Filomat 35:3 (2021), 973–993
https://doi.org/10.2298/FIL2103973G

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

An Extension of Karapinar and Sadaranganis' Result Through the C-Class Functions by Using α-Admissible Mapping with Applications

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Abstract. The main objective of this work is to introduce a new type of non-linear contraction via C-class functions by using α-admissible mapping. Our new results extend and generalize the very recent results of Karapinar and Sadarangani (2015. RACSAM. [37]). Illustrative examples are given to support our new findings. We have shown that our results satisfy the periodic fixed point results after modifying the contraction. Next, we extend our main findings from a self-mapping T to two self-mappings T; S. Also, an example is provided to justify the effectiveness of our new result on two self mappings, where the partially ordered structure fails. Finally, we apply our new findings to solve ordinary differential and non-linear integral equations.

1. Introduction and preliminaries

The concept of metric space was proposed by Ḟrechet [27] in 1905. The fundamental theorem of fixed point theory, i.e., Banach−Cacciopoli theorem [(1922) [21]], was established in the context of complete metric space. This theorem became very famous due to its versatile application in the non-linear analysis (like integral equations, fractional derivatives, matrix equations, dynamic programming, differential equations, and many more). In the year 2002, Branciari [23] established the Banach−Cacciopoli theorem in the integral form. On the other side, Ran and Reurings [46] found the existence and uniqueness results of a fixed point for operators satisfying a particular type of contraction in an ordered metric space structure. In that paper, the authors applied their work in solving the non-linear matrix equations. Later in 2015, Karapinar and Sadarangani [37] extended the result of [48] in the setting of partially ordered sets. They considered an example for its applicability, where the main results of Samet and Vetro [48] are not applicable. Also, the authors in [37] applied their new findings to solve a dynamic programming problem. In the year 2012, Samet et al.[49] introduced the concept of α-admissible mapping. In the context of complete metric space, they proved the fixed point results under α-ψ-contractive type condition. Again, the results of Samet et al.[49] were extended and generalized by Karapinar and Samet in [38]. Since then, many authors have used this notion to extend the fixed point results in different ways. We now briefly discuss some important results

2010 *Mathematics Subject Classification*. Primary 47H09; Secondary 47H10, 54H25
Keywords. $\alpha-$admissible mapping, C-class function, Berinde type mappings, Hölder's inequality, Ordinary differential equations, Non-linear integral equations.
Received: 17 March 2020; Revised: 23 September 2020; Accepted: 18 October 2020
Communicated by Erdal Karapınar
The first author (SKG) would like to thank University Grants Commission, Govt. of India (Sr. No. 2121540966, Ref. No: 20/12/2015(ii)EU-V), for financial support.
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of the fixed point theory that have been studied by using admissible mapping. Aydi et al.[19] extended the idea of F-contraction involving α-admissible mapping in the setup of complete metric space. Alsulami et al. [14] defined the notion of weak triangular α-admissible mapping and investigated (α-ψ-ϕ) type contraction in the platform of complete metric space. It is known that Meir-Keeler(MK)-contraction is one of the famous contraction in fixed point theory. Authors established different types of generalized MK-contractions in various setups such as metric like space, G-metric space via α-admissible mapping to extend and generalize fixed point results (see [13], [32]). Simultaneously, researchers worked on multi-valued mappings as well as on nonself multi-valued mappings for certain contractions by using the concept of α-admissible mapping (see [7], [8], [9], [11]). In [10], the α-ψ-type contraction has been investigated in the framework of uniform space.

Again, Popescu [44] introduced the idea of triangular α-orbital admissible as a refinement of the triangular α-admissible notion, described in [36]. Many researchers have utilized this notion to generalize several well-known results of fixed point theory. Recently, two famous contractions, namely, Geraghty contraction and Istrăţescu type Contraction, were studied in the context of b-metric space by applying triangular α-orbital admissible mapping in [3], [35], respectively. Arshad et al.[18] established some interesting results on θ-contraction involving triangular α-orbital admissible mapping in the framework of Branciari metric space. In [6], Alharbi et al. studied a new type of contractive condition by combining the concept simulation function with the α-orbital admissible function in the setting of b-metric space. By using the simulation function, Karapinar and Chifu investigated MK-type contractions and Geraghty type contractions in the platform of wt-distance via triangular α-orbital admissible mapping in [34]. For more details on admissible mapping, we refer the reader to [4], [20], [24], [29], and the related references therein.

However, in this article, we aim to focus our study on α-admissible mapping to generalize some significant fixed point theory results through C-class function. Again, the concept of the C-class functions proposed by Ansari (2014), contained a large class of contractions (see [12], [16], [45]). In recent times many interesting results related to fixed point theory have been established (see for example [5], [15], [22], [25], [26], [31], [33], [40], [41], [43], [46], [47], [51]). For the sake of completeness, we now recall some basic results, including definitions, lemmas, which will be needed in our new investigations. Throughout this paper, $\mathbb{R}^+$ denotes $[0,\infty)$.

Definition 1.1. *[49] Let α be a mapping from $X \times X$ to $\mathbb{R}^+$. Then a mapping $T : X \to X$ is said to α-admissible if*

$$\alpha(x,y) \geq 1 \text{ implies } \alpha(Tx,Ty) \geq 1, \text{ for all } x,y \in X.$$

Definition 1.2. *[2] Let α be a mapping from $X \times X$ to $\mathbb{R}^+$. Let $T, S : X \to X$ be two mappings. Then the pair (T,S) is said to be α-admissible if*

$$\alpha(x,y) \geq 1 \text{ implies } \alpha(Tx,Sy) \geq 1 \text{ and } \alpha(Sx,Ty) \geq 1, \text{ for all } x,y \in X.$$

Definition 1.3. *[36] Let $\alpha : X \times X \to \mathbb{R}^+$ and $T : X \to X$ be two mappings. Then the mapping T is said to be a triangular α-admissible mapping if*

1. $\alpha(x,y) \geq 1$ *implies* $\alpha(Tx,Ty) \geq 1$, *for all* $x,y \in X$;
2. $\alpha(x,y) \geq 1, \alpha(y,z) \geq 1$ *implies* $\alpha(x,z) \geq 1$, *for all* $x,y,z \in X$.

Definition 1.4. *[2] Let $\alpha : X \times X \to \mathbb{R}^+$ and $T, S : X \to X$ be three mappings. Then the pair (T,S) is said to be a triangular α-admissible mapping if*

1. $\alpha(x,y) \geq 1$ *implies* $\alpha(Tx,Sy) \geq 1$ *and* $\alpha(Sx,Ty) \geq 1$, *for all* $x,y \in X$;
2. $\alpha(x,y) \geq 1, \alpha(y,z) \geq 1$ *implies* $\alpha(x,z) \geq 1$, *for all* $x,y,z \in X$.

Note: The mapping α is said to be symmetric if $\alpha(x,y) = \alpha(y,x)$ holds for all $x,y \in X$.

Definition 1.5. *[39] A function $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ is said to be an altering distance function if the following two properties are satisfied:*

1. *ψ is non-decreasing and continuous;*
2. *$\psi(t) = 0$ if and only if $t = 0$.*

Let Ψ denotes the class of all altering distance functions. From now, Φ^* denotes the collection of all functions $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ such that φ is continuous, and $\varphi(t) > 0$ for $t > 0$. Next, we come to the definition of C-class function.

Definition 1.6. *[45] A continuous mapping $F : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ is called a C-class function if it satisfies the following two properties:*

1. *$F(k, l) \leq k$;*
2. *$F(k, l) = k$ implies either $k = 0$ or $l = 0$, for all $k, l \in \mathbb{R}^+$.*

From now we write C to denote the collection of all C-class functions. Next, we give some examples of C-class function, which we have taken from [16], [45].

Example 1.7. *The following functions $F : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ belong to C, for all $k, l \in \mathbb{R}^+$.*
(1) $F(k, l) = k - l$; (2) $F(k, l) = \lambda k, 0 < \lambda < 1$; (3) $F(k, l) = \frac{k}{(1+l)^r}, r \in (0, \infty)$; (4) $F(k, l) = \frac{log(l+a^k)}{1+l}, a > 1$; (5) $F(k, l) = \frac{ln(1+a^k)}{2}, a > e$; (6) $F(k, l) = \frac{k}{\Gamma(1/2)} \int_0^\infty \frac{e^{-x}}{\sqrt{x}+l} dx$, where Γ is the Euler-gamma function; (7) $F(k, l) = kH(k, l)$, where $H(k, l) : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous function with $H(k, l) < 1$, for all $k, l \in \mathbb{R}^+$; (8) $F(k, l) = \phi(k)$, where $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ is a upper semi-continuous function with $\phi(0) = 0$, and $\phi(t) < t$ for $t > 0$; (9) $F(k, l) = k - \frac{l}{k+l}$; (10) $F(k, l) = k - (\frac{2+l}{1+l})l$.

Definition 1.8. *[17] Let $\psi \in \Psi$, $\varphi \in \Phi^*$, and $F \in C$, then F is said to be monotone if*

$$p, q \in \mathbb{R}^+ \text{ with } p \leq q \text{ implies } F(\psi(p), \varphi(p)) \leq F(\psi(q), \varphi(q)).$$

Let Φ denotes the collections of all functions $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ which satisfy the following properties:

1. all $\phi \in \Phi$ are Lebesgue-integrable, summable on each compact subset of $\mathbb{R}^+$;
2. $\int_0^\rho \phi(t)dt > 0$, for each $\rho > 0$.

The following two lemmas will be useful later.

Lemma 1.9. *[42] Let $\{b_n\}$ be a sequence such that $b_n \geq 0$, for every $n \in \mathbb{N}$, with $\lim_{n\to\infty} b_n = b$, then*

$$\lim_{n\to\infty} \int_0^{b_n} \phi(t)dt = \int_0^b \phi(t)dt.$$

Lemma 1.10. *[42] Let $\{b_n\}$ be a sequence such that $b_n \geq 0$, for every $n \in \mathbb{N}$, then*

$$\lim_{n\to\infty} \int_0^{b_n} \phi(t)dt = 0 \text{ if and only if } \lim_{n\to\infty} b_n = 0.$$

Definition 1.11. *[37] Let (X, d) be a metric space, and $T : X \to X$ be an operator. Then the orbit of T at a point x is denoted by $O(x, T)$ and is defined as $O(x, T) = \{x, Tx, T^2x, \cdots, T^nx, \cdots\}$.*

Next, we rewrite Definition 1.1 in [50] by considering T, S as single-valued mappings, and f as identity mapping.

Definition 1.12. *[50] Let (X, d) be a metric space, and $T, S : X \to X$ be two operators. Then the orbit of T, S at a point x_0, is denoted by $O(x_0, T, S)$ and is defined as $O(x_0, T, S) = \{x_n : x_{2n+1} = Tx_{2n}$ if n is even $, x_{2n} = Sx_{2n-1}$ if n is odd $\}$.*

Definition 1.13. *If for $x_0 \in X$, there exists a sequence $\{x_n\}$ in X such that every Cauchy sequence in the form $O(x_0, T, S)$ converges in X, then $O(x_0, T, S)$ is called an orbitally complete metric space.*

Remark 1.14. *The class of all orbitally complete metric spaces contains the class of all complete metric spaces, and it is a proper subset (see [50]).*

Let (X, d) be a metric space. Suppose $T : X \to X$ is an operator. For any $x, y \in X$, we write
$U_1(x, y) = d(y, Ty)\frac{1+d(x,Tx)}{1+d(x,y)}$,
$U_2(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(x,Ty)+d(y,Tx)}{2}\}$, $U_3(x, y) = \frac{d(x,Ty)d(y,Tx)}{1+d(x,y)}$,
$U_4(x, y) = \frac{d(x,Tx)d(x,Ty)}{2[1+d(x,y)]}$, $U_5(x, y) = \frac{d(y,Ty)d(y,Tx)}{2[1+d(x,y)]}$ and $U_6(x, y) = \min\{d(x, Tx), d(y, Tx)\}$.
Rest of the paper, we will write $I(a)$ to mean $I(a) = \int_0^a \phi(t)dt$, where $\phi \in \Phi$. Next, we state the main result of [37].

Theorem 1.15. *Let $(X, d, \preceq)$ be a partially ordered metric space. Let $T : X \to X$ be a non-decreasing, continuous mapping such that (X, d) is a $T-$orbitally complete metric space. Suppose that there exist $\gamma_1, \gamma_2 \in [0, 1)$ with $\gamma_1 + \gamma_2 < 1$, and $\phi \in \Phi$ for which T satisfies the following contraction*

$$I(d(Tx, Ty)) \leq \gamma_1 I(U_1(x, y)) + \gamma_2 I(U_2(x, y)) + LI(U_6(x, y)), \tag{1}$$

whenever $x \preceq y$ in X and $L \geq 0$. Also, suppose that there exists a point $x_0 \in X$ such that $x_0 \preceq Tx_0$. Then T has at least one fixed point.

Now, we arrive at a situation to state and prove our main results.

2. Main results

Denote:
$M(x, y) = \max\Big\{I(U_1(x, y)), I(U_2(x, y)), I(U_3(x, y)), I(U_4(x, y)), I(U_5(x, y))\Big\}$.

Theorem 2.1. *Let (X, d) be a metric space. Let $\psi, \varphi : \mathbb{R}^+ \to \mathbb{R}^+$ be two mappings such that $\psi \in \Psi$ and $\varphi \in \Phi^*$. Suppose that $\alpha : X \times X \to \mathbb{R}^+$ and $T : X \to X$ be two mappings such that T is α-admissible. Assume that there exists a $\phi \in \Phi$, for which T satisfies the following contraction*

$$\psi(I(d(Tx, Ty))) \leq F(\psi(M(x, y)), \varphi(M(x, y))) + \theta(U_6(x, y))I(\psi(U_6(x, y))), \tag{2}$$

for all $x, y \in X$ with $\alpha(x, y) \geq 1$, where $F \in \mathcal{C}$ and $\theta : \mathbb{R} \to \mathbb{R}$ be a mapping such that $\limsup_{t \longrightarrow r^+} \theta(t)$ exists for all $r \in \mathbb{R}^+$. Suppose that there exists a $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$. Suppose that α satisfies the transitivity property, i.e., for $x, y, z \in X$ with $\alpha(x, y) \geq 1$, $\alpha(y, z) \geq 1$ implies $\alpha(x, z) \geq 1$. Also, suppose that T is a continuous function with (X, d) is a $T-$orbitally complete metric space. Then T has a fixed point.

Proof. Let $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$. Since T is α-admissible mapping, by using this fact, one can easily construct a Picard iterative sequence $\{x_n\}$ such that $x_{n+1} = Tx_n$ and $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$. Now suppose that there exists $n^*(\in \{0\} \cup \mathbb{N})$, such that $x_{n^*} = x_{n^*+1} = Tx_{n^*}$, then x_{n^*} becomes a fixed point of T, and consequently, the proof is over. From now we assume that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$, i.e., $d(x_n, x_{n+1}) > 0$, for all $n \in \mathbb{N}$. Our first aim is to show that

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0.$$

Now if we put $x = x_{n-1}, y = x_n$ in (2), then we have

$$\psi(I(d(Tx_{n-1}, Tx_n))) \leq F(\psi(M(x_{n-1}, x_n)), \varphi(M(x_{n-1}, x_n))) + \theta(U_6(x_{n-1}, x_n))I(\psi(U_6(x_{n-1}, x_n))). \tag{3}$$

Now, we get
$U_1(x_{n-1}, x_n) = d(x_n, x_{n+1})\frac{1+d(x_{n-1},x_n)}{1+d(x_{n-1},x_n)} = d(x_n, x_{n+1})$,

$$\begin{aligned} U_2(x_{n-1}, x_n) &= \max\{d(x_{n-1}, x_n), d(x_n, x_{n+1}), \frac{d(x_{n-1}, x_{n+1}) + d(x_n, x_n)}{2}\} \\ &= \max\{d(x_{n-1}, x_n), d(x_n, x_{n+1}), \frac{d(x_{n-1}, x_{n+1})}{2}\} \\ &\leq \max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\}, \end{aligned}$$

$U_3(x_{n-1}, x_n) = \frac{d(x_{n-1},x_{n+1})d(x_n,x_n)}{1+d(x_{n-1},x_n)} = 0$,
$U_4(x_{n-1}, x_n) = \frac{d(x_{n-1},x_n)d(x_{n-1},x_{n+1})}{2[1+d(x_{n-1},x_n)]} \leq \max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\}$,
$U_5(x_{n-1}, x_n) = \frac{d(x_n,x_{n+1})d(x_n,x_n)}{2[1+d(x_{n-1},x_n)]} = 0$ and
$U_6(x_{n-1}, x_n) = \min\{d(x_{n-1}, x_n), d(x_n, x_n)\} = 0$.
Thus, from (3), we have

$$\psi(I(d(x_n, x_{n+1}))) \leq F(\psi(I(\max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\})), \varphi(I(\max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\}))). \tag{4}$$

Now, suppose $\max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\} = d(x_n, x_{n+1})$, then from (4), we have

$$\psi(I(d(x_n, x_{n+1}))) \leq F(\psi(I(d(x_n, x_{n+1}))), \varphi(I(d(x_n, x_{n+1}))) \leq \psi(I(d(x_n, x_{n+1})). \tag{5}$$

Again, from the second property of C-class function, we have either $\psi(I(d(x_n, x_{n+1})) = 0$ or $\varphi(I(d(x_n, x_{n+1})) = 0$. From both the cases we have $I(d(x_n, x_{n+1}) = 0$, which implies $d(x_n, x_{n+1}) = 0$. Hence $x_n = x_{n+1}$, which contradicts to our assumption. Hence, we obtain $\max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\} = d(x_{n-1}, x_n)$. Thus, from (3), we have

$$\psi(I(d(x_n, x_{n+1}))) \leq F(\psi(I(d(x_{n-1}, x_n))), \varphi(I(d(x_{n-1}, x_n))) \leq \psi(I(d(x_{n-1}, x_n)). \tag{6}$$

Since ψ is non-decreasing, we have

$$I(d(x_n, x_{n+1})) \leq I(d(x_{n-1}, x_n)).$$

Let us put $\beta_n = I(d(x_{n-1}, x_n))$. Consequently $\beta_{n+1} \leq \beta_n$, i.e., $\{\beta_n\}$ becomes a non-increasing sequence. Therefore, there exists a non-negative real number $\beta^* \geq 0$ such that $\lim_{n\to\infty} \beta_n = \beta^*$. Now considering limit as $n \to \infty$ in (6), we obtain

$$\psi(\beta^*) \leq F(\psi(\beta^*)), \varphi(\beta^*)) \leq \psi(\beta^*), \tag{7}$$

since F, ψ, φ all are continuous functions. Clearly second property of C-class function forces β^* to be equal with 0, i.e., $\lim_{n\to\infty} \beta_n = 0$. Now, by applying Lemma 1.10, we have $\lim_{n\to\infty} d(x_{n-1}, x_n) = 0$. Our next aim is to show that $\{x_n\}$ is a Cauchy sequence, which will be shown by the method of contradiction. Suppose $\{x_n\}$ is not a Cauchy sequence. Then, there exists a $\epsilon > 0$ for which we can obtain two sub-sequence $\{x_{m(i)}\}$ and $\{x_{n(i)}\}$ of $\{x_n\}$ such that $1 \leq i \leq m(i) < n(i)$ and

$$d(x_{m(i)}, x_{n(i)}) \geq \epsilon. \tag{8}$$

Let $n(i)$ be the smallest positive integer, for which (8) holds. Thus, from (8), we have

$$d(x_{m(i)}, x_{n(i)-1}) \leq \epsilon. \tag{9}$$

Now, using (8) and (9), one can obtain the following limits

$$\lim_{i\to\infty} d(x_{m(i)}, x_{n(i)}) = \epsilon, \lim_{i\to\infty} d(x_{m(i)}, x_{n(i)-1}) = \epsilon, \lim_{i\to\infty} d(x_{m(i)-1}, x_{n(i)-1}) = \epsilon, \lim_{i\to\infty} d(x_{m(i)-1}, x_{n(i)}) = \epsilon. \tag{10}$$

Again, using transitivity property of α, we have $\alpha(x_{m(i)}, x_{n(i)}) \geq 1$. Hence, from (2), we have

$$\begin{aligned}\psi(I(d(x_{m(i)}, x_{n(i)}))) &= \psi(I(d(Tx_{m(i)-1}, Tx_{n(i)-1}))) \\ &\leq F(\psi(M(x_{m(i)-1}, x_{n(i)-1})), \varphi(M(x_{m(i)-1}, x_{n(i)-1}))) \\ &+ \theta(U_6(x_{m(i)-1}, x_{n(i)-1}))I(\psi(U_6(x_{m(i)-1}, x_{n(i)-1}))).\end{aligned} \tag{11}$$

Now, we have the following relations
$U_1(x_{m(i)-1}, x_{n(i)-1}) = d(x_{n(i)-1}, Tx_{n(i)})\frac{1+d(x_{m(i)-1}, Tx_{m(i)})}{1+d(x_{m(i)-1}, x_{n(i)-1})}$,
$U_2(x_{m(i)-1}, x_{n(i)-1})$
$= \max\left\{d(x_{m(i)-1}, x_{n(i)-1}), d(x_{m(i)-1}, x_{m(i)}), d(x_{n(i)-1}, x_{n(i)}), \frac{d(x_{m(i)-1}, x_{n(i)})+d(x_{n(i)-1}, x_{m(i)})}{2}\right\}$,
$U_3(x_{m(i)-1}, x_{n(i)-1}) = \frac{d(x_{m(i)-1}, x_{n(i)})d(x_{n(i)-1}, x_{m(i)})}{1+d(x_{m(i)-1}, x_{n(i)-1})}$, $U_4(x_{m(i)-1}, x_{n(i)-1}) = \frac{d(x_{m(i)-1}, x_{m(i)})d(x_{m(i)-1}, x_{n(i)})}{2[1+d(x_{m(i)-1}, x_{n(i)-1})]}$,
$U_5(x_{m(i)-1}, x_{n(i)-1}) = \frac{d(x_{n(i)-1}, x_{n(i)})d(x_{n(i)-1}, x_{m(i)})}{2[1+d(x_{m(i)-1}, x_{n(i)-1})]}$ and
$U_6(x_{m(i)-1}, x_{n(i)-1}) = \min\{d(x_{m(i)-1}, x_{m(i)}), d(x_{n(i)-1}, x_{m(i)})\}$.
Taking limit as $i \to \infty$ in (11), we obtain

$$\psi(I(\epsilon)) \leq F(\psi(I(\epsilon)), \varphi(I(\epsilon))) \leq \psi(I(\epsilon)), \tag{12}$$

which implies either $\psi(I(\epsilon)) = 0$ or $\varphi(I(\epsilon)) = 0$. Consequently, we have $\epsilon = 0$, a contradiction. Hence, our assumption that $\{x_n\}$ is not a Cauchy sequence is incorrect. Thus, $\{x_n\}$ is a Cauchy sequence. Since (X, d) is a $T-$orbitally complete metric space, consequently there exists $x^* \in X$ such that $\lim_{n\to\infty} x_n = x^*$. Lastly, we have

$$x^* = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} Tx_n = T(\lim_{n\to\infty} x_n) = Tx^*(\text{ since } T \text{ is continuous }),$$

which shows that x^* is a fixed point of T. □

In our next theorem, we replace the continuity assumption by a suitable condition.

Theorem 2.2. *Suppose that all the conditions of Theorem 2.1 are satisfied except continuity. Also, assume that (X, d) fulfills the α-regularity property, i.e., if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ and $x_n \to x^* \in X$ as $n \to \infty$ implies $\alpha(x_n, x^*) \geq 1$ for all $n \in \mathbb{N}$. Then T has a fixed point.*

Proof. From Theorem 2.1, we have $\lim_{n\to\infty} x_n = x^*$, where $x_{n+1} = Tx_n$. Now, using $\alpha(x_n, x^*) \geq 1$, inequality (2) becomes

$$\begin{aligned}\psi(I(d(x_{n+1}, Tx^*))) &= \psi(I(d(Tx_n, Tx^*))) \\ &\leq F(\psi(M(x_n, x^*)), \varphi(M(x_n, x^*)) \\ &+ \theta(U_6(x_n, x^*)I(\psi(U_6(x_n, x^*)).\end{aligned} \tag{13}$$

Also, we have the followings
$U_1(x_n, x^*) = d(x^*, Tx^*)\frac{1+d(x_n, x_{n+1})}{1+d(x_n, x^*)}$,
$U_2(x_n, x^*) = \max\{d(x_n, x^*), d(x_n, x_{n+1}), d(x^*, Tx^*), \frac{d(x_n, Tx^*)+d(x^*, x_{n+1})}{2}\}$, $U_3(x_n, x^*) = \frac{d(x_n, Tx^*)d(x^*, x_{n+1})}{1+d(x_n, x^*)}$,
$U_4(x_n, x^*) = \frac{d(x_n, x_{n+1})d(x_n, Tx^*)}{2[1+d(x_n, x^*)]}$, $U_5(x_n, x^*) = \frac{d(x^*, Tx^*)d(x^*, x_{n+1})}{2[1+d(x_{n+1}, x^*)]}$ and $U_6(x_n, x^*) = \min\{d(x_n, x_{n+1}), d(x^*, x_{n+1})\}$.
Taking limit as $n \to \infty$ in both sites of (13), we obtain

$$\psi(I(d(x^*, Tx^*))) \leq F(\psi(I(d(x^*, Tx^*)), \varphi(I(d(x^*, Tx^*))) \leq \psi(I(d(x^*, Tx^*))), \tag{14}$$

which implies either $\psi(I(d(x^*, Tx^*))) = 0$ or $\varphi(I(d(x^*, Tx^*))) = 0$. This gives $d(x^*, Tx^*) = 0$ implies $x^* = Tx^*$. Hence, our proof is completed. □

Remark 2.3. *The above all results still hold if one replace the $U_6(x, y)$ by*

$$U_6^*(x, y) = \min\{d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\}.$$

Our next theorem deals with the uniqueness of fixed point of the mapping T.

Theorem 2.4. *In addition to the hypotheses of Theorem 2.1 and Theorem 2.2 if we add $\alpha(x^*, y^*) \geq 1$ for all $x^*, y^* \in$ Fix (T)(collection of all fixed points of T). Then T has a unique fixed point.*

Proof. Let $x^*, y^* \in$ Fix (T) with $\alpha(x^*, y^*) \geq 1$. Consequently, from (2), we obtain

$$\psi(I(d(x^*, y^*))) = \psi(I(d(Tx^*, Ty^*))) \leq F(\psi(M(x^*, y^*)), \varphi(M(x^*, y^*))) + \theta(U_6(x^*, y^*))I(\psi(U_6(x^*, y^*))). \tag{15}$$

Now, we have

$U_1(x^*, y^*) = 0, U_2(x^*, y^*) = d(x^*, y^*), U_3(x^*, y^*) = \frac{d(x^*,y^*)^2}{1+d(x^*,y^*)} = \left[\frac{d(x^*,y^*)}{1+d(x^*,y^*)}\right] d(x^*, y^*)$,
$U_4(x^*, y^*) = 0, U_5(x^*, y^*) = 0$ and $U_6(x^*, y^*) = 0$. Thus, from (15), we obtain

$$\psi(I(d(x^*, y^*))) = \psi(I(d(Tx^*, Ty^*))) \leq F(\psi(I(d(x^*, y^*))), \varphi(I(d(x^*, y^*))) \leq \psi(I(d(x^*, y^*))), \tag{16}$$

i.e., (16) gives $x^* = y^*$. □

In our next theorem, we introduce a new condition to obtain a unique fixed point of the operator T, where we assume an extra condition on $F(\in \mathcal{C})$, i.e., F is a monotone mapping.

Theorem 2.5. *In addition to the hypotheses of Theorem 2.1 and Theorem 2.2, suppose that for every $x, y \in X$, there exists $z \in X$, such that $\alpha(z, x) \geq 1$, and $\alpha(z, y) \geq 1$ with $\lim_{n\to\infty} d(Tz_{n-1}, Tz_n) = 0$, where $z_{n+1} = Tz_n$ is a Picard iterative sequence with initial point $z_0 = z$. Then T has a unique fixed point.*

Proof. Let $x^*, y^* \in$ Fix (T). Now, by our assumption, there exists a $z(\in X)$ such that $\alpha(z, x^*) \geq 1$ and $\alpha(z, y^*) \geq 1$. Since T is α-admissible, consequently we have $\alpha(z, x^*) \geq 1$ implies $\alpha(Tz, Tx^*) = \alpha(Tz, x^*) = \alpha(Tz_0, x^*) = \alpha(z_1, x^*) \geq 1$. Again, $\alpha(z_1, x^*) \geq 1$ implies $\alpha(Tz_1, Tx^*) = \alpha(z_2, x^*) \geq 1$. Proceeding in this manner, one can easily obtain $\alpha(z_{n+1}, x^*) = \alpha(Tz_n, Tx^*) \geq 1$. Now, from (2), we have

$$\begin{aligned}\psi(I(d(z_{n+1}, x^*))) &= \psi(I(d(Tx^*, Tz_n))) \\ &\leq F(\psi(M(x^*, z_n)), \varphi(M(x^*, z_n))) + \theta(U_6(x^*, z_n))I(\psi(U_6(x^*, z_n))).\end{aligned} \tag{17}$$

Now, we have the following

$U_1(x^*, z_n) = 0, U_2(x^*, z_n) = \max\left\{d(z_n, x^*), 0, d(z_n, z_{n+1}), \frac{d(z_{n+1},x^*)+d(z_n,x^*)}{2}\right\}$,
$U_3(x^*, z_n) = \frac{d(z_{n+1},x^*)d(z_n,x^*)}{1+d(z_n,x^*)}, U_4(x^*, z_n) = 0, U_5(x^*, z_n) = \frac{d(z_n,z_{n+1})d(z_n,x^*)}{2[1+d(z_n,x^*)]}$ and
$U_6(x^*, z_n) = \min\left\{0, d(z_n, x^*)\right\}$.
Putting all the values in (17) and using the monotonicity of F, we obtain

$$\begin{aligned}&\psi(I(d(z_{n+1}, x^*))) \\ &= \psi(I(d(Tx^*, Tz_n))) \\ &\leq F(\psi(I(\max\{d(z_n, z_{n+1}), d(z_n, x^*), d(z_{n+1}, x^*)\})), \varphi(I(\max\{d(z_n, z_{n+1}), d(z_n, x^*), d(z_{n+1}, x^*)\}))).\end{aligned} \tag{18}$$

Next to show that $\lim_{n\to\infty} z_n = x^*$. Without loss of generality, we may assume that $d(z_n, x^*) > 0$, for all $n \in \mathbb{N}$. If there exists a $n^* \in \mathbb{N}$ such that $d(z_{n^*}, x^*) = 0$ implies $z_{n^*} = x^*$, which means $z_k = x^*$, for all $k \geq n^*$. Consequently, we have $\lim_{n\to\infty} z_n = x^*$. From now, we proceed by considering $d(z_n, x^*) > 0$, for all $n \in \mathbb{N}$. Since, by our assumption $\lim_{n\to\infty} d(z_n, z_{n+1}) = 0$, so for sufficiently large value of n, we have

$$\begin{aligned}&\psi(I(d(z_{n+1}, x^*))) \\ &= \psi(I(d(Tx^*, Tz_n))) \\ &\leq F(\psi(I(\max\{d(z_n, x^*), d(z_{n+1}, x^*)\})), \varphi(I(\max\{d(z_n, x^*), d(z_{n+1}, x^*)\}))).\end{aligned} \tag{19}$$

If $\max\{d(z_n, x^*), d(z_{n+1}, x^*)\} = d(z_{n+1}, x^*)$, then by second property of F, we arrive at a contradiction from (19) (since $d(z_n, x^*) > 0$, for all $n \in \mathbb{N}$). Thus, we have $\max\{d(z_n, x^*), d(z_{n+1}, x^*)\} = d(z_n, x^*)$, for sufficiently large value of $n \in \mathbb{N}$ say $\tilde{n}$. Consequently from (19), we get $I(d(z_{n+1}, x^*)) \le I(d(z_n, x^*))$, for all $n \ge \tilde{n}$. Thus $\{a_n\}$ becomes a decreasing sequence for $n \ge \tilde{n}$, where $a_n = I(d(z_n, x^*))$. Hence, there exists a non negative real number say, $a^*(\ge 0)$ such that $\lim_{n\to\infty} a_n = a^*$. Taking limit in both sites of (19), we have

$$\psi(a^*) \le F(\psi(a^*), \varphi(a^*)) \le \psi(a^*). \tag{20}$$

Clearly (20) implies $a^* = 0$. Now, by using Lemma 1.10, we obtain $\lim_{n\to\infty} d(z_n, x^*) = 0$. Consequently, we achieve our first goal. Similarly, for the point y^*, we can show that $\lim_{n\to\infty} d(z_n, y^*) = 0$. Also, we know that, for a convergent sequence in a metric space can have at most one limit point, i.e., $x^* = y^*$ and our proof is completed. □

Next, we give an example to support our main result, where the main result of Karapinar and Sadarangani [37] is not applicable.

Example 2.6. *Let $(X, d, \preceq)$ be a partially ordered metric space, where $X = \mathbb{R}$, d denotes the usual metric, and we define the partial order " $\preceq$ " as $x \preceq y$ if and only if $x \ge y \ge -3$. Now we define two mappings $T : X \to X$ and $\alpha : X \times X \to [0, \infty)$, as follows*

$$T(x) = \begin{cases} \frac{x}{8} & \text{if } x \in \mathbb{R}^+, \\ x & \text{otherwise,} \end{cases}$$

and

$$\alpha(x, y) = \begin{cases} \cosh(x + y) + 1 & \text{if } x, y \in \mathbb{R}^+, \\ 0 & \text{otherwise.} \end{cases}$$

Now consider $\psi(t) = 2t$, and $\phi(t) = t$ for every $t \in \mathbb{R}^+$. Suppose $\alpha(x, y) \ge 1$ implies $x, y \ge 0$. Then, we have

$$\begin{aligned} &\psi\Big(\int_0^{d(Tx,Ty)} \phi(t)dt\Big) \\ &= \psi\Big(\frac{d(Tx, Ty)^2}{2}\Big) \\ &= d(Tx, Ty)^2 \\ &= \frac{|x - y|^2}{8^2} \\ &\le \Big(\frac{1}{16}\Big)\Big(2\int_0^{|x-y|} tdt\Big) \\ &= \frac{1}{16}\psi\Big(\int_0^{d(x,y)} \phi(t)dt\Big) \\ &\le \frac{1}{16}\psi\Big(\int_0^{\max\left\{d(x,y),\ d(x,Tx),\ d(y,Ty),\ \frac{d(x,Ty)+d(y,Tx)}{2}\right\}} \phi(t)dt\Big) \\ &\le \frac{1}{16}\psi(M(x, y)) + L^*I(\psi(U_6(x, y))). \end{aligned} \tag{21}$$

Clearly, (21) satisfies inequality (2). Also, by our construction, T is continuous, T is α-admissible mapping, and α satisfies the transitivity property. Here $\alpha(2, T2) = \alpha(2, \frac{1}{4}) \ge 1$ and (X, d) is complete, so it is $T-$orbitally complete. Thus, all the conditions of Theorem 2.1 are satisfied with $F(k, l) = \lambda k, 0 < \lambda < 1$, and $\theta(t) = L^*(\ge 0)$, for all $t \in \mathbb{R}$.

Now we verify that the contraction stated in Theorem 1.15, i.e., inequality (1), is not satisfied whenever "$x \preceq y$". For this let us take $x = -\frac{3}{2}$ and $y = -2$, then $x \preceq y$. Now we calculate $U_1(x,y), U_2(x,y)$, and $U_6(x,y)$ for the point $(-\frac{3}{2}, -2)$. A simple calculation gives us $U_1(-\frac{3}{2},-2) = 0$, $U_2(-\frac{3}{2},-2) = \frac{1}{2}$, $U_6(-\frac{3}{2},-2) = 0$ and $d(T(-\frac{3}{2}), T(-2)) = \frac{1}{2}$. Clearly, there does not exist any $\phi \in \Phi$, $\lambda_1, \lambda_2 \in [0,\infty)$ with $\lambda_1 + \lambda_2 < 1$, $L \geq 0$ such that the following inequality

$$I(d(T(-\frac{3}{2}), T(-2))) \leq \gamma_1 I(U_1(-\frac{3}{2},-2)) + \gamma_2 I(U_2(-\frac{3}{2},-2)) + LI(U_6(-\frac{3}{2},-2))$$

holds. Thus, we have extended the main result of Karapinar and Sadarangani [37] successfully.

Note: By considering different types of C-class functions in inequality (2) with $\phi \equiv 1$ in the integral part of the contraction, we can derive many well known classical results of fixed point theory. But the converse is not possible. To verify this, next, we consider an example to show that if one chooses $\phi \equiv 1$ in a particular type of C-class function, then the contraction becomes invalid.

Example 2.7. *Let $X = \{y_1, y_2, y_3, y_4\}$. Let $d : X \times X \to [0,\infty)$ be a mapping defined by $d(y_i, y_i) = 0$ for $i = 1,2,3,4$; $d(y_1,y_4) = d(y_4,y_1) = d(y_3,y_4) = d(y_4,y_3) = d(y_2,y_3) = d(y_3,y_2) = 2$; $d(y_1,y_3) = d(y_3,y_1) = d(y_2,y_4) = d(y_4,y_2) = 5$; $d(y_1,y_2) = d(y_2,y_1) = 4$. Let $T : X \to X$ be a mapping, given by $Ty_1 = y_2, Ty_2 = y_3, Ty_3 = y_3, Ty_4 = y_2$. Let $\alpha : X \times X \to [0,\infty)$ be a mapping, defined by*

$$\alpha(x,y) = \begin{cases} 1 & \text{if } x, y \in \{y_1, y_2, y_3\}, \\ 0 & \text{otherwise.} \end{cases}$$

Now we define two functions $\psi, \varphi : \mathbb{R}^+ \to \mathbb{R}^+$ by $\psi(t) = \frac{t}{2}, \varphi(t) = \frac{1}{32}$ if $t \in [0,1]$ and $\varphi(t) = -\frac{31}{32} + t^2$, if $t \in (1,\infty)$. Consider $\phi \in \Phi$ as $\phi(t) = e^{-t}$. Next we show that inequality (2) satisfies for $x = y_i$ and $y = y_j$, where $i, j \in \{1,2,3\}$ with $F(k,l) = \frac{k}{1+l}$ and $\theta(t) = L = 1$. It is clear that if $i = j$, then we have nothing to show. So we verify (2) for all points where $i \neq j$. First we consider $x = y_1$ and $y = y_2$, then $\int_0^{d(Ty_1,Ty_2)} e^{-t}dt = \int_0^{d(Ty_2,Ty_1)} e^{-t}dt = 1 - e^{-2}$. By taking $x = y_1$ and $y = y_2$ in (2), we obtain $U_1(y_1,y_2) = 2, U_2(y_1,y_2) = 4, U_3(y_1,y_2) = 0, U_4(y_1,y_2) = 2, U_5(y_1,y_2) = 0$ and $U_6(y_1,y_2) = 0$. Thus, we have

$$\frac{1-e^{-2}}{2} \leq \frac{1-e^{-4}}{2[1+\frac{1}{32}]}.$$

Similarly if we take $x = y_2$ and $y = y_1$, then we have $U_1(y_2,y_1) = \frac{12}{5}, U_2(y_2,y_1) = 4, U_3(y_2,y_1) = 0, U_4(y_2,y_1) = 0, U_5(y_2,y_1) = 2$ and $U_6(y_2,y_1) = 2$. Then, we get

$$\frac{1-e^{-2}}{2} \leq \frac{1-e^{-4}}{2[1+\frac{1}{32}]} + (1-e^{-1}).$$

Next, if we take $x = y_1$ and $y = y_3$, then $\int_0^{d(Ty_1,Ty_3)} e^{-t}dt = \int_0^{d(Ty_3,Ty_1)} e^{-t}dt = 1 - e^{-2}$. Also, we have $U_1(y_1,y_3) = 0, U_2(y_1,y_3) = 5, U_3(y_1,y_3) = \frac{5}{3}, U_4(y_1,y_3) = \frac{5}{3}, U_5(y_1,y_3) = 0$ and $U_6(y_1,y_3) = 2$. Hence, we obtain

$$\frac{1-e^{-2}}{2} \leq \frac{1-e^{-5}}{2[1+\frac{1}{32}]} + (1-e^{-1}).$$

In a similar way, if we take $x = y_3$, $y = y_1$, then it gives $U_1(y_3,y_1) = \frac{2}{3}, U_2(y_3,y_1) = 5, U_3(y_3,y_1) = \frac{5}{3}, U_4(y_3,y_1) = 0, U_5(y_3,y_1) = \frac{5}{3}$ and $U_6(y_3,y_1) = 0$. Consequently, we get

$$\frac{1-e^{-2}}{2} \leq \frac{1-e^{-4}}{2[1+\frac{1}{32}]}.$$

Next we consider $x = y_2$ and $y = y_3$, then $\int_0^{d(Ty_2,Ty_3)} e^{-t}dt = \int_0^{d(Ty_3,Ty_2)} e^{-t}dt = 0$. Also, one can calculate $U_1(y_2, y_3) = 0, U_2(y_2, y_3) = 2, U_3(y_2, y_3) = 0, U_4(y_2, y_3) = \frac{2}{3}, U_5(y_2, y_3) = 0$ and $U_6(y_2, y_3) = 0$. Thus, we have

$$0 \leq \frac{1 - e^{-2}}{2[1 + \frac{1}{32}]}.$$

Similarly for the point $x = y_3$ and $y = y_2$, we have

$$0 \leq \frac{1 - e^{-2}}{2[1 + \frac{1}{32}]}.$$

Therefore all the conditions of Theorem 2.1 are satisfied with $F(k, l) = \frac{k}{1+l}, \phi(t) = e^{-t}, \theta(t) = L = 1, \psi(t) = \frac{t}{2}, \varphi(t) = \frac{1}{32}$ if $t \in [0, 1]$, and $\varphi(t) = -\frac{31}{32} + t^2$, if $t \in (1, \infty)$. Clearly T has a fixed point. Here y_3 is that point.

Remark 2.8. *Based on the above example, here we have some observations. Clearly $\psi \in \Psi$ and $\varphi \in \Phi^*$ but $\varphi \notin \Psi$ since $\varphi(0) > 0$. In example 2.2, we do not consider any ordered structure, which shows that our results are more general. Lastly, one can not put here $\phi \equiv 1$. To verify our claim, consider $x = y_1$ and $y = y_2$ in inequality (2) with $\phi \equiv 1$. Then, we have*

$$\psi(\int_0^2 1dt) = \frac{\int_0^2 1dt}{2} \nleq \frac{\int_0^4 1dt}{2[1 - \frac{31}{32} + (\int_0^4 1dt)^2]} = \frac{\psi(\int_0^4 1dt)}{1 + \varphi(\int_0^4 1dt)}.$$

Our next example supports Theorem 2.2, where we drop the assumption of continuity of the mapping T.

Example 2.9. *Let $(X, d, \preceq)$ be a partially ordered metric space, where $X = \mathbb{R}^+$, d denotes the usual metric, and we define the partial order " $\preceq$ " as $x \preceq y$ if and only if $x \geq y$. Next, we define two mappings $\alpha : X \times X \to [0, \infty)$, and $T : X \to X$ as follows*

$$\alpha(x, y) = \begin{cases} x^2 + y^2 + 1 & \text{if } x, y \in \mathbb{R}^+ \setminus [0, 2), \\ 0 & \text{otherwise,} \end{cases}$$

and

$$T(x) = \begin{cases} \frac{x}{4} + 3 & \text{if } x \in \mathbb{R}^+ \setminus [0, 2), \\ x^2 + 1 & \text{otherwise.} \end{cases}$$

Take $\psi(t) = t$, and

$$\phi(t) = \begin{cases} 0 & \text{if } t = 0, \\ \frac{1}{2\sqrt{t}} & \text{if } t > 0. \end{cases}$$

Here, our constructed function T is not continuous. Hence, Theorem 2.1 is not applicable. If $\alpha(x, y) \geq 1$ implies $x, y \geq 2$. Thus, for $\alpha(x, y) \geq 1$, we have

$$\int_0^{d(Tx,Ty)} \phi(t)dt = \sqrt{\frac{|x - y|}{4}} = \frac{1}{2}\int_0^{d(x,y)} \phi(t)dt \leq \frac{1}{2}\int_0^{U_2(x,y)} \phi(t)dt \leq \frac{1}{2}M(x, y). \tag{22}$$

Take $x_0 = 5$ then $\alpha(x_0, Tx_0) \geq 1$. Clearly the contractive condition of Theorem 2.1 is satisfied with $F(k, l) = \lambda k$, $\lambda = \frac{1}{2}$. One can easily observe that α-regularity property is also satisfied. Moreover, all the conditions of Theorem 2.2 are satisfied. Thus, T has a fixed point, i.e., 4 is that point.

Remark 2.10. *In this example, we consider a partially ordered set, and T is not continuous still Theorem 6 in [37] is not applicable. Since "$2.5 \preceq 1.8$" $\not\Longrightarrow$ "$T(2.5) \preceq T(1.8)$", i.e., T is not non-decreasing.*

3. Some consequences

3.1. Fixed point results involving binary relation

Let (X, d) be a metric space, and ρ be a binary relation defined on X. Denote $R := \rho \cup \rho^{-1}$. It is clear that $x, y \in X$ with xRy if and only if $x\rho y$ or $y\rho x$. A mapping $T : X \to X$ is said to be comparable mapping w.r.t "ρ" if xRy implies $TxRTy$ for every $x, y \in X$.

Definition 3.1. *Let (X, d) be a metric space, and ρ be a binary relation defined on X. Then the space X is said to be R-transitive, if the following condition is satisfied*

$$\text{for } x, y, z \in X \text{ with } xRy \text{ and } yRz \text{ implies } xRz.$$

Definition 3.2. *Let (X, d) be a metric space, and ρ be a binary relation defined on X. Then we say, the space X enjoys the property $(\clubsuit)_R$, if for any sequence $\{x_n\}$ in X with $x_n \to x$ as $n \to \infty$ for some $x \in X$ and x_nRx_{n+1} for all $n \in \mathbb{N}$, then x_nRx for all $n \in \mathbb{N}$.*

Corollary 3.3. *Let (X, d) be a metric space, and ρ be a binary relation defined on X. Let $\psi, \varphi : \mathbb{R}^+ \to \mathbb{R}^+$ be two mappings with $\psi \in \Psi$ and $\varphi \in \Phi^*$. Let $T : X \to X$ be a given mapping. Assume that there exists a $\phi \in \Phi$, for which T satisfies the following contraction*

$$\psi(I(d(Tx, Ty))) \leq F(\psi(M(x, y)), \varphi(M(x, y))) + \theta(U_6(x, y))I(\psi(U_6(x, y))), \tag{23}$$

for all $x, y \in X$ with xRy, where $F \in \mathcal{C}$ and $\theta : \mathbb{R} \to \mathbb{R}$ be a mapping such that $\limsup\limits_{t \longrightarrow r^+} \theta(t)$ exists for all $r \in \mathbb{R}^+$. Assume that X is R-transitive, and it satisfies $(\clubsuit)_R$ condition. Also, assume that there exists a $x_0 \in X$ with x_0RTx_0. Furthermore, suppose that T is a comparable and continuous mapping with (X, d) is a $T-$orbitally complete metric space. Then T has a fixed point.

Proof. First, we define the mapping $\alpha : X \times X \to \mathbb{R}^+$ in the following manner

$$\alpha(x, y) = \begin{cases} 1 & \text{if } xRy, \\ 0 & \text{otherwise.} \end{cases}$$

Since $\alpha(x, y) \geq 1$ implies "xRy". Consequently, inequality (23) satisfies for all those points, for which $\alpha(x, y) \geq 1$. T is α-admissible since $\alpha(x, y) \geq 1$ implies "xRy" and T is comparable mapping, i.e., we have "$TxRTy$", which gives $\alpha(Tx, Ty) \geq 1$. Since X is R transitive, i.e., α is also a transitive mapping. By the given condition, there exists a point x_0 such that x_0RTx_0, which implies $\alpha(x_0, Tx_0) \geq 1$. Thus, all the conditions of Theorem 2.1 are satisfied. Therefore, T has a fixed point. □

Remark 3.4. *Every partial order is a binary relation.*

Remark 3.5. *In Corollary 3.3 if we consider $\psi(t) = t$, for all $t \in \mathbb{R}^+$, $F(k, l) = \lambda k$, and $\theta(t) = L(\geq 0)$, for all $t \in \mathbb{R}$ with binary relation as partial ordering, then inequality (1) implies (23) with $0 \leq \lambda_1 + \lambda_2 \leq \lambda < 1$.*

Corollary 3.6. *Let (X, d) be a metric space. Let $\psi, \varphi : \mathbb{R}^+ \to \mathbb{R}^+$ be two mappings with $\psi \in \Psi$ and $\varphi \in \Phi^*$. Suppose that $\alpha : X \times X \to \mathbb{R}^+$ and $T : X \to X$ be two mappings such that T is α-admissible. Assume that there exists a $\phi \in \Phi$, for which T satisfies the following contraction*

$$\psi(I(d(Tx, Ty))) \leq \frac{\psi(M(x, y))\varphi(M(x, y))}{1 + \varphi(M(x, y))} + \theta(U_6(x, y))I(\psi(U_6(x, y))), \tag{24}$$

for all $x, y \in X$ with $\alpha(x, y) \geq 1$, where $F \in \mathcal{C}$ and $\theta : \mathbb{R} \to \mathbb{R}$ be a mapping such that $\limsup\limits_{t \longrightarrow r^+} \theta(t)$ exists for all $r \in \mathbb{R}^+$. Suppose that there exists a $x_0 \in X$ with $\alpha(x_0, Tx_0) \geq 1$. Suppose that α satisfies the transitivity property, i.e., for $x, y, z \in X$ with $\alpha(x, y) \geq 1$, $\alpha(y, z) \geq 1$ implies $\alpha(x, z) \geq 1$. Also, suppose that T is a continuous function with (X, d) is a $T-$orbitally complete metric space. Then T has a fixed point.

Proof. Take $F(k,l) = \frac{kl}{1+l}$ in Theorem 2.1. □

Corollary 3.7. *Let (X,d) be a metric space. Let $\psi, \varphi : \mathbb{R}^+ \to \mathbb{R}^+$ be two mappings with $\psi \in \Psi$ and $\varphi \in \Phi^*$. Suppose that $\alpha : X \times X \to \mathbb{R}^+$ and $T : X \to X$ be two mappings with T is α-admissible. Assume that there exists a $\phi \in \Phi$, for which T satisfies the following contraction*

$$\psi(I(d(Tx,Ty))) \leq \psi(M(x,y)) - \varphi(M(x,y)) + \theta(U_6(x,y))I(\psi(U_6(x,y))), \tag{25}$$

for all $x,y \in X$ with $\alpha(x,y) \geq 1$, where $F \in \mathcal{C}$ and $\theta : \mathbb{R} \to \mathbb{R}$ be a mapping such that $\limsup_{t \longrightarrow r^+} \theta(t)$ exists for all $r \in \mathbb{R}^+$. Suppose that there exists a $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$. Suppose that α satisfies the transitivity property, i.e., for $x,y,z \in X$ with $\alpha(x,y) \geq 1$, $\alpha(y,z) \geq 1$ implies $\alpha(x,z) \geq 1$. Also, suppose that T is a continuous function with (X,d) is a $T-$orbitally complete metric space. Then T has a fixed point.

Proof. Take $F(k,l) = k - l$ in Theorem 2.1. □

3.2. Periodic fixed point result

In this section, we study periodic fixed point results (for details see [1], [28]). It is clear that if z is a fixed point of T, then it is also a fixed point of T^n for every $n \in \mathbb{N}$. But the converse statement is not true.

Definition 3.8. *Let $T : X \to X$ be a mapping. Then, T is said to satisfy property(P) if Fix(T) = Fix(T^n), for each $n \in \mathbb{N}$, where $Fix(T) := \{z \in X \mid Tz = z\}$.*

Next, we slightly change inequality (2) by adding $b \in (1,\infty)$ and $c \in (0,\infty)$ in the following way

$$\psi(b^{1+c}I(d(Tx,Ty))) \leq F(\psi(M(x,y)), \varphi(M(x,y))) + \theta(U_6(x,y))I(\psi(U_6(x,y))). \tag{26}$$

Theorem 3.9. *Suppose that all the hypotheses of Theorem 2.1 are satisfied with inequality (26) instead of (2). Furthermore, we also assume that the condition (★) holds, i.e., if $z \in$Fix(T^n) and $z \notin$Fix(T), then $\alpha(T^{n-1}z, T^n z) \geq 1$. Then T satisfies property (P).*

Proof. By considering (26), we get a unique fixed point of T, since inequality (26) implies inequality (2). Consequently, T has a fixed point, and Fix(T) = Fix(T^n) is true for $n = 1$. Now assume $n > 1$, and we prove property(P) of T using the contradiction method. Now take $z \in$Fix(T^n) and $z \notin$Fix(T), then by using condition (★), we have $\alpha(T^{n-1}z, T^n z) \geq 1$. Thus, from (26), we have

$$\begin{aligned}\psi(b^{1+c}I(d(z,Tz))) &= \psi(b^{1+c}I(d(T(T^{n-1}z), T(T^n z)))) \\ &\leq F(\psi(M(T^{n-1}z, T^n z)), \varphi(M(T^{n-1}z, T^n z))) \\ &\quad + \theta(U_6(T^{n-1}z, T^n z))I(\psi(U_6(T^{n-1}z, T^n z))).\end{aligned} \tag{27}$$

Now, we have
$U_1(T^{n-1}z, T^n z) = d(T^n z, T^{n+1}z)$, $U_2(T^{n-1}z, T^n z) = \max\{d(T^{n-1}z, T^n z), d(T^n z, T^{n+1}z)\}$,
$U_3(T^{n-1}z, T^n z) = 0$, $U_4(T^{n-1}z, T^n z) = \frac{d(T^{n-1}z, T^n z)d(T^{n-1}z, T^{n+1}z)}{2[1+d(T^{n-1}z, T^n z)]}$,
$U_5(T^{n-1}z, T^n z) = 0$ and $U_6(T^{n-1}z, T^n z) = 0$.
From (27), we obtain

$$\begin{aligned}&\psi(b^{1+c}I(d(z,Tz))) \\ &= \psi(b^{1+c}I(d(T^n z, T^{n+1}z))) \\ &\leq F(\psi(I(\max\{d(T^{n-1}z, T^n z), d(T^n z, T^{n+1}z)\})), \varphi(I(\max\{d(T^{n-1}z, T^n z), d(T^n z, T^{n+1}z)\}))) \\ &\leq \psi(I(\max\{d(T^{n-1}z, T^n z), d(T^n z, T^{n+1}z)\})).\end{aligned} \tag{28}$$

Since ψ is non-decreasing, we have

$$I(d(z,Tz)) = I(d(T^n z, T^{n+1}z)) \leq \frac{1}{b^{1+c}} I(\max\{d(T^{n-1}z, T^n z), d(T^n z, T^{n+1}z)\}). \tag{29}$$

Clearly, if $\max\{d(T^{n-1}z, T^n z), d(T^n z, T^{n+1}z)\} = d(T^n z, T^{n+1}z)$, then we arrive at a contradiction. Thus, we must have

$$I(d(z,Tz)) = I(d(T^n z, T^{n+1}z)) \leq \frac{1}{b^{1+c}} I(d(T^{n-1}z, T^n z)).$$

Continuing in this manner, we have

$$I(d(z,Tz)) = I(d(T^n z, T^{n+1}z)) \leq \frac{1}{b^{1+c}} I(d(T^{n-1}z, T^n z)) \leq \cdots \leq \left(\frac{1}{b^{1+c}}\right)^n I(d(z,Tz)) < I(d(z,Tz)),$$

which is a contradiction. Hence, we have $Tz = z$, and consequently the proof is completed. □

4. Fixed point results involving two self mappings

First, we define Theorem 2.1 for two self mappings $T, S : X \to X$. Here, we use $U_6^*(x,y)$ instead of $U_6(x,y)$. Let (X,d) be a metric space. Suppose $T, S : X \to X$ be two operators. Then, for any $x, y \in X$, we write

$V_1(x,y) = d(y,Sy)\frac{1+d(x,Tx)}{1+d(x,y)}$, $V_2(x,y) = \max\{d(x,y), d(x,Tx), d(y,Sy), \frac{d(x,Sy)+d(y,Tx)}{2}\}$,
$V_3(x,y) = \frac{d(x,Sy)d(x,Ty)}{1+d(x,y)}$, $V_4(x,y) = \frac{d(x,Tx)d(x,Sy)}{2[1+d(x,y)]}$, $V_5(x,y) = \frac{d(y,Sy)d(y,Tx)}{2[1+d(x,y)]}$ and
$V_6(x,y) = \min\{d(x,Tx), d(y,Sy), d(x,Sy), d(y,Tx)\}$.
Define:

$$M^*(x,y) = \max\Big\{\beta(V_1(x,y)), \beta(V_2(x,y)), \beta(V_3(x,y)), \beta(V_4(x,y)), \beta(V_5(x,y))\Big\}. \tag{30}$$

Before proceeding further, first, we need the following lemma.

Lemma 4.1. *Let (X,d) be a metric space. Let $\alpha : X \times X \to [0,\infty)$ and $\beta : \mathbb{R}^+ \to \mathbb{R}^+$ be two mappings such that β is a continuous and non-decreasing function with $\beta(t) = 0$ if and only if $t = 0$. Now we define two sets Fix $(T) = \{z \in X \mid z = Tz\}$, and Fix $(S) = \{z \in X \mid z = Sz\}$ where T, S are two mappings from X into itself. Assume that for every $x, y \in X$ with $\alpha(x,y) \geq 1$, the following inequality is satisfied*

$$\psi(\beta(d(Tx,Sy))) \leq F(\psi(M^*(x,y)), \varphi(M^*(x,y))) + \theta(V_6(x,y))\beta(\psi(V_6(x,y))), \tag{31}$$

where ψ, φ, θ as defined in Theorem 2.1. If Fix (T) is non-empty set, and $\alpha(z_1,z_2) \geq 1$ for any $z_1, z_2 \in$ Fix (T), then Fix (T) is a singleton set and Fix $(T) =$ Fix (S).

Proof. Let $z \in$ Fix (T) with $\alpha(z,z) \geq 1$, then from (31), we have

$$\psi(\beta(d(z,Sz))) = \psi(\beta(d(Tz,Sz))) \leq F(\psi(M^*(z,z)), \varphi(M^*(z,z))) + \theta(V_6(z,z))\beta(\psi(V_6(z,z))). \tag{32}$$

Now, we have
$V_1(z,z) = d(z,Sz)$, $V_2(z,z) = d(z,Sz)$, $V_3(z,z) = 0$, $V_4(z,z) = 0$, $V_5(z,z) = 0$ and $V_6(z,z) = 0$. From (32), we have

$$\psi(\beta(d(z,Sz))) \leq F(\psi(\beta(d(z,Sz))), \varphi(\beta(d(z,Sz)))) \leq \psi(\beta(d(z,Sz))),$$

which implies either $\psi(\beta(d(z,Sz))) = 0$ or $\varphi(\beta(d(z,Sz))) = 0$. Thus, from both the cases, we have $\beta(d(z,Sz)) = 0$ implies $d(z,Sz) = 0$ implies $z = Sz$. Consequently, we obtain $z \in$ Fix (S). Let $z^* \in$ Fix (T) be any arbitrary point. Then, by our observation $z^* \in$ Fix (S) with $\alpha(z,z^*) \geq 1$, and by (31), we have

$$\psi(\beta(d(z,z^*))) = \psi(\beta(d(Tz,Sz^*))) \leq F(\psi(M^*(z,z^*)), \varphi(M^*(z,z^*))) + \theta(V_6(z,z^*))\beta(\psi(V_6(z,z^*))). \tag{33}$$

Also, we obtain

$V_1(z,z^*) = 0$, $V_2(z,z^*) = d(z,z^*)$, $V_3(z,z^*) = \left[\frac{d(z,z^*)}{1+d(z,z^*)}\right]d(z,z^*)$, $V_4(z,z^*) = 0$,
$V_5(z,z^*) = 0$ and $V_6(z,z^*) = 0$. Consequently from (33), we get

$$\psi(\beta(d(z,z^*))) \leq F(\psi(\beta(d(z,z^*))), \varphi(\beta(d(z,z^*)))) \leq \psi(\beta(d(z,z^*))),$$

which implies either $\psi(\beta(d(z,z^*))) = 0$ or $\varphi(\beta(d(z,z^*))) = 0$. Thus, from both the cases, we have $\beta(d(z,z^*)) = 0$ implies $d(z,z^*) = 0$ implies $z = z^*$. Consequently, we have Fix $(T) =$ Fix (S) and it is singleton. □

Remark 4.2. *If Fix (S) is a non-empty set and $\alpha(z_1, z_2) \geq 1$, for any $z_1, z_2 \in$ Fix (S), then Fix (S) is a singleton set and Fix $(T) =$ Fix (S).*

Denote:

$$M^{**}(x, y) = \max\Big\{I(V_1(x, y)), I(V_2(x, y)), I(V_3(x, y)), I(V_4(x, y)), I(V_5(x, y))\Big\}.$$

Theorem 4.3. *Let (X, d) be a metric space. Let $\psi, \varphi : \mathbb{R}^+ \to \mathbb{R}^+$ be two mappings with $\psi \in \Psi$ and $\varphi \in \Phi^*$. Suppose that $\alpha : X \times X \to \mathbb{R}^+$ and $T, S : X \to X$ be three mappings with α is symmetric, and the pair (T, S) is triangular α-admissible mapping. Assume that there exists a $\phi \in \Phi$, for which the mappings T, S satisfy the following contraction*

$$\psi(I(d(Tx, Sy))) \leq F(\psi(M^{**}(x, y)), \varphi(M^{**}(x, y))) + \theta(V_6(x, y))I(\psi(V_6(x, y))), \tag{34}$$

for all $x, y \in X$ with $\alpha(x, y) \geq 1$, where $F \in \mathcal{C}$ and $\theta : \mathbb{R} \to \mathbb{R}$ be a mapping such that $\limsup\limits_{t \longrightarrow r^+} \theta(t)$ exists for all $r \in \mathbb{R}^+$. Suppose that there exists a $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$. Also, suppose that either T is a continuous function, or α satisfies the regularity property with (X, d) is a $O(T, S)$−orbitally complete metric space and $\alpha(z_1, z_2) \geq 1$, for any $z_1, z_2 \in$ Fix (T) (assuming Fix $(T) \neq \emptyset$). Then T and S have a unique common fixed point.

Proof. Let $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$. By using the triangular property of α, one can easily construct a sequence $\{x_n\}_{n=0}^{\infty}$ with $x_{2n+1} = Tx_{2n}$ and $x_{2n} = Sx_{2n-1}$, where $n = 0, 1, 2, \cdots$ and also $\alpha(x_m, x_n) \geq 1$, for all $m, n \in \mathbb{N}$ with $m < n$. If $x_n = x_{n+1}$, for some $n \in \mathbb{N}$ with if n is even, then T has a fixed point, and we use Lemma 4.1 to show that it is also a fixed point of S or if n is odd, then it becomes a fixed point of S, and consequently it is also a fixed point of T by Remark 4.2. From now we assume that $c_n = d(x_n, x_{n+1}) > 0$, for all $n \in \mathbb{N}$. Take $x = x_{2n-2}$ and $y = x_{2n-1}$ in (34), we have

$$\begin{aligned}
&\psi(I(d(x_{2n-1}, x_{2n}))) \\
&= \psi(I(d(Tx_{2n-2}, Sx_{2n-1}))) \\
&\leq F(\psi(M^{**}(x_{2n-2}, x_{2n-1})), \varphi(M^{**}(x_{2n-2}, x_{2n-1}))) \\
&+ \theta(V_6(x_{2n-2}, x_{2n-1}))I(\psi(V_6(x_{2n-2}, x_{2n-1}))).
\end{aligned} \tag{35}$$

It can be easily shown that (35) implies the following inequality

$$\begin{aligned}
&\psi(I(d(x_{2n-1}, x_{2n}))) \\
&\leq F(\psi(I(\max\{d(x_{2n-1}, x_{2n}), d(x_{2n-2}, x_{2n-1})\})), \varphi(I(\max\{d(x_{2n-1}, x_{2n}), d(x_{2n-2}, x_{2n-1})\}))) \\
&\leq \psi(I(\max\{d(x_{2n-1}, x_{2n}), d(x_{2n-2}, x_{2n-1})\})).
\end{aligned} \tag{36}$$

If $\max\{d(x_{2n-1}, x_{2n}), d(x_{2n-2}, x_{2n-1})\} = d(x_{2n-1}, x_{2n})$, then we get a contradiction, since it implies $d(x_{2n-1}, x_{2n}) = 0$. Thus, we must have $\max\{d(x_{2n-1}, x_{2n}), d(x_{2n-2}, x_{2n-1})\} = d(x_{2n-2}, x_{2n-1})$. Thus, we have $I(c_{2n-1}) \leq I(c_{2n-2})$. Similarly, we can show that $I(c_{2n}) \leq I(c_{2n-1})$. Hence, $\{I(c_n)\}_{n=0}^{\infty}$ becomes a decreasing sequence. So it converges to some non-negative real number, say $c^*(\geq 0)$. Thus, from (36), we have

$$\begin{aligned}
&\psi(I(d(x_{2n-1}, x_{2n}))) \\
&\leq F(\psi(I(d(x_{2n-2}, x_{2n-1}))), \varphi(I(d(x_{2n-2}, x_{2n-1})))).
\end{aligned} \tag{37}$$

Taking limit as $n \to \infty$, in (37), we obtain

$$\psi(c^*) \leq F(\psi(c^*), \varphi(c^*)) \leq \psi(c^*),$$

which implies $c^* = 0$. Consequently, by Lemma 1.10, we get that c_n tends to 0 as $n \to \infty$. Next, we show that $\{x_n\}$ is a Cauchy-sequence. It is sufficient to show $\{x_{2n}\}$ is Cauchy. Suppose on the contrary, i.e., there exists a $\epsilon > 0$ such that for which we can get two subsequence $\{x_{2m(i)}\}$ and $\{x_{2n(i)}\}$ such that $n(i)$ is the smallest integer for which $n(i) > m(i) \geq i \geq 1$ with

$$d(x_{2m(i)}, x_{2n(i)}) \geq \epsilon,\ d(x_{2m(i)}, x_{2n(i)-2}) < \epsilon. \tag{38}$$

Using (38), one can easily derive

$$\lim_{i\to\infty} d(x_{2m(i)}, x_{2n(i)}) = \epsilon, \lim_{i\to\infty} d(x_{2m(i)-1}, x_{2n(i)}) = \epsilon, \lim_{i\to\infty} d(x_{2m(i)}, x_{2n(i)+1}) = \epsilon, \lim_{i\to\infty} d(x_{2m(i)-1}, x_{2n(i)+1}) = \epsilon. \tag{39}$$

Putting $x = x_{2n(i)}$ and $y = x_{2m(i)-1}$ in (34), we have

$$\begin{aligned} &\psi(I(d(x_{2n(i)+1}, x_{2m(i)}))) \\ &= \psi(I(d(Tx_{2n(i)}, Sx_{2m(i)-1}))) \\ &\leq F(\psi(M^{**}(x_{2n(i)}, x_{2m(i)-1})), \varphi(M^{**}(x_{2n(i)}, x_{2m(i)-1}))) \\ &+ \theta(V_6(x_{2n(i)}, x_{2m(i)-1}))I(\psi(V_6(x_{2n(i)}, x_{2m(i)-1}))). \end{aligned} \tag{40}$$

Next, we calculate all the values of V_i, for $i = 1, 2, 3, 4, 5, 6$ and taking limit as $i \to \infty$ in (40), one can get the following

$$\psi(I(\epsilon)) \leq F(\psi(I(\epsilon)), \varphi(I(\epsilon))) \leq \psi(I(\epsilon)),$$

which implies $\epsilon = 0$. Consequently, $\{x_n\}$ becomes a Cauchy sequence. Since (X, d) is a $O(T, S)-$orbitally complete metric space, so there exists a x^* such that $\lim_{n\to\infty} x_n = x^*$ implies $\lim_{n\to\infty} x_{2n} = x^*$ and $\lim_{n\to\infty} x_{2n-1} = x^*$. Next, we consider T is a continuous mapping, then $\lim_{n\to\infty} x_{2n} = x^*$ implies $\lim_{n\to\infty} Tx_{2n} = Tx^*$ implies $\lim_{n\to\infty} x_{2n+1} = Tx^*$ implies $x^* = Tx^*$. Thus, we obtain a fixed point of T. Hence, by using Lemma 4.1, we have Fix (T) is a singleton set and Fix $(T) =$ Fix (S) with $\beta \equiv I$, where $I(a) = \int_0^a \phi(t)dt$. Next, consider T is not continuous, then by the regularity property of α gives $\alpha(x_n, x^*) \geq 1$, for all $n \in \mathbb{N}$ implies $\alpha(x_{2n-1}, x^*) \geq 1$, for all $n \in \mathbb{N}$ implies $\alpha(x^*, x_{2n-1}) \geq 1$, for all $n \in \mathbb{N}$(since α is symmetric). Now put $x = x^*$ and $y = x_{2n-1}$ in (34), and keeping in mind that $\lim_{n\to\infty} x_n = x^*$, one can show that $x^* = Tx^*$. Again, by using Lemma 4.1 we obtain Fix (T) is a singleton set and Fix $(T) =$ Fix (S) with $\beta \equiv I$, where $I(a) = \int_0^a \phi(t)dt$, and hence the proof is completed. □

Note: It is very easy to see that one can list a further outcome of our main results by letting the mappings ψ, φ, F, θ in a suitable way.

Example 4.4. *Let $X = \{1, 2, 3, 4, 5\}$ and $d : X \times X \to \mathbb{R}^+$ be a mapping given by*
$d(x, x) = 0$, for all $x \in X$;
$d(4, 5) = d(5, 4) = d(1, 5) = d(5, 1) = d(1, 4) = d(4, 1) = d(1, 3) = d(3, 1) = \frac{6}{7}$;
$d(2, 5) = d(5, 2) = d(2, 4) = d(4, 2) = d(1, 2) = d(2, 1) = 1$;
$d(3, 5) = d(5, 3) = d(3, 4) = d(4, 3) = d(2, 3) = d(3, 2) = \frac{4}{7}$. Also, suppose that for any $x, y \in X$ with "$x \preceq y$" means x is divisible by y. Then $(X, d, \preceq)$ is a partially ordered metric space. Let $T, S : X \to X$ be two mappings defined by

$$T(1) = T(2) = T(3) = 1, T(4) = 2, T(5) = 3 \text{ and}$$

$$S(x) = 1, \text{ for all } x \in X.$$

Let $g : \mathbb{R}^+ \to [0, 1]$ be a mapping given by $g(t) = 0.6 + \frac{0.4}{1+t}$. Then g is continuous and $g(t) = 1$ if and only if $t = 0$. Let us define $F(k, l) : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ by $F(k, l) = kg(l)$. Also we define a map $\alpha : X \times X \to \mathbb{R}^+$ by

$$\alpha(x, y) = \begin{cases} 1 & \text{if } x, y \in \{1, 2, 3\}, \\ 0 & \text{otherwise.} \end{cases}$$

Now consider $\psi(t) = t, \varphi(t) = t$, where $\psi \in \Psi, \varphi \in \Phi^*$, and $\phi(\in \Phi)$ as $\phi(t) = e^t$. Then, we can easily verify that for $\alpha(x, y) \geq 1$, inequality (34) is satisfied with $F(k, l) = kg(l)$. Here all the conditions of Theorem 4.3 are satisfied, and 1 is the common fixed point of T and S.

Now, if one considers the order structure instead of α-admissible mapping, then inequality (34) does not satisfy. To check this take $x = 4$ and $y = 2$ (since $4 \preceq 2$). By some easy calculation, one can get the following inequality

$$\int_0^1 \phi(t)dt \le g\Big(\int_0^1 \phi(t)dt\Big)\Big(\int_0^1 \phi(t)dt\Big).$$

The above inequality does not satisfy for $\phi(t) = e^t$, even there does not exist any $\phi \in \Phi$, which satisfy the above inequality.

5. Application

In this section, we apply our theoretical results to study the existence of a solution for an ordinary differential equation and non-linear integral equations.

5.1. Application to ordinary differential equation

Motivated by the papers [30], [43] here, we present our first example. Let us consider the first-order periodic differential equation,

$$\begin{aligned} &\frac{dy}{dt} = h(t, y(t)), t \in [0, L] \\ &y(0) = y(L), \text{ where } L > 0. \end{aligned} \tag{41}$$

Let us put $[0, L] = I$. Here $h : I \times \mathbb{R} \to \mathbb{R}$ is a continuous function. From now, we write $C(I, \mathbb{R})$ to denote the class of all continuous functions defined on I. Also, it is well known that the class $C(I, \mathbb{R})$ is complete with respect to the metric d given by $d(x, y) = \sup\{|x(t) - y(t)| : t \in I\}$, for $x, y \in C(I, \mathbb{R})$. Now, we prove the following existence and uniqueness theorem.

Theorem 5.1. *Consider the first order periodic differential equation given by (41). Let $\eta : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a given mapping and $\rho \in \mathbb{R}^+$. Suppose that the following conditions are satisfied:*

1. *there exists a scalar $\beta > 0$ such that $\beta \le \left(\frac{3\rho(e^{\rho L}-1)^2}{K(e^{2\rho L}+e^{\rho L}+1)}\right)^{\frac{1}{3}}$ where $K = L^2$;*
2. *there exists $x_0 \in C(I, \mathbb{R})$ such that $\eta(x_0(t), Ax_0(t)) \ge 0$ for all $t \in I$, where A is function from $C(I, \mathbb{R})$ into itself given by $Ax(t) = \int_0^L G(t, u)\Big[h(u, x(u)) + \rho x(u)\Big]du$, where $G(t, u)$ is a Green's function defined by*

$$G(t, u) = \begin{cases} \frac{e^{\rho(L+u-t)}}{e^{\rho L}-1} & \text{if } 0 \le u < t \le L \\ \frac{e^{\rho(u-t)}}{e^{\rho L}-1} & \text{if } 0 \le t < u \le L \end{cases};$$

3. *for every $t \in I$ and $x, y \in C(I, \mathbb{R})$, $\eta(x(t), y(t)) \ge 0$ implies that $\eta(Ax(t), Ay(t)) \ge 0$;*
4. *for all $t \in I$ the continuous function h given in eq (41) satisfies the following inequality*

$$\Big|h(t, p) - h(t, q) + \rho(p - q)\Big| \le \beta\left[ln\left(\frac{1 + a^{|p-q|^{\frac{3}{2}}}}{2}\right)\right]^{\frac{2}{3}}$$

for every $p, q \in \mathbb{R}$ with $\eta(p, q) \ge 0$, where $a > e + 5$ and $\rho > 0$;

5. *for every $t \in I$, if $\{x_n\}$ be a sequence in $C(I, \mathbb{R})$, such that $x_n \to x$ in $C(I, \mathbb{R})$ and $\eta(x_n(t), x_{n+1}(t)) \ge 0$ for all $n \in \mathbb{N}$, then $\eta(x_n(t), x(t)) \ge 0$ for all $n \in \mathbb{N}$.*

Then the differential eq (41) has a solution, i.e., Fix(A) $\neq \emptyset$. Furthermore, if $\eta(z_1(t), z_2(t)) \ge 0$, for all $z_1, z_2 \in$Fix(A), then the solution is unique.

Proof. First, we observe that $y \in C(I, \mathbb{R})$ is a solution of eq (41) if and only if it is a solution of the corresponding integral equation, given by

$$y(t) = \int_0^L G(t,u)\Big[h(u, y(u)) + \rho y(u)\Big]du.$$

Then, the problem (41) is equivalent to find a fixed point $y^*(\in C(I, \mathbb{R}))$ of the operator A. Let $x, y \in C(I, \mathbb{R})$ such that $\eta(x(t), y(t)) \geq 0$ for all $t \in I$. Now, we have

$$\begin{aligned}
& d(Ax, Ay) \\
&= \sup_{t\in I} |Ax(t) - Ay(t)| \\
&= \sup_{t\in I} \Big| \int_0^L \Big(G(t,u)\Big)\Big(h(u, x(u)) - h(u, y(u)) + \rho(x(u) - y(u))\Big)du\Big| \\
&\leq \sup_{t\in I} \int_0^L \Big|\Big(G(t,u)\Big)\Big(h(u, x(u)) - h(u, y(u)) + \rho(x(u) - y(u))\Big)\Big|du \\
&\leq \sup_{t\in I} \int_0^L \Big|G(t,u)\Big|\Big|h(u, x(u)) - h(u, y(u)) + \rho(x(u) - y(u))\Big|du \\
&\leq \sup_{t\in I} \int_0^L \Big|G(t,u)\Big|\Bigg(\beta\Bigg[ln\Bigg(\frac{1 + a^{|x(u)-y(u)|^{\frac{3}{2}}}}{2}\Bigg)\Bigg]^{\frac{2}{3}}\Bigg)du \quad \text{(by condition (4) of Theorem 5.1).}
\end{aligned} \tag{42}$$

Now, by applying Hölder's inequality in the R.H.S of (42), we have

$$\begin{aligned}
& \int_0^L \Big|G(t,u)\Big|\Bigg(\beta\Bigg[ln\Bigg(\frac{1 + a^{|x(u)-y(u)|^{\frac{3}{2}}}}{2}\Bigg)\Bigg]^{\frac{2}{3}}\Bigg)du \\
&\leq \Bigg[\int_0^L \Big|G(t,u)\Big|^3 du\Bigg]^{\frac{1}{3}} \Bigg[\int_0^L \Bigg(\beta\Bigg[ln\Bigg(\frac{1 + a^{|x(u)-y(u)|^{\frac{3}{2}}}}{2}\Bigg)\Bigg]^{\frac{2}{3}}\Bigg)^{\frac{3}{2}} du\Bigg]^{\frac{2}{3}} \\
&\leq \Bigg(\int_0^L \Big|G(t,u)\Big|^3 du\Bigg)^{\frac{1}{3}} \Bigg(\int_0^L \beta^{\frac{3}{2}}\Bigg[ln\Bigg(\frac{1 + a^{|x(u)-y(u)|^{\frac{3}{2}}}}{2}\Bigg)\Bigg]du\Bigg)^{\frac{2}{3}}.
\end{aligned} \tag{43}$$

Now, one can easily calculate the following

$$\int_0^L \Big|G(t,u)\Big|^3 du = \frac{(e^{2\rho L} + e^{\rho L} + 1)}{3\rho(e^{\rho L} - 1)^2}. \tag{44}$$

Using (42), and (43), we have

$$d(Ax, Ay) \leq \Bigg(\int_0^L \Big|G(t,u)\Big|^3 du\Bigg)^{\frac{1}{3}} \Bigg(\beta^{\frac{3}{2}}\Bigg[ln\Bigg(\frac{1 + a^{d(x,y)^{\frac{3}{2}}}}{2}\Bigg)\Bigg]L\Bigg)^{\frac{2}{3}}. \tag{45}$$

Now, from condition (1) of Theorem 5.1, we have
$\beta \leq \left(\frac{3\rho(e^{\rho L}-1)^2}{K(e^{2\rho L}+e^{\rho L}+1)}\right)^{\frac{1}{3}}$ implies $\beta^{\frac{3}{2}} \leq \left(\frac{3\rho(e^{\rho L}-1)^2}{K(e^{2\rho L}+e^{\rho L}+1)}\right)^{\frac{1}{2}}$. Thus, we get

$$\Bigg(\beta^{\frac{3}{2}}\Bigg[ln\Bigg(\frac{1 + a^{d(x,y)^{\frac{3}{2}}}}{2}\Bigg)\Bigg]L\Bigg) \leq \Bigg(\Bigg(\frac{3\rho(e^{\rho L} - 1)^2}{(e^{2\rho L} + e^{\rho L} + 1)}\Bigg)^{\frac{1}{2}}\Bigg[ln\Bigg(\frac{1 + a^{d(x,y)^{\frac{3}{2}}}}{2}\Bigg)\Bigg]\Bigg). \tag{46}$$

From (45) and using (44), (46), we get

$$d(Ax,Ay) \le \left[ln\left(\frac{1 + a^{d(x,y)^{\frac{3}{2}}}}{2} \right) \right]^{\frac{2}{3}}. \tag{47}$$

This implies for every $x, y \in C(I, \mathbb{R})$ with $\eta(x(t), y(t)) \ge 0$ for $t \in I$, we have

$$\begin{aligned} & d(Ax,Ay)^{\frac{3}{2}} \le ln\left(\frac{1 + a^{d(x,y)^{\frac{3}{2}}}}{2} \right) \\ & \text{implies } \sqrt{\int_0^{d(Ax,Ay)} 3\lambda^2 d\lambda} \le ln\left(\frac{1 + a^{\sqrt{\int_0^{d(x,y)} 3\lambda^2 d\lambda}}}{2} \right) \le ln\left(\frac{1 + a^{\sqrt{M(x,y)}}}{2} \right). \end{aligned} \tag{48}$$

Next, we define a mapping $\tilde{\alpha} : X \times X \to \mathbb{R}^+$, where $X := C(I, \mathbb{R})$, in the following manner

$$\tilde{\alpha}(x,y) = \begin{cases} 1 & \text{if } \eta(x(t), y(t)) \ge 0 \text{ for all } t \in I, \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, $\tilde{\alpha}$ is a α-admissible mapping, which ensures condition (3) of Theorem 5.1. From condition (2), there exists a point $x_0 \in X$ such that $\tilde{\alpha}(x_0, Ax_0) \ge 1$. Now compare (48) with inequality (2) by considering $\psi(s) = \sqrt{s}, \phi(s) = 3s^2, F(k,l) = ln(\frac{1+a^k}{2})$, where $a > e + 5$. Since $\tilde{\alpha}(x_0, Ax_0) \ge 1$, consequently, we get sequence $x_n (\in X)$ such that $x_n \to x$. Using condition (5) of Theorem 5.1, we obtain the regularity property of $\tilde{\alpha}$. Thus, Theorem 2.2 guarantees that x is a fixed point of A, i.e., Fix$(A) \neq \emptyset$. Further suppose that, $\eta(z_1(t), z_2(t)) \ge 0$ for all $z_1, z_2 \in$Fix(A) implies $\tilde{\alpha}(z_1, z_2) \ge 1$ for all $z_1, z_2 \in$Fix(A). Thus, we obtain a unique fixed point of A, i.e., a unique solution of the periodic differential eq (41). □

5.2. Application to non-linear integral equation

In this section, we apply our new findings, which already obtain in section 4, to solve non-linear integral equations. Next, we consider the following integral equations

$$\begin{aligned} x(t) &= \int_0^L Q(t,u) K_1(t,u,x(u)) du + P(t) \\ x(t) &= \int_0^L Q(t,u) K_2(t,u,x(u)) du + P(t), \text{ for all } t \in [0, L], \end{aligned} \tag{49}$$

where $L > 0$. Let us put $[0, L] = I$. From now, we write $X = C(I, \mathbb{R})$ to denote the class of all continuous functions defined on I. Also, it is well known that the class $C(I, \mathbb{R})$ is complete with respect to the metric d given by $d(x, y) = \sup\{|x(t) - y(t)| : t \in I\}$, for $x, y \in C(I, \mathbb{R})$. Next, we suppose that the following conditions are satisfied:

(c_1) $K_1, K_2 : I \times I \times \mathbb{R} \to \mathbb{R}$ are continuous functions;

(c_2) $Q : I \times I \to \mathbb{R}^+$ is a continuous function;

(c_3) $P \in C(I, \mathbb{R})$;

(c_4) For all $t, u \in I$ and $x, y \in X$, we have:

$$\left| K_1(t,u,x(u)) - K_2(t,u,y(u)) \right| \le \left[ln\left(\frac{1 + a^{|x(u)-y(u)|^{\frac{3}{2}}}}{2} \right) \right]^{\frac{2}{3}} \text{ where } a > e + 5;$$

(c_5) $\displaystyle\int_0^L Q(t,u)^3 du \le \frac{1}{L^2}.$

Now, we prove the following existence and uniqueness theorem.

Theorem 5.2. *Under the assumptions* $(c_1) - (c_5)$*, the eq (49) has a unique solution in X.*

Proof. First, we define two operators $A, B : X \to X$ as follows

$$Ax(t) = \int_0^L Q(t,u)K_1(t,u,x(u))du + P(t),$$
$$Bx(t) = \int_0^L Q(t,u)K_2(t,u,x(u))du + P(t).$$

Now, for all $t, u \in I$ and $x, y \in X$, we have

$$\begin{aligned}|Ax(t) - By(t)| &= |\int_0^L Q(t,u)K_1(t,u,x(u))du - \int_0^L Q(t,u)K_2(t,u,y(u))du| \\ &\le \int_0^L |Q(t,u)||K_1(t,u,x(u))du - K_2(t,u,y(u))|du \\ &\le \Big(\int_0^L |Q(t,u)|^3 du\Big)^{\frac{1}{3}} \Big(\int_0^L |K_1(t,u,x(u))du - K_2(t,u,y(u))|^{\frac{3}{2}} du\Big)^{\frac{2}{3}} \\ &\le \frac{1}{L^{\frac{2}{3}}}\Big(\int_0^L ln\Big(\frac{1 + a^{|x(u)-y(u)|^{\frac{3}{2}}}}{2}\Big)du\Big)^{\frac{2}{3}} \\ &\le \frac{1}{L^{\frac{2}{3}}}\Big(ln\Big(\frac{1 + a^{d(x,y)^{\frac{3}{2}}}}{2}\Big)\Big)^{\frac{2}{3}} L^{\frac{2}{3}} \\ &= \Big(ln\Big(\frac{1 + a^{d(x,y)^{\frac{3}{2}}}}{2}\Big)\Big)^{\frac{2}{3}}.\end{aligned}$$

Now, considering supremum in L.H.S of the above inequality, we obtain

$$\begin{aligned}&d(Ax, By)^{\frac{3}{2}} \le ln\Big(\frac{1 + a^{d(x,y)^{\frac{3}{2}}}}{2}\Big), \\ &\text{implies } \sqrt{\int_0^{d(Ax,By)} 3\lambda^2 d\lambda} \le ln\Big(\frac{1 + a^{\sqrt{\int_0^{d(x,y)} 3\lambda^2 d\lambda}}}{2}\Big) \le ln\Big(\frac{1 + a^{\sqrt{M^*(x,y)}}}{2}\Big).\end{aligned} \tag{50}$$

Next, compare (50) with (34) by considering $\psi(s) = \sqrt{s}, \phi(s) = 3s^2, F(k,l) = ln(\frac{1+a^k}{2})$, where $a > e + 5$. Thus, all the conditions of Theorem 4.3 are satisfied with $\alpha(x, y) = 1$ for all $x, y \in X$. Consequently, the mappings A, B have a unique common fixed point, i.e., the eq (49) has a unique solution in $C(I, \mathbb{R})$.

□

6. Acknowledgement

The authors are very grateful to Professor Dr. Vladimir Rakočević (the editor in chief), Professor Dr. Erdal Karapinar (the associate editor) and the anonymous reviewers for their valuable comments and several useful suggestions which improved the presentation of the paper. The first author (SKG) would like to thank University Grants Commission, Govt. of India (Sr. No. 2121540966, Ref. No. 20/12/2015(ii)EU-V), for financial support.

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