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An Observation About Pseudospectra

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Abstract. For $\varepsilon > 0$ and a bounded linear operator T acting on some Hilbert space, the ε-pseudospectrum of T is $\sigma_\varepsilon(T) = \{z \in \mathbb{C} : \|(zI - T)^{-1}\| > \varepsilon^{-1}\}$. This note provides a characterization of those operators T satisfying $\sigma_\varepsilon(T) = \sigma(T) + B(0, \varepsilon)$ for all $\varepsilon > 0$. Here $B(0, \varepsilon) = \{z \in \mathbb{C} : |z| < \varepsilon\}$. In particular, such operators on finite dimensional spaces must be normal.

1. Introduction

As usual, we let $\mathbb{N}, \mathbb{C}$ denote respectively the set of positive integers and the set of complex numbers. $\mathcal{H}$ will always denote a complex separable infinitely dimensional Hilbert space. Denote by $\mathcal{B}(\mathcal{H})$ the Banach algebra of all bounded linear operators on $\mathcal{H}$.

This paper is a continuation of a previous paper of the authors [8], where pseudospectral radii of Hilbert space operators are studied. The *spectrum* of an operator $T \in \mathcal{B}(\mathcal{H})$ is

$$\sigma(T) = \{z \in \mathbb{C} : zI - T \text{ is not invertible in } \mathcal{B}(\mathcal{H})\}.$$

Given $\varepsilon > 0$, the *ε-pseudospectrum* of T is defined as

$$\sigma_\varepsilon(T) = \{z \in \mathbb{C} : \|(zI - T)^{-1}\| > \varepsilon^{-1}\}.$$

Conventionally, it is assumed that $\|(zI - T)^{-1}\| = \infty$ if $z \in \sigma(T)$. The reader is referred to [10] for other equivalent definitions of the ε-pseudospectrum.

The behaviors of pseudospectra of operators are quite different from that of their spectra. It is obvious that the ε-pseudospectrum of T is always open. However, pseudospectra can be used to give effective estimations of spectra. In fact, one can check that

$$\bigcap_{\varepsilon>0} \sigma_\varepsilon(T) = \sigma(T).$$

Moreover, it is known that the map $(\varepsilon, T) \mapsto \sigma_\varepsilon(T)$ is continuous (see [4, Prop. 2.7]).

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The aim of this note is to discuss the relation between spectra and pseudospectra. For $T \in \mathcal{B}(\mathcal{H})$ and $\varepsilon > 0$, it is known that $\sigma(T) + B(0,\varepsilon) \subset \sigma_\varepsilon(T)$, and the converse inclusion in general does not hold. Here $B(0,\varepsilon)$ denotes the set $\{z \in \mathbb{C} : |z| < \varepsilon\}$. For example, if an operator $A \in \mathcal{B}(\mathcal{H})$ is nilpotent of order 2, then $\sigma_\varepsilon(A) = B(0, \sqrt{\varepsilon^2 + \|A\|\varepsilon})$ (see [4, Proposition 2.4]), and

$$\sigma(A) + B(0,\varepsilon) = B(0,\varepsilon) \subsetneq \sigma_\varepsilon(A).$$

Thus a natural question arises.

Question 1.1. *When does an operator T satisfy*

$$\sigma_\varepsilon(T) = \sigma(T) + B(0,\varepsilon) \quad \text{for all} \quad \varepsilon > 0\ ? \tag{1}$$

For any $\varepsilon > 0$, note that

$$\sigma(T) + B(0,\varepsilon) = \{z \in \mathbb{C} : \operatorname{dist}(z,\sigma(T)) < \varepsilon\} = \{z \in \mathbb{C} : 1/\operatorname{dist}(z,\sigma(T)) > \varepsilon^{-1}\}.$$

Thus an operator T satisfies (1) if and only if $\|(zI - T)^{-1}\| = 1/\operatorname{dist}(z,\sigma(T))$ for $z \in \mathbb{C} \setminus \sigma(T)$.

We remark that a von Neumann operator T always satisfies (1). Recall that T is called a *von Neumann operator* if $\|f(T)\| = \sup\{|f(z)| : z \in \sigma(T)\}$ for rational functions f with poles off $\sigma(T)$. Note that if T is a von Neumann operator, then $\|T\| = r(T)$, where $r(T)$ denotes the *spectral radius* of T. The class of von Neumann operators includes some special classes of operators, such as normal operators and subnormal operators. Thus if T is von Neumann, then, for any $z \in \mathbb{C} \setminus \sigma(T)$, we have

$$\|(zI - T)^{-1}\| = \sup\{|(z-\lambda)^{-1}| : \lambda \in \sigma(T)\} = 1/\operatorname{dist}(z,\sigma(T)).$$

It follows that $\sigma_\varepsilon(T) = \sigma(T) + B(0,\varepsilon)$ for all $\varepsilon > 0$. So it is natural to ask whether the converse holds. Here we give a counterexample.

Example 1.2. *Let S be the unilateral shift on $l^2(\mathbb{N})$ defined by*

$$(\alpha_1, \alpha_2, \alpha_3, \cdots) \longmapsto (0, \alpha_1, \alpha_2, \alpha_3, \cdots).$$

Since S is subnormal (and hence a von Neumann operator, see [3, Proposition 9.2]), we have

$$\sigma_\varepsilon(S) = \sigma(S) + B(0,\varepsilon) = B(0, 1+\varepsilon).$$

Let $R \in \mathcal{B}(\mathbb{C}^2)$ be the operator on $\mathbb{C}^2$ determined by the following matrix

$$\begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}.$$

Then, by [4, Proposition 2.4], $\sigma_\varepsilon(R) = B(0, \sqrt{\varepsilon^2 + 2\varepsilon})$. So $\sigma_\varepsilon(R) \subset \sigma_\varepsilon(S)$.

Set $T = S \oplus R$. Then $\sigma(T) = \sigma(S) \cup \sigma(R) = B(0,1)^-$ and

$$\sigma_\varepsilon(T) = \sigma_\varepsilon(S) \cup \sigma_\varepsilon(R) = \sigma_\varepsilon(S) = \sigma(S) + B(0,\varepsilon) = \sigma(T) + B(0,\varepsilon).$$

However, noting that $\|T\| = 2 > 1 = r(T)$, T is not a von Neumann operator.

In this note, we give a characterization of those operators T satisfying (1). To state our main result, we need an extra definition. Two operators A and B are called *approximately unitarily equivalent*, denoted as $A \simeq_a B$, if there exist a sequence of unitary operators U_n such that $\lim_n U_n^* A U_n = B$ (see [2, Definition 39.9]). If $A \simeq_a B$, then it is easy to check that $\sigma(A) = \sigma(B)$ and $\sigma_\varepsilon(A) = \sigma_\varepsilon(B)$ for all $\varepsilon > 0$.

The main result of this paper is the following theorem, which gives an answer to Question 1.1.

Theorem 1.3. *For $T \in \mathcal{B}(\mathcal{H})$, the following are equivalent:*

(i) $\sigma_\varepsilon(T) = \sigma(T) + B(0, \varepsilon)$ *for all* $\varepsilon > 0$.

(ii) *T is approximately unitarily equivalent to an operator of form* $N \oplus A$, *where N is normal with* $\sigma(N) = \partial\sigma(T)$ *and*

$$\|(zI - A)^{-1}\| \leq \|(zI - N)^{-1}\|, \ \ \forall z \in \mathbb{C} \setminus \sigma(T).$$

Remark 1.4. *Let* $T \in \mathcal{B}(\mathcal{H})$ *and suppose* $\sigma_\varepsilon(T) = \sigma(T) + B(0, \varepsilon)$ *for all* $\varepsilon > 0$. *By definition, the norm of the resolvent function* $(zI - T)^{-1}$ *of T coincides with that of a normal operator. By Theorem 1.3 (ii), this means that T "has" (up to approximate unitary equivalence) a normal part N with* $\sigma(N) = \partial\sigma(T)$. *Thus each point in* $\partial\sigma(T)$ *is a normal approximate eigenvalue of T. Recall that a complex number* λ *is called a normal approximate eigenvalue [5] of* $A \in \mathcal{B}(\mathcal{H})$ *if there exists a sequence* $\{x_n\}_{n\geq 1}$ *of unit vectors such that*

$$\|(A - \lambda)x_n\| + \|(A - \lambda)^* x_n\| \to 0.$$

Remark 1.5. *We remark that the result of Theorem 1.3 is sharp. That is, approximate unitary equivalence can not be replaced by unitary equivalence, since the operator T in Example 1.2 is abnormal, that is, T admits no nonzero reducing subspace M such that* $T|_M$ *is normal.*

Example 1.2 shows that the equality (1) in general does not imply the normality of T. However, if T acts on some finite dimensional Hilbert space, then we shall prove in Section 2 the following result.

Theorem 1.6. *Let A be a bounded linear operator acting on a finite dimensional Hilbert space* $\mathcal{K}$. *Then* $\sigma_\varepsilon(A) = \sigma(A) + B(0, \varepsilon)$ *for all* $\varepsilon > 0$ *if and only if A is normal.*

The proof of main result will be provided in Section 2. In the rest of this section, we fix some notations and terminology.

Let $T \in \mathcal{B}(\mathcal{H})$. We denote by $\ker T$ and ran T the kernel of T and the range of T respectively. If ran T is closed and either $\ker T$ or $\ker T^*$ is of finite dimension, then T is called a *semi-Fredholm operator*. The following set

$$\sigma_{lre}(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ is not semi-Fredholm}\}$$

is called the *Wolf spectrum* of T.

Let $T \in \mathcal{B}(\mathcal{H})$. If Δ is a nonempty clopen subset of $\sigma(T)$, then there exists an analytic Cauchy domain Ω such that $\Delta \subset \Omega$ and $[\sigma(T) \setminus \Delta] \cap \overline{\Omega} = \emptyset$. We let $E(\Delta; T)$ denote the *Riesz idempotent* of T corresponding to Δ, that is,

$$E(\Delta; T) = \frac{1}{2\pi \mathrm{i}} \int_\Gamma (\lambda - T)^{-1} \mathrm{d}\lambda,$$

where $\Gamma = \partial\Omega$ is positively oriented with respect to Ω in the sense of complex variable theory. By the Riesz Decomposition Theorem ([9, Theorem 2.10]), $\mathcal{H}(\Delta; T) := \mathrm{ran}(E(\Delta; T))$ is an invariant subspace of T and $\sigma(T|_{\mathcal{H}(\Delta;T)}) = \Delta$. If λ is an isolated point of $\sigma(T)$, then $\{\lambda\}$ is a clopen subset of $\sigma(T)$; if, in addition, $\dim \mathcal{H}(\{\lambda\}; T) < \infty$, then λ is called a *normal eigenvalue* of T. We denote by $\sigma_0(T)$ the set of all normal eigenvalues of T. The reader is referred to [1, page 210] or [7, Chapter 1] for more details.

2. Proof of Theorem 1.3

We first introduce some useful lemmas.

Lemma 2.1 ([1], page 366). *Let* $T \in \mathcal{B}(\mathcal{H})$. *Then* $\partial\sigma(T) \subseteq [\sigma_0(T) \cup \sigma_{lre}(T)]$.

Recall that an operator T is said to be *normaloid* if $\|T\| = r(T)$ (see [6, page 117]).

Lemma 2.2 ([11], Theorem 3.1). *Let* $T \in \mathcal{B}(\mathcal{H})$ *be normaloid. If N is a normal operator on some Hilbert space with* $\sigma(N) \subseteq \{z \in \sigma_{lre}(T) : |z| = \|T\|\}$, *then* $T \simeq_a T \oplus N$.

Lemma 2.3 ([11], Corollary 3.2). *Let $T \in \mathcal{B}(\mathcal{H})$ and Γ be a compact subset of $\mathbb{C}$. If $T \simeq_a T \oplus \lambda I$ for any $\lambda \in \Gamma$ and N is a normal operator on some Hilbert space with $\sigma(N) = \Gamma$, where I is the identity operator on $\mathcal{H}$, then $T \simeq_a T \oplus N$.*

Lemma 2.4 ([11], Corollary 3.5). *Let $T \in \mathcal{B}(\mathcal{H})$. If $\lambda \in \sigma_0(T)$ and there exists $z_0 \in \mathbb{C} \setminus \sigma(T)$ such that*

$$|(z_0 - \lambda)^{-1}| = \|(z_0 - T)^{-1}\|,$$

then $\ker(\lambda - T)$ *reduces T.*

Proof. [Proof of Theorem 1.6] We need only prove the necessity. Assume that $\sigma(A) = \{\lambda_i : i = 1, 2, \cdots, k\}$ and choose a positive number $\delta > \|A\|$. Thus $\delta \notin \sigma(A)$. Put $T = \delta I \oplus A$, where I is the identity operator on $\mathcal{H}$ with $\dim \mathcal{H} = \infty$. Thus $\sigma_0(T) = \{\lambda_i : i = 1, 2, \cdots, k\}$.

On the other hand, it is easy to see $\sigma_\varepsilon(T) = \sigma(T) + B(0, \varepsilon)$ for all $\varepsilon > 0$. Then

$$\|(z - T)^{-1}\| = 1/\mathrm{dist}(z, \sigma(T)), \quad \forall z \in \mathbb{C} \setminus \sigma(T).$$

So, for each λ_i, there exists z_i such that $|(z_i - \lambda_i)^{-1}| = \|(z_i - T)^{-1}\|$. By Lemma 2.4, $\ker(A - \lambda_i) = \ker(T - \lambda_i)$ reduces T. Set $M = \vee_{i=1}^k \ker(A - \lambda_i)$. Then $M \subset \mathcal{K}$ and reduces A. So

$$A = \begin{bmatrix} \lambda_1 I_1 & & & \\ & \ddots & & \\ & & \lambda_k I_k & \\ & & & A_0 \end{bmatrix} \begin{matrix} \ker(A - \lambda_1) \\ \vdots \\ \ker(A - \lambda_k) \\ \mathcal{K} \ominus M \end{matrix},$$

where the entries not shown are zero. Since $\sigma(A_0) \subset \sigma(A) = \{\lambda_i : i = 1, 2, \cdots, k\}$, it follows that $M = \mathcal{K}$. So A is normal. □

The following result is a mild improvement of Theorem 4.4 in [11].

Proposition 2.5. *Let $T \in \mathcal{B}(\mathcal{H})$ and suppose $\|(zI - T)^{-1}\| = 1/\mathrm{dist}(z, \sigma(T))$ for all $z \in \mathbb{C} \setminus \sigma(T)$. If N is a normal operator with $\sigma(N) = \sigma_{lre}(T) \cap \partial\sigma(T)$, then $T \simeq_a T \oplus N$.*

Proof. According to Lemma 2.3, we only need to prove that $T \simeq_a T \oplus \lambda I$ for any $\lambda \in \sigma_{lre}(T) \cap \partial\sigma(T)$.

Let $\lambda_0 \in \sigma_{lre}(T) \cap \partial\sigma(T)$. We can find $\{z_n\}_{n=1}^\infty \subseteq \mathbb{C} \setminus \sigma(T)$ such that $z_n \to \lambda_0$. For $n \geq \mathbb{N}$, there exist $\lambda_n \in \sigma(T)$ such that $\mathrm{dist}(z_n, \sigma(T)) = |z_n - \lambda_n|$. So $\lambda_n \in \partial\sigma(T)$ and $\lambda_n \to \lambda_0$.

By the hypothesis, we have

$$\|(z_n - T)^{-1}\| = 1/\mathrm{dist}(z_n, \sigma(T)) = 1/|z_n - \lambda_n|.$$

By the spectral mapping theorem, $(z_n - \lambda_n)^{-1} \in \sigma((z_n - T)^{-1})$ for $n \geq 1$.

Case 1. There exist $n_1 < n_2 < n_3 < \cdots$ such that $\lambda_{n_k} \in \sigma_{lre}(T)$.

If so, then $(z_{n_k} - \lambda_{n_k})^{-1} \in \sigma_{lre}((z_{n_k} - T)^{-1})$. By Lemma 2.2, we obtain

$$(z_{n_k} - T)^{-1} \simeq_a (z_{n_k} - T)^{-1} \oplus (z_{n_k} - \lambda_{n_k})^{-1} I, \quad \forall k \geq 1,$$

yielding $T \simeq_a T \oplus \lambda_{n_k} I$ for all $k \geq 1$. Since $\lambda_{n_k} \to \lambda_0$, one can see $T \simeq_a T \oplus \lambda_0 I$.

Case 2. There exists $m > 0$ such that $\lambda_n \notin \sigma_{lre}(T)$ for $n \geq m$.

Since $\lambda_n \in \partial\sigma(T)$, it follows from Lemma 2.1 that $\lambda_n \in \sigma_0(T)$ for $n \geq m$. Noting that $\|(z_n - T)^{-1}\| = 1/|z_n - \lambda_n|$, it follows from Lemma 2.4 that $\ker(T - \lambda_n)$ reduces T for $n \geq m$. Noting that $\lambda_n \to \lambda_0 \in \sigma_{lre}(T)$, it can be assumed that $\{\lambda_n : n \geq m\}$ are pairwise distinct. So T can be written as

$$T = A \oplus \mathrm{diag}\{\lambda_m, \lambda_{m+1}, \lambda_{m+2}, \cdots\}.$$

Since $\lambda_n \to \lambda_0$, one can see

$$\lambda_0 \in \sigma_{lre}(\mathrm{diag}\{\lambda_m, \lambda_{m+1}, \lambda_{m+2}, \cdots\}).$$

Then, by a corollary of the Weyl-von Neumann-Berg Theorem (see [2, Proposition 39.10]), we have

$$\mathrm{diag}\{\lambda_m, \lambda_{m+1}, \lambda_{m+2}, \cdots\} \simeq_a \mathrm{diag}\{\lambda_m, \lambda_{m+1}, \lambda_{m+2}, \cdots\} \oplus \lambda_0 I.$$

So

$$\begin{aligned} T &= A \oplus \mathrm{diag}\{\lambda_m, \lambda_{m+1}, \lambda_{m+2}, \cdots\} \\ &\simeq_a A \oplus \Big(\mathrm{diag}\{\lambda_m, \lambda_{m+1}, \lambda_{m+2}, \cdots\} \oplus \lambda_0 I\Big) \\ &\simeq_a \Big(A \oplus \mathrm{diag}\{\lambda_m, \lambda_{m+1}, \lambda_{m+2}, \cdots\}\Big) \oplus \lambda_0 I \\ &= T \oplus \lambda_0 I. \end{aligned}$$

Hence the proof is complete. □

Remark 2.6. *We remark that the proof of the preceding theorem is inspired by that of Theorem 4.3 in [11].*

Now we are going to prove Theorem 1.3.

Proof. [Proof of Theorem 1.3] "$\Longleftarrow$". Since $T \simeq_a N \oplus A$, it follows that $(z-T)^{-1} \simeq_a (z-N)^{-1} \oplus (z-A)^{-1}$ for $z \in \mathbb{C} \setminus \sigma(T)$. Then

$$\begin{aligned} \|(zI-T)^{-1}\| &= \max\{\|(zI-N)^{-1}\|, \|(zI-A)^{-1}\|\} = \|(zI-N)^{-1}\| \\ &= 1/\mathrm{dist}(z, \sigma(N)) = 1/\mathrm{dist}(z, \partial\sigma(T)) = 1/\mathrm{dist}(z, \sigma(T)), \end{aligned}$$

proving the sufficiency.

"$\Longrightarrow$". Set $\Gamma_0 = \partial\sigma(T) \cap \sigma_0(T)$ and $\Gamma_1 = \partial\sigma(T) \cap \sigma_{lre}(T)$. By Lemma 2.1, we have $\partial\sigma(T) = \Gamma_0 \cup \Gamma_1$. Since $\sigma_0(T)$ is at most denumerable, without loss of generality, it can be assumed that $\Gamma_0 = \{\lambda_1, \lambda_2, \lambda_3, \cdots\}$.

For each $n \geq 1$, since $\lambda_n \in \sigma_0(T)$ is an isolated point of $\sigma(T)$, we can find $z_n \in \mathbb{C} \setminus \sigma(T)$ such that $|z_n - \lambda_n| < \mathrm{dist}(z_n, \sigma(T) \setminus \{\lambda_n\})$. Then

$$\|(z_n - T)^{-1}\| = 1/\mathrm{dist}(z_n, \sigma(T)) = 1/|\lambda_n - z_n|.$$

Since $\lambda_n \in \sigma_0(T)$, it follows from Lemma 2.4 that $\ker(T - \lambda_n)$ reduces T. Note that $\{\lambda_n : n \geq 1\}$ are pairwise distinct. Thus T can be written as

$$T = A \oplus \mathrm{diag}\{\lambda_1, \lambda_2, \lambda_3, \cdots\}. \tag{2}$$

Choose a normal operator N_1 on $\mathcal{H}$ with $\sigma(N_1) = \Gamma_1$. Then, by Proposition 2.5, we have $T \simeq_a T \oplus N_1$. It follows that

$$T \simeq_a T \oplus N_1 = A \oplus \mathrm{diag}\{\lambda_1, \lambda_2, \lambda_3, \cdots\} \oplus N_1.$$

Set $N = \mathrm{diag}\{\lambda_1, \lambda_2, \lambda_3, \cdots\} \oplus N_1$. Then N is normal, $T \simeq_a A \oplus N$ and

$$\sigma(N) = \sigma(N_1) \cup \{\lambda_n : n \geq 1\} = \partial\sigma(T).$$

Thus, for each $z \in \mathbb{C} \setminus \sigma(T)$, we have

$$\|(zI-A)^{-1}\| \leq \|(zI-T)^{-1}\| = 1/\mathrm{dist}(z, \sigma(T)) = 1/\mathrm{dist}(z, \partial\sigma(T)) = 1/\mathrm{dist}(z, \sigma(N)) = \|(zI-N)^{-1}\|.$$

The proof is complete. □

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