Filomat 35:3 (2021), 963–972
https://doi.org/10.2298/FIL2103963A

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Generalized Quasi-Regular Representation and its Applications for Shearlet Transforms

Z. Amiri[a], R. A. Kamyabi-Gol[b]

[a]*Department of Mathematics, Ferdowsi University of Mashhad, P.O.Box 1159- 91775 Mashhad, Iran*
[b]*Department of Mathematics, Ferdowsi University of Mashhad and*
Center of Excellence in Analysis on Algebraic Structures (CEAAS), P.O.Box 1159- 91775 Mashhad, Iran

Abstract. The construction of continuous shearlet transform has been extended to higher dimensions. It was generalized to a group that is topologically isomorphic to a group of semidirect product of locally compact groups. In this paper, by a unified theoretical linear algebra approach to the representation theory, a class of continuous shearlet transforms obtained from the generalized quasi-regular representation is presented. In order to develop such representation, we utilize a homogeneous space with a relatively invariant Radon measure as tool from computational and abstract harmonic analysis.

1. Introduction

Over the last decade shearlet system has achieved significant popularity, see [4, 13]. These innovative systems are made modeling for any kinds of applications such as signal processing, compressed sensing and machine learning. Application of shearlet in compressive sensing problems have been studied in [7]. In addition, the mathematical theory of shearlet systems have outstanding roles and applications in computer science and communication engineering. This theory is based on the shear matrix, translation and dilation of a given signal. Kutyniok et al. proved that the shearlet transform can be derived from a square-integrable representation $\pi : \mathbb{S} \to U(L^2(\mathbb{R}^2)$ of certain group S, see [12]. The analyzing 2-dimensional data set have been completely well studied and extended for higher dimensions. For instance, in [6] Dahlke et al. extended shearlet theory to n-dimensional signals and introduced the multivariate continuous shearlet transform and investigated its properties. This paper aim to investigate some other aspects of generalization of continuous shearlet transforms related to homogeneous spaces.These kinds of shearlet transforms can be contained in a class of continuous shearlet transforms obtained from a continuous unitary representation of a locally compact group on $L^2(X)$, where X is a homogeneous space that group acts on it. More precisely, we consider a unified approach to the theoretical aspects of the generalized quasi- regular representation over the product of these spaces. We then proceed to obtain the generalized quasi- regular representation for a large non-Abelian classes of groups, in particular for semidirect product groups, as well as a class called shearlet groups. To obtain purpose, we need to prove some results of homogeneous space.

The outline of the rest of this paper is as follows: Section 2 is devoted to fix notations and recall a brief summary of representation theory which we need in the sequel. In Section 3, a necessary and

2010 *Mathematics Subject Classification*. Primary 42C40; Secondary 42C15, 65J22, 65T60
Keywords. Homogeneous space, strongly quasi invariant measure, unitary representation, continuous shearlet transforms
Received: 14 March 2020; Revised: 17 April 2021; Accepted: 19 April 2021
Communicated by Ljubiša D.R. Kočinac
Email addresses: za_am10@um.ac.ir (Z. Amiri), kamyabi@um.ac.ir (R. A. Kamyabi-Gol)

sufficient condition to obtain generalized representation is provided and this kind of representations are characterized. In the last section, we propose some examples to show that in this way we can obtain the n-dimensional continuous shearlet transforms which have been introduced in [6] and make a few examples of continuous shearlet transforms that cannot be gotten by the procedure has been discussed in [6, 12].

2. Preliminaries and Notations

To provide a first insight into our results, we start by recalling mathematical framework of homogeneous space.

Through the world of harmonic analysis and after locally compact groups, we have objects like G-spaces, and in special case transitive G-spaces which are well known as follows.

Let G be a locally compact group and X be a locally compact Hausdorff space. A left action of G on X is a continuous mapping $(x, s) \to xs$ from $G \times X$ into X which satisfies the following statements.

(i) For all $x \in G$, the mapping $s \to xs, s \in X$, is a homeomorphism on X.

(ii) For all $x, y \in G$ and $s \in X$, $(xy)s = x(ys)$.

A space X equipped with an action of G is called a G- space. X is said to be transitive G-space if for all $t, s \in X$ there exists some $x \in G$ for which $xs = t$. Proposition 2.44 of [8] guarantees that most of transitive G-spaces can be considered as a quotient space G/H for some closed subgroup H of G. Although G/H is not group when H is not normal, but crucial part of the classical harmonic analysis on G carries over homogeneous spaces. It is well known $C_c(G/H)$ consists of all Pf functions, where f is a continuous function on G with compact support and

$$pf(xH) = \int_H f(x\xi)d\xi.$$

The theory of classical harmonic analysis on coset space G/H is investigated in [8, 9, 14]. We will now present collection of tools for analysis on homogeneous space that are essential in the theory of quasi-regular representations.

Suppose that μ is a Radon measure on G/H, for each $x \in G$ the transitive μ_x of μ is defined by $\mu_x(E) = \mu(xE)$ for each Borel subset E of G/H. The measure μ is said to be G-invariant if $\mu_x = \mu$ for all $x \in G$ and is said to be strongly quasi-invariant if there is a continuous function $\lambda : G \times G/H \to (0, \infty)$ which satisfies $d\mu_x(yH) = \lambda(x, yH)d\mu(yH)$, for $x, y \in G$. If the function $\lambda(x, .)$ reduces to a constant, μ is called relatively invariant under G. It is well known that any homogeneous space G/H possesses a strongly quasi invariant measure and all of them are constructed by a rho-function ρ. We recall that a rho-function for the pair (G, H) is defined to be a continuous function $\rho : G \to (0, \infty)$ which satisfies

$$\rho(xh) = \frac{\Delta_H(h)}{\Delta_G(h)}\rho(x) \quad (x \in G, h \in H), \tag{1}$$

where Δ_G, Δ_H are the modular functions on G and H, respectively (see [8, 11]). For any pair (G, H) and each rho-function ρ, there is a strongly quasi invariant measure μ in G/H such that $d\mu_x(yH) = \lambda(x, yH)d\mu(yH)$, for each $x, y \in G$, in which

$$\lambda(x, yH) = \frac{\rho(xy)}{\rho(y)}. \tag{2}$$

The homogeneous space G/H has a G- invariant measure if and only if the constant function $\rho(x) = 1, x \in G$, is rho-function for the pair (G, H), or equivalently $\Delta_{G|H} = \Delta_H$(cf. [8], Subsection 2.6).

The class of locally compact semidirect product groups, as a large class of non-Abelian groups, has significant roles in theories connecting mathematical physics and mathematical theory of coherent states analysis [2, 5, 14]. To introduce 3-fold semidirect product block shearlet group, we need to review some of the standard facts on semidirect product groups and summarize some results on them. Let H, K and

L be three locally compact groups with identity elements e_H, e_K and e_L, respectively. Moreover, assume $\tau : H \to Aut(K)$ and $\theta : H \ltimes_\tau K \to Aut(L)$ be continuous homomorphisms which, $(h,k) \to \tau_h(k)$ from $H \times K$ to K and $((h,k),l) \to \theta_{(h,k)}(l)$ from $H \times K \times L$ to L. The semidirect product $(H \ltimes_\tau K) \ltimes_\theta L$ is the locally compact topological group with the underlying set $(H \times K) \times L$ which is equipped by the product topology and the group operation is defined by

$$(h,k,l)(h',k',l') = (hh', k\tau_h(k'), l\theta_{(h,k)}(l')),$$
$$(h,k,l)^{-1} = (h^{-1}, \tau_{h^{-1}}(k^{-1}), \theta_{(h,k)^{-1}}(l^{-1})).$$

With the assumption as above, any locally compact group in the form of $(H \ltimes_\tau K) \ltimes_\theta L$ denoted by $\mathbb{S}_{(\theta,\tau)}$, is called an abstract shearlet group associated to homomorphisms θ and τ. In what follows, we use the abstract shearlet group with the abbreviation $\mathbb{S}$ for it. The left Haar measure of semidirect product group $\mathbb{S}$ is

$$d\mu(h,k,l) = \delta_\theta(h,k)\delta_\tau(h)d\mu_H(h)d\mu_K(k)d\mu_L(l),$$

in which, $d\mu_H(h)$, $d\mu_K(k)$ and $d\mu_L(l)$ are the left Haar measures of the locally compact groups H, K and L where the positive continuous homomorphisms $\delta_\tau : H \to (0,\infty)$ and $\delta_\theta : H \ltimes_\tau K \to (0,\infty)$ are given by

$$d\mu_K(k) = \delta_\tau(h)d\mu_K(\tau_h(k)), \quad d\mu_L(l) = \delta_\theta(h,k)d\mu_L(\theta_{(h,k)}(l)).$$

We refer the reader to [3] and the comprehensive list of references therein.

3. The Construction of Generalized Quasi-Regular Representations

Through this paper, let H and K be closed subgroups of the locally compact group G such that $K \subset H$. Then the left coset space

$$G/H = \{gH; g \in G\},$$

is a transitive $G-$ space and the left coset space

$$H/K = \{hK; h \in H\},$$

is a transitive $H-$ space. Due to transition spaces G/H and H/K, $G/H \times H/K$ is a transitive $G \times H$ -space. To achieve our result, we need the next lemma.

Lemma 3.1. *With notation as above, if G is a σ-compact group, then the spaces $(G \times H)/(H \times K)$ and $G/H \times H/K$ are homeomorphic.*

Proof. Let $H_0 = \{g \in G : gs = s\}$ and $K_0 = \{h \in H : ht = t\}$ where $s \in G/H$ and $t \in H/K$ are fixed. It is easy to check that

$$H_0 \times K_0 = \{(g,h) \in G \times H : (gs, ht) = (s,t)\}.$$

Define $\phi : G \times H \to G/H \times H/K$ by

$$\phi(g,h) = (gH, hK), \quad g \in G, h \in H.$$

Obviously ϕ is a continuous surjection of $G \times H$ onto $G/H \times H/K$ that is the multiplication of left cosets of H in G and K in H, respectively. Hence ϕ induces a continuous bijection

$$\Phi : (G \times H)/(H_0 \times K_0) \to G/H \times H/K,$$

such that $\Phi o q = \phi$, where $q : G \times H \to (G \times H)/(H_0 \times K_0)$ is the quotient map. Now we apply [8, Proposition 2.44] i.e. if G is a σ-compact, then Φ is a homeomorphism. If we choose a different base point $(s', t') = (g_0 s, h_0 t)$ the only effect is to replace $H_0 \times K_0$ with $H_0' \times K_0'$ where $H_0' = g_0 H_0 g_0^{-1}, K_0' = h_0 K_0 h_0^{-1}$ and the map $(g,h) \to (g_0 g g_0^{-1}, h_0 h h_0^{-1})$ induces a homeomorphism between $(G \times H)/(H_0 \times K_0)$ and $(G \times H)/(H_0' \times K_0')$. Hence we consider arbitrary closed subgroups H_0 and K_0 of locally compact group G and complete the proof as above. □

0Note that a transitive G-space is known as a homogeneous space [8].

Remark 3.2. In [8], it has been justified that if G is σ-compact, then

$$L^2(G/H)\otimes L^2(H/K)\cong L^2(G/H\times H/K)\cong L^2\Big((G\times H)/(H\times K)\Big).$$

Let functions $\rho_G : G \to (0,\infty)$ and $\rho_H : H \to (0,\infty)$ be rho-functions for pairs (G,H) and (H,K), respectively. In the next proposition we show that $(G\times H)/(H\times K)$ has a strongly quasi invariant Radon measure which arise from $\rho_G\rho_H$.

Proposition 3.3. *Given rho-functions ρ_H and ρ_K for the pairs (G,H) and (H,K) respectively, there is a strongly quasi invariant Radon measure μ on $(G\times H)/(H\times K)$ such that*

$$\frac{d\mu_{(g_1,h_1)}}{d\mu}\Big((g_2,h_2)(H\times K)\Big)=\frac{\rho_G(g_1g_2)}{\rho_G(g_2)}\frac{\rho_H(h_1h_2)}{\rho_H(h_2)}\quad (g_1,g_2\in G,h_1,h_2\in H).$$

Proof. Let ρ_G and ρ_H be the rho-functions defined in (1) for pairs (G,H) and (H,K), respectively. Then $\rho(g,h)=\rho_G(g)\rho_H(h)$ is a rho-function for $(G\times H)/(H\times K)$. It follows from this and (2) that $\lambda=\lambda_G\times\lambda_H$. Since G is a σ compact group

$$d\mu_{(g_1,h_1)}\Big((g_2,h_2)H\times K\Big)=\lambda((g_1,h_1),(g_2,h_2)H\times K)d\mu\Big((g_2,h_2)H\times K\Big),$$

which implies that μ is strongly quasi invariant measure. □

Using the action of a group G on itself, one can define a representation $\pi : G\to U(L^2(G))$, the left regular representation, as follows:

$$\pi(x)(f)=L_xf\quad (x\in G, f\in L^2(G)).$$

For a closed subgroup H of G, if G/H has a G- invariant Radon measure μ which arises from the constant function $\rho(x)=1$, $x\in G$, then there exists a representation $\pi : G\to U(L^2(G/H))$, the quasi-regular representation, such that

$$\pi(x)\varphi(yH)=\varphi(x^{-1}yH),$$

where $\varphi\in L^2(G/H)$(cf.[8, subsection 6.1]). In the case H is the trivial subgroup $\{e\}$, the quasi-regular representation coincides with the left regular representation. Now, our aim is to define a generalized representation Π of $G\times H$. Since forming tensor product is the most important tool to produce new representation, it is quite natural to ask how these two processes are related. Let $\Gamma : G\times H\to\mathbb{C}-\{0\}$ be defined by $\Gamma(g,h)=\Gamma(g)\Gamma(h)$, we will denote Γ of G and Γ of H by Γ_G and Γ_H, respectively. In the general case, via each $g\in G$ and $h\in H$, define

$$\begin{cases}\Pi(g,h): L^2(G/H)\otimes L^2(H/K)\longrightarrow L^2(G/H)\otimes L^2(H/K)\\ \Pi(g,h)\varphi\otimes\psi(xH,yK)=\Gamma(g,h)\varphi\otimes\psi(g^{-1}xH,h^{-1}yK),\end{cases}$$

for all $xH\in G/H$, $yK\in H/K$. To prove Π of $G\times H$ is a unitary representation, we show that $\Pi_G(g)=\Gamma_G(g)\varphi(g^{-1}xH)$ of G and $\Pi_H(h)=\Gamma_H(h)\psi(h^{-1}yK)$ of H are unitary representations. For each $\varphi\in C_c(G/H)$, take $f\in C_c^+(G)$ so that $|\varphi|^2=Pf$. Then

$$\begin{aligned}\|\Pi_G(g)\varphi\|_2^2 &= \int_{G/H}|\Pi_G(g)\,\varphi\,(xH)|^2\,d\mu(xH)\\ &= \int_{G/H}|\Gamma_G(g)|^2\,Pf(g^{-1}xH)\,d\mu(xH)\\ &= \int_G f(g^{-1}x)\,|\Gamma_G(g)|^2\,\rho_G(x)\,dx\\ &= \int_G f(x)\,|\Gamma_G(g)|^2\,\rho_G(gx)\,dx.\end{aligned}$$

Due to $\| \varphi \|_2^2 = \int_G f(x)\,\rho_G(x)\,dx$, $\Pi_G(g)$ will be unitary if and only if

$$\int_G f(x)\,(\,|\Gamma_G(g)|^2 \rho_G(gx) - \rho_G(x)\,)\,dx = 0,$$

for all $f \in C_c^+(G)$. This leads to $|\Gamma_G(g)|^2 \rho_G(gx) = \rho_G(x)$ for all $x \in G$. So $\Pi_G(g)$ will be unitary if and only if $\frac{\rho_G(x)}{\rho_G(gx)} = |\Gamma_G(g)|^2$ only depends on g. Similarly, $\Pi_H(h)$ will be unitary if and only if $\frac{\rho_H(y)}{\rho_H(hy)} = |\Gamma_H(h)|^2$ only depends on h. It follows that we can define such a unitary representation Π if and only if $\frac{\rho(x,y)}{\rho(gx,hy)} = |\Gamma(g,h)|^2$ only depends on (g,h). The argument above can be summarized as follows.

Theorem 3.4. *There exists such a representation Π of $G \times H$ if and only if measure μ of $(G \times H)/(H \times K)$ is relatively invariant. In other words, The existence of a homomorphism rho-function for the pair $(G \times H, H \times K)$ is a necessary and sufficient condition to have a representation Π with*

$$\Pi(g,h)\varphi \otimes \psi(xH, yK) = \Gamma(g,h)\varphi \otimes \psi(g^{-1}xH, h^{-1}yK), \tag{3}$$

for some constants $\Gamma(g,h) \in C$.

The representation $\Pi : G \times H \to L^2(G/H) \otimes L^2(H/K)$ which satisfies (3) is called a generalized quasi-regular representation of $G \times H$. More precisely, in [8] the generalized quasi-regular representations has been specified with a continuous homomorphism $\Gamma : G \times H \to (\mathbb{C} - \{0\}, \cdot)$ for which $|\Gamma(g,h)| = \sqrt{\frac{\rho(e,e)}{\rho(g,h)}}$. In fact, let μ be a relatively invariant measure on $(G \times H)/(H \times K)$ which arises from the rho-function ρ. Then the left regular representation of $G \times H$ is the continuous unitary representation Π of $G \times H$ with

$$[\Pi(g,h)\varphi \otimes \psi](xH, yK) = \sqrt{\frac{\rho(e,e)}{\rho(g,h)}}\varphi \otimes \psi(g^{-1}xH, h^{-1}yK), \tag{4}$$

for $xH \in G/H$, $yK \in H/K$ and $\varphi \in L^2(G/H)$, $\psi \in L^2(H/K)$. General theoretical results can be found in [9, 11, 15].

Let H be a closed subgroup of G and χ be a unitary representation of H. We mean by $Ind_H^G \chi$ the unitary representation of G which is induced by χ. We also recall that a character of group G is a continuous homomorphism from group G to the unit circle $\mathbb{T}$, see [8]. Through the next theorem we illustrate that a generalized quasi-regular representation of $G \times H$ is tensor product of a character of $G \times H$ and induced representations $Ind_H^G 1$ and $Ind_K^H 1$, where 1 is the trivial representation.

Theorem 3.5. *Let μ be a relatively invariant Radon measure on $(G \times H)/(H \times K)$ which arises from rho-function ρ. Then a generalized quasi-regular representation of $G \times H$ can be uniquely determined by a character of $G \times H$. More precisely, all generalized quasi-regular representations of $G \times H$ can be written in the form $\sigma \otimes (\Pi_G \times \Pi_H)$ which $\sigma : G \times H \to \mathbb{T}$ is a character of $G \times H$ and Π_G , Π_H are the induced representations $Ind_H^G 1$, $Ind_K^H 1$ respectively.*

Proof. Let $\Pi : G \times H \to U(L^2(G/H) \otimes L^2(H/K))$ be a generalized quasi-regular representation of $G \times H$. Base on [15, Theorem 3.2], there exists a continuous homomorphism $\Gamma : G \times H \to (\mathbb{C} - 0)$ such that for each $(g,h) \in G \times H$,

$$|\Gamma(g,h)| = \sqrt{\frac{\rho(e_G, e_H)}{\rho(g,h)}} = \sqrt{\frac{\rho_G(e_G)\rho_H(e_H)}{\rho_G(g)\rho_H(h)}} = |\Gamma_G(g)||\Gamma_H(h)|,$$

and

$$\Pi(g,h)(\varphi \otimes \psi)(x,y) = \Gamma(g,h)\varphi(g^{-1}xH)\psi(h^{-1}yK),$$

where $\varphi \in L^2(G/H)$ and $\psi \in L^2(H/K)$. We can write $\Gamma = \sigma|\Gamma|$ in which $\sigma : (G \times H) \to \mathbb{T}$ is a character of $G \times H$. If Π_G is the induced representation $Ind_H^G 1$ of G and Π_H is the induced representation $Ind_K^H 1$ of H, then

$$\Pi_G(g)\varphi(xH) = |\Gamma_G(g)|\varphi(g^{-1}xH),$$
$$\Pi_H(h)\psi(yK) = |\Gamma_H(h)|\psi(h^{-1}yK),$$

where $g \in G, h \in H, \varphi \in L^2(G/H)$ and $\psi \in L^2(H/K)$ [8, subsection 6.1] . Let $\Pi_\sigma : G \times H \to \mathbb{C} \otimes \mathbb{C}$ be the representation defined by σ, i.e. $\Pi_\sigma(g,h) = \sigma(g,h), \quad (g,h) \in G \times H$. We denote this representation by σ. Assume that unitary $Ins : L^2(G/H) \otimes L^2(H/K) \to (\mathbb{C} \otimes \mathbb{C}) \otimes \left(L^2(G/H) \otimes L^2(H/K)\right)$ is defined by $Ins(\varphi \otimes \psi) = \underbrace{1 \otimes 1}_{I} \otimes (\varphi \otimes \psi)$, where $\varphi \in L^2(G/H)$ and $\psi \in L^2(H/K)$, then we get

$$\begin{aligned}
&\left(Ins \circ \Pi(g,h)\right)(\varphi \otimes \psi)\left((\alpha_1 \otimes xH) \otimes (\alpha_2 \otimes yK)\right) \\
&= \left(I \otimes \Pi(g,h)(\varphi \otimes \psi)\right)\left((\alpha_1 \otimes xH) \otimes (\alpha_2 \otimes yK)\right) \\
&= (\alpha_1 \otimes \alpha_2) \otimes \Pi(g,h)(\varphi \otimes \psi)(xH \otimes yK) \\
&= (\alpha_1, \otimes \alpha_2) \otimes \Gamma(g,h)\varphi(g^{-1}xH)\psi(h^{-1}yK) \\
&= (\alpha_1 \otimes \alpha_2) \otimes \sigma(g,h)|\Gamma_G(g)||\Gamma_H(h)|\varphi(g^{-1}xH)\psi(h^{-1}yK) \\
&= \sigma(g,h)(\alpha_1 \otimes \alpha_2) \otimes \Pi_G(g)\varphi(xH)\Pi_H(h)\psi(yK) \\
&= \left(\sigma(g,h) \otimes (\Pi_G \times \Pi_H)(g,h)\right)\left(I \otimes (\varphi \otimes \psi)\right)(\alpha_1 \otimes \alpha_2) \otimes (xH, yK) \\
&= \left((\sigma \otimes (\Pi_G \times \Pi_H))(g,h) \circ Ins\right)(\varphi \otimes \psi)\left((\alpha_1 \otimes xH) \otimes (\alpha_2 \otimes yK)\right),
\end{aligned}$$

for all $\alpha_1, \alpha_2 \in \mathbb{C}$ and for $xH \in G/H, yK \in H/K$. Therefore $Ins \circ \Pi(g,h) = \sigma \otimes (\Pi_G \times \Pi_H)(g,h) \circ Ins$, for all $(g,h) \in G \times H$. Hence $\Pi \cong \sigma \otimes (\Pi_G \times \Pi_H)$. For the reverse direction, let σ be a character of $G \times H$. Then $\Gamma : G \times H \to \mathbb{C} - \{0\}$, is a continuous homomorphism, where $\Gamma(g,h) = \sigma(g,h)\sqrt{\frac{\rho(e_G, e_H)}{\rho(g,h)}}, (g,h) \in G \times H$. $\Pi : G \times H \to U(L^2(G/H) \otimes L^2(H/K))$ is a representation of $G \times H$, where

$$\Pi(g,h)(\varphi \otimes \psi)(xH, yK) = \sigma(g,h)\sqrt{\frac{\rho(e_G, e_H)}{\rho(g,h)}}\varphi(g^{-1}xH)\psi(h^{-1}yK).$$

Trivially,

$$|\Gamma(g,h)| = \sqrt{\frac{\rho(e_G, e_H)}{\rho(g,h)}},$$

for all $(g,h) \in G \times H$, and by the first part of the proof we have

$$\Pi \cong \sigma \otimes (\Pi_G \times \Pi_H).$$

□

4. Application and Examples

Through this section, we assume that H, K and L are locally compact groups with given left Haar measures μ_H, μ_K and μ_L, respectively, and $\tau : H \to Aut(K), \theta : H \ltimes_\tau K \to Aut(L)$ are continuous homomorphisms. Let $G_\tau = H \ltimes_\tau K$ be the semi-direct product of H and K with respect to τ. Let abstract shearlet group $\mathbb{S} = G_\tau \ltimes_\theta L$ be the semi-direct product of G_τ and L with respect to θ.

The functions $\rho_\tau : G_\tau \to (0,\infty)$ and $\rho_\theta : \mathbb{S} \to (0,\infty)$ given by

$$\begin{aligned} \rho_\tau(h,k) &= \frac{\Delta_H(h)}{\Delta_{G_\tau}(h)} = \delta_\tau^{-1}(h), \quad (h,k) \in G_\tau, \\ \rho_\theta(h,k,l) &= \frac{\Delta_{G_\tau}(h,k)}{\Delta_{\mathbb{S}}(h,k)} = \delta_\theta^{-1}(h,k), \quad (h,k,l) \in \mathbb{S}, \end{aligned} \tag{5}$$

are rho-functions for pairs $(\mathbb{S}, G_\tau)$ and (G_τ, H), respectively. The induced strongly quasi-invariant measure μ via the rho-function $\rho = \rho_\tau\rho_\theta$, is a relatively invariant measure on the homogeneous space $(\mathbb{S}\times G_\tau)/(G_\tau \times H)$. Furthermore, we can offer a continuous shearlet transform via the representation Π as follows:

$$\begin{cases} \Pi : \mathbb{S} \times G_\tau \to L^2(\mathbb{S}/G_\tau) \otimes L^2(G_\tau/H) \\ \Pi(s,g)\Big((\psi\otimes\varphi)(x,y)\Big) = \delta_\theta^{\frac{1}{2}}(h,k)\delta_\tau^{\frac{1}{2}}(\mathbf{h})\psi(s^{-1}xG_\tau))\varphi(g^{-1}yH), \end{cases}$$

for all $(s,g) \in (\mathbb{S} \times G_\tau)$ where $s = (h,k,l)$ and $g = (\mathbf{h},\mathbf{k})$. Since $L^2(\mathbb{S}/G_\tau)$ and $L^2(G_\tau/H)$ are isometrically isomorphic to $L^2(L)$ and $L^2(K)$, respectively, as well as $s^{-1}x = (h^{-1}h_1, \tau_{h^{-1}}(k^{-1}k_1), \theta_{(h,k)^{-1}}(l^{-1}l_1))$, $g^{-1}y = (\mathbf{h}^{-1}\mathbf{h}_1), \tau_{\mathbf{h}^{-1}}(\mathbf{k}^{-1}\mathbf{k}_1)$ for all $x = (h_1,k_1,l_1), y = (\mathbf{h}_1,\mathbf{k}_1)$, we can redefine the representation $\Pi : \mathbb{S}\times G_\tau \to U(L^2(L)\otimes L^2(K))$ by

$$\Pi(s,g)\Big((\psi\otimes\varphi)(l_1,\mathbf{k}_1)\Big) = \delta_\theta^{\frac{1}{2}}(h,k)\delta_\tau^{\frac{1}{2}}(\mathbf{h})\psi(\theta_{(h,k)^{-1}}(l^{-1}l_1))\varphi(\tau_{\mathbf{h}^{-1}}(\mathbf{k}^{-1}\mathbf{k}_1)),$$

where $l_1 \in L$ and $k_1 \in K$. The continuous shearlet transform which is obtained via this representation is the same as the shearlet transform that has been defined in [3, 6, 12].

Example 4.1. The most important example of abstract shearlet group is the shearlet group $\mathbb{S} = (\mathbb{R}^*\ltimes_\tau\mathbb{R})\ltimes_\theta\mathbb{R}^2$ which is introduced in [12]. Let homomorphism $\tau : \mathbb{R}^* \to Aut(\mathbb{R})$ and $\theta : (\mathbb{R}^*\ltimes_\tau\mathbb{R}) \to Aut(\mathbb{R}^2)$ be given by $\tau_a(b) = \sqrt{|a|}b$ and $\theta_{(a,b)}(t) = S_bA_at$, where

$$S_b = \begin{pmatrix} 1 & b \\ \circ & 1 \end{pmatrix}, \quad A_a = \begin{pmatrix} a & \circ \\ \circ & \sqrt{|a|} \end{pmatrix},$$

are respectively, shear and anisotropic dilation matrices for $(a,b) \in (\mathbb{R}^*\ltimes_\tau\mathbb{R})$, $t \in \mathbb{R}^2$. Note that $G_\tau = (\mathbb{R}^*\ltimes_\tau\mathbb{R})$ is known as the Affine group. By (5), the pairs $(\mathbb{S}, G_\tau)$ and (G_τ, H) admit a homomorphism rho-functions $\rho_\theta(a,b,t) = \delta_\theta^{-1}(a,b)$ and $\rho_\tau(\mathbf{a},\mathbf{b}) = \delta_\tau^{-1}(\mathbf{a})$, where $s = (a,b,t) \in \mathbb{S}$, $g = (\mathbf{a},\mathbf{b}) \in G_\tau$, respectively. Therefore we can offer a continuous transform via the representation $\Pi : \mathbb{S}\times G_\tau \to U(L^2(\mathbb{S}/G_\tau)\otimes L^2(G_\tau/H))$ defined by

$$[\Pi(s,g)(\varphi\otimes\psi)](xG_\tau, yH) = \delta_\theta^{\frac{1}{2}}(a,b)\delta_\tau^{\frac{1}{2}}(\mathbf{a})\varphi(s^{-1}xG_\tau)\psi(g^{-1}yH),$$

for $s = (a,b,t) \in \mathbb{S}$, $g = (\mathbf{a},\mathbf{b}) \in G_\tau$, $\varphi \in L^2(\mathbb{S}/G_\tau)$ and $\psi \in L^2(G_\tau/H)$. By [8, Proposition 2.21], $\delta_\theta(a,b) = |a|^{\frac{-3}{2}}$ and $\delta_\tau(\mathbf{a}) = |\mathbf{a}|^{\frac{-1}{2}}$. Now we can redefine the representation Π as follows:

$$\begin{aligned} &\Pi : \mathbb{S}\times G_\tau \to U(L^2(\mathbb{R}^2)\otimes L^2(\mathbb{R})) \\ [\Pi(s,g)(\varphi\otimes\psi)](t_1,\mathbf{b}_1) &= \delta_\theta^{\frac{1}{2}}(a,b)\delta_\tau^{\frac{1}{2}}(\mathbf{a})\varphi(\theta_{(a,b)^{-1}}(t_1-t))\psi(\tau_{\mathbf{a}^{-1}}(\mathbf{b_1}-\mathbf{b})) \\ &= |a|^{\frac{-3}{4}}|\mathbf{a}|^{-\frac{1}{4}}\varphi(A_a^{-1}S_b^{-1}(t_1-t))\psi(\frac{\mathbf{b_1}-\mathbf{b}}{\sqrt{|\mathbf{a}|}}), \end{aligned}$$

for almost all $t_1 \in \mathbb{R}^2$ and $\mathbf{b_1} \in \mathbb{R}$.

Note that in examples 4.1 and 4.2, The continuous shearlet transforms which are obtained via generalized quasi-regular representation are the same as the shearlet transform that have been defined in [6, 12]

Example 4.2. Higher dimensional shearlets are very useful tools for analyzing signals with higher dimensional domain, especially three dimensional domain such as seismic waves. We utilize the approach take in [6] for the locally compact group $\mathbb{S} = (\mathbb{R}^*\ltimes_\tau\mathbb{R}^{n-1})\ltimes_\theta\mathbb{R}^n$ in which, τ and θ are given by $\tau_a(b) = |a|^{1-\frac{1}{n}}b$ and $\theta_{(a,b)}(t) = S_bA_at$, where anisotoropic dilation matrix

$$A_a = \begin{pmatrix} a & \circ_{n-1} \\ \circ_{n-1}^T & sgn|a|^{\frac{1}{n}} I_{n-1} \end{pmatrix}, \quad a \in \mathbb{R}^*,$$

and the shear matrix

$$S_b = \begin{pmatrix} 1 & b \\ \circ_{n-1}^T & I_{n-1} \end{pmatrix}, \quad b \in \mathbb{R}^{n-1},$$

in which I_n denote the $n \times n$ identity and 0_n the vector with n entries. Since $\delta_\theta(a,b) = |detA_a|^{-1}$ and $\delta_\tau(\mathbf{a}) = |\mathbf{a}|^{(1-\frac{1}{n})(1-n)}$, by using a procedure as in the preceding example, one can compute the representation Π as

$$\Pi : \mathbb{S} \times G_\tau \to U(L^2(\mathbb{R}^n) \otimes L^2(\mathbb{R}^{n-1}))$$

$$[\Pi(s,g)(\varphi \otimes \psi)](t_1, \mathbf{b}_1) = \delta_\theta^{\frac{1}{2}}(a,b)\delta_\tau^{\frac{1}{2}}(\mathbf{a})\varphi(\theta_{(a,b)^{-1}}(t_1 - t))\psi(\tau_{\mathbf{a}^{-1}}(\mathbf{b_1} - \mathbf{b}))$$
$$= |a|^{\frac{1}{2n}-1}|\mathbf{a}|^{(1-\frac{1}{n})(\frac{1-n}{2})}\varphi(A_a^{-1}S_b^{-1}(t_1 - t))\psi(\frac{\mathbf{b_1} - \mathbf{b}}{\sqrt{|\mathbf{a}|}}),$$

for almost all $t_1 \in \mathbb{R}^n$, $\mathbf{b_1} \in \mathbb{R}^{n-1}$, where $\varphi \in L^2(\mathbb{R}^n)$ and $\psi \in L^2(\mathbb{R}^{n-1})$.

Example 4.3. The similitude group of plane defined by

$$Sim_2(\mathbb{R}) = \underbrace{(\mathbb{R}^+ \ltimes_\tau SO_2(\mathbb{R}))}_{G_\tau} \ltimes_\theta \mathbb{R}^2,$$

for any $s = (a, M, t) \in Sim_2(\mathbb{R})$ is a kind of abstract shearlet group. This group is the group theoretical framework of two dimensional continuous wavelet transform. By the positive continuous homomorphisms $\tau_a(M) = M$ and $\theta_{(a,M)} = aMt$, we can define a unitary representation $\Pi : Sim_2(\mathbb{R}) \times G_\tau \to U(L^2(\mathbb{R}^2) \otimes L^2(SO_2(\mathbb{R}))$ as follows:

$$\Pi(s,g)(\varphi \otimes \psi)(y,B) = \frac{1}{a}\varphi(\frac{1}{a}M^{-1}(y-t))\psi(BA^{-1}),$$

for $g = (b, A) \in G_\tau$, $B \in SO_2(\mathbb{R})$ and $y \in \mathbb{R}^2$.

Example 4.4. We consider the vector space

$$\mathbb{R}^4 \cong \mathbb{R}^3 \times \mathbb{R},$$

as the space of pairs (q,t) describing events in a four-dimensional space time. Here q stands for the spatial coordinate of the event and t for the time of the event. There are three types of symmetries of this space time:

1. Special Galilei transformations:

$$\mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}^3 \times \mathbb{R} \qquad (q,t) \to (q + vt, t), \quad v \in \mathbb{R}^3.$$

2. Rotations

$$\mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}^3 \times \mathbb{R} \qquad (q,t) \to (At, t), \quad A \in SO_3.$$

3. Space translation

$$\mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}^3 \times \mathbb{R} \quad (q,t) \to (q + v, t), \quad v \in \mathbb{R}^3.$$

and time translations

$$\mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}^3 \times \mathbb{R} \quad (q,t) \to (q, t + b), \quad b \in \mathbb{R}.$$

All these maps are affine maps on $\mathbb{R}^4$. The subgroup $\Gamma \subseteq Aff_4(\mathbb{R})$ generated by the maps in (1), (2) and (3) is called the Galilei group [10]. Roughly speaking $\Gamma$is isomorphic to the group

$$\mathcal{G} = \underbrace{(SO_3(\mathbb{R}) \ltimes_\tau \mathbb{R}^3)}_{G_\tau} \ltimes_\lambda \mathbb{R}^4,$$

where G_τ acts on $\mathbb{R}^4$ by $(A, v) \cdot (q, t) = (Aq + vt, t)$ and $SO_3(\mathbb{R})$ acts on $\mathbb{R}^3$ by $A \cdot v = Av$, these deduce that $\delta_\lambda = \delta_\tau = 1$. Furthermore, the quasi-regular representation $\Pi : \mathcal{G} \times G_\tau \to U(L^2(\mathbb{R}^4) \otimes L^2(\mathbb{R}^3))$ reduce to:

$$\Pi(A, v, (q, t))(\varphi \otimes \psi)((p, \delta), v) = \varphi(A^{-1}(p - q - (\delta - t)v)\psi(B^{-1}(\omega - v)).$$

In the next example, we offer group G with two closed subgroups H and L that none of them is normal in $G, G = HL$, and $H \cap L = \langle e \rangle$, we also consider $H = KM, K \cap M = \langle e \rangle$, that two closed subgroups K and M are not normal in H.

Example 4.5. Let N be an odd number with $N \geq 5$. Suppose that $G = B_N$, the alternating group of degree $N, B = B_{N-1}$, and $L = \langle \beta \rangle$ where $\beta \in B_N$ is the N-cycle $(1 \quad 2 \quad \cdots \quad N) \in B_N$. Then G is a simple group and none of the closed subgroups H and L is normal in G. Therefore, $G = HL$ and $H \cap L = \langle e \rangle$ (cf. [1]). So for all $\gamma \in G$, there exists a unique $\alpha \in H$ that $\gamma = \beta^t \alpha$, in which $i = \gamma(N)$. Furthermore,

$$\gamma H = \beta^{\gamma(N)} H \qquad (\gamma \in G). \tag{6}$$

Let $\Pi : G \to U(L^2(G/H))$ be the quasi-regular representation. Also, let $\psi \in L^2(G/H)$ and $g \in G$. Then there exists some $\alpha \in H$ and $1 \leq i \leq N$ for which $g = \beta^t \alpha$. By using (6), for all $1 \leq n \leq N$, we obtain

$$\begin{aligned}\Pi(g)\psi(\beta^n H) &= \psi(\alpha^{-1}\beta^{n-i}H) \\ &= \psi(\beta^{\alpha^{-1}\beta^{n-i}}H) \\ &= \begin{cases} \psi(\beta^{\alpha^{-1}(N+n-i)}H) & n \leq i, \\ \psi(\beta^{\alpha^{-1}(n-i)}H) & i < n. \end{cases}\end{aligned}$$

It is easy to see that $L^2(G/H)$ is isometrically isomorphic to $L^2(L)$ and hence we can rewrite the quasi-regular representation $\Pi : G \to U(L^2(L))$ as follows:

$$\Pi(\beta^i\alpha)f(\beta^n) = \begin{cases} f(\beta^{\alpha^{-1}(N+n-i)}) & n \leq i, \\ f(\beta^{\alpha^{-1}(n-i)}) & i < n, \end{cases}$$

for all $1 \leq i, i \leq N$ and $\alpha \in H$.

Now take $K = B_{N-2}$ the alternating group of degree $N - 2$, and $M = \langle \beta_0 \rangle$, where $\beta_0 \in B_{N-1}$ is the $(N-1)$-cycle $(1 \quad 2 \quad \cdots \quad N-1) \in B_{N-1}$. Then $H = KM$ is a simple group constructed of the closed subgroups K and M such that $K \cap M = \langle e \rangle$.

As it has been done before, the quasi-regular representation $\Pi : H \to U(L^2(H/K))$ can be defined by

$$\Pi(h)\varphi(\beta_0^n H) = \begin{cases} \varphi(\beta_0^{\alpha_0^{-1}(N-1+n-i)}K) & n \leq i, \\ \varphi(\beta_0^{\alpha_0^{-1}(n-i)}K) & i < n, \end{cases}$$

for all $1 \leq i, n \leq N - 1$ and $\alpha_0 \in K$. We redefine the representation Π of H as $\Pi : H \to U(L^2(M))$ by

$$\Pi(\beta_0^i\alpha_0)f(\beta_0^n) = \begin{cases} f(\beta_0^{\alpha_0^{-1}(N+n-i)}) & n \leq i, \\ f(\beta_0^{\alpha_0^{-1}(n-i)}) & i < n, \end{cases}$$

for all $1 \leq i, n \leq N - 1$. Now we can compute the generalized quasi-regular representation Π of $G \times H$ and define a continuous shearlet transform by using a square integrable sub representation of Π.

References

[1] A.L. Agore, A. Chirvasitu, B. Ion, G. Militaru, Factorization problems for finite groups, Algebras Represent. Theory 12 (2009) 481–488.
[2] A.A. Arefijamaal, R.A. Kamyabi-Gol, On the square integrability of quasi regular representation on semidirect product groups, J. Geom. Anal. 19 (2009) 541–552.
[3] V. Atayi, R. A. Kamyabi-Gol, Abstract shearlet transform, Bull. Belg. Math. Soc. Simon Stevin. 22 (2015) 669–681.
[4] T.A. Bubba, G. Kutyniok, M. Lassas, M. März, W. Samek, S. Siltanen, V. Srinivasan, Learning the invisible: A hybrid deep learning-shearlet framework for limited angle computed tomography, Inverse Probl., to appear (doi.org/10.1088/1361-6420/ab10ca).
[5] G. Chirikjian, A. Kyatkin, Engineering Applications of Non Commutative Harmonic Analysis with Emphasis on Rotation and Motion Groups, CRC Press, 2001.
[6] S. Dahlke, G. Steidl, G.Teschke, The continuous shearlet transform in arbitrary space dimensions, J. Fourier Anal. Appl. 16 (2010) 340–354.
[7] Y. Eldar, G. Kutyniok, Compressed Sensing: Theory and Applications, Cambridge University Press, 2012.
[8] G.B. Folland, A Course in Abstract Harmonic Analysis, CRC Press, 1995.
[9] E. Hewitt, K.A. Ross, Absrtact Harmonic Analysis I, Springer-Verlag, 1963.
[10] J. Hilgert, K. Neeb, Structure and Geometry of Lie group, Springr, 2012.
[11] E. Kaniuth, K. F. Taylor, Induced Representations of Locally Compact Groups, Cambridge University Press, 2013.
[12] G. Kutyniok, D. Labate, Shearlets, Multiscale Analysis for Multivariate Data, Birkhauser-Springer, 2012.
[13] J. Ma, M. März, S. Funk, J. Schulz-Menger, G. Kutyniok, T. Schaeffter, C. Kolbitsch, Shearlet-based compressed sensing for fast 3D cardiac MR imaging using iterative reweighting, Phys. Med. Biol. 63 (2018) 235004.
[14] H. Reiter, J.D. Stegeman, Classical Harmonic Analysis, 2nd edition, Oxford University Press, New York, 2000.
[15] N. Tavallaei, R.A. Kamyabi-Gol, Wavelet transforms via generalized quasi-regular representations, Comput. Harmon. Anal. 26 (2009) 291–300.