Filomat 35:3 (2021), 911–917
https://doi.org/10.2298/FIL2103911H

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Quasicontinuous Functions and the Topology of Uniform Convergence on Compacta

Ľubica Holá[a], Dušan Holý[b]

[a]*Academy of Sciences, Institute of Mathematics Štefánikova 49, 81473 Bratislava, Slovakia*
[b]*Department of Mathematics and Computer Science, Faculty of Education, Trnava University, Priemyselná 4, 918 43 Trnava, Slovakia*

Abstract. Let X be a Hausdorff topological space, $Q(X,\mathbb{R})$ be the space of all quasicontinuous functions on X with values in $\mathbb{R}$ and τ_{UC} be the topology of uniform convergence on compacta. If X is hemicompact, then $(Q(X,\mathbb{R}),\tau_{UC})$ is metrizable and thus many cardinal invariants, including weight, density and cellularity coincide on $(Q(X,\mathbb{R}),\tau_{UC})$. We find further conditions on X under which these cardinal invariants coincide on $(Q(X,\mathbb{R}),\tau_{UC})$ as well as characterizations of some cardinal invariants of $(Q(X,\mathbb{R}),\tau_{UC})$. It is known that the weight of continuous functions $(C(\mathbb{R},\mathbb{R}),\tau_{UC})$ is $\aleph_0$. We will show that the weight of $(Q(\mathbb{R},\mathbb{R}),\tau_{UC})$ is $2^{\mathfrak{c}}$.

1. Introduction

Quasicontinuous functions were introduced by Kempisty in 1932 in [18]. However as far as we know the first mention of the condition of quasicontinuity can be found in the paper of R. Baire [1] in the study of continuity points of separately continuous functions from $\mathbb{R}^2$ into $\mathbb{R}$. Quasicontinuous functions were studied in many papers, see for example [3], [8, 13, 14], [19], [24] and others. They are important in many areas of mathematics. Quasicontinuous functions are selections of minimal usco and minimal cusco maps [7, 9–12]. They found applications in the study of topological groups [4, 21, 23], in the study of dynamical systems [5], in the study of extensions of densely defined continuous functions [17], etc. The quasicontinuity is also used in the study of CHART groups [22].

In our paper we will study cardinal invariants of the space of quasicontinuous functions equipped with the topology of uniform convergence on compacta. Notice that the properties of the first countability, metrizability and complete metrizability of this space were studied in [15] and Arzela-Ascoli type theorems for the space of quasicontinuous functions were proved in [13, 14].

Let X be a Hausdorff topological space, $Q(X,\mathbb{R})$ be the space of all quasicontinuous functions on X with values in $\mathbb{R}$ and τ_{UC} be the topology of uniform convergence on compacta. We will study cardinal invariants of $(Q(X,\mathbb{R}),\tau_{UC})$. $Q(X,\mathbb{R})$ is not a subgroup of $\mathbb{R}^X$, which makes the work with the space $Q(X,\mathbb{R})$ more difficult. However $Q(X,\mathbb{R})$ shares some properties with topological groups, for example the coincidence of a character and π-character, or weight and π-weight, etc. If X is hemicompact, then by [15] $(Q(X,\mathbb{R}),\tau_{UC})$ is metrizable, thus all cardinal invariants $c, d, nw, s, e, L, \pi w, w$ coincide on $(Q(X,\mathbb{R}),\tau_{UC})$. We find further

2010 *Mathematics Subject Classification*. Primary 54C35; Secondary 54C08, 54C30
Keywords. Quasicontinuous functions, topology of uniform convergence on compacta, density, weight, netweight, cellularity
Received: 11 March 2020; Revised: 02 June 2020; Accepted: 06 June 2020
Communicated by Ljubiša D.R. Kočinac
Research was supported by the grant Vega 2/0016/20
Email addresses: hola@mat.savba.sk (Ľubica Holá), dusan.holy@truni.sk (Dušan Holý)

conditions on X under which these cardinal invariants coincide on $(Q(X,\mathbb{R}), \tau_{UC})$ as well as characterizations of some cardinal invariants of $(Q(X,\mathbb{R}), \tau_{UC})$. We also show that the cardinality of $Q(\mathbb{R},\mathbb{R})$ is $2^{\mathfrak{c}}$ and that the weight of $(Q(\mathbb{R},\mathbb{R}), \tau_{UC})$ is $2^{\mathfrak{c}}$.

2. Preliminaries

Let X and Y be topological spaces. A function $f : X \to Y$ is quasicontinuous [24] at $x \in X$ if for every open set $V \subset Y$, $f(x) \in V$ and every open set $U \subset X$, $x \in U$ there is a nonempty open set $W \subset U$ such that $f(W) \subset V$. If f is quasicontinuous at every point of X, we say that f is quasicontinuous.

We say that a subset of X is quasi-open (or semi-open) [24] if it is contained in the closure of its interior. Then a function $f : X \to Y$ is quasicontinuous if and only if $f^{-1}(V)$ is quasi-open for every open set $V \subset Y$.

We denote by $\mathbb{N}$ the set of positive integers and by $\mathbb{R}$ the space of real numbers with the usual metric. The symbol $\overline{A}$ will stand for the closure of the set A in a topological space.

Let X be a Hausdorff topological space. Denote by $F(X,\mathbb{R})$ the set of all functions from X to $\mathbb{R}$, by $Q(X,\mathbb{R})$ the set of all quasicontinuous functions in $F(X,\mathbb{R})$ and by $C(X,\mathbb{R})$ the space of all continuous functions in $F(X,\mathbb{R})$.

By $K(X)$ we mean the family of all nonempty compact subsets of X.

Denote by τ_{UC} the topology of uniform convergence on compact sets on $F(X,\mathbb{R})$. This topology is induced by the uniformity $\mathfrak{U}_{UC}$ which has a base consisting of sets of the form

$$W(K,\varepsilon) = \{(f,g) : \ \forall\, x \in K \ \ |f(x) - g(x)| < \varepsilon\},$$

where $K \in K(X)$ and $\varepsilon > 0$. The general τ_{UC}-basic neighborhood of $f \in F(X,\mathbb{R})$ will be denoted by $W(f,K,\varepsilon)$, where

$$W(f,K,\varepsilon) = \{g : \ \forall\, x \in K \ \ |f(x) - g(x)| < \varepsilon\}.$$

Denote by τ_p the topology of pointwise convergence on $F(X,\mathbb{R})$. This topology is induced by the uniformity $\mathfrak{U}_p$ which has a base consisting of sets of the form

$$W(A,\varepsilon) = \{(f,g) : \ \forall\, x \in A \ \ |f(x) - g(x)| < \varepsilon\},$$

where A is a finite set and $\varepsilon > 0$. The general τ_p-basic neighborhood of $f \in F(X,\mathbb{R})$ will be denoted by $W(f,A,\varepsilon)$, where

$$W(f,A,\varepsilon) = \{g : \ \forall\, x \in A \ \ |f(x) - g(x)| < \varepsilon\}.$$

Of course $F(X,\mathbb{R}) = \mathbb{R}^X$ and the topology τ_p of the pointwise convergence on $F(X,\mathbb{R})$ is just the product topology on $\mathbb{R}^X$.

3. Cardinal Invariants of $(Q(X,\mathbb{R}), \tau_{UC})$

First we will remind definitions of cardinal invariants of a topological space Z [6]. We define the weight of Z:

$$w(Z) = \aleph_0 + \min\{|\mathcal{B}| : \ \mathcal{B} \text{ is a base in } Z\},$$

the density of Z:

$$d(Z) = \aleph_0 + \min\{|D| : \ D \text{ is a dense set in } Z\},$$

the cellularity of Z:

$$c(Z) = \aleph_0 + \sup\{|\mathcal{U}| : \ \mathcal{U} \text{ is a pairwise disjoint family of nonempty open sets in } Z\},$$

the network weight of Z:

$$nw(Z) = \aleph_0 + \min\{|\mathcal{N}| : \ \mathcal{N} \text{ is a network in } Z\}.$$

They are in general related by the inequalities

$$c(Z) \leq d(Z) \leq nw(Z) \leq w(Z).$$

When Z is metrizable

$$c(Z) = d(Z) = nw(Z) = w(Z).$$

The character of a point z in Z is defined as:

$$\chi(Z, z) = \aleph_0 + \min\{|\mathcal{O}| : \mathcal{O} \text{ is a base at } z\},$$

and the character of Z is defined as:

$$\chi(Z) = \sup\{\chi(Z, z) : z \in Z\}.$$

To define the π-character of Z, we first need a notion of a local π-base. If $z \in Z$, a collection $\mathcal{V}$ of nonempty open subsets of Z is called a local π-base at z provided that for each open neighborhood U of z, there exists a $V \in \mathcal{V}$ which is contained in U.

The π-character of a point z in Z is defined as:

$$\pi_\chi(Z, z) = \min\{|\mathcal{V}| : \mathcal{V} \text{ is a local } \pi\text{-base at } z\},$$

and the π-character of Z is defined as:

$$\pi_\chi(Z) = \aleph_0 + \sup\{\pi_\chi(Z, z) : z \in Z\}.$$

The k-cofinality of a topological space Z is defined to be

$$kcof(Z) = \min\{|\beta| : \beta \text{ is a cofinal family in } K(Z)\}.$$

A topological space Z is hemicompact if $kcof(Z) = \aleph_0$; i.e. there is a countable family $\{K_n : n \in \mathbb{N}\}$ in $K(Z)$ such that for every $K \in K(Z)$ there is $n \in \mathbb{N}$ with $K \subset K_n$.

In what follows let X be a Hausdorff nontrivial topological space, i.e., X is at least countable. We will consider the cardinal invariants of the space $(Q(X, \mathbb{R}), \tau_{UC})$. Because of simplicity we will omit the specification of the topology τ_{UC}.

First we prove that the character and the π-character of $Q(X, \mathbb{R})$ coincide.

Theorem 3.1. *Let X be a topological space. Then $\chi(Q(X, \mathbb{R})) = \pi_\chi(Q(X, \mathbb{R})) = kcof(X)$.*

Proof. First we prove that $kcof(X) \leq \pi_\chi(Q(X, \mathbb{R}))$. Let f be the zero function on X. Then f is a quasicontinuous function. Let $\{W(f_t, A_t, \varepsilon_t) : A_t \in K(X), \varepsilon_t > 0, t \in T\}$ be a local π-base of f in $Q(X, \mathbb{R})$ with $|T| \leq \pi_\chi(Q(X, \mathbb{R}))$.

We claim that $\{A_t : t \in T\}$ is a cofinal family in $K(X)$. Let $A \in K(X)$. There must exist $t \in T$ with

$$W(f_t, A_t, \varepsilon_t) \subset W(f, A, 1).$$

We show that $A \subset A_t$. Suppose there is $a \in A \setminus A_t$. Let U be an open set such that $a \in U$ and $\overline{U} \cap A_t = \emptyset$.

Let $g : X \to \mathbb{R}$ be defined as follows:

$$g(z) = \begin{cases} 1, & z \in \overline{U}; \\ f_t(z), & \text{otherwise.} \end{cases}$$

Then g is a quasicontinuous function and $g(s) = f_t(s)$ for every $s \notin \overline{U}$; thus also for every $s \in A_t$. Then $g \in W(f_t, A_t, \varepsilon_t)$, but $g \notin W(f, A, 1)$, a contradiction. Thus

$$kcof(X) \leq \pi_\chi(Q(X, \mathbb{R})) \leq \chi(Q(X, \mathbb{R}).$$

To prove that $\chi(Q(X, \mathbb{R}) \leq kcof(X)$, let $f \in Q(X, \mathbb{R})$ and let β be a cofinal subfamily of $K(X)$ with $|\beta| = kcof(X)$. It is easy to verify that the family $\{W(f, K, 1/n) : K \in \beta, n \in \mathbb{N}\}$ is a local base at f. $\square$

Corollary 3.2. ([15]) *Let X be a topological space. The following are equivalent:*

1. *X is hemicompact,*
2. *$Q(X, \mathbb{R})$ is first countable.*

For a Tychonoff space Z we define the uniform weight of Z [6]:

$$u(Z) = \aleph_0 + \min\{m : \text{there is a uniformity on } Z \text{ of weight } \leq m\}.$$

It is known (see [6]) that $w(Z) = c(Z) \cdot u(Z)$, $w(Z) = e(Z) \cdot u(Z)$, where $e(Z)$ is extent of Z defined as follows:

$$e(Z) = \aleph_0 + \sup\{|E| : E \text{ is a closed discrete set in } Z\}.$$

Theorem 3.3. *Let X be a topological space. Then $u(Q(X, \mathbb{R})) = kcof(X)$.*

Proof. Let β be a cofinal family in $K(X)$ such that $kcof(X) = |\beta|$. It is easy to verify that the family $\{W(K, 1/n) : K \in \beta, n \in \mathbb{N}\}$ is a base of the uniformity $\mathfrak{U}_{UC}$. Thus $u(Q(X, \mathbb{R})) \leq kcof(X)$. For every Tychonoff space Z we have $\chi(Z) \leq u(Z)$. Since by Theorem 3.1 $kcof(X) = \chi(Q(X, \mathbb{R}))$, we have $u(Q(X, \mathbb{R})) = kcof(X)$. □

Corollary 3.4. ([15]) *Let X be a topological space. The following are equivalent.*

1. *X is hemicompact,*
2. *$Q(X, \mathbb{R})$ is metrizable,*
3. *$Q(X, \mathbb{R})$ is first countable.*

The following result will show that also the weight and the π-weight of $Q(X, \mathbb{R})$ coincide.

To define the π-weight of a topological space Z, we first need a notion of a π-base. A collection $\mathcal{V}$ of nonempty open subsets of Z is called a π-base [20] provided that for each open set U in Z, there exists a $V \in \mathcal{V}$ which is contained in U.

Define the π-weight of Z by:

$$\pi w(Z) = \aleph_0 + \min\{|\mathcal{B}| : \mathcal{B} \text{ is a } \pi - \text{base in } Z\}.$$

Corollary 3.5. *For every space X, $\pi w(Q(X, \mathbb{R})) = w(Q(X, \mathbb{R}))$. In fact,*
$\pi w(Q(X, \mathbb{R})) = kcof(X) \cdot d(Q(X, \mathbb{R})$ and $w(Q(X, \mathbb{R})) = kcof(X) \cdot c(Q(X, \mathbb{R})) = kcof(X) \cdot e(Q(X, \mathbb{R}))$.

Proof. It is known (see [6]) that for a Tychonoff space Z, $w(Z) = c(Z) \cdot u(Z)$ and $w(Z) = e(Z) \cdot u(Z)$. Thus by Theorem 3.3 $w(Q(X, \mathbb{R})) = kcof(X) \cdot c(Q(X, \mathbb{R}))$ and $w(Q(X, \mathbb{R})) = kcof(X) \cdot e(Q(X, \mathbb{R}))$. Since $\pi w(Q(X, \mathbb{R}) \geq \pi\chi(Q(X, \mathbb{R})) = \chi(Q(X, \mathbb{R}) = kcof(X)$ and $\pi w(Q(X, \mathbb{R})) \geq d(Q(X, \mathbb{R})$ we have

$$\pi w(Q(X, \mathbb{R})) \geq kcof(X) \cdot d(Q(X, \mathbb{R})) \geq w(Q(X, \mathbb{R}).$$

□

Theorem 3.6. *For a locally compact space X, $w(Q(X, \mathbb{R})) = nw(Q(X, \mathbb{R}))$.*

Proof. To prove that $w(Q(X, \mathbb{R})) = nw(Q(X, \mathbb{R}))$ by Corollary 3.5 it suffices to show that $kcof(X) \leq nw(Q(X, \mathbb{R}))$. Because X is locally compact, it has a base $\mathcal{B}$ of relatively compact sets such that $|\mathcal{B}| = w(X)$. Then the family of all finite unions of members of $\{\overline{B} : B \in \mathcal{B}\}$ is cofinal in $K(X)$ and has cardinality $w(X)$. So $kcof(X) \leq w(X)$. It is known that $w(X) = nw(C(X, \mathbb{R}))$ [20]. Since $nw(C(X, \mathbb{R})) \leq nw(Q(X, \mathbb{R}))$, we have $kcof(X) \leq nw(Q(X, \mathbb{R}))$. □

Using the same idea as in the proof of Theorem 3.6 we can show that for a locally compact space X we also have the equality $w(C(X, \mathbb{R})) = nw(C(X, \mathbb{R}))$, which seems not to be known in the literature.

In the following lemma we will use the notion of the discrete cellularity introduced in [2]. To define the discrete cellularity of a topological space Z we need a notion of a discrete family of subsets of Z. We say that a family $\mathcal{U}$ of subsets of a topological space Z is discrete if each point $z \in Z$ has a neighborhood that meets at most one set of the family $\mathcal{U}$.

The discrete cellularity of Z is defined as:

$$dc(Z) = \aleph_0 + \sup\{|\mathcal{U}| : \mathcal{U} \text{ is a discrete family of nonempty open sets in } Z\}.$$

Remark 3.7. For every topological space Z, $dc(Z) \leq c(Z)$ and $dc(Z) \leq e(Z)$ [2].

Lemma 3.8. *Let X be a topological space which contains an infinite compact set. Then* $dc(Q(X,\mathbb{R})) \geq \mathfrak{c}$.

Proof. Let K be an infinite compact set in X. There is a pairwise disjoint sequence $\{U_n : n \in \mathbb{N}\}$ of open sets such that $U_n \cap K \neq \emptyset$ for every $n \in \mathbb{N}$. Let $2^{\mathbb{N}}$ denote the set of all functions from $\mathbb{N}$ to $\{0,1\}$. For every $\varphi \in 2^{\mathbb{N}}$ denote by $\mathbb{N}_\varphi$ the set of all $n \in \mathbb{N}$ where $\varphi(n) = 1$ and let $f_\varphi : X \to \mathbb{R}$ be a function defined as follows:

$$f_\varphi(x) = \begin{cases} 1, & \text{if } x \in \overline{\bigcup_{n\in\mathbb{N}_\varphi} U_n}; \\ 0, & \text{otherwise.} \end{cases}$$

Then f_φ is a quasicontinuous function.

For every $\varphi \in 2^{\mathbb{N}}$ define $B_\varphi = W(f_\varphi, K, 1/4)$. Let $g \in Q(X,\mathbb{R})$. Then $W(g, K, 1/4)$ intersect at most one set of $\{B_\varphi : \varphi \in 2^{\mathbb{N}}\}$. So $\{B_\varphi : \varphi \in 2^{\mathbb{N}}\}$ is a discrete family of open subset of $(Q(X,\mathbb{R}), \tau_{UC})$. □

For a topological space Z we define the Lindelöf degree of Z:

$$L(Z) = \aleph_0 + \min\{\kappa : \text{every open cover of } Z \text{ has a subcover of cardinality at most } \kappa\}$$

and the spread of Z:

$$s(Z) = \aleph_0 + \sup\{|E| : E \text{ is a discrete set in } Z\}.$$

If X is hemicompact, i.e. $kcof(X) = \aleph_0$, then by Corollary 3.4 $Q(X,\mathbb{R})$ is metrizable, thus all cardinal invariants $c, d, nw, s, e, L, \pi w, w$ coincide on $Q(X,\mathbb{R})$. The following theorem gives other conditions on X under which the cardinal invariants coincide on $Q(X,\mathbb{R})$.

Theorem 3.9. *Let X be a topological space which contains an infinite compact set and let* $kcof(X) \leq \mathfrak{c}$. *Then we have*

$$c(Q(X,\mathbb{R})) = d(Q(X,\mathbb{R})) = e(Q(X,\mathbb{R})) = L(Q(X,\mathbb{R})) =$$
$$s(Q(X,\mathbb{R})) = nw(Q(X,\mathbb{R})) = \pi w(Q(X,\mathbb{R})) = w(Q(X,\mathbb{R})) \geq \mathfrak{c}.$$

Proof. By Corollary 3.5, $kcof(X).e(Q(X,\mathbb{R})) = w(Q(X,\mathbb{R})) = kcof(X).c(Q(X,\mathbb{R}))$. By Lemma 3.8 we have $e(Q(X,\mathbb{R})) = w(Q(X,\mathbb{R})) = c(Q(X,\mathbb{R}))$. Since other cardinal invariants are between c, w and e, we are done. □

Corollary 3.10. *Let X be a discrete topological space. Then* $c(Q(X,\mathbb{R})) = \aleph_0$ *and* $w(Q(X,\mathbb{R})) = kcof(X) = |X|$.

Proof. If X is a discrete topological space, then the topology τ_{UC} coincides with the topology τ_p on $Q(X,\mathbb{R}) = C(X,\mathbb{R}) = \mathbb{R}^X$ and τ_p on $\mathbb{R}^X$ is the product topology. Thus by [6] $c(Q(X,\mathbb{R})) = \aleph_0$. Since $w(Q(X,\mathbb{R})) = kcof(X) \cdot c(Q(X,\mathbb{R})) = |X|$. □

Proposition 3.11. *Let X be a discrete topological space such that* $|X| = \mathfrak{c}$. *Then* $d(Q(X,\mathbb{R})) = \aleph_0$.

Proof. If X is a discrete topological space, then the topology τ_{UC} coincides with the topology τ_p on $Q(X,\mathbb{R}) = C(X,\mathbb{R}) = \mathbb{R}^X$. Thus by [6] $d(Q(X,\mathbb{R})) = \aleph_0$. □

Theorem 3.12. *Let X be a Tychonoff topological space. The following are equivalent:*

1. $w(Q(X,\mathbb{R})) = \aleph_0$,
2. *X is countable and every compact set in X is finite.*

Proof. (1) ⇒ (2) If $w(Q(X,\mathbb{R})) = \aleph_0$, then $dc(Q(X,\mathbb{R})) = \aleph_0$, thus by Lemma 3.8 every compact set in X must be finite. Then the topology τ_{UC} coincides with the topology τ_p on $Q(X,\mathbb{R})$. Thus also $w(C(X,\mathbb{R})) = \aleph_0$ in the topology τ_p. By Corollary 4.5.4 in [20] X must be countable.

(2) ⇒ (1) If every compact set in X is finite, the topology $\tau_{UC} = \tau_p$ on $Q(X,\mathbb{R})$. Since $Q(X,\mathbb{R}) \subset \mathbb{R}^X$, $w(Q(X,\mathbb{R})) \leq w(\mathbb{R}^X)$. X is countable, thus $w(\mathbb{R}^X) = \aleph_0$. □

Lemma 3.13. *Let X be a first countable space. If* $dc(Q(X, \mathbb{R})) = \aleph_0$*, then X is discrete.*

Proof. If $dc(Q(X, \mathbb{R})) = \aleph_0$, then by Lemma 3.8 every compact set in X must be finite. Suppose there is a non-isolated point $x \in X$. Then we can find a sequence $\{x_n : n \in \mathbb{N}\}$ of different points which converges to x. The set $K = \{x\} \cup \{x_n : n \in \mathbb{N}\}$ is an infinite compact set in X, a contradiction. □

Theorem 3.14. *Let X be a first countable space. The following are equivalent:*

1. $dc(Q(X, \mathbb{R})) = \aleph_0$,
2. $c(Q(X, \mathbb{R})) = \aleph_0$,
3. *X is discrete.*

Proof. (1) ⇒ (3) By Lemma 3.13. (3) ⇒ (2) by Corollary 3.10. (2) ⇒ (1) by Remark 3.7. □

Theorem 3.15. *Let X be a topological space.The following are equivalent:*

1. $dc(Q(X, \mathbb{R})) = \aleph_0$,
2. $c(Q(X, \mathbb{R})) = \aleph_0$,
3. *Every compact set in X is finite.*

Proof. (1) ⇒ (3) is clear from Lemma 3.8. (3) ⇒ (2) If every compact set in X is finite, then the topology τ_{UC} coincides with the topology τ_p on $Q(X, \mathbb{R})$. By [16] $Q(X, \mathbb{R})$ is dense in $\mathbb{R}^X$ equipped with the product topology. Thus $c(Q(X, \mathbb{R})) = c(\mathbb{R}^X) = \aleph_0$. (2) ⇒ (1) By Remark 3.7. □

Theorem 3.16. *Let X be a first countable topological space. The following are equivalent:*

1. $w(Q(X, \mathbb{R})) = \aleph_0$,
2. $nw(Q(X, \mathbb{R})) = \aleph_0$,
3. $L(Q(X, \mathbb{R})) = \aleph_0$,
4. *X is countable and discrete.*

Proof. (4) ⇒ (1) By Corollary 3.10 $w(Q(X, \mathbb{R})) = \aleph_0$, so we are done. The implications (1) ⇒ (2) and (2) ⇒ (3) are trivial. We prove (3) ⇒ (4). If $L(Q(X, \mathbb{R})) = \aleph_0$, then also $dc(Q(X, \mathbb{R})) = \aleph_0$, thus by Lemma 3.13, X must be discrete. Then $Q(X, \mathbb{R}) = C(X, \mathbb{R})$ and the topology τ_{UC} coincides with the topology τ_p on $Q(X, \mathbb{R})$. By 3.(b), page 68 in [20] X must be countable. □

4. Comparison of Cardinal Invariants of $Q(\mathbb{R}, \mathbb{R})$ and $C(\mathbb{R}, \mathbb{R})$

Example 4.1. Let $\mathbb{R}$ be equipped with the usual Euclidean metric d. Denote by C the Cantor set. The Cantor set is a closed and nowhere dense set in $\mathbb{R}$ with $|C| = \mathfrak{c}$. Let 2^C denote the set of all functions from C to $\{0, 1\}$. For every $\phi \in 2^C$ by C_ϕ we denote the set of all $x \in C$ where $\phi(x) = 1$. For every $\phi \in 2^C$ let $f_\phi : \mathbb{R} \to \mathbb{R}$ be a function defined as follows:

$$f_\phi(x) = \begin{cases} \sin(1/d(x, C)), & \text{if } x \notin C; \\ 1, & \text{if } x \in C_\phi; \\ 0, & \text{if } x \in C \setminus C_\phi, \end{cases}$$

where $d(x, C) = \inf\{d(x, c) : c \in C\}$. Then f_ϕ is a quasicontinuous function for every $\phi \in 2^C$. From this follows that $|Q(\mathbb{R}, \mathbb{R})| \geq 2^{\mathfrak{c}}$. Thus $|Q(\mathbb{R}, \mathbb{R})| = 2^{\mathfrak{c}}$.

For every $\phi \in 2^C$ define $B_\phi = W(f_\phi, C, 1/4)$. Then $\{B_\phi : \phi \in 2^C\}$ is a pairwise disjoint family of open sets in $(Q(\mathbb{R}, \mathbb{R}), \tau_{UC})$. Thus $c(Q(\mathbb{R}, \mathbb{R})) = 2^{\mathfrak{c}}$.

Remark 4.2. Notice also that the set $\{f_\phi : \phi \in 2^C\}$ from the previous Example is a closed discrete set. Let $g \in Q(\mathbb{R}, \mathbb{R}) \setminus \{f_\phi : \phi \in 2^C\}$. Suppose that $f_\psi \in W(g, C, 1/4)$ for some $\psi \in 2^C$. Since there may be at most one $\psi \in 2^C$ such that $f_\psi \in W(g, C, 1/4)$, the set $W(g, C, 1/4) \setminus \{f_\psi\}$ is an open neighborhood of g such that $\{f_\phi : \phi \in 2^C\} \cap (W(g, C, 1/4) \setminus \{f_\psi\}) = \emptyset$.

Now we can compare the cardinal invariants of $(Q(\mathbb{R},\mathbb{R}),\tau_{UC})$ and $(C(\mathbb{R},\mathbb{R}),\tau_{UC})$. Since $\mathbb{R}$ is hemicompact, both $(Q(\mathbb{R},\mathbb{R}),\tau_{UC})$ and $(C(\mathbb{R},\mathbb{R}),\tau_{UC})$ are metrizable. Thus all cardinal invariants $c, d, nw, s, e, L, \pi w, w$ coincide on these spaces. By Proposition 1.2 (3) in [2] also $dc(Q(\mathbb{R},\mathbb{R})) = c(Q(\mathbb{R},\mathbb{R}))$ and $dc(C(\mathbb{R},\mathbb{R})) = c(C(\mathbb{R},\mathbb{R}))$.

By the previous example we have

$$dc(Q(\mathbb{R},\mathbb{R})) = c(Q(\mathbb{R},\mathbb{R})) = d(Q(\mathbb{R},\mathbb{R})) = e(Q(\mathbb{R},\mathbb{R})) = L(Q(\mathbb{R},\mathbb{R})) =$$
$$s(Q(\mathbb{R},\mathbb{R})) = nw(Q(\mathbb{R},\mathbb{R})) = \pi w(Q(\mathbb{R},\mathbb{R})) = w(Q(\mathbb{R},\mathbb{R})) = 2^{\mathfrak{c}},$$

and $|Q(\mathbb{R},\mathbb{R})| = 2^{\mathfrak{c}}$.

By [20] and [2] we have

$$dc(C(\mathbb{R},\mathbb{R})) = c(C(\mathbb{R},\mathbb{R})) = d(C(\mathbb{R},\mathbb{R})) = e(C(\mathbb{R},\mathbb{R})) = L(C(\mathbb{R},\mathbb{R})) =$$
$$s(C(\mathbb{R},\mathbb{R})) = nw(C(\mathbb{R},\mathbb{R})) = \pi w(C(\mathbb{R},\mathbb{R})) = w(C(\mathbb{R},\mathbb{R})) = \aleph_0,$$

and $|C(\mathbb{R},\mathbb{R})| = \mathfrak{c}$.

Acknowledgements

The authors would like to thank the referee for his comments.

References

[1] R. Baire, Sur les functions des variables reelles, Ann. Mat. Pura Appl. 3 (1899) 1–122.
[2] T. Banakh, A. Ravsky, Verbal covering properties of topological spaces, Topology Appl. 201 (2016) 181–205.
[3] J.M. Borwein, Minimal cuscos and subgradients of Lipschitz functions, In: Fixed point theory and applications (Marseille, 1989), Pitman Res. Notes Math. Ser. 252, Longman Sci. Tech., Harlow 1991, 57–81.
[4] A. Bouziad, Every Čech-analytic Baire semitopological group is a topological group, Proc. Amer. Math. Soc. 124 (1996) 953–959.
[5] A. Crannell, M. Frantz, M. LeMasurier, Closed relations and equivalence classes of quasicontinuous functions, Real Anal. Exchange 31 (2006/2007) 409-424.
[6] R. Engelking, General Topology, Heldermann Verlag, Berlin, 1989.
[7] Ľ. Holá, D. Holý, Minimal usco maps, densely continuous forms and upper semicontinuous functions, Rocky Mount. Math. J. 39 (2009) 545–562.
[8] Ľ. Holá, D. Holý, Pointwise convergence of quasicontinuous mappings and Baire spaces, Rocky Mount. Math. J. 41 (2011) 1883–1894.
[9] Ľ. Holá, D. Holý, New characterization of minimal CUSCO maps, Rocky Mount. Math. J. 44 (2014) 1851–1866.
[10] Ľ. Holá, D. Holý, Relation betwen minimal USCO and minimal CUSCO maps, Portug. Math. 70 (2013) 211–224.
[11] Ľ. Holá, D. Holý, Minimal usco and minimal cusco maps, Khayyam J. Math. 1 (2015) 125–150.
[12] Ľ. Holá, D. Holý, Minimal usco and minimal cusco maps and compactness, J. Math. Anal. Appl. 439 (2016) 737–744.
[13] Ľ. Holá, D. Holý, Quasicontinuous subcontinuous functions and compactness, Mediterr. J. Math. 13 (2016) 4509–4518.
[14] Ľ. Holá, D. Holý, Quasicontinuous functions and compactness, Mediterr. J. Math. 14 (2017) Art. No. 219.
[15] Ľ. Holá, D. Holý, Metrizability of the space of quasicontinuous functions, Topology Appl. 246 (2018) 137–143.
[16] Ľ. Holá, D. Holý, Quasicontinuous functions and the topology of pointwise convergence, Topology Appl. 282 (2020) Art. No. 107301.
[17] Ľ. Holá, Spaces of densely continuous forms, USCO and minimal USCO maps, Set-Valued Anal. 11 (2003) 133–151.
[18] S. Kempisty, Sur les fonctions quasi-continues, Fund. Math. 19 (1932) 184–197.
[19] S. Marcus, Sur les fonctions quasi-continues au sense Kempisty, Colloq. Math. 8 (1961) 45–53.
[20] R.A. McCoy, I. Ntantu, Topological Properties of Spaces of Continuous Functions, Lecture Notes in Mathematics 1315, Springer-Verlag, Berlin, 1988.
[21] W.B. Moors, Any semitopological group that is homeomorphic to a product of Čech-complete spaces is a topological group, Set-Valued Var. Anal. 21 (2013) 627–633.
[22] W.B. Moors, Fragmentable mappings and CHART groups, Fund. Math. 234 (2016) 191–200
[23] W.B. Moors, Semitopological groups, Bouziad spaces and topological groups, Topology Appl. 160 (2013) 2038–2048.
[24] T. Neubrunn, Quasi-continuity, Real Anal. Exchange 14 (1988) 259–306.