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On Weakly B Symmetric Pseudo Riemannian Manifolds and its Applications

Carlo Alberto Mantica[a], Uday Chand De[b], Young Jin Suh[c]

[a] *Physics Department, Università degli Studi di Milano, Via Celoria 16, 20133, Milano, Italy, I. I. S. Lagrange, Via L., Modignani 65, 20161 Milano, Italy*
[b] *Department of Pure Mathematics, University of Calcutta, 35, Ballygaunge Circular Road, Kolkata 700019, West Bengal, India*
[c] *Department of Mathematics and RIRCM, Kyungpook National University, Daegu, 41566, South Korea*

Abstract. The object of the present paper is to study weakly B symmetric manifolds $(WBS)_n$. At first some geometric properties of $(WBS)_n (n > 2)$ have been studied. Finally, we consider $(WBS)_4$ spacetimes. They turn out to be both perfect and imperfect fluids Robertson-Walker space-times : an equation of state is provided in the first case, and in the second the nature of the bulk viscosity pressure is pointed out. Also, we construct an example of a $(WBS)_4$.

1. Introduction

Let $(M^n, g), (n = dim M)$ be a pseudo Riemannian manifold, i.e., a manifold M with the pseudo Riemannian metric g, and let ∇ be the pseudo Riemannian connection of (M^n, g). A pseudo Riemannian manifold is called locally symmetric [6] if $\nabla R = 0$, where R is the Riemannian curvature tensor of (M^n, g). This condition of local symmetry is equivalent to the fact that at every point $P \in M$, the local geodesic symmetry $F(P)$ is an isometry [41] . The class of symmetric manifolds is a very natural generalization of the class of manifolds of constant curvature.

A non-flat pseudo-Riemannian manifold $(M^n, g)(n > 2)$ is called weakly symmetric [50] if the curvature tensor R of type (0,4) satisfies the condition

$$\begin{aligned}&(\nabla_X R)(Y, Z, U, V) = A(X)R(Y, Z, U, V) + D(Y)R(X, Z, U, V)\\&+E(Z)R(Y, X, U, V) + G(U)R(Y, Z, X, V) + J(V)R(Y, Z, U, X),\end{aligned} \tag{1}$$

where $R(Y, Z, U, V) = g(\mathcal{R}(Y, Z)U, V)$, $\mathcal{R}$ is the curvature tensor of type (1,3) and A, D, E, G and J are 1-forms respectively which are non-zero simultaneously. Such a manifold is denoted by $(WS)_n$. It was proved in [16] that the 1-forms must be related as follows

$D = E$ and $G = J$.

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Email addresses: carloalberto.mantica@libero.it (Carlo Alberto Mantica), uc_de@yahoo.com (Uday Chand De), yjsuh@knu.ac.kr (Young Jin Suh)

That is, the weakly symmetric manifold is characterized by the condition

$$(\nabla_X R)(Y,Z,U,V) = A(X)R(Y,Z,U,V) + D(Y)R(X,Z,U,V)$$
$$+D(Z)R(Y,X,U,V) + G(U)R(Y,Z,X,V) + G(V)R(Y,Z,U,X). \tag{2}$$

The 1-forms A, D and G are called the associated 1-forms. If in (2) the 1-form A is replaced by $2A$; D and G are replaced by A, then the manifold (M^n, g) reduces to a pseudo symmetric manifold in the sense of Chaki [10].

Again if $A = D = G = 0$, the manifold defined by (2) reduces to a symmetric manifold in the sense of Cartan. The existence of a $(WS)_n$ was proved by Prvanović [44] and a concrete example is given by De and Bandyopadhyay ([16],[17]).

Weakly symmetric manifolds have been studied by several authors ([4], [20], [21], [34], [35], [36], [42], [43]) and many others.

In 1993, Tamassy and Binh [51] introduced weakly Ricci symmetric manifolds. It may be mentioned that a pseudo Ricci symmetric manifold is a particular case of a weakly Ricci symmetric manifold. In 2012, Mantica and Molinari ([33],[34]) introduced weakly $\mathcal{Z}$ symmetric manifolds which is denoted by $(W\mathcal{Z}S)_n$. It is a generalization of the notion of weakly Ricci symmetric manifolds[51]. A (0,2) symmetric tensor is a generalized $\mathcal{Z}$ tensor if

$$\mathcal{Z}(X,Y) = S(X,Y) + \phi g(X,Y), \tag{3}$$

where ϕ is an arbitrary scalar function and S denotes the Ricci tensor of type $(0,2)$. The scalar $\tilde{Z}$ is obtained by contracting (3) over X, Y as follows:

$$\tilde{Z} = r + n\phi. \tag{4}$$

In a subsequent paper [45] De et al. introduced a (0,2) symmetric tensor B as follows:

$$B(X,Y) = aS(X,Y) + brg(X,Y), \tag{5}$$

where a and b are non-zero arbitrary scalar functions and r is the scalar curvature. The authors [45] have studied pseudo B symmetric manifolds with applications to relativity. The pseudo B-symmetric manifold is the generalized notion of pseudo Z-symmetric manifolds. The scalar $\tilde{B}$ is obtained by contracting (5) over X, Y as follows:

$$\tilde{B} = (a + nb)r. \tag{6}$$

Pseudo $\mathcal{Z}$ symmetric, weakly $\mathcal{Z}$ symmetric and recurrent $\mathcal{Z}$ forms on pseudo-Riemannian manifolds have been studied in ([34], [35] and [36]) respectively.

Inspired by these works we introduce a new type of manifold called weakly B-symmetric manifolds. A manifold is called weakly B-symmetric and denoted by $(WBS)_n$, if the B tensor of type (0,2) is non-zero and satisfies the condition

$$(\nabla_X B)(Y,Z) = A(X)B(Y,Z) + D(Y)B(X,Z) + E(Z)B(X,Y), \tag{7}$$

where A, D, E are 1-form which not simultaneously zero. Such an n-dimensional manifold will be called weakly B-symmetric manifold and denoted by $(WBS)_n$.

A non-flat pseudo-Riemannian manifold $(M^n, g)(n > 2)$ is defined to be a quasi Einstein manifold [9] if its Ricci tensor S of type (0,2) is not identically zero and satisfies the following condition:

$$S(X,Y) = \alpha g(X,Y) + \beta\eta(X)\eta(Y),$$

where α, β are scalars and η is a non-zero 1-form for all vector fields X. The quasi Einstein manifold is denoted by $(QE)_n$.

On the other hand, in a pseudo-Riemannian manifold quasi Einstein manifolds arose during the study of exact solutions of the Einstein's field equations as well as during considerations of quasi-umbilical hypersurfaces of semi-Euclidean spaces.

A. Gray [29] introduced a class of Riemannian manifold determined by covariant derivative of the Ricci tensor. The class consisting of all Riemannian manifolds whose Ricci tensor S is a Codazzi tensor, i.e,

$$(\nabla_X S)(Y, Z) = (\nabla_Y S)(X, Z). \tag{8}$$

The class B consisting of all Riemannian manifolds whose Ricci tensor is cyclic parallel, i.e.,

$$(\nabla_X S)(Y, Z) + (\nabla_Y S)(Z, X) + (\nabla_Z S)(X, Y) = 0. \tag{9}$$

A pseudo-Riemannian manifold is said to satisfy Codazzi type of Ricci tensor if the Ricci tensor is non zero and satisfies the condition (8).

The present paper is organized as follows:
After introduction, in Section 2, some curvature properties of $(WBS)_n$ have been studied. Among others it is proved that a $(WBS)_n$ is a quasi-Einstein manifold. Moreover, in a $(WBS)_n$ with divergence free conformal curvature tensor, the B tensor is of Codazzi type provided the scalars a, b, r are constants. Next we prove that a $(WBS)_n$ admits cyclic parallel B tensor if and only if the sum of the associated 1-forms vanishes. Finally, in Section 4, we consider $(WBS)_4$ space-times. It is shown that if the conformal curvature is divergence free, then the space-time is both a perfect and imperfect fluid Robertson-Walker space-time : an equation of state is provided in the first case, and in the second the nature of the bulk viscosity pressure is pointed out. Finally, we construct an example of a $(WBS)_4$.

2. Some curvature properties of $(WBS)_n$

Let S and r denote the Ricci tensor of type (0,2) and the scalar curvature respectively and L denote the symmetric endomorphism of the tangent space at each point corresponding to the Ricci tensor S , that is ,

$$g(LX, Y) = S(X, Y), \tag{10}$$

for any vector field X, Y.

Now we state the following Lemma which will be used later:

Walker's Lemma [53]: If a_{ij}, b_k are numbers satisfying

$$a_{ij} = a_{ji}, \tag{11}$$

and

$$a_{ij}b_k + a_{jk}b_i + a_{ki}b_j = 0, \tag{12}$$

for $i, j, k = 1, 2, ..., n$, then either all $a_{ij} = 0$ or, all b_i are zero.

Let us consider a $(WBS)_n$. Then from (7) we have

$$(\nabla_X B)(Y, Z) = A(X)B(Y, Z) + D(Y)B(X, Z) + E(Z)B(X, Y). \tag{13}$$

Interchanging Y, Z in (13) we have

$$(\nabla_X B)(Z, Y) = A(X)B(Z, Y) + D(Z)B(X, Y) + E(Y)B(X, Z). \tag{14}$$

Since B is symmetric, subtracting (14) from (13) we obtain

$$[D(Y) - E(Y)]B(X, Z) = [D(Z) - E(Z)]B(X, Y). \tag{15}$$

We define $\omega(X) = D(X) - E(X) = g(X, \tilde{\rho})$, for all vector field X, where $\tilde{\rho}$ is a vector field associated with the 1-form ω. Therefore the above equation reduces to

$$\omega(Y)B(X, Z) = \omega(Z)B(X, Y). \tag{16}$$

Putting $Z = \tilde{\rho}$ in (16), we get

$$\omega(Y)B(X, \tilde{\rho}) = \omega(\tilde{\rho})B(X, Y). \tag{17}$$

Again putting $X = Y = e_i$ in (16), where $\{e_i\}$ is a pseudo orthonormal basis, we have

$$B(Z, \tilde{\rho}) = \omega(Z)\tilde{B}, \tag{18}$$

where $\tilde{B}$ is defined by (6). Thus (17) and (18) yields

$$\omega(\tilde{\rho})B(X, Z) = \tilde{B}\omega(Z)\omega(X). \tag{19}$$

If $\omega(\tilde{\rho}) \neq 0$ it follows that

$$B(X, Y) = \tilde{B}\frac{\omega(X)}{\sqrt{\omega(\tilde{\rho})}}\frac{\omega(Y)}{\sqrt{\omega(\tilde{\rho})}}, \tag{20}$$

in case of $\omega(\tilde{\rho}) > 0$, and

$$B(X, Y) = \tilde{B}\frac{\omega(X)}{\sqrt{-\omega(\tilde{\rho})}}\frac{\omega(Y)}{\sqrt{-\omega(\tilde{\rho})}}, \tag{21}$$

if $\omega(\tilde{\rho}) < 0$. From (5) we have thus

$$S(X, Y) = \alpha g(X, Y) + \beta\eta(X)\eta(Y), \tag{22}$$

where $\alpha = \frac{-\tilde{B}r}{a}$, $\beta = \frac{\tilde{B}}{a}$, $\eta(X) = \frac{\omega(X)}{\sqrt{\omega(\tilde{\rho})}}$ if $\omega(\tilde{\rho}) > 0$ and $\eta(X) = \frac{\omega(X)}{\sqrt{-\omega(\tilde{\rho})}}$ if $\omega(\tilde{\rho}) < 0$. Therefore we can state the following:

Proposition 2.1. *A $(WBS)_n$ manifold is quasi-Einstein provided $\omega(\tilde{\rho}) \neq 0$*

Again contracting (16) over X and Z, we get

$$\omega(LY) = \frac{r}{a}\{a + (n-1)b\}\omega(Y). \tag{23}$$

This can be rewritten as

$$S(Y, \tilde{\rho}) = \frac{r}{a}\{a + (n-1)b\}g(Y, \tilde{\rho}). \tag{24}$$

Thus $\frac{r}{a}\{a + (n-1)b\}$ is an eigenvalue of the Ricci tensor S corresponding to the eigenvector $\tilde{\rho}$.
Thus we conclude the following:

Proposition 2.2. *In a $(WBS)_n$, $\frac{r}{a}\{a + (n-1)b\}$ is an eigenvalue of the Ricci tensor S corresponding to the eigenvector $\tilde{\rho}$ defined by $\omega(X) = g(X, \tilde{\rho})$ for all vector field X.*

Moreover for a $(WBS)_n$ we have

$$(\nabla_X B)(Z, Y) = A(X)B(Z, Y) + D(Z)B(X, Y) + E(Y)B(X, Z), \tag{25}$$

where

$$B(X, Y) = aS(X, Y) + brg(X, Y). \tag{26}$$

We define $F(X) = A(X) - D(X) = g(X, \rho_1)$, for all vector field X, where ρ_1 is a vector field associated with the 1-form F. Therefore

$$(\nabla_Z B)(X, Y) - (\nabla_X B)(Z, Y) = F(Z)B(X, Y) - F(X)B(Z, Y). \tag{27}$$

Also it is known that

$$(\nabla_Z B)(X, Y) = (Za)S(X, Y) + a(\nabla_Z S)(X, Y) + Z(br)g(X, Y). \tag{28}$$

Assume that $divC = 0$, where C denotes the Weyl conformal curvature tensor and 'div' denotes divergence. Hence we have [54]

$$(\nabla_X S)(Y, U) - (\nabla_U S)(Y, X) = \frac{1}{2(n-1)}[g(Y, U)dr(X) - g(X, Y)dr(U)]. \tag{29}$$

Using (28), (29) in (27) we have

$$\begin{aligned} & F(Z)B(X, Y) - F(X)B(Z, Y) \\ = \; & (Za)S(X, Y) - (Xa)S(Y, Z) + Z(br)g(X, Y)X(br)g(Y, Z), \\ & -\frac{a}{2(n-1)}[g(Z, Y)dr(X) - g(X, Y)dr(Z)]. \end{aligned} \tag{30}$$

It follows that

$$F(Z)B(X, Y) - F(X)B(Z, Y) = 0, \tag{31}$$

provided a, b and r are constants. Making use of (31) in (27) we have

$$(\nabla_Z B)(X, Y) = (\nabla_X B)(Z, Y). \tag{32}$$

Thus we can state the following:

Proposition 2.3. *If in a $(WBS)_n$, the conformal curvature tensor is divergence free, then the B tensor is of Codazzi type provided the scalars a, b, r are constants.*

A pseudo-Riemannian manifold is said to satisfy cylic parallel Ricci tensor [29] if its Ricci tensor S of type $(0, 2)$ is non zero and satisfies the condition

$$(\nabla_X S)(Y, Z) + (\nabla_Y S)(X, Z) + (\nabla_Z S)(X, Y) = 0. \tag{33}$$

Analogous to the definition in (33) we define cyclic B-tensor as follows: An n-dimensional pseudo-Riemannian manifold is said to be satisfy cylic parallel B tensor if the following condition holds:

$$(\nabla_X B)(Y, Z) + (\nabla_Y B)(X, Z) + (\nabla_Z B)(X, Y) = 0. \tag{34}$$

Now from (7) we obtain

$$\begin{aligned} & (\nabla_X B)(Y, Z) + (\nabla_Y B)(X, Z) + (\nabla_Z B)(X, Y) \\ & = H(X)B(Y, Z) + H(Y)B(X, Z) + H(Z)B(X, Y), \end{aligned} \tag{35}$$

where $H(X) = A(X) + D(X) + E(X)$. Using (34) in (35) yields

$$H(X)B(Y, Z) + H(Y)B(X, Z) + H(Z)B(X, Y) = 0.$$

Then by Walke's Lemma we can see that either $H(X) = 0$ or $B(Y, Z) = 0$ for all X, Y, Z. But B is non zero in $(WBS)_n$. Thus $H(X) = 0$, that is, $A(X) + D(X) + E(X) = 0$. The converse is obvious. Thus we have the following:

Proposition 2.4. *A $(WBS)_n$ admits cyclic parallel B tensor if and only if the sum of the associated 1-forms vanishes.*

3. $(WBS)_n$, $n > 3$ with divergence free Weyl tensor

In this section we assume that the $(WBS)_n$ has divergence free Weyl tensor, that is, $divC = 0$, where C denotes the Weyl conformal curvature tensor and 'div' denotes divergence. Hence we have [54]

$$(\nabla_X S)(Y,U) - (\nabla_U S)(Y,X) = \frac{1}{2(n-1)}[g(Y,U)dr(X) - g(X,Y)dr(U)]. \tag{36}$$

Since $(WBS)_n$ is a quasi-Einstein manifold, using (22) in (36) we have

$$\begin{aligned}&d\alpha(X)g(Y,U) + d\beta(X)\eta(Y)\eta(U) + \beta[(\nabla_X\eta)(Y)\eta(U) + \eta(Y)(\nabla_X\eta)(U)]\\&-d\alpha(U)g(Y,X) - d\beta(U)\eta(Y)\eta(X) - \beta[(\nabla_U\eta)(Y)\eta(X) + \eta(Y)(\nabla_U\eta)(X)]\\&= \frac{1}{2(n-1)}[g(Y,U)dr(X) - g(X,U)dr(U)].\end{aligned} \tag{37}$$

We define a unit vector field ρ by $g(X,\rho) = \eta(X)$ for all vector field X. Taking a frame field and contracting over X and Y we get

$$\begin{aligned}&(1-n)d\alpha(U) + d\beta(\rho)\eta(U) + \beta(\nabla_\rho\eta)(U)\\&+\beta\eta(U)(\delta\eta) + d\beta(U) = -\frac{1}{2}dr(U),\end{aligned} \tag{38}$$

where $\delta\eta = \sum_{i=1}^{n}(\nabla_{e_i}\eta)(e_i)$.

Putting $X = Y = \rho$ in (38) implies

$$\begin{aligned}&d\alpha(\rho)\eta(U) - d\beta(\rho)\eta(U) - \beta(\nabla_\rho\eta)(U) + d\alpha(U) - d\beta(U)\\&= \frac{1}{2(n-1)}[dr(\rho)\eta(U) + dr(U)].\end{aligned} \tag{39}$$

Substituting (39) in (38) we get

$$\begin{aligned}&(2-n)d\alpha(U) + d\alpha(\rho)\eta(U) + \beta\eta(U)(\delta\eta)\\&-\frac{1}{2(n-1)}dr(\rho)\eta(U) = \frac{(2-n)}{2(n-1)}dr(U).\end{aligned} \tag{40}$$

Putting $U = \rho$ in (40) we obtain

$$(1-n)d\alpha(\rho) - \beta(d\eta) = -\frac{1}{2}dr(\rho). \tag{41}$$

Using (41) in (40) we have

$$\begin{aligned}&(2-n)d\alpha(U) + (2-n)d\alpha(\rho_2)\eta(U) + \frac{(n-2)}{2(n-1)}dr(\rho_2)\eta(U)\\&= \frac{(2-n)}{2(n-1)}dr(U).\end{aligned} \tag{42}$$

Let us consider $r = \alpha$, then from (22) we have

$$d\alpha(U) = dr(U) \text{ and } d\beta(U) = -(n-1)d\alpha(U). \tag{43}$$

Using (43) in (42) we obtain

$$d\alpha(U) = -d\alpha(\rho)\eta(U). \tag{44}$$

Putting $Y = \rho$ in (37) and using (44) we have

$$(\nabla_U\eta)(X) - (\nabla_X\eta)(U) = 0. \tag{45}$$

Therefore the 1-form η defined in (22) is closed and hence $d\eta(X, Y) = 0$. It follows that

$$g(\nabla_X\rho, Y) = g(\nabla_Y\rho, X), \tag{46}$$

for all X and Y, which implies that ρ is irrotational. Now putting $Y = \rho$ in (46) we get

$$g(\nabla_X\rho, \rho) = g(\nabla_\rho\rho, X). \tag{47}$$

Since $g(\nabla_X\rho, \rho) = 0$, from (47) it follows that

$$g(\nabla_\rho\rho, X) = 0, \tag{48}$$

for all X. Hence

$$\nabla_\rho\rho = 0. \tag{49}$$

This means the integral curves of the vector field ρ are geodesic.

Therefore we can state the following:

Theorem 3.1. *In a $(WBS)_n$ with divergence free conformal curvature tensor, the integral curves of the vector field ρ are geodesic and irrotational, provided $r = \alpha$.*

Using (44) in (39) we obtain

$$\begin{aligned}\beta(\nabla_\rho\eta)(U) &= -(n-1)d\alpha(\rho)\eta(U) - (n-1)d\alpha(U)\\ &\quad -\frac{1}{2(n-1)}[dr(\rho)\eta(U) + d\alpha(U)].\end{aligned} \tag{50}$$

Again using (44) and $r = \alpha$ in (50) we have

$$(\nabla_\rho\eta)(U) = 0. \tag{51}$$

Now we consider the scalar function

$$f = \frac{1}{2(n-1)}\frac{dr(\rho)}{\beta^2}. \tag{52}$$

Using $r = \alpha$ and (43) we have

$$\nabla_X f = -\frac{2nr}{2(n-1)}\frac{dr(\rho)}{\beta^2}dr(X) + \frac{1}{2\beta(n-1)}d^2r(\rho, X). \tag{53}$$

From (44) we also have

$$dr(U) = -dr(\rho)\eta(U). \tag{54}$$

On the other hand

$$\begin{aligned}d^2r(Y, X) &= -d^2r(\rho, Y)\eta(X) - dr(\rho)(\nabla_Y\eta)(X)\\ &\quad -dr(\rho)\eta(\nabla_Y\rho)\eta(X),\end{aligned} \tag{55}$$

that is,

$$d^2r(Y, X) = -d^2r(\rho, Y)\eta(X) - dr(\rho)(\nabla_Y\eta)(X). \tag{56}$$

Since $(\nabla_X\eta)(Y) = (\nabla_Y\eta)(X)$ and $d^2r(Y, X) = d^2r(X, Y)$.

Putting $X = \rho$ in (56) we have

$$d^2r(Y, \rho) = -d^2r(\rho, \rho)\eta(Y). \tag{57}$$

Thus using (53) in (57) we obtain

$$\nabla_X f = -\frac{rn}{n-1}\frac{dr(\rho)}{\beta^2}dr(X) - \frac{1}{2\beta(n-1)}d^2r(\rho,\rho)\eta(X), \tag{58}$$

which implies that

$$\nabla_X f = K\eta(X), \tag{59}$$

where $K = \frac{1}{2\beta(n-1)}[-d^2r(\rho,\rho) - \frac{2rn}{\beta}dr(\rho)dr(X)]$. Using (59), it is easy to see that $\tilde{\omega}(X) = \frac{1}{2(n-1)}\frac{dr(\rho)}{\beta^2}\eta(X) = f\eta(X)$ is closed. In fact, $d\tilde{\omega}(X,Y) = 0$.
Using (44), (45), (54) in (37) we have

$$\begin{aligned} & -dr(\rho)\eta(X)g(Y,U) - 2n\alpha dr(\rho)\eta(X)\eta(Y)\eta(U) \\ & +\beta[(\nabla_X\eta)(Y)\eta(U) + \eta(Y)(\nabla_X\eta)(U)] \\ & +dr(\rho)\eta(U)g(Y,X) + 2n\alpha dr(\rho)\eta(X)\eta(Y)\eta(U) \\ & -\beta[(\nabla_U\eta)(Y)\eta(X) + \eta(Y)(\nabla_U\eta)(X)] \\ = & \ \frac{1}{2(n-1)}[-g(Y,U)dr(\rho)\eta(X) + g(X,Y)dr(\rho)\eta(U)]. \end{aligned} \tag{60}$$

Putting $U = \rho$ in (60) and using (49) we obtain

$$\begin{aligned} (\nabla_X\eta)(Y) = \ & fg(X,Y) + \tilde{\omega}(X)\eta(Y) \\ & -\frac{dr(\rho)}{\beta}[g(X,Y) + \eta(X)\eta(Y)]. \end{aligned} \tag{61}$$

From (61) we can write

$$(\nabla_X\eta)(Y) = \lambda g(X,Y) + \gamma(X)\eta(Y), \tag{62}$$

where $\lambda = f - \frac{dr(\rho)}{\beta}$, $\gamma(X) = \tilde{\omega}(X) - \frac{dr(\rho)}{\beta}\eta(X)$. Obviously $\gamma(X)$ is closed. Therefore the vector field ρ defined by $g(X,\rho) = \eta(X)$ is a unit concircular vector field ([40]).

Hence we can state the following:

Theorem 3.2. *In a $(WBS)_n$ with divergence free conformal curvature tensor and satisfying an additional condition $r = \alpha$, the vector field ρ defined by $g(X,\rho) = \eta(X)$ is a unit concircular vector field.*

4. Physical applications: $(WBS)_4$ spacetimes

A 4-dimensional Lorentzian manifold admitting a global timelike vector field is called time orientable Lorentzian manifold, physically known as spacetime. A spacetime is the stage of present modeling of the physical world: a torsion-less, time oriented Lorentzian manifold. For details of spacetime, we cite ([5],[18], [22]-[25],[27], [28],[39],[46], [55]).

It is to be noted that the basic geometric features of $(WBS)_n$ are also being mentioned in the Lorentzian manifold which is necessarily a pseudo Riemannian manifold. Hence Proposition 2.1, Theorem 3.1 and Theorem 3.2 are also true for $(WBS)_4$ spacetimes.

In cosmology, the observation that the space is isotropic and homogeneous on an astronomically immense scale chooses the Robertson-Walker (RW) metric. In 1995, Aliás, Romero and Sánchez [1] generalized the notion of RW metric and called it a generalized Robertson-Walker (GRW) metric. A Lorentzian manifold M of dimension $n \geq 3$ endowed with the Lorentzian metric g defined by

$$ds^2 = g_{ab}dx^a dx^b = -(dt)^2 + \varphi(t)^2 g^*_{lm}(\overrightarrow{x})dx^l dx^m,$$

where t is the time and $g^*_{lm}(\overrightarrow{x})dx^l dx^m$ is the metric tensor of a Riemannian manifold, is a GRW spacetime. In other words, a GRW spacetime is the warped product $-I \times \varphi^2 M^*$, where I is an open interval of the real line, φ is a smooth warping function or scale factor such that $\varphi > 0$ and M^* is an $(n-1)-$ dimensional Riemannian manifold. In particular, if M^* is a 3− dimensional Riemannian space of constant curvature, then the warped product $-I \times \varphi^2 M^*$ is said to be a RW spacetime. A four dimensional conformally flat almost pseudo Ricci symmetric spacetime ($R_{bc,a} = (\alpha_a + \beta_a)R_{bc} + \alpha_b R_{ac} + \alpha_c R_{ba}$) is a Robertson-Walker spacetime [19]. Throughout the paper, we denote the comma ", " as the covariant differentiation. A Robertson-Walker spacetime complies the cosmological principle, that is, the spacetime is spatially isotropic and spatially homogeneous, although the GRW spacetime is not necessarily spatially homogeneous [7]. In [5] Brozos-Vázquez, Garcia-Rio and Vázquez-Lorenzo bridged the gap between RW spacetime and GRW spacetime by providing the following: "*A GRW spacetime is conformally flat if and only if it is a RW spacetime*." For more details of GRW spacetimes, we call [2],[33],[38] and their references.

Lorentzian manifolds with a Ricci tensor of the form

$$R_{ij} = \alpha g_{ij} + \beta \eta_i \eta_j. \tag{63}$$

where α, β are scalars and η_i is a unit time like vector, are often named perfect fluid space-times. It is well known that any Robertson-Walker space-time is a perfect fluid space-time [41] and for $n = 4$, a GRW space-time is a perfect fluid if and only if it is a Robertson-Walker space-time.

Form (63) of the Ricci tensor is implied by Einstein's equation if the energy-matter content of space-time is a perfect fluid with velocity vector field ρ. The scalars α and β are linearly related to the pressure p and the energy density μ measured in the locally comoving inertial frame. They are not independent because of the Bianchi identity $\nabla^m R_{im} = \frac{1}{2}\nabla_i R$, which translates into

$$\nabla^m (Bu_j u_m) = \frac{1}{2}\nabla_j[(n-2)\alpha - \beta]. \tag{64}$$

Geometers identify special form (63) of the Ricci tensor as the defining property of quasi-Einstein manifolds (with any metric signature). Pseudo-Riemannian quasi-Einstein spaces arose in the study of exact solutions of Einstein's equations. Robertson-Walker space-times are quasi-Einstein ([47] and references therein).

Considering the above facts from Proposition 2.1, we can state the following:

Theorem 4.1. *A $(WBS)_4$ spacetime is the perfect fluid spacetime.*

Recently, Bang Yen-Chen proved the following deep result (see [7] and [8]): A Lorentzian manifold of dimension $n \geq 4$ is a GRW space-time if and only if it admits a time-like vector, $X^j X_j < 0$, such that

$$\nabla_k X_j = \rho g_{kj}. \tag{65}$$

Vector fields satisfying equation (65) are called concircular: nice properties of such time-like vector fields were pointed out in [32]. Concircular vector fields have an important role in general relativity, e.g. trajectories of time-like concircular fields in the de Sitter model determine the world lines of receding or colliding galaxies satisfying the Weyl hypothesis (see [49]).

Mantica et al. ([33] and [37]) considered Lorentzian manifolds (of dimension $n > 3$) with Ricci tensor of the form $R_{ij} = \alpha g_{ij} + \beta \eta_i \eta_j$, where α and β are scalar fields and η_i is a unit time like vector. They proved that if the condition $divC = 0$ is satisfied, then the manifold is a generalized Robertson-Walker space-time whose sub-manifold is a Riemannian Einstein space. In Theorem 3.2, we prove that if a $(WBS)_n$ satisfies $divC = 0$ with $r = \alpha$, then ρ defined by $g(X, \rho) = \eta(X)$ is a proper concircular vector field. Also we prove that a $(WBS)_n$ is a quasi-Einstein manifold.

Hence we state the following:

Theorem 4.2. *A $(WBS)_4$ spacetime with divergence free conformal curvature tensor with the condition $r = \alpha$ is a perfect fluid generalized Robertson-Walker spacetime.*

In Theorem 3.1 we prove that a $(WBS)_4$ satisfying the condition $divC = 0$, the integral curves of the vector field ρ are geodesic and irrotational, provided $r = \alpha$. The Roy Choudhury equation [11] for the fluid in $(WBS)_4$ spacetime can be written as

$$\nabla_i \eta_j = \omega_{ij} + \tau_{ij} + f[g_{ij} + \eta_i \eta_j], \tag{66}$$

where ω_{ij} is the vorticity tensor and τ_{ij} is the shear tensor respectively. Since the vector field η_i is a unit concircular vector (see Theorem 4.2), we get

$$\nabla_i \eta_j = f[g_{ij} + \eta_i \eta_j]. \tag{67}$$

Comparing (66) and (67) we get

$$\omega_{ij} + \tau_{ij} = 0. \tag{68}$$

Since ρ is irrotational, hence the vorticity of the fluid vanishes. Therefore $\omega_{ij} = 0$ and consequently (68) implies that $\tau_{ij} = 0$. Thus we can state the following:

Theorem 4.3. *If a $(WBS)_4$ spacetime satisfies $divC = 0$ with the additional condition $r = \alpha$, then the spacetime has vanishing vorticity and vanishing shear.*

Also in [37] the authors proved that under the same condition mentioned above the vector η_i is a concircular vector and it is rescalable to a unit time like $\nabla_k X_j = \rho g_{jk}$. Hence $\nabla_k X_j + \nabla_j X_k = 2\rho g_{jk}$ which implies that the vector field ρ is a conformal Killing vector field. It has been proved by Sharma [48] that if a spacetime with divergence free conformal curvature tensor admits conformal Killing vector field, then the spacetime is either conformally flat or of Petrov type N. In view of the above result we can state the following:

Theorem 4.4. *A $(WBS)_4$ spacetime satisfying divergence free conformal curvature tensor and satisfying an additional condition $r = \alpha$ is either conformally flat or of Petrov type N.*

The condition $r = \alpha$ allows us to define an equation of state for the perfect fluid . If we consider Einstein's field equations without cosmological constant i.e.

$$R_{ij} - \frac{R}{2} g_{ij} = \kappa T_{ij}, \tag{69}$$

being $\kappa = \frac{8\pi G}{c^4}$ the Einstein gravitational constant, an energy momentum tensor T_{ij} (see [13] and [41]) describing the matter content of the space-time is defined. Then , if the condition (63) is specified, being $u = \eta$ a time-like unit vector field, and recalling that $\beta = n\alpha - r$, Einstein's equations give

$$\kappa T_{ij} = (n\alpha - r) u_i u_j + (\alpha - \frac{r}{2}) g_{ij}.$$

This matches with the expression of a perfect fluid energy momentum tensor (see [13], [41], [52]) $T_{ij} = (\mu + p) u_i u_j + p g_{ij}$, where $\kappa\mu = (n-1)\alpha - \frac{r}{2}$ is the energy density and $\kappa p = \alpha - \frac{r}{2}$ is the isotropic pressure and u_j the fluid flow velocity. Usually in a perfect fluid p and μ are related by an equation of state of the form $p = p(\mu, \Theta)$ being Θ the absolute temperature. In the situation in which the state reduces to the form $p = p(\mu)$ the fluid is named isentropic . In our case the condition $r = \alpha$ gives $\kappa\mu = \frac{r(2n-3)}{2}$, $\kappa p = \frac{r}{2}$ so that $p = \frac{\mu}{2n-3}$. In order to ensure $\mu > 0$ we should have $r > 0$ so that the pressure is positive. In $n = 4$ dimensions the condition $divC = 0$ implies that the space is conformally flat and the space-time reduces to an ordinary Robertson-Walker space-time. The equation of state is $p = \frac{\mu}{5}$. We have the following:

Theorem 4.5. *A $(WBS)_4$ space-time with divergence free conformal curvature tensor represents an isentropic perfect fluid Robertson-Walker space-time with $p = \frac{\mu}{5}$.*

We recall that a four dimensional RW spacetime may be characterized by the metric

$$ds^2 = -dt^2 + a^2(t)[d\chi^2 + \sigma^2(\chi)\{d\theta^2 + \sin^2\theta d\varphi^2\}], \tag{70}$$

being $\sigma(\chi) = \frac{\sin(\sqrt{k}\chi)}{\sqrt{k}}$ where $k = \frac{R^*}{6}$ is the normalized spatial curvature; a(t) is named the scale factor of the Universe. Following [3] it is possible to get the behaviour of the scale factor in terms of hypergeometric functions. As a first consider the Friedmann equations, namely

$$\frac{\ddot{a}}{a} = -\kappa\frac{3p+\mu}{6},$$

$$\frac{R^*}{2a^2} + 3(\frac{\dot{a}}{a})^2 = \kappa\mu, \tag{71}$$

being R^* the spatial curvature, and a dot means a derivation with respect to the time in the co-moving frame. If a state equation $p = (\gamma - 1)\mu$ is supposed, then it is inferred

$$a\ddot{a} + \frac{3\gamma-2}{2}(\dot{a})^2 + k\frac{3\gamma-2}{2} = 0. \tag{72}$$

In our case it is $\gamma = \frac{7}{5}$. A first integral of (72) is provided by

$$(\dot{a})^2 = (\frac{a_0}{a})^{3\gamma-2} - k, \tag{73}$$

with $a_0 > 0$. Now we introduce the auxiliary variable $w = k(\frac{a_0}{a})^{3\gamma-2}$ so that (73) becomes

$$\dot{w} = \frac{2}{(2A-1)a_0}k^A w^{1-A}\sqrt{(1-w)}, \tag{74}$$

where $A = \frac{1}{2}(\frac{3\gamma}{3\gamma-2})$. Equation (74) may be integrated to

$$t - t_0 = \frac{2A-1}{A}a_0 k^{-A} w^A F(\frac{1}{2}; A; A+1; w), \tag{75}$$

where F is the hypergeometric function which converges for $|w| < 1$ and it is not defined if $A = 1-n$, $n = 1, 2, 3, ...$. But in our case $A = \frac{21}{22}$ so that (75) is a well defined solution. It gives the cosmic time as a function of $w = k(\frac{a_0}{a})^{3\gamma-2}$ so that the behaviour of the scale factor is defined only implicitly.

At this point it should be noted that the perfect fluid energy momentum tensor is not the only one compatible with the form (63) of the Ricci tensor . A more general one is given by $T_{ij} = (\mu + p + \Pi)u_i u_j + (p + \Pi)g_{ij}$ (see for example [30]), and represents an imperfect fluid (without heat transfer and shear viscosity) , being Π a dissipative term called bulk viscous pressure. In such a case we have $\kappa\mu = (n-1)\alpha - \frac{r}{2}$ and $\kappa(p + \Pi) = \alpha - \frac{r}{2}$. Again in our case the condition $r = \alpha$ gives $\kappa\mu = \frac{r(2n-3)}{2}$ and $\kappa(p+\Pi) = \frac{r}{2}$ so that $p + \Pi = \frac{\mu}{(2n-3)}$. Theorem 4.5 could refined as follows:

Theorem 4.6. *A $(WBS)_4$ space-time with harmonic conformal curvature tensor represents an imperfect fluid Robertson- Walker space-time with $p + \Pi = \frac{\mu}{5}$, being Π the bulk viscous pressure.*

The geometric form (63) of quasi-Einstein space cannot distinguish between perfect fluids and imperfect fluids with bulk viscous pressure (but without heat transfer and shear viscosity), so that Π is not determined by geometry. Thus it can be only determined by thermodynamics . This is usually done by writing the divergence of the entropy 4-current (see [30]) and then imposing that this quantity is not negative according to the second law of thermodynamics. In this way constitutive (or transport) equations are imposed for the dissipative quantities. In simplest approach , called the Eckart theory, for the bulk viscosity pressure, it is $\Pi = -\zeta\nabla_i u_i$ being ζ the bulk viscosity. For a generalized Robertson-Walker space-time (see [31]) it is $\nabla_i u^i = 3H$, being H the Hubble's parameter in standard cosmology: we have thus $\Pi = -3\zeta H$. The algebraic nature of the Eckart constitutive equations leads to severe problems because they violate relativistic causality. In a more refined approach, called the Israel-Stewart theory, the constitutive equations satisfy causality: for the bulk viscosity pressure it is (in the Maxwell-Cattaneo truncated form ([30]) $\tau_0\dot{\Pi} + \Pi = -\zeta\nabla_i u^i$, being τ_0 is a relaxational time restoring causality and $\dot{\Pi} = u^k\nabla_k\Pi$. For a generalized Robertson-Walker space-time it is thus $\tau_0\dot{\Pi} + \Pi = -3\zeta H$. In particularly simple cases (τ_0, ζ constants, Π depending only on time) this last equation ensures that the bulk viscosity pressure evolves towards a limit value.

More recently a new model for the bulk viscosity was proposed by Disconzi et al. in [15] (see also [12]and [14]). It is nearly as simple as Eckart viscosity but does not have the causality problems of that model. Moreover it is much simpler than the Israel-Stewart theory and it is possible to conjecture that the model is causal for all physical systems of interest, like the Israel-Stewart one, as stated in [15]. More explicitly in this model it is $\Pi = -\zeta\nabla_j C^j$, where C_j is the dynamic velocity, defined by $C_j = Fu_j$ and F is called the index of the fluid and depends on the nature of the fluid. In particular Disconzi's model uses $F = \frac{p+\mu}{n}$, where n is the rest mass density, satisfying $\nabla_j(nu^j) = 0$ (conservation along the flow lines). We have thus $\Pi = -\zeta(\dot{F} + 3HF)$ and $\dot{n} = -3Hn$ so that $\Pi = -\frac{\zeta}{n}[\dot{\mu} + \dot{p} + 6H(\mu + p)]$. Now from Theorem 4.6 we have $\dot{p} + \dot{\Pi} = \frac{\dot{\mu}}{5}$; the time evolution equation for the energy density $\dot{\mu} + 3H(\mu + p + \Pi) = 0$ [30] gives $\dot{\mu} = -\frac{18}{5}H\mu$ so that with a bit of algebra we get $\frac{\zeta}{n}\dot{\Pi} - \Pi(1 + 6\frac{\zeta H}{n}) = \frac{\zeta H\mu}{25n}$. Unlike the Eckart and the Israel-Stewart models, in this case the time evolution of the bulk viscosity pressure depends on the equation of state of the imperfect fluid given in Theorem 4.6. Finally it should be noted that in this model the entropy production results to be non-negative for a wide range of possible cases [15].

5. Example of a $(WBS)_4$

In this section we construct an example of a weakly B-symmetric pseudo Riemannian manifold.
We consider a pseudo Riemannian manifold (M^4, g) endowed with the Lorentzian metric g given by

$$ds^2 = g_{ij}dx^i dx^j = -(dx^1)^2 + e^{x^1}[(dx^2)^2 + (dx^3)^2 + (dx^4)^2], \tag{76}$$

where $i, j = 1, 2, 3, 4$.
The only non-vanishing components of the Christoffel symbols, the curvature tensor and the Ricci tensor are

$$\Gamma^1_{22} = \Gamma^1_{33} = \Gamma^1_{44} = \frac{1}{2}e^{x^1}, \ \ \Gamma^2_{12} = \Gamma^3_{13} = \Gamma^4_{14} = \frac{1}{2},$$

$$R_{1221} = R_{1331} = R_{1441} = -\frac{1}{4}e^{x^1}, R_{2332} = R_{2442} = R_{3443} = -\frac{1}{4}e^{2x^1},$$

$$S_{11} = -\frac{1}{2}, \ \ S_{22} = S_{33} = S_{44} = -\frac{1}{4}e^{x^1}.$$

It can be easily shown that the scalar curvature r of the resulting manifold (M^4, g) is $-\frac{1}{4}$ which is non-vanishing and constant.
We shall now show that this M^4 is a $(WBS)_4$ spacetime i.e., it satisfies the defining relation (7).
Let us take the non-zero arbitrary scalar functions a and b as follows:

$$a = 4e^{2x^1}, \ \ b = 4e^{x^1}.$$

Then only the non vanishing component for B tensor and its covarient derivatives are given by

$$B_{11} = -2e^{2x^1} + e^{x^1}, \ \ B_{22} = B_{33} = B_{44} = -e^{3x^1} - e^{2x^1}$$

$$B_{11,1} = -4e^{2x^1} + e^{x^1}, \ \ B_{22,1} = B_{33,1} = B_{44,1} = -2e^{3x^1} - e^{2x^1}.$$

We choose the 1-forms as follows:

$$A_i(x) = \begin{cases} \frac{2e^{x^1}+1}{e^{x^1}+1}, & \text{for i=1} \\ \frac{1}{3x^2}, & \text{for i=2} \\ e^{x^1}, & \text{for i=3,4} \end{cases}$$

$$D_i(x) = \begin{cases} \frac{1}{-2e^{x^1}+1}, & \text{for i=1} \\ \frac{1}{3x^2}, & \text{for i=2} \\ x^1, & \text{for i=3,4} \end{cases}$$

$$E_i(x) = \begin{cases} -\frac{1}{e^{x^1}+1}, & \text{for i=1} \\ \frac{1}{5x^2}, & \text{for i=2} \\ x^1x^2, & \text{for i=3,4} \end{cases}$$

at any point $x \in M$. In our (M^4, g), (7) reduces with these 1-forms to the following equations:

$$B_{11,1} = A_1B_{11} + D_1B_{11} + E_1B_{11} \tag{77}$$

$$B_{22,1} = A_1B_{22} + D_2B_{12} + E_2B_{12} \tag{78}$$

$$B_{33,1} = A_1B_{33} + D_3B_{13} + E_3B_{13} \tag{79}$$

and

$$B_{44,1} = A_1B_{44} + D_4B_{14} + E_4B_{14} \tag{80}$$

It can be easily verified that the equations (77), (78), (79) and (80) are true.
So, the manifold under consideration is a weakly B-symmetric pseudo Riemannian spacetime, that is, $(WBS)_4$.

6. Conclusion

Quasi Einstein manifolds arose during the study of exact solutions of the Einstein field equations. It is proved that a $(WBS)_4$ spacetime is a quasi-Einstein spactime. So a $(WBS)_4$ spacetime can be taken as a model of the perfect fluid spacetime in general relativity. Also it is shown that a $(WBS)_4$ spacetime with divergence free conformal curvature tensor under certain condition is a perfect fluid generalized Robertson-Walker spacetime and the nature of the spacetime is of vanishing vorticity and vanishing shear. Moreover a $(WBS)_4$ spacetime satisfying divergence free conformal curvature tensor under certain condition is either conformally flat or of Petrov type N. Again we prove that a $(WBS)_4$ space-time with divergence free conformal curvature tensor represents a isentropic perfect fluid Robertson-Walker spacetime and represents an imperfect fluid Robertson-Walker spacetime.

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