports 21(71) (2019) 15–25.
[21] E. Polati, C. Kizilates, S. Kesim, On split *k*-Fibonacci and *k*-Lucas quaternions, Advances in Applied Clifford Algebras 26 (2016) 353–362.
[22] J. L. Ramírez, Some combinatorial properties of the *k*-Fibonacci and the *k*-Lucas quaternions, Analele Stiintifice ale Universitatii Ovidius Constanta 23 (2015) 201–212.
[23] N. J. A. Sloane, S. Plouffe, The Encyclopedia of Integer Sequences, Academic Press, San Diego, 1995.
[24] H. M. Srivastava, Some Generalizations of a Combinatorial Identity of L. Vietoris, Discrete Mathematics 65(1) (1987) 99–102.
[25] H. M. Srivastava, P. E. Ricci, P. Natalini, A family of complex Appell polynomial sets, Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales. Serie A. Matemáticas 113 (2019), 2359 – 2371.
[26] H. M. Srivastava, B. Y. Yaşar, M. A. Özarslan, A Class of Big (p; q)-Appell Polynomials and Their Associated Difference Equations, Filomat 33(10) (2019) 3085–3121.
[27] H. M. Srivastava, G. Yasmin, A. Muhyi, S. Araci, Certain Results for the Twice-Iterated 2D q-Appell Polynomials, Symmetry 11(10) (2019) 1307, 1–23.
[28] A. Szynal-Liana, I. Włoch, The Pell quaternions and the Pell octonions, Advances in Applied Clifford Algebras 26 (2016) 435–440.
[29] A. Szynal-Liana, I. Włoch, A note on Jacobsthal quaternions, Advances in Applied Clifford Algebras 26 (2016) 441–447.
[30] D. Tasci, On *k*-Jacobsthal and *k*-Jacobsthal-Lucas quaternions, Journal of Science and Arts 40 (2017) 469–476.
[31] U. Tokeşer, Z. Ünal, G. Bilgici, Split Pell and Pell-Lucas quaternions, Advances in Applied Clifford Algebras 27 (2017) 1881–1893.
[32] L. Vietoris, Über das Vorzeichen gewisser trigonometrischer Summen, Sitzungsber. Österr. Akad. Wiss 167 (1958) 125–135.
[33] L. Vietoris, Eine Verallgemeinerung der Gleichung $(n + 1)! = n!(n + 1)$ und zugehörige vermutete Ungleichungen, Monatshefte für Mathematik, 97(1984), 157–160
[34] T. Yağmur, Split Jacobsthal and Jacobsthal-Lucas quaternions, Communications in Mathematics and Applications 10 (2019) 429–438.