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On Extended Commuting Operators

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Abstract. In this paper, we study properties of extended commuting operators. In particular, we provide the polar decomposition of the product of (λ, μ)-commuting operators where λ and μ are real numbers with $\lambda\mu > 0$. Furthermore, we find the restriction of μ for the product of (λ, μ)-commuting quasihyponormal operators to be quasihyponormal. We also give spectral and local spectral relations between λ-commuting operators. Moreover, we show that the operators λ-commuting with a unilateral shift are representable as weighted composition operators.

1. Introduction

Let $\mathcal{H}$ be a separable complex Hilbert space and let $\mathcal{L}(\mathcal{H})$ denote the algebra of all bounded linear operators on $\mathcal{H}$. For $T \in \mathcal{L}(\mathcal{H})$, we write $\sigma(T)$, $\sigma_p(T)$, $\sigma_{ap}(T)$, $\sigma_{le}(T)$, and $r(T)$ for the spectrum, the point spectrum, the approximate point spectrum, the left essential spectrum, and the spectral radius of T, respectively.

For operators $S, T \in \mathcal{L}(\mathcal{H})$ and a complex number λ, we say that S is *λ-commuting* with T if $ST = \lambda TS$. Different classes of operators can be specified depending on the restriction on λ (see [16]). The λ-commuting relation of operators on Hilbert spaces is often useful in quantum mechanics as a tool for the analysis of their spectra. For example, there is an anti-commuting relation (i.e., $\lambda = -1$) between Pauli spin matrices which are complex matrices arising in the study of spin in quantum mechanics ([3]). It has been studied further on the λ-commuting property in the context of quantum groups (see [3] and [11]).

In [4], S. Brown showed that operators λ-commuting with nonzero compact operators have nontrivial hyperinvariant subspaces, as one of the generalizations of the famous Lomonosov's theorem about the invariant subspace problem for operators commuting with compact operators (see [14]). Since then, many mathematicians have been interested in λ-commuting operators.

For $\lambda, \mu \in \mathbb{C}$, two operators $S, T \in \mathcal{L}(\mathcal{H})$ are said to be *(λ, μ)-commuting* if S is λ-commuting and S^* is μ-commuting with T, namely $ST = \lambda TS$ and $S^*T = \mu TS^*$. If $S, T \in \mathcal{L}(\mathcal{H})$ are (λ, μ)-commuting, then S^*S is $\lambda\mu$-commuting with T. In particular, if S is an isometry, then $\lambda\mu = 1$.

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By Fuglede-Putnam Theorem, if $A, B \in \mathcal{L}(\mathcal{H})$ are normal and $AX = XB$ for some $X \in \mathcal{L}(\mathcal{H})$, then $A^*X = XB^*$ (see [8]). Hence, if S is normal and λ-commuting with T, then S and T are $(\lambda, \overline{\lambda})$-commuting. This observation gives several examples of (λ, μ)-commuting operators. For a simple example, given any fixed complex constant λ with $|\lambda| \le 1$, suppose D is a diagonal operator given by $De_n = \lambda^n e_n$ for $n \ge 0$, where $\{e_n\}_{n=0}^{\infty}$ is an orthonormal basis for $\mathcal{H}$. Then, every weighted shift W on $\mathcal{H}$ determined by $We_n = \alpha_n e_{n+1}$ for $n \ge 0$ satisfies $DW = \lambda WD$. Since D is normal, the operators D and W are $(\lambda, \overline{\lambda})$-commuting by Fuglede-Putnam Theorem; we also observe that W and D are (λ^{-1}, λ)-commuting. For another example, the 2×2 matrices $S = \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}$ and $T = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$ are $(\frac{1}{3}, 3)$-commuting.

In this paper, we study properties of extended commuting operators. In particular, we provide the polar decomposition of the product of (λ, μ)-commuting operators where λ and μ are real numbers with $\lambda\mu > 0$. Furthermore, we find the restriction of μ for the product of (λ, μ)-commuting quasihyponormal operators to be quasihyponormal. We also give spectral and local spectral relations between λ-commuting operators. Moreover, we show that the operators λ-commuting with a unilateral shift are representable as weighted composition operators.

2. Preliminaries

An operator $T \in \mathcal{L}(\mathcal{H})$ is said to have *the single-valued extension property* (or SVEP) if for every open set G in $\mathbb{C}$ and every analytic function $f : G \to \mathcal{H}$ with $(T - z)f(z) \equiv 0$ on G, we have $f(z) \equiv 0$ on G. For an operator $T \in \mathcal{L}(\mathcal{H})$ and a vector $x \in \mathcal{H}$, the set $\rho_T(x)$, called the *local resolvent* of T at x, consists of elements z_0 in $\mathbb{C}$ such that there exists an $\mathcal{H}$-valued analytic function $f(z)$ defined in a neighborhood of z_0 which verifies $(T - z)f(z) \equiv x$. The *local spectrum* of T at x is given by $\sigma_T(x) := \mathbb{C} \setminus \rho_T(x)$. Moreover, we define the *local spectral subspace* of T as $H_T(F) := \{x \in \mathcal{H} : \sigma_T(x) \subset F\}$, where F is a subset of $\mathbb{C}$. An operator $T \in \mathcal{L}(\mathcal{H})$ is said to have *Dunford's property* (C) if $H_T(F)$ is closed for each closed subset F of $\mathbb{C}$. We say that $T \in \mathcal{L}(\mathcal{H})$ is said to have *Bishop's property* (β) if for every open subset G of $\mathbb{C}$ and every sequence $f_n : G \to \mathcal{H}$ of $\mathcal{H}$-valued analytic functions such that $(T - z)f_n(z)$ converges uniformly to 0 in norm on compact subsets of G, then $f_n(z)$ converges uniformly to 0 in norm on compact subsets of G. The following implications are well known (see [2], [6], or [13] for more details):

Bishop's property (β) $\Rightarrow$ Dunford's property (C) $\Rightarrow$ SVEP.

3. Main results

In this section, we first consider (λ, μ)-commuting operators. As remarked at section one, Fugled-Putnam theorem gives examples of $(\lambda, \overline{\lambda})$-commuting operators. We remark that if $S, T \in \mathcal{L}(\mathcal{H})$ are $(\lambda, \overline{\lambda})$-commuting for some $\lambda \in \mathbb{C}$ and $T \neq 0$ does not have dense range, then $\mathrm{ran}(T)$ is a common nontrivial invariant subspace for S and S^*, since $S(Tx) = \lambda TSx \in \mathrm{ran}(T)$ and $S^*(Tx) = \overline{\lambda} TS^*x \in \mathrm{ran}(T)$ for all $x \in \mathcal{H}$.

In order to provide the polar decomposition of the product of (λ, μ)-commuting operators, we show that their partial isometric parts and positive parts satisfy the following extended commuting relationships.

Lemma 3.1. *Let $S, T \in \mathcal{L}(\mathcal{H})$ be (λ, μ)-commuting where λ and μ are real numbers with $\lambda\mu > 0$. If $S = U_S|S|$ and $T = U_T|T|$ denote the polar decompositions, then the following statements hold:*
(i) $|T|S = (\lambda^{-1}\mu)^{\frac{1}{2}} S|T|$ and $|S|T = (\lambda\mu)^{\frac{1}{2}} T|S|$;
(ii) $|S|U_T = (\lambda\mu)^{\frac{1}{2}} U_T|S|$ and $|T|U_S = (\lambda^{-1}\mu)^{\frac{1}{2}} U_S|T|$;
(iii) $|S||T| = |T||S|$, $|S^*||T| = |T||S^*|$, and $|S||T^*| = |T^*||S|$;
(iv) $U_SU_T = U_TU_S$ and $U_S^*U_T = U_TU_S^*$ if λ and μ are positive, and $U_SU_T = -U_TU_S$ and $U_S^*U_T = -U_TU_S^*$ if λ and μ are negative.

Proof. (i) Set $\delta = \lambda^{-1}\mu > 0$. It is easy to see that $p(|T|^2)S = Sp(\delta|T|^2)$ for any polynomial p. Since there exists a sequence $\{p_n\}$ of polynomials convergent uniformly to $f(s) = s^{\frac{1}{2}}$ on the compact interval $[0,(1+\delta)\|T\|^2]$ from Stone-Weierstrass theorem, it follows that $|T|S = \delta^{\frac{1}{2}}S|T| = (\lambda^{-1}\mu)^{\frac{1}{2}}S|T|$.

Similarly, we infer that $|S|T = (\lambda\mu)^{\frac{1}{2}}T|S|$, since T and S are (λ^{-1},μ)-commuting.

(ii) Since $ST = \lambda TS$ and $|T|S = (\lambda^{-1}\mu)^{\frac{1}{2}}S|T|$ by (i), we obtain that

$$SU_T|T| = \lambda U_T|T|S = \lambda(\lambda^{-1}\mu)^{\frac{1}{2}}U_TS|T| = \tau(\lambda\mu)^{\frac{1}{2}}U_TS|T|$$

where $\tau = 1$ if $\lambda,\mu > 0$, and $\tau = -1$ if $\lambda,\mu < 0$. If $x \in \ker(|T|)$, then $|T|Sx = (\lambda^{-1}\mu)^{\frac{1}{2}}S|T|x = 0$. Since $\ker(|T|) = \ker(U_T)$, we have $U_TSx = 0 = SU_Tx$. Hence, it holds that

$$SU_T = \tau(\lambda\mu)^{\frac{1}{2}}U_TS \text{ on } \mathcal{H} = \overline{\operatorname{ran}(|T|)}\oplus\ker(|T|). \tag{1}$$

In a similar manner, use the identities $S^*T = \mu TS^*$ and $S^*|T| = (\lambda^{-1}\mu)^{\frac{1}{2}}|T|S^*$ obtained from (i) to verify that

$$S^*U_T|T| = \mu U_T|T|S^* = \mu(\lambda\mu^{-1})^{\frac{1}{2}}U_TS^*|T| = \tau(\lambda\mu)^{\frac{1}{2}}U_TS^*|T|$$

and $U_TS^*x = S^*U_Tx = 0$ for all $x \in \ker(|T|)$, which yields that

$$S^*U_T = \tau(\lambda\mu)^{\frac{1}{2}}U_TS^*. \tag{2}$$

Combining (2) with (1), one can deduce that $(S^*S)U_T = \lambda\mu U_T(SS^*)$. As in the proof of (i), we can show that $|S|U_T = (\lambda\mu)^{\frac{1}{2}}U_T|S|$. Since T and S are (λ^{-1},μ)-commuting, we see that $|T|U_S = (\lambda^{-1}\mu)^{\frac{1}{2}}U_S|T|$.

(iii) Since $|T||S|^2 = (|T|S^*)S = (\lambda^{-1}\mu)^{-\frac{1}{2}}S^*(|T|S) = |S|^2|T|$ from (i), the positive parts $|S|$ and $|T|$ commute. Moreover, observe that S^* and T are (μ,λ)-commuting, while S and T^* are (μ^{-1},λ^{-1})-commuting. Hence, we also see that $|S^*||T| = |T||S^*|$ and $|S||T^*| = |T^*||S|$.

(iv) Applying (ii) and (iii), we obtain that

$$ST = U_S(|S|U_T)|T| = (\lambda\mu)^{\frac{1}{2}}U_SU_T|S||T| \tag{3}$$

and

$$\lambda TS = \lambda U_T(|T|U_S)|S| = \lambda U_T\Big((\lambda^{-1}\mu)^{\frac{1}{2}}U_S|T|\Big)|S| = \tau(\lambda\mu)^{\frac{1}{2}}U_TU_S|S||T|$$

where $\tau = 1$ if $\lambda,\mu > 0$, and $\tau = -1$ if $\lambda,\mu < 0$. Since $ST = \lambda TS$, we have $U_SU_T = \tau U_TU_S$ on $\overline{\operatorname{ran}(|S||T|)}$. In addition, if $x \in \ker(|S||T|) = \ker(|T||S|)$, then $U_S|T|x = U_T|S|x = 0$, and so $|T|U_Sx = |S|U_Tx = 0$ from (ii). Therefore, both U_SU_T and U_TU_S are identically zero on $\ker(|S||T|)$, concluding that $U_SU_T = \tau U_TU_S$ on $\mathcal{H} = \overline{\operatorname{ran}(|S||T|)} \oplus \operatorname{ran}(|S||T|)^\perp$. Since S^* and T are (μ,λ)-commuting and the adjoint S^* has the polar decomposition $S^* = U_S^*|S^*|$, we derive the equality for U_S^* and U_T, as well. □

Theorem 3.2. *Assume that $S,T \in \mathcal{L}(\mathcal{H})$ are (λ,μ)-commuting where λ and μ are real numbers with $\lambda\mu > 0$. If $ST = U_{ST}|ST|$ is the polar decomposition, then*

$$U_{ST} = U_SU_T \text{ and } |ST| = (\lambda\mu)^{\frac{1}{2}}|S||T|.$$

In addition, if $TS = U_{TS}|TS|$ is the polar decomposition, then

$$U_{TS} = U_TU_S \text{ and } |TS| = (\lambda^{-1}\mu)^{\frac{1}{2}}|S||T|.$$

Proof. Since T and S are (λ^{-1},μ)-commuting and $|S||T| = |T||S|$ by Lemma 3.1, it suffices to consider the product ST. We note from (3) that the product ST is factorized as follows:

$$ST = (U_SU_T)\Big((\lambda\mu)^{\frac{1}{2}}|S||T|\Big).$$

Since $|S||T| = |T||S|$, we obtain that

$$(|S||T|)^2 = S^*(ST^*)T = \mu^{-1}(S^*T^*)ST = (\lambda\mu)^{-1}T^*S^*ST = (\lambda\mu)^{-1}|ST|^2.$$

This ensures that

$$|ST| = (\lambda\mu)^{\frac{1}{2}}|S||T|. \tag{4}$$

If $x \in \ker(ST) = \ker(|ST|)$, then $|S||T|x = 0$ from (4). Since $|T|U_Sx = (\lambda^{-1}\mu)^{\frac{1}{2}}U_S|T|x = 0$ by Lemma 3.1, we have $U_Sx \in \ker(|T|) = \ker(U_T)$, namely $x \in \ker(U_TU_S)$. Conversely, if $x \in \ker(U_TU_S)$, then $|T|U_Sx = 0$ and so $U_S|T|x = 0$, which implies with (4) that $x \in \ker(|S||T|) = \ker(ST)$. Since $\ker(U_SU_T) = \ker(U_TU_S)$ by Lemma 3.1, it holds that

$$\ker(U_SU_T) = \ker(U_TU_S) = \ker(ST) = \ker(|ST|).$$

Now, it remains to show U_SU_T is partial isometric on $\ker(U_SU_T)^\perp = \overline{\mathrm{ran}(|ST|)}$. Equations (1), (2), and (4) give that

$$\begin{aligned}(U_SU_T)^*(U_SU_T)|ST| &= (U_TU_S)^*(U_TU_S)|ST| = U_S^*U_T^*U_TU_S\Big((\lambda\mu)^{\frac{1}{2}}|S||T|\Big)\\ &= U_S^*U_T^*\Big((\lambda\mu)^{\frac{1}{2}}U_TS\Big)|T| = U_S^*(\tau U_T^*S)U_T|T|\\ &= U_S^*\Big((\lambda\mu)^{\frac{1}{2}}SU_T^*\Big)U_T|T| = (\lambda\mu)^{\frac{1}{2}}(U_S^*U_S|S|)(U_T^*U_T|T|)\\ &= (\lambda\mu)^{\frac{1}{2}}|S||T| = |ST|,\end{aligned}$$

completing the proof. □

For an operator $T \in \mathcal{L}(\mathcal{H})$ with polar decomposition $T = U|T|$, we define the *Aluthge transform* of T, denoted by $\widetilde{T}$, as

$$\widetilde{T} = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}.$$

Recall that an operator $T \in \mathcal{L}(\mathcal{H})$ is called *p-hyponormal* if $(T^*T)^p \geq (TT^*)^p$, where $0 < p < \infty$. It is well known that if $0 < q < p < \infty$, then p-hyponormal operators are q-hyponormal. In addition, $T \in \mathcal{L}(\mathcal{H})$ is p-hyponormal for some $0 < p < 1$, then $\widetilde{T}$ is $(p + \frac{1}{2})$-hyponormal (see [7]). Hence, $\widetilde{T}$ is always hyponormal. In [10], the authors gave several connections between operators and their Aluthge transforms.

Corollary 3.3. *If $S, T \in \mathcal{L}(\mathcal{H})$ are (λ, μ)-commuting operators where λ and μ are real numbers with $\lambda\mu > 0$, then the following statements hold:*
(i) $\widetilde{S}$ and $\widetilde{T}$ are (λ, μ)-commuting and $\widetilde{ST} = |\mu|^{\frac{1}{2}}\widetilde{S}\widetilde{T} = \lambda|\mu|^{\frac{1}{2}}\widetilde{T}\widetilde{S}$.
(ii) $\widetilde{S}$ and T are (λ, μ)-commuting.
(iii) S and $\widetilde{T}$ are (λ, μ)-commuting.

Proof. Let $S = U_S|S|$ and $T = U_T|T|$ be the polar decompositions of S and T.

(i) Lemma 3.1 implies that $|S|^{\frac{1}{2}}U_T = (\lambda\mu)^{\frac{1}{4}}U_T|S|^{\frac{1}{2}}$, $|T|^{\frac{1}{2}}U_S = (\lambda^{-1}\mu)^{\frac{1}{4}}U_S|T|^{\frac{1}{2}}$, and $(|S||T|)^{\frac{1}{2}} = |S|^{\frac{1}{2}}|T|^{\frac{1}{2}} = |T|^{\frac{1}{2}}|S|^{\frac{1}{2}}$. Thus, we obtain from Theorem 3.2 that

$$\begin{aligned}\widetilde{ST} &= (\lambda\mu)^{\frac{1}{2}}|S|^{\frac{1}{2}}(|T|^{\frac{1}{2}}U_S)(U_T|S|^{\frac{1}{2}})|T|^{\frac{1}{2}} = |\mu|^{\frac{1}{2}}|S|^{\frac{1}{2}}U_S|T|^{\frac{1}{2}}|S|^{\frac{1}{2}}U_T|T|^{\frac{1}{2}}\\ &= |\mu|^{\frac{1}{2}}(|S|^{\frac{1}{2}}U_S|S|^{\frac{1}{2}})(|T|^{\frac{1}{2}}U_T|T|^{\frac{1}{2}}) = |\mu|^{\frac{1}{2}}\widetilde{S}\widetilde{T}.\end{aligned}$$

Since $U_SU_T = \tau U_TU_S$ where $\tau = 1$ if $\lambda, \mu > 0$, and $\tau = -1$ if $\lambda, \mu < 0$, we get that

$$\begin{aligned}\widetilde{ST} &= \tau(\lambda\mu)^{\frac{1}{2}}|T|^{\frac{1}{2}}(|S|^{\frac{1}{2}}U_T)(U_S|T|^{\frac{1}{2}})|S|^{\frac{1}{2}} = \lambda|\mu|^{\frac{1}{2}}|T|^{\frac{1}{2}}U_T|S|^{\frac{1}{2}}|T|^{\frac{1}{2}}U_S|S|^{\frac{1}{2}}\\ &= \lambda|\mu|^{\frac{1}{2}}(|T|^{\frac{1}{2}}U_T|T|^{\frac{1}{2}})(|S|^{\frac{1}{2}}U_S|S|^{\frac{1}{2}}) = \lambda|\mu|^{\frac{1}{2}}\widetilde{T}\widetilde{S}.\end{aligned}$$

Hence, we see that $\widetilde{ST} = \lambda\widetilde{TS}$. Furthermore, since $U_S^*U_T = \tau U_T U_S^*$ by Lemma 3.1, it holds that

$$\begin{aligned}(\widetilde{S})^*\widetilde{T} &= (|S|^{\frac{1}{2}}U_S^*|S|^{\frac{1}{2}})(|T|^{\frac{1}{2}}U_T|T|^{\frac{1}{2}}) = |S|^{\frac{1}{2}}(U_S^*|T|^{\frac{1}{2}})(|S|^{\frac{1}{2}}U_T)|T|^{\frac{1}{2}}\\ &= |\mu|^{\frac{1}{2}}|S|^{\frac{1}{2}}|T|^{\frac{1}{2}}U_S^*U_T|S|^{\frac{1}{2}}|T|^{\frac{1}{2}} = \tau|\mu|^{\frac{1}{2}}|T|^{\frac{1}{2}}(|S|^{\frac{1}{2}}U_T)(U_S^*|T|^{\frac{1}{2}})|S|^{\frac{1}{2}}\\ &= \mu|T|^{\frac{1}{2}}U_T|S|^{\frac{1}{2}}|T|^{\frac{1}{2}}U_S^*|S|^{\frac{1}{2}} = \mu\widetilde{T}(\widetilde{S})^*.\end{aligned}$$

(ii) Using Lemma 3.1, we see that

$$\begin{aligned}\widetilde{S}T &= |S|^{\frac{1}{2}}U_S(|S|^{\frac{1}{2}}T) = (\lambda\mu)^{\frac{1}{4}}|S|^{\frac{1}{2}}U_ST|S|^{\frac{1}{2}}\\ &= (\lambda\mu)^{\frac{1}{4}}|S|^{\frac{1}{2}}(U_SU_T)|T||S|^{\frac{1}{2}} = \tau(\lambda\mu)^{\frac{1}{4}}(|S|^{\frac{1}{2}}U_T)(U_S|T|)|S|^{\frac{1}{2}}\\ &= \tau(\lambda\mu)^{\frac{1}{2}}(\lambda^{-1}\mu)^{-\frac{1}{2}}U_T(|S|^{\frac{1}{2}}|T|)U_S|S|^{\frac{1}{2}} = \lambda T\widetilde{S}\end{aligned}$$

where $\tau = 1$ if $\lambda, \mu > 0$, and $\tau = -1$ if $\lambda, \mu < 0$. This means that $\widetilde{S}$ and T are λ-commuting. Similarly, one can derive that $(\widetilde{S})^*$ and T are μ-commuting.

(iii) Since T and S are (λ^{-1}, μ)-commuting, we obtain from (ii) that $\widetilde{T}$ and S are (λ^{-1}, μ)-commuting, or equivalently, S and $\widetilde{T}$ are (λ, μ)-commuting. □

Remark 3.4. *Let $S, T \in \mathcal{L}(\mathcal{H})$ be λ-commuting for some nonzero real number λ. If $\widetilde{S}$ is hyponormal and T is normal, then $\widetilde{S}$ and $\widetilde{T} = T$ are (λ, λ^{-1})-commuting by Theorem 3.3 and Fuglede-Putnam Theorem. Since $\sigma(\widetilde{ST}) = \sigma(ST)$ due to [10], we obtain from [18] that $\widetilde{ST}$ is hyponormal if and only if $\sigma(ST) \neq \{0\}$, which holds exactly when $\lambda = \pm 1$.*

Recall that an operator $T \in \mathcal{L}(\mathcal{H})$ is said to be *quasinormal* if T^*T commutes with T.

Corollary 3.5. *Let $S, T \in \mathcal{L}(\mathcal{H})$ be (λ, μ)-commuting quasinormal operators such that $ST \neq 0$, where λ and μ are real numbers with $\lambda\mu > 0$. Then ST is quasinormal if and only if $\mu = \pm 1$. In particular, if one of S and T is normal and the product ST is quasinormal, then $\lambda = \mu = \pm 1$.*

Proof. Assume $S = U_S|S|$, $T = U_T|T|$, and $ST = U_{ST}|ST|$ are the polar decompositions. We will use the equalities $U_{ST} = U_SU_T = \pm U_TU_S$, $|ST| = (\lambda\mu)^{\frac{1}{2}}|S||T|$, $|S|U_T = (\lambda\mu)^{\frac{1}{2}}U_T|S|$, and $|T|U_S = (\lambda^{-1}\mu)^{\frac{1}{2}}U_S|T|$ obtained from Lemma 3.1 and Theorem 3.2. Since $U_S|S| = |S|U_S$ and $U_T|T| = |T|U_T$ due to the quasinormality of S and T, it follows that

$$\begin{aligned}U_{ST}|ST| &= ST = U_S|S|U_T|T| = |S|(U_S|T|)U_T\\ &= (\lambda^{-1}\mu)^{-\frac{1}{2}}|S||T|U_SU_T = (\lambda^{-1}\mu)^{-\frac{1}{2}}(\lambda\mu)^{-\frac{1}{2}}|ST|U_{ST} = |\mu|^{-1}|ST|U_{ST}.\end{aligned}$$

Hence, ST is quasinormal if and only if $\mu = \pm 1$.

If S or T is normal and ST is quasinormal, then $S^*T = \mu TS^* = \pm TS^*$. Fuglede-Putnam theorem implies that $ST = \pm TS$, and thus $\lambda = \mu = \pm 1$. □

An operator $T \in \mathcal{L}(\mathcal{H})$ is called *quasihyponormal* if $T^*(T^*T - TT^*)T \geq 0$, or $\|T^2x\| \geq \|T^*Tx\|$ for all $x \in \mathcal{H}$. In the following theorem, we show that if $|\mu| \leq 1$, then the product of two (λ, μ)-commuting quasihyponormal operators is again quasihyponormal.

Theorem 3.6. *Let S and T be quasihyponormal operators in $\mathcal{L}(\mathcal{H})$ that are (λ, μ)-commuting. If $|\mu| \geq 1$, then ST is quasihyponormal. Furthermore, if $\lambda \neq 0$ and $|\mu| \geq 1$, then TS is quasihyponormal.*

Proof. Suppose that $ST = \lambda TS$ and $S^*T = \mu TS^*$. Since S and T are quasihyponormal, we obtain that

$$\begin{aligned}\|(ST)^2x\| &= \|\lambda^2T(ST)Sx\| = |\lambda|^3\|T^2S^2x\|\\ &\geq |\lambda|^3\|T^*TS^2x\| = |\lambda|^2\|T^*(\lambda TS)Sx\|\\ &= |\lambda|^2\|(T^*S)TSx\| = |\lambda|\ \|\overline{\mu}ST^*(\lambda TS)x\|\end{aligned}$$

$$
\begin{aligned}
&= |\lambda||\mu|\,\|S(T^*S)Tx\| = |\lambda||\mu|^2\|S^2T^*Tx\| \\
&\geq |\lambda||\mu|\,\|S^*(\overline{\mu}ST^*)Tx\| = |\mu|\,\|(\overline{\lambda}S^*T^*)STx\| \\
&= |\mu|\,\|T^*S^*STx\| = |\mu|\,\|(ST)^*(ST)x\|
\end{aligned}
$$

for all $x \in \mathcal{H}$. Thus, if $|\mu| \geq 1$, then ST is quasihyponormal.

If $\lambda \neq 0$, then T and S are $(\lambda^{-1}, \overline{\mu})$-commuting. As in the above argument, one can deduce that if $|\mu| \geq 1$, then TS is quasihyponormal. □

An operator T in $\mathcal{L}(\mathcal{H})$ is said to be *nilpotent* if $T^n = 0$ for some positive integer n; in this case, the smallest positive integer n with $T^n = 0$ is referred to as the order of T. We say that $T \in \mathcal{L}(\mathcal{H})$ is *quasinilpotent* if $\sigma(T) = \{0\}$.

Corollary 3.7. *Let S and T be quasihyponormal operators in $\mathcal{L}(\mathcal{H})$ that are (λ, μ)-commuting and $ST \neq 0$. If $|\lambda| \neq 1$ and $|\mu| \geq 1$, then ST is nilpotent of order 2 and one of S and T has a nontrivial invariant subspace.*

Proof. Let $|\lambda| \neq 1$ and $|\mu| \geq 1$. The product ST is quasihyponormal by Theorem 3.6. Applying [12], we write $ST = \begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix}$ on $\mathcal{H} = \mathcal{M} \oplus \mathcal{M}^\perp$, where A is hyponormal and $\mathcal{M} = \overline{\mathrm{ran}(ST)}$. It is easy to see that if $\alpha \neq 0$ and $ST - \alpha$ is invertible, then so is $A - \alpha$. Indeed, if $X = \begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \end{pmatrix}$ is the inverse of $ST - \alpha$, then we have $X_1(A - \alpha) = I_{\mathcal{M}}$, $(A - \alpha)X_1 + BX_3 = I_{\mathcal{M}}$, and $\alpha X_3 = 0$ where $I_{\mathcal{M}}$ denotes the identity operator on $\mathcal{M}$. Since α is nonzero, $X_3 = 0$ so that $A - \alpha$ is invertible. Hence $\sigma(A) \setminus \{0\} \subset \sigma(ST) \setminus \{0\}$. Since ST is quasinilpotent from the proof of Theorem 3.9, it follows that $\sigma(A) \setminus \{0\} = \emptyset$, i.e., $\sigma(A) = \{0\}$. Since A is hyponormal, it should be the zero operator. Thus $ST = \begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix}$ is nilpotent of order 2. Applying [5, Theorem 5], we get that $\overline{\mathrm{ran}(S)} \neq \mathcal{H}$ or $\ker(T) \neq \{0\}$. Accordingly, S or T has a nontrivial invariant subspace. □

Corollary 3.8. *Let $S \in \mathcal{L}(\mathcal{H})$ be normal and $T \in \mathcal{L}(\mathcal{H})$ be quasihyponormal with $ST \neq 0$. If $ST = \lambda TS$ for some $|\lambda| \geq 1$, then both ST and TS are quasihyponormal; in particular, if $|\lambda| > 1$, then ST and TS are nilpotent of order 2.*

Proof. Since $S^*T = \overline{\lambda}TS^*$ by Fuglede-Putnam Theorem, the operators S and T are $(\lambda, \overline{\lambda})$-commuting. Moreover, T and S are (λ^{-1}, λ)-commuting. Therefore, the products ST and TS are quasihyponormal due to Theorem 3.6. If $|\lambda| > 1$, then we obtain from Corollary 3.7 that ST and TS are nilpotent of order 2. □

We next give several properties of λ-commuting operators. We first consider the product of λ-commuting operators.

Theorem 3.9. *Let S and T be operators in $\mathcal{L}(\mathcal{H})$ such that $ST = \lambda TS$ for some $\lambda \in \mathbb{C}$. Then the following statements hold:*
(i) $r(ST) \leq r(S)r(T)$ and $r(TS) \leq r(S)r(T)$.
(ii) Suppose that $\lambda \neq 0$ and $dist(\frac{z}{\lambda}, \sigma(T)) > 0$ for $z \in \rho(T)$. If f is analytic in a neighborhood of $\sigma(T)$, then $Sf(T) = f(\lambda T)S$.

Proof. (i) Assume that $ST = \lambda TS$ for some $\lambda \in \mathbb{C}$. If $\lambda = 0$, then $ST = 0$ and hence $\sigma(ST) = \{0\}$. Since $\sigma(ST) \setminus \{0\} = \sigma(TS) \setminus \{0\}$, both ST and TS are quasinilpotent. Thus $r(ST) = r(TS) = 0 \leq r(S)r(T)$.

If $\lambda \neq 0$ and $|\lambda| \neq 1$, then $\lambda\sigma(TS) \cup \{0\} = \sigma(ST) \cup \{0\} = \sigma(TS) \cup \{0\}$ where $\lambda\sigma(TS) := \{\lambda\alpha : \alpha \in \sigma(TS)\}$. Since $|\lambda| \neq 1$, we have $\sigma(TS) = \{0\}$. Then ST and TS are quasinilpotent, and so we obtain the given inequalities, obviously.

Assume that $|\lambda| = 1$. We will show that

$$(ST)^n = \lambda^{\frac{n(n+1)}{2}} T^n S^n \tag{5}$$

for any positive integer n, using induction on n. Since $ST = \lambda TS$, equation (5) is true for $n = 1$. If (5) holds for $n = k$, then

$$\begin{aligned}(ST)^{k+1} &= (ST)(ST)^k = \lambda^{\frac{k(k+1)}{2}}(ST^{k+1})S^k \\ &= \lambda^{\frac{k(k+1)}{2}}(\lambda^{k+1}T^{k+1}S)S^k = \lambda^{\frac{(k+1)(k+2)}{2}}T^{k+1}S^{k+1}.\end{aligned}$$

Hence (5) holds for all positive integers n. Since $|\lambda| = 1$, it follows from (5) that

$$r(ST) = \lim_{n\to\infty} \|(ST)^n\|^{\frac{1}{n}} \leq \lim_{n\to\infty} \|S^n\|^{\frac{1}{n}} \lim_{n\to\infty} \|T^n\|^{\frac{1}{n}} = r(S)r(T).$$

Since $TS = \frac{1}{\lambda}ST = \overline{\lambda}ST$, we also obtain $r(TS) \leq r(S)r(T)$.

(ii) Since $ST = \lambda TS$, we get that $S(z-T) = zS - ST = zS - \lambda TS = (z - \lambda T)S$ for $z \in \mathbb{C}$. Since $\mathrm{dist}(\frac{z}{\lambda}, \sigma(T)) > 0$ for any $z \in \rho(T)$, the inclusion $\lambda\sigma(T) \subset \sigma(T)$ holds, and so $(z - \lambda T)^{-1}S = S(z - T)^{-1}$ for any $z \in \rho(T)$. Let Γ be a closed curve surrounding $\sigma(T)$. Then

$$\begin{aligned}Sf(T) &= S\Big[\frac{1}{2\pi i}\int_\Gamma f(z)(z-T)^{-1}dz\Big] = \frac{1}{2\pi i}\int_\Gamma f(z)(z-\lambda T)^{-1}Sdz \\ &= \Big[\frac{1}{2\pi i}\int_\Gamma f(z)(z-\lambda T)^{-1}dz\Big]S = f(\lambda T)S,\end{aligned}$$

which completes the proof. □

Recall that an operator T in $\mathcal{L}(\mathcal{H})$ is called *normaloid* if $\|T\| = r(T)$. An operator $T \in \mathcal{L}(\mathcal{H})$ is said to belong to *class A* if $|T^2| \geq |T|^2$. Every operator which belongs to class A is normaloid, and hyponormal operators belong to class A (see [7]).

Corollary 3.10. *Let S and T be operators in $\mathcal{L}(\mathcal{H})$ such that $ST = \lambda TS$ for some $\lambda \in \mathbb{C}$ and ST belongs to class A. If S or T is quasinilpotent, then $ST = TS = 0$.*

Proof. The products ST and TS are quasinilpotent by Theorem 3.9 (i). Since $TS = \frac{1}{\lambda}ST$ as well as ST belong to class A and every class A operator is normaloid, we get that $ST = TS = 0$. □

We next provide spectral properties of λ-commuting operators.

Theorem 3.11. *Suppose that $S, T \in \mathcal{L}(\mathcal{H})$ satisfy $ST = \lambda TS$ for some $\lambda \in \mathbb{C}$. For $\sigma_\Delta \in \{\sigma_p, \sigma_{ap}, \sigma_{le}\}$, the following assertions hold:*
(i) either $0 \in \sigma_\Delta(T)$ or else $\lambda\sigma_\Delta(S) \subset \sigma_\Delta(S)$;
(ii) either $0 \in \sigma_\Delta(S)$ or else $\sigma_\Delta(T) \subset \lambda\sigma_\Delta(T)$.

Proof. We only consider the left essential spectrum; the proofs for the others are similar.

(i) Suppose that $0 \notin \sigma_{le}(T)$. If $\alpha \in \lambda\sigma_{le}(S) = \sigma_{le}(\lambda S)$, choose a sequence $\{x_n\}$ of unit vectors in $\mathcal{H}$ such that $x_n \to 0$ weakly and $\lim_{n\to\infty} \|(\lambda S - \alpha)x_n\| = 0$. Then $\lim_{n\to\infty} \|(S - \alpha)Tx_n\| = \lim_{n\to\infty} \|T(\lambda S - \alpha)x_n\| = 0$. Note that $\lim_{n\to\infty} \|Tx_n\| \neq 0$ since $0 \notin \sigma_{le}(T)$. This implies that $\alpha \in \sigma_{le}(S)$. Thus $\lambda\sigma_{le}(S) \subset \sigma_{le}(S)$. Hence, we conclude that $\lambda\sigma_{le}(S) \subset \sigma_{le}(S)$ or $0 \in \sigma_{le}(T)$.

(ii) Let $0 \notin \sigma_{le}(S)$. If $\alpha \in \sigma_{le}(T)$, then there exists a sequence $\{x_n\}$ of unit vectors in $\mathcal{H}$ such that $x_n \to 0$ weakly and $\lim_{n\to\infty} \|(T-\alpha)x_n\| = 0$. Since $ST = \lambda TS$, we have $\lim_{n\to\infty} \|(\lambda T-\alpha)Sx_n\| = \lim_{n\to\infty} \|S(T-\alpha)x_n\| = 0$. Since $0 \notin \sigma_{le}(S)$, the sequence $\{Sx_n\}$ does not converge to 0 in norm, and so $\alpha \in \sigma_{le}(\lambda T) = \lambda\sigma_{le}(T)$. Therefore $\sigma_{le}(T) \subset \lambda\sigma_{le}(T)$. □

Applying the proof of Theorem 3.11, we infer that $T\ker(S-\mu) \subset \ker(S-\lambda\mu)$ and $S\ker(T-\mu) \subset \ker(\lambda T-\mu)$ for each $\mu \in \mathbb{C}$. Hence, $\ker(S)$ and $\ker(T)$ are common invariant subspaces for S and T.

Corollary 3.12. *For $\sigma_\Delta \in \{\sigma_p, \sigma_{ap}, \sigma_{le}\}$, the following assertions hold:*
(i) If S and T are (λ, μ)-commuting operators in $\mathcal{L}(\mathcal{H})$ such that $0 \notin \sigma_\Delta(S) \cup \sigma_\Delta(S^*)$ and $\mu \neq 0$, then

$$\mu^{-1}\sigma_\Delta(T) \subset \sigma_\Delta(T) \subset \lambda\sigma_\Delta(T).$$

(ii) If $S \in \mathcal{L}(\mathcal{H})$ is normal and T is any operator in $\mathcal{L}(\mathcal{H})$ such that $ST = \lambda TS$ for some λ with $|\lambda| = 1$ and $0 \notin \sigma_\Delta(S)$, then

$$\sigma_\Delta(T) = \lambda\sigma_\Delta(T).$$

Proof. (i) Since $ST = \lambda TS$ and $S^*T = \mu TS^*$, we know from Theorem 3.11 that $\sigma_\Delta(T) \subset \lambda\sigma_\Delta(T)$ and $\sigma_\Delta(T) \subset \mu\sigma_\Delta(T)$, which completes the proof.

(ii) Since S is normal, the operators S and T are $(\lambda, \overline{\lambda})$-commuting. In addition, since $\|Sx\| = \|S^*x\|$, we infer that $0 \notin \sigma_\Delta(S^*)$ as well as $0 \notin \sigma_\Delta(S)$. According to (i), we get that $(\overline{\lambda})^{-1}\sigma_\Delta(T) \subset \sigma_\Delta(T) \subset \lambda\sigma_\Delta(T)$. Since $|\lambda| = 1$, we have $\overline{\lambda}^{-1} = \lambda$, and so it follows that $\sigma_\Delta(T) = \lambda\sigma_\Delta(T)$. □

If λ is a root of unity, then the inclusions in Theorem 3.11 become equalities, as follows:

Corollary 3.13. *Let $S, T \in \mathcal{L}(\mathcal{H})$ satisfy that $ST = \lambda TS$ where λ is a root of unity. Then the following statements hold for $\sigma_\Delta \in \{\sigma_p, \sigma_{ap}, \sigma_{le}\}$:*
(i) If $0 \notin \sigma_\Delta(T)$, then $\sigma_\Delta(S) = \lambda\sigma_\Delta(S)$;
(ii) If $0 \notin \sigma_\Delta(S)$, then $\sigma_\Delta(T) = \lambda\sigma_\Delta(T)$.

Proof. Assume that $0 \notin \sigma_\Delta(T)$. If $\lambda^k = 1$, then it follows from Proposition 3.11 that $\sigma_\Delta(S) \supset \lambda\sigma_\Delta(S) \supset \lambda^k\sigma_\Delta(S) = \sigma_\Delta(S)$. Thus $\sigma_\Delta(S) = \lambda\sigma_\Delta(S)$. Similarly, one can obtain the result (ii). □

Recall that $T \in \mathcal{L}(\mathcal{H})$ is said to be an *m-isometry* if $\sum_{j=0}^{m}(-1)^j\binom{m}{j}T^{*j}T^j = 0$, where m is a positive integer. In [1], it turned out that every m-isometry has approximate point spectrum contained in the unit circle.

Corollary 3.14. *Suppose that S and T are operators in $\mathcal{L}(\mathcal{H})$ such that $ST = \lambda TS$ for some $\lambda \in \mathbb{C}$. Then the following assertions hold:*
(i) If $0 \notin \sigma_{ap}(T)$, then $\sigma_{ap}(S) = \{0\}$ or $|\lambda| \leq 1$.
(ii) If $0 \notin \sigma_{ap}(S)$, then $\sigma_{ap}(T) = \{0\}$ or $|\lambda| \geq 1$. Hence, if $0 \notin \sigma_{ap}(S) \cup \sigma_{ap}(T)$, then $|\lambda| = 1$.
(iii) If $|\lambda| \neq 1$ and S is an m-isometry for some positive integer m, then $0 \in \sigma_p(T)$.

Proof. (i) Let $0 \notin \sigma_{ap}(T)$. If $\sigma_{ap}(S) \neq \{0\}$, select $\alpha \in \sigma_{ap}(S) \setminus \{0\}$. Since $\lambda\sigma_{ap}(S) \subset \sigma_{ap}(S)$ by Theorem 3.11, we get that $\lambda^k\alpha \in \sigma_{ap}(S)$ for each nonnegative integer k. Since $\sigma_{ap}(T)$ is compact, we have $|\lambda| \leq 1$.

(ii) Assume that $0 \notin \sigma_{ap}(S)$ and there is $\alpha \in \sigma_{ap}(T) \setminus \{0\}$. Then Theorem 3.11 implies that $\alpha \in \lambda^k\sigma_{ap}(T)$ for each nonnegative integer k. Write $\alpha = \lambda^k\beta_k$ with $\{\beta_k\} \subset \sigma_{ap}(T)$. Since $\{\beta_k\}$ is bounded and $\alpha \neq 0$, we see that $|\lambda| \geq 1$.

(iii) Since $\sigma_{ap}(S) \subset \{z \in \mathbb{C} : |z| = 1\}$ and $|\lambda| \neq 1$, we have $\lambda\sigma_p(S) \not\subset \sigma_p(S)$. Accordingly, $0 \in \sigma_p(T)$ from Theorem 3.11. □

We now consider local spectral properties of λ-commuting operators. Let $H_0(T) = \{x \in \mathcal{H} : \lim_{n\to\infty} \|T^n x\|^{\frac{1}{n}} = 0\}$ be the *quasinilpotent part* of $T \in \mathcal{L}(\mathcal{H})$.

Proposition 3.15. *Let $S, T \in \mathcal{L}(\mathcal{H})$. If $ST = \lambda TS$ for some $\lambda \in \mathbb{C}$, then the following statements hold:*
(i) $\sigma_S(Tx) \subset \lambda\sigma_S(x)$ and $\lambda\sigma_T(Sx) \subset \sigma_T(x)$ for all $x \in \mathcal{H}$.
(ii) $TH_S(F) \subset H_S(\lambda F)$ for any subset F of $\mathbb{C}$.
(iii) If $\lambda \neq 0$, then $SH_T(\lambda F) \subset H_T(F)$ for any subset F of $\mathbb{C}$.
(iv) If $\lambda \neq 0$, then $H_0(S)$ is invariant for T.

Proof. (i) Let $x \in \mathcal{H}$ be given. If $z_0 \in \rho_{\lambda S}(x)$, select a neighborhood D of z_0 and an analytic function $f : D \to \mathcal{H}$ such that $(\lambda S - z)f(z) \equiv x$ on D. Using the identity $ST = \lambda TS$, we get that $(S - z)Tf(z) = T(\lambda S - z)f(z) = Tx$ for all $z \in D$, and so $z_0 \in \rho_S(Tx)$. Thus $\rho_{\lambda S}(x) \subset \rho_S(Tx)$, or $\sigma_S(Tx) \subset \sigma_{\lambda S}(x)$.

If $z_0 \in \rho_T(x)$, choose an analytic function $f : D \to \mathcal{H}$ on a neighborhood D of z_0 such that $(T - z)f(z) \equiv x$ on D, implying that $(\lambda T - z)Sf(z) = S(T - z)f(z) = Sx$ for all $z \in D$. Then $z_0 \in \rho_{\lambda T}(Sx)$. Hence $\sigma_{\lambda T}(Sx) \subset \sigma_T(x)$.

Applying [13, Theorem 3.3.8], we see that $\sigma_S(Tx) \subset \lambda\sigma_S(x)$ and $\lambda\sigma_T(Sx) \subset \sigma_T(x)$.

(ii) Let F be any subset of $\mathbb{C}$. If $x \in H_S(F)$, then $\sigma_S(Tx) \subset \lambda\sigma_S(x) \subset \lambda F$ by (i), which means that $Tx \in H_S(\lambda F)$. Accordingly, we have $TH_S(F) \subset H_S(\lambda F)$.

(iii) If $x \in H_T(\lambda F)$, then $\lambda\sigma_T(Sx) \subset \sigma_T(x) \subset \lambda F$ from (i), and so either $\lambda = 0$ or else $SH_T(\lambda F) \subset H_T(F)$.

(iv) Since $\|S^nTx\|^{\frac{1}{n}} = \|\lambda^n TS^n x\|^{\frac{1}{n}} \le |\lambda|\|T\|^{\frac{1}{n}}\|S^n x\|^{\frac{1}{n}}$ for $x \in \mathcal{H}$ and $\lambda \neq 0$, we observe that $x \in H_0(S)$ implies $Tx \in H_0(S)$. Thus $H_0(S)$ is invariant for T. □

Corollary 3.16. *Suppose that $S, T \in \mathcal{L}(\mathcal{H})$ are λ-commuting where λ is a root of unity with order k. If λ is a root of unity with order k and S has Dunford's property (C), then $H_S(F)$ is a common invariant subspace of S and T^k, where F is any closed subset of $\mathbb{C}$.*

Proof. If S has Dunford's property (C), then $H_S(F)$ is a hyperinvariant subspace for S (see [2], [6], or [13]). Since

$$T^k H_S(F) \subset T^{k-1}H_S(\lambda F) \subset \cdots \subset TH_S(\lambda^{k-1}F) \subset H_S(\lambda^k F) = H_S(F)$$

by Proposition 3.15. Hence $H_S(F)$ is invariant under T^k. □

Remark. Assume $S \in \mathcal{L}(\mathcal{H})$ is hyponormal. It turned out in [17, Theorem 2.4] that if S is λ-commuting with $T \in \mathcal{L}(\mathcal{H})$ and $\sigma(ST)$ consists of k distinct nonzero elements, then $\lambda^k = 1$. Since S has Bishop's property (β) (see [15]), it has Dunford's property (C), and so $H_S(F)$ is a common invariant subspace of S and T^k by Corollary 3.16.

Let $H^2 = H^2(\mathbb{D})$ be the canonical Hardy space of the open unit disk $\mathbb{D}$, and let H^∞ be the space of bounded functions in H^2. For an analytic map φ from $\mathbb{D}$ into itself and $u \in \mathbb{D}$, the *weighted composition operator* $W_{f,\varphi} : H^2 \to H^2$ is defined by $W_{u,\varphi}h = u \cdot (h \circ \varphi)$. In particular, we say that $C_\varphi := W_{1,\varphi}$ is a *composition operator*. We next prove that the operators λ-commuting with the unilateral shift U on H^2 given by $(Uf)(z) = zf(z)$ are weighted composition operators.

Theorem 3.17. *Let U be the unilateral shift on H^2 given by $(Uf)(z) = zf(z)$, and $\lambda \in \partial\mathbb{D}$. Assume that $S \in \mathcal{L}(H^2)$. Then $SU = \lambda US$ if and only if $S = W_{u,\lambda z}$ for some $u \in H^\infty$.*

Proof. We first note that $(W_{u,\lambda z}Uh)(z) = \lambda zu(z)h(\lambda z) = \lambda(UW_{u,\lambda z}h)(z)$ for all $h \in H^2$ and $z \in \mathbb{D}$, namely $W_{u,\lambda z}$ is λ-commuting with U.

If $S \in \mathcal{L}(H^2)$ satisfies $SU = \lambda US$, then $Sz^j = SU^j(1) = \lambda^j U^j S(1) = \lambda^j z^j S(1)$ for $j = 0, 1, 2, \cdots$. Hence, it is easy to see that if p is a polynomial, then $Sp = uC_{\lambda z}p$ where $u = S(1) \in H^2$. Let $f \in H^2$. Choosing a sequence $\{p_n\}$ of polynomials such that $p_n \to f$ in H^2 as $n \to \infty$, we observe that $p_n \to f$ uniformly on compact sets in $\mathbb{D}$; indeed,

$$|p_n(\alpha) - f(\alpha)| = |\langle p_n - f, K_\alpha\rangle| \le \|p_n - f\|\|K_\alpha\| = \frac{\|p_n - f\|}{\sqrt{1 - |\alpha|^2}}$$

for $\alpha \in \mathbb{D}$, where $K_\alpha(z) = \frac{1}{1-\overline{\alpha}z}$ is the reproducing kernel of H^2 at the point α. So, $(Sp_n)(z) = u(z)p_n(\lambda z) \to u(z)f(\lambda z) = u(z)(C_{\lambda z}f)(z)$ pointwise on $\mathbb{D}$. Since $Sp_n \to Sf$ in H^2, we obtain that $Sf = uC_{\lambda z}f$.

Finally, we want to show that $u \in H^\infty$. For a positive integer n, consider the set $E_n = \{\zeta \in \partial\mathbb{D} : |u(\zeta)| > n\}$, and define a function ω_n on $\partial\mathbb{D}$ by $\omega(\zeta) = \chi_n(\overline{\lambda}\zeta)$ where χ_n is the characteristic function on E_n. Then

$$\|S\omega_n\|^2 = \frac{1}{2\pi}\int_{\partial\mathbb{D}} |u(\zeta)|^2|\chi_n(\zeta)|^2 dm(\zeta) \ge \frac{n^2}{2\pi}\int_{E_n} dm(\zeta) = n^2 m(E_n)$$

for all positive integers n. Since $C_{\lambda z}$ is unitary with $C^*_{\lambda z} = C_{\overline{\lambda} z}$, we obtain that $\|\omega_n\| = \|C_{\lambda z}\omega_n\| = \|\chi_n\| = m(E_n)$ for all positive integers n. Thus $\|S\omega_n\| \geq n\|\omega_n\|$ for all positive integers n. If $m(E_n) \neq 0$ for infinitely many n, then $\|S\| \geq n$ for infinitely many n, which is a contradiction. Therefore, $m(E_n) = 0$ for all but finitely many n, i.e., $u \in H^\infty$. Hence $S = W_{u,\lambda z}$. □

For a bounded sequence $\{\alpha_n\}_{n=0}^\infty$ in $\mathbb{C}$, a *weighted shift* on $\mathcal{H}$ with weights $\{\alpha_n\}$ is an operator T such that $Te_n = \alpha_n e_{n+1}$ for $n \geq 0$, where $\{e_n\}_{n=0}^\infty$ denotes an orthonormal basis for $\mathcal{H}$. We next give some examples for λ-commuting operators.

Example 3.18. *Consider the weighted shift S, so-called the Bergman shift, determined by the weights $\{\sqrt{\frac{n+1}{n+2}}\}_{n=0}^\infty$. Then S is hyponormal. Let T be any weighted shift with positive weights $\{\beta_n\}$, and let $\lambda \in \mathbb{C} \setminus \{0\}$. Then $ST = \lambda TS$ if and only if $\beta_{n+1} = \frac{n+2}{\lambda\sqrt{(n+1)(n+3)}}\beta_n$ for $n \geq 0$, that is, $\beta_n = \frac{1}{\lambda^n}\sqrt{\frac{2(n+1)}{n+2}}\beta_0$ for $n \geq 0$.*

For a positive integer $n > 1$, define J_r and J_l on $\bigoplus_1^n \mathcal{H}$ by

$$J_r = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ I & 0 & \cdots & 0 & 0 \\ 0 & I & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & I & 0 \end{pmatrix} \text{ and } J_l = \begin{pmatrix} 0 & I & 0 & \cdots & 0 \\ 0 & 0 & I & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & I \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

Theorem 3.19. *Let $T \in \mathcal{L}(\bigoplus_1^n \mathcal{H})$ and $\lambda \in \mathbb{C} \setminus \{0\}$. Then the following statements hold:*
(i) If $TJ_r = \lambda J_r T$, then T has Bishop's property (β) if and only if P_1TP_1 has Bishop's property (β) where P_1 denotes the orthogonal projection of $\bigoplus_1^n \mathcal{H}$ onto $\mathcal{H}\oplus\{0\}\oplus\cdots\oplus\{0\}$.
(ii) If $TJ_l = \lambda J_l T$, then T has Bishop's property (β) if and only if P_nTP_n has Bishop's property (β) where P_n denotes the orthogonal projection of $\bigoplus_1^n \mathcal{H}$ onto $\{0\}\oplus\cdots\oplus\{0\}\oplus\mathcal{H}$.

Proof. (i) Suppose that $TJ_r = \lambda J_r T$. Then we can express T as

$$T = \begin{pmatrix} T_1 & 0 & \cdots & \cdots & 0 & 0 \\ T_2 & \lambda T_1 & \ddots & \ddots & 0 & 0 \\ T_3 & \lambda T_2 & \lambda^2 T_1 & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ T_{n-1} & \lambda T_{n-2} & \ddots & \ddots & \lambda^{n-2}T_1 & 0 \\ T_n & \lambda T_{n-1} & \cdots & \cdots & \lambda^{n-2}T_2 & \lambda^{n-1}T_1 \end{pmatrix}$$

where $\{T_j\}_{j=1}^n \subset \mathcal{L}(\mathcal{H})$. Hence, it suffices to show that T has Bishop's property (β) if and only if T_1 does. If T_1 has Bishop's property (β), take any sequence $\{f_k\}$ of $\bigoplus_1^n \mathcal{H}$-valued functions analytic on an open set G in $\mathbb{C}$ such that $\|(T - z)f_k(z)\| \to 0$ uniformly on compact sets in G as $k \to \infty$. Set $f_k(z) = \bigoplus_{j=1}^n f_{k,j}(z)$ where each $f_{k,j} : G \to \mathcal{H}$ is an analytic function. Then

$$\begin{cases} \lim_{k\to\infty} \|(T_1 - z)f_{k,1}(z)\| = 0 \\ \lim_{k\to\infty} \|\sum_{j=1}^{\ell-1} \lambda^{j-1}T_{\ell-j+1}f_{k,j}(z) + (\lambda^{\ell-1}T_1 - z)f_{k,\ell}(z)\| = 0, \ 1 < \ell \leq n, \end{cases} \tag{6}$$

uniformly on compact sets in G. Since T_1 has Bishop's property (β), we get that $\lim_{k\to\infty} \|f_{k,1}(z)\| = 0$ uniformly on compact sets in G. If $\lim_{k\to\infty} \|f_{k,j}(z)\| = 0$ uniformly on compact sets in G for all $j = 1, 2, \cdots, \ell - 1$, then (6) implies that $\lim_{k\to\infty}\|(\lambda^{\ell-1}T_1 - z)f_{k,\ell}(z)\| = 0$ uniformly on compact sets in G. Since T_1 has Bishop's property (β), we obtain that $\lim_{k\to\infty}\|f_{k,\ell}(z)\| = 0$ uniformly on compact sets in G. By induction, $\{f_{k,j}\}$ converges to 0 uniformly on compact sets in G for each $j = 1, 2, \cdots, n$. Therefore, T has Bishop's property (β).

Conversely, assume T has Bishop's property (β) and $\{f_{k,1}\}$ is a sequence of analytic functions on an open set G for which $\lim_{k\to\infty}\|(T_1 - z)f_{k,1}(z)\| = 0$ uniformly on compact sets in G. Setting the function $g_k(\zeta) := 0\oplus\cdots\oplus 0\oplus f_{k,1}(\lambda^{1-n}\zeta)$ analytic for $\zeta \in \lambda^{n-1}G$, we have $\lim_{k\to\infty}\|(T-\zeta)g_k(\zeta)\| = 0$ uniformly on compact sets in $\lambda^{n-1}G$. Hence $\lim_{k\to\infty}\|g_k(\zeta)\| = 0$ uniformly on compact sets in $\lambda^{n-1}G$, meaning that $\lim_{k\to\infty}\|f_{k,1}(z)\| = 0$ uniformly on compact sets in G. Thus T_1 has Bishop's property (β).

(ii) Since $TJ_l = \lambda J_l T$ if and only if

$$T = \begin{pmatrix} \lambda^{n-1}T_n & \lambda^{n-2}T_{n-1} & \cdots & \cdots & \lambda T_2 & T_1 \\ 0 & \lambda^{n-2}T_n & \cdots & \cdots & \lambda T_3 & T_2 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \ddots & \ddots & \lambda T_{n-1} & T_{n-2} \\ 0 & 0 & \cdots & \cdots & \lambda T_n & T_{n-1} \\ 0 & 0 & \cdots & \cdots & 0 & T_n \end{pmatrix}$$

where $\{T_j\}_{j=1}^n \subset \mathcal{L}(\mathcal{H})$, we derive (ii) in a similar fashion to the proof of (i). □

For an operator $T \in \mathcal{L}(\mathcal{H})$, a T-invariant subspace $\mathcal{M}$ is said to be a *spectral maximal* space of T if $\mathcal{M}$ contains any T-invariant subspace $\mathcal{N}$ with $\sigma(T|_{\mathcal{N}}) \subset \sigma(T|_{\mathcal{M}})$.

Corollary 3.20. *For $T \in \mathcal{L}(\bigoplus_1^n \mathcal{H})$, suppose that one of the following conditions holds:*
(i) $TJ_r = \lambda J_r T$ for some $\lambda \in \mathbb{C}\setminus\{0\}$ and P_1TP_1 has Bishop's property (β);
(ii) $TJ_l = \lambda J_l T$ for some $\lambda \in \mathbb{C}\setminus\{0\}$ and P_nTP_n has Bishop's property (β)
where P_1 and P_n are the orthogonal projections given in Theorem 3.19. Then T has Dunford's property (C) and the single-valued extension property. Moreover, $H_T(F)$ is a spectral maximal space of T for any closed subset F of $\mathbb{C}$.

Proof. Since T has Bishop's property (β) due to Theorem 3.19, it has Dunford's property (C) and the single-valued extension property. Then $H_T(F)$ is spectral maximal from [13, Proposition 1.2.20]. □

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