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Representation of Slim Lattice by Poset

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Abstract. In this paper, we present the necessary and sufficient conditions for a poset to be a poset of the union of join and meet irreducible elements of the slim lattice. Slim lattices are special finite lattices that are intensively investigated recently. The problem that we solved in this paper is a generalization of the problem proposed very recently by Czédli.

1. Introduction

The study of slim lattices began in 2007 with the work of Grätzer and Knapp [14], who first defined them. Shortly afterwards, Czédli and Schmidt [8] proposed a slightly different definition of these lattices. According to Czédli and Schmidt [8], a finite lattice L is *slim* if J_L, the set of join-irreducible elements of L, contains no three-element antichain. Equivalently, a finite lattice L is slim if and only if J_L is the union of two disjoint chains. Since then the theory of slim lattices, especially the theory of slim semimodular lattices, has been intensively developed by the works of Czédli [1–4], Grätzer and Knapp [15, 16], Schmidt [17], Czédli and Schmidt [9–11], Czédli, Osvárt, Udvari [6], Czédli and Grätzer [7], Czédli, Dékány, Oszvárt, Szakács, Udvari [5] and others.

Recently, Czédli [3] posed a problem of representation of slim lattices by posets (partially ordered sets) of their meet- irreducible elements. The problem of representability is solved for slim semimodular lattices [3]. Slim semimodular lattices are mutually determined by quasiplanar diagrams of their meet irreducibles. The solution proposed by Czedli [3] is not suitable for slim lattices in general, because slim lattices are not uniquely determined by poset of their meet-irreducible elements. In other words, there are more slim lattices with a same poset of meet- irreducibles (see Example 1). Nevertheless, there are other suitable posets of elements of the slim lattice with quasiplanar diagram, which we consider in this paper. Namely, it is proved here that a slim lattice is uniquely determined by the poset $J_L \cup M_L$ of the union of their join- and meet-irreducible elements.

In this paper, we investigate the necessary and sufficient conditions under which a given poset is isomorphic to a poset $J_L \cup M_L$ of some slim lattice L.

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2. Preliminaries

A *complete lattice* L is a partially ordered set (poset $(L, \leq)$) in which every subset has a least upper bound and a greatest lower bound. A complete lattice has the top and the bottom element which are usually denoted by 1 and 0 (or 1_L and 0_L to avoid confusion), respectively. A finite lattice L is *slim* if J_L, the set of join-irreducible elements of L, contains no three-element antichain.

With $x \parallel y$ we denote that lattice (poset) elements x and y are not comparable.

Let $(L, \leq)$ be a lattice. Then $C \subseteq L$ is a *chain* if any two elements of C are comparable. Subset A of the lattice L is an *antichain* if any two elements of A are incomparable. The *width* of L, denoted by $w(L)$, is the size (number of elements in finite case) of the largest antichain in L.

Theorem 2.1. *[12] [Dilworth's theorem]*
Let P be a poset of width k. Then P is a union of k disjoint chains.

The symbol $\downarrow a = \{x \in L \mid x \leq a\}$ denotes the *principal ideal* generated by an element a. The principal filter is the dual notion denoted by $\uparrow a$.

An injection $f : P \to Q$ is an *order-embedding* from the lattice $(P, \leq)$ into the lattice $(Q, \leq)$ if
$x \leq y$ if and only if $f(x) \leq f(y)$, for all $x, y \in P$.

A bijection which is order embedding is *isomorphism* of posets P and Q.

We say that an order-embedding which preserves all infima is a *meet-embedding* and dually, an order-embedding which preserves all suprema is a *join-embedding*. We say that $L_1 \subseteq L$ is a *complete meet-sublattice* of L if L and L_1 are complete lattices and all infima in L and L_1 coincide.

Meet-between lattices have also been introduced recently and applied in fuzzy set theory [13].

Let L be a lattice and $a, b, c \in L$. We say that b is *meet-between* ($\wedge$-between) a and c, and denote it by $abc_\wedge$ if:

$$(a \wedge b) \vee (b \wedge c) = b \text{ and } b \geq a \wedge c. \tag{1}$$

A lattice L is a *meet-between lattice* ($\wedge$-between lattice) if for all $a, b, c \in L$, if they are two by two incomparable elements, then they are in a relation $\wedge$-between.

It is proved in [13] that if L is a meet-between lattice, then $w(J_L) \leq 2$. Thus, meet-between lattices are a generalization of slim lattices.

A finitely spatial lattice is another notion recently introduced by Wehrung [18]. A lattice L is *finitely spatial* if every element of L is a join of join-irreducible elements of L. Dually, a lattice L is *dually finitely spatial* if every element of L is a meet of meet-irreducible elements of L. A lattice L is *finitely bi-spatial* if it is both, finitely spatial and dually finitely spatial.

We need the following results:

Lemma 2.2. *[13] A complete meet-sublattice L_1 of a complete meet-between lattice L is a complete meet-between lattice.*

Proposition 2.3. *[13] Let C_1, C_2 be complete chains. The lattice $C_1 \times C_2$ is a complete finitely bi-spatial meet-between lattice.*

Proposition 2.4. *[13] A complete lattice L is a finitely spatial meet-between planar lattice if and only if it can be meet-embedded into a direct product of two complete chains.*

3. Theorem of representation of slim lattices

Representability of a slim lattice L by the poset $J_L \cup M_L$, is based on the fact that slim lattices are meet-embeddable into a product of finite chains $C_1 \times C_2$, which will be proven in the sequel, and that each element of the lattice L is representable by the suprema of all join-irreducible elements of the lattice L below it, and by the infima of all meet-irreducible elements of the lattice L above it, since L is a finite lattice.

Theorem 3.1. *Let P be finite poset and let P_1 and P_2 be finite nonempty subposets of P such that $P = P_1 \cup P_2$ and $w(P_1) \leq 2$. Let C_1 and C_2 be the disjoint chains such that $P_1 = C_1 \cup C_2$ and let $C_1' = C_1 \cup \{0\}$, $C_2' = C_2 \cup \{0\}$ and $P' = P \cup \{0\}$ such that 0 is the bottom element of C_1' and C_2', where 0 is the bottom element of P if P has the bottom element. If P does not have the bottom element, then $0 \notin P$ is added as the bottom element in P'.*

Let the following conditions be satisfied:

1) For all $x \in P$

if x is the bottom element of P or if there exist $a, b \in P$ such that

$$(a \parallel b \ \text{ and } \ x = a \vee b), \tag{2}$$

then $x \in P_2 \setminus P_1$;

2) For all $x \in P$,

if x is the top element of P or if

$$x = \bigwedge\{a \mid a > x\}, \{a \mid a > x\} \neq \emptyset, \tag{3}$$

then $x \in P_1 \setminus P_2$;

3) For all $x \in P$, $x = p \vee q$, where:

$$p = \bigvee\{y \in C_1' \mid y \leq x\}, \ \ q = \bigvee\{z \in C_2' \mid z \leq x\}; \tag{4}$$

4) For all $x_3 \in P_2 \setminus P_1$, $x_1, x_2 \in P$, $x_i = p_i \vee q_i$, $i = 1, 2, 3$, where p_i and q_i are defined by (4). If $x_1 \parallel x_2$ and $x_3 < x_i$, $i = 1, 2$ then at most one is valid from the following:

$$p_1 = p_3, \ \ q_1 = q_3, \ \ p_2 = p_3, \ \ q_2 = q_3.$$

Then there exists a slim lattice L, such that poset $J_L \cup M_L$ is isomorphic with poset P. Moreover, poset J_L is isomorphic with subposet P_1 and poset M_L is isomorphic with subposet P_2.

Remark 3.2. *If $x = p_1 \vee q_1$ and ($x \in C_1'$ or $x \in C_2'$), then $x = x \vee q_1$ or $x = p_1 \vee x$, respectively, where p_1, q_1 are determined by equalities (4) in condition 3) of this theorem. If $x \in P_2 \setminus P_1$ and $x = p_1 \vee q_1$, then $x > p_1$, $x > q_1$ and $p_1 \parallel q_1$. From conditions 1) and 2) it follows that the top element in P, if it exists, is not a supremum of two incomparable elements and that the bottom element in P is not the infimum of incomparable elements.*

Proof. Let $P = P_1 \cup P_2$ be a finite poset which fulfills the conditions of the theorem and let $D = C_1' \times C_2'$.

Mapping $f : P' \to D$ is defined in the following way:

$$f(x) = (p_1, q_1), \tag{5}$$

for all $x \in P'$, $x = p_1 \vee q_1$, where p_1, q_1 are determined by equalities (4) in the condition 3) of this theorem. For every element $x \in P'$ the elements defined with (4) exist and are determined uniquely, because C_i', $i = 1, 2$ are finite chains. According to the conditions the mapping has been well defined, and $x \leq y$ is valid if and only if $f(x) \leq f(y)$, for each $x, y \in P'$. Hence the mapping is also injective.

Further on, let $f(P') = \{f(x) \mid x \in P'\}$ and

$$L = \{\wedge_D S \mid \text{ for all } S \subseteq f(P')\}. \tag{6}$$

Let us prove that poset L, defined in this way, is a meet-sublattice of the lattice D having the same top elements. Let us first notice that $f(P') \subseteq L$, and for $x = 0$, $p_1 = q_1 = 0$. If P' has the top element 1, then $f(1)$ is the top element of D. If P' does not have the top element, then $\bigwedge_D \emptyset = 1_D \in L$, according to the definition of L. Thus, the bottom and the top elements of the poset L coincide, respectively, with the bottom and the top elements of the lattice D.

According to the construction, L is closed under infima, and hence it is a complete lattice. Mapping f preserves order on the poset $f(P') \subseteq L$, and according to the construction, the infima of the elements of the lattice L coincide with infima of the same elements in the lattice D. Thus, L is a complete meet-sublattice of the lattice D, and according to Lemma 2.2 and Proposition 2.3, it can be concluded that L is a meet-between lattice and $w(J_L) \leq 2$. Since L is a finite lattice, it follows that L is a slim lattice.

Before we prove that the mentioned isomorphisms are valid, we notice that since the lattice L is finite, each element $(a, b) \in L$ that is not join-irreducible can uniquely be represented as suprema of two particular join-irreducible elements.

Namely, an arbitrary element $(a, b) \in L$ is supremum of the following elements of the lattice L:

$$\wedge_{(a,y)\in L}(a, y) = (a, \wedge_{(a,y)\in L}y), \tag{7}$$

whose first coordinate is a and an element of the lattice L whose second coordinate is b:

$$\wedge_{(x,b)\in L}(x, b) = (\wedge_{(x,b)\in L}x, b). \tag{8}$$

In case one of these elements is equal to (a, b) this element is join-irreducible. In case both elements are equal to (a, b), this element is the bottom element of the lattice L. Now, we suppose that (a, b) is different from any of these elements.

We want to prove that $f(a) = \wedge_{(a,y)\in L}(a, y)$, $f(b) = \wedge_{(x,b)\in L}(x, b)$ and that these elements are join- irreducible.

Since $(a, b) \in L \subseteq D$, then $a \in C_1'$, $b \in C_2'$ and $f(a), f(b) \in L$. Further, we consider $f(a)$, the proof is analogous for $f(b)$. Let $f(a) = (a, z)$. We claim that $z = \wedge_{(a,y)\in L}y$, meaning that in lattice L there is no element smaller than (a, z), and whose first coordinate is a. It is clear that there is no $a' \in P'$ where $f(a') = (a, t) < f(a) = (a, z)$. Indeed, from this assumption it can be concluded that $a' <_{P'} a$, but also that $a' = a \vee_{P'} t \geq a$, which is a contradiction. Further on, we want to prove that for each $S \subseteq f(P')$, we have that $\wedge_D S = (a, t) \geq (a, z)$. Here we prove that for each element $(x, y) \in S$ we have that $x \geq a$ and $y \geq z$. Since $\wedge_D S = (a, t)$, it is clear that $x \geq a$ for each element $(x, y) \in S$.

Since $S \subseteq f(P')$, there is a $p \in P'$ such that $f(p) = (x, y)$ and $p = x \vee y$. Hence it is possible to conclude that $p \geq x \geq a$. Thus $f(p) \geq f(a)$, i.e. $(x, y) \geq (a, z)$. It follows that $y \geq z$.

Thus, it is proved that there are no elements smaller than $(a, z) = f(a)$ in the lattice L whose first coordinate is a. Moreover, if there is an element $(a, b) \in L$, then there are also $a \in C_1'$, $b \in C_2'$ such that $f(a) \vee f(b) = (a, b)$.

Now, we want to prove that $f(a)$ is a join- irreducible element for all $a \in P_1$. Let $f(a) = (a, z)$. Suppose that there are $(u, q), (p, v) \in L$ such that $(u, q) \vee (p, v) = (a, z) = f(a)$ and $(u, q) \parallel (p, v)$. Thus $u, p \in C_1'$, $q, v \in C_2'$, ($u > p$ and $q < v$) or ($u < p$ and $q > v$). According to previously proven, $f(u) \vee f(q) = (u, q)$ and $f(p) \vee f(v) = (p, v)$. Thus $f(a) = f(u) \vee f(q) \vee f(p) \vee f(v)$. Without loss of generality, we suppose that $u > p$ and $q < v$. It follows that $f(u) > f(p)$, $f(q) < f(v)$ and $f(a) = f(u) \vee f(v)$. If $u > v$, then $f(u) > f(v)$ and $f(a) = f(u)$. Thus $f(a) = (u, q)$. Similarly, if $u < v$, then $f(a) = (p, v)$. If $u \parallel v$, then $a = u \vee v$. Indeed, from $f(a) = f(u) \vee f(v)$ it follows that $a \geq u \vee v$. Thus $f(a) \geq f(u \vee v) \geq f(u) \vee f(v) = f(a)$. It follows that $f(a) = f(u \vee v)$. Thus $a = u \vee v$ and $a \in P_2 \setminus P_1$, by condition 1) of the theorem, which leads into contradiction with the assumption $a \in P_1$.

Thence it follows that all the elements $f(P_1)$ are join- irreducible in the lattice L. On basis the construction of the lattice L and the above conclusions, it is clear that they are the only join- irreducible elements of the lattice L. Thereby it is proved that $f(P_1) = J_L$.

Finally, we claim that the elements $f(P_2)$ are exactly meet-irreducible elements of the lattice L.

Suppose that an element $(a, b) = f(t)$ with $t \in P_2$ is not meet-irreducible, i.e., that $(a, b) = \bigwedge(\{(x, y) \in L \mid (a, b) < (x, y)\})$. Since each element in L is either from $f(P')$ or a meet of some elements from $f(P')$, we have that $(a, b) = \bigwedge(\{(x, y) \in f(P') \mid (a, b) < (x, y)\})$. By the properties of the mapping f, we have that $t = \bigwedge(\{y \mid t < y\})$. Thus $t \in P_1 \setminus P_2$, by condition 2) of this theorem, which is a contradiction to the assumption $t \in P_2$. So, it is proved that each element $f(t)$ for $t \in P_2$ is meet-irreducible. To prove the converse, suppose that (a, b) is a meet-irreducible element of L. Since, each element from L is either a meet of elements from $f(P')$ and hence not meet-irreducible, or it is an element of $f(P')$. If this element is from $f(P')$ then $(a, b) = f(t)$ and t can not be element from $P_1 \setminus P_2$ (by 2), condition (3)), so we have that $t \in P_2$. □

Theorem 3.3. *Let L be a finite slim lattice with the bottom element 0 and the top element 1. Let C_1 and C_2 be disjoint chains such that $J_L = C_1 \cup C_2$ and let $C_1' = C_1 \cup \{0\}$ and $C_2' = C_2 \cup \{0\}$. Let $J_L' = J_L \cup \{0\}$ and let $P' = J_L' \cup M_L$. Then the following properties are satisfied:*

1) For all $x \in P'$, $x = p_1 \vee_{P'} q_1$, where:

$$p_1 = \bigvee \{p \in C_1' \mid p \leq x\}, \quad q_1 = \bigvee \{q \in C_2' \mid q \leq x\}.$$

2) For all $x \in P'$, $x \neq 0$, $x \in M_L \setminus J_L$, if and only if there exist $a, b \in P'$ such that

$$a \parallel b \ \text{ and } \ x = a \vee_{P'} b.$$

3) For all $x \in P'$, $x \neq 1$, $x \in J_L \setminus M_L$ if and only if

$$x = \wedge_{P'} \{a \mid a > x\}.$$

4) For all $x_3 \in M_L \setminus J_L$, $x_1, x_2 \in P'$, $x_i = p_i \vee q_i$, $i = 1, 2, 3$, if $x_1 \parallel x_2$ and $x_3 < x_i$, $i = 1, 2$ then at most one is valid from the following:

$$p_1 = p_3, \quad q_1 = q_3, \quad p_2 = p_3, \quad q_2 = q_3.$$

Proof. Let L be a slim lattice and $P = J_L \cup M_L$. According to the definition of the slim lattice, $w(J_L) \leq 2$. Let $J_L = C_1 \cup C_2$, where C_1 and C_2 are disjoint chains. Further, let $C_1' = C_1 \cup \{0\}$ and $C_2' = C_2 \cup \{0\}$, $J_L' = J_L \cup \{0\}$ and $P' = J_L' \cup M_L$.

Since every element in a finite lattice is a join of all join-irreducible elements below it, we have that for all $x \in L$, $x = (\vee_L(\downarrow x \cap C_1')) \vee_L (\vee_L(\downarrow x \cap C_2'))$.

Let $p_1 = \vee(\downarrow x \cap C_1')$ and $q_1 = \vee(\downarrow x \cap C_2')$.

1) From $x = p_1 \vee_L q_1$, $x \in P'$ it follows that $x = p_1 \vee_{P'} q_1$, since all three elements x, p_1 and p_2 belong to P' and the order in P' is inherited from L.

2) Let $x \neq 0$ and $x = p_1 \vee_L q_1$. Then $x \neq p_1$, $x \neq q_1$, $p_1 \parallel q_1$ if and only if $x \in M_L \setminus J_L$, thence the proof of property 2) follows straightforwardly.

3) Suppose that $x \in J_L \setminus M_L$ and $x \neq 1$. This means that x is not a meet-irreducible element in L. Hence, $x = \wedge_L\{a \in M_L \mid a > x\}$. Since x is not the top element 1, set $\{a \mid a > x\}$ is not empty.

Now, $x = \wedge_L\{a \in M_L \mid a > x\}$ is equivalent to $x = \wedge_{P'}\{a \in M_L \mid a > x\}$ since $x \in P'$ and $M_L \subseteq P'$.

Let us prove that the property 4) is valid. Let $x_3 \in M_L \setminus J_L$, $x_1, x_2 \in J_L \cup M_L \cup \{0_L\}$ be arbitrary different elements such that $x_i = p_i \vee q_i$ (as above), $p_i \in C_1'$, $q_i \in C_2'$ $(i = 1, 2, 3)$, $x_1 \parallel x_2$ and $x_3 < x_i$, $i = 1, 2$. We are going to prove that at most one of the following equalities is valid: $p_1 = p_3$, $q_1 = q_3$, $p_2 = p_3$, $q_2 = q_3$.

Without loss of generality, we assume that $p_2 = p_3$. Then $q_2 > q_3$, otherwise x_2 and x_3 would be the same. Bearing in mind that $q_i \in C_2'$ $(i = 1, 2, 3)$, if $p_1 = p_3$ then the elements x_1, x_2 would be comparable, which is in contradiction with the assumption.

Thus, $p_1 > p_3$. In case $q_1 = q_3$, we would have that $x_1 \wedge x_2 = x_3$, which would be contradiction with the assumption that x_3 is meet-irreducible. □

Theorems 3.1 and 3.3 provide necessary and sufficient conditions for a poset P to be isomorphic with a poset $J_L \cup M_L$ of a slim lattice L. Moreover, as a direct consequence of these theorems and proposition 2.4, we have the following theorem.

Theorem 3.4. *Let L be a finite lattice. Then the following properties of the lattice L are equivalent.*

1.) L is a meet- between lattice.

2.) L is a slim lattice.

3.) L can be meet-embedded into a direct product of two finite chains.

Proof. Properties 1) and 3) are equivalent by Proposition 2.4, and from 1) it follows 2). According to Theorems 3.1 and 3.3, from 2) follows 3), which concludes the proof. □

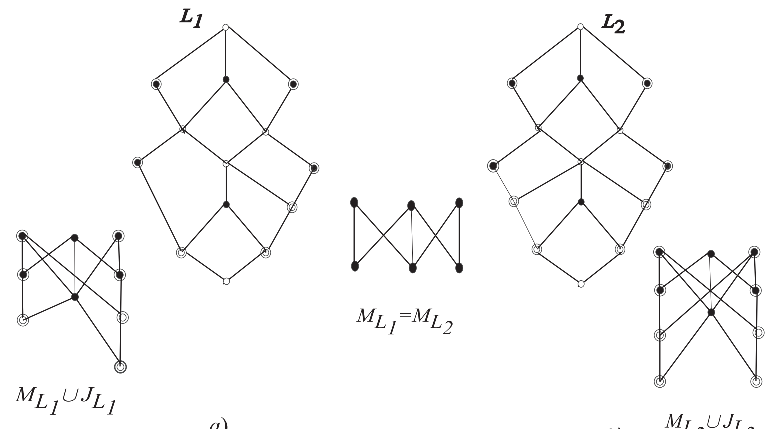

Figure 1: Lattices L_1, L_2 and the corresponding posets $M_{L_1} \cup J_{L_1}$, $M_{L_2} \cup J_{L_2}$ and $M_{L_1} = M_{L_2}$.

Example 3.5. *Posets of meet irreducible elements of the slim lattice L_1 and the slim semimodular lattice L_2 are same, but their posets $M_{L_1} \cup J_{L_1}$ (Fig. 1a)) and $M_{L_2} \cup J_{L_2}$ (Fig. 1b)) are different. In Figure 1, elements of poset $M_{L_i} \setminus J_{L_i}$ are marked by black filled circles, elements of poset $J_{L_i} \setminus M_{L_i}$ are marked by double circles, and elements of $J_{L_i} \cap M_{L_i}$ ($i = 1, 2$) are marked by double circles with the black filled center.*

4. Conclusion

In this paper we give a representation of a finite slim lattice by a poset of its meet and join irreducibles. A representation of a finite distributive lattices by the set of its join irreducible elements (or meet-irreducible elements) is well known. A recent result by Czédli gives a representation of slim semimodular lattices by a quasiplanar diagram of their meet irreducible elements.

Further investigation is planned in two directions. The first one is a generalization of obtained results to infinite spatial (or dually spatial) meet-between lattices. Another line of investigation can be further development of representation theorems by poset which is a union of meet and join-irreducible elements for other types of lattices.

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